

Dynamic Indeterminacy and Welfare in Credit Economies*

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Abstract

We characterize the equilibrium set and constrained-efficient allocations of a pure credit economy with limited commitment under both pairwise and centralized meetings. We show that the set of equilibria derived under "not-too-tight" solvency constraints (Alvarez and Jermann, 2000; Gu et al., 2013b) is of measure zero in the whole set of Perfect Bayesian Equilibria. There exist a continuum of endogenous credit cycles of any periodicity and a continuum of sunspot equilibria, irrespective of the assumed trading mechanism. Moreover, any equilibrium allocation of the corresponding monetary economy is an equilibrium allocation of the pure credit economy but the reverse is not true. On the normative side, we establish conditions under which constrained-efficient allocations cannot be implemented with "not-too-tight" solvency constraints.

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1 Introduction

The inability of individuals to commit to honor their future obligations is a key friction of decentralized economies that jeopardizes the Arrow-Debreu apparatus based on promises to deliver goods at different dates and in different states. Economies with limited commitment have been studied predominantly in monetary theory. Stark examples are pure currency economies where anonymity and lack of commitment make credit infeasible. Arguably, pure currency economies have become less relevant due to advances in record-keeping technologies that facilitate the use of credit. Yet, monitoring technologies do not purge economies from the limited commitment problem—they do not make individuals entirely trustworthy. Hence, this paper investigates the full set of equilibria and constrained-efficient allocations of a pure credit economy taking seriously the limited commitment friction.¹

There are two recent contributions, one normative and one positive, that shed some light on economies with limited commitment. On the normative side, Alvarez and Jermann (2000), AJ thereafter, establish a Second Welfare Theorem for a pure exchange, one-good economy where agents are subject to endowment shocks and have limited commitment—a special case of the environment in Kehoe and Levine (1993). They prove that constrained-efficient allocations can be implemented by competitive trades subject to "not-too-tight" solvency constraints. These constraints specify that in every period agents can issue the maximum amount of debt that is incentive-compatible with no default, thereby allowing as much risk sharing as possible. From a positive perspective Gu et al. (2013b), GMMW thereafter, study a pure credit economy subject to the same "not-too-tight" solvency constraints and show the possibility of endogenous credit cycles.² The conditions for such cycles, however, are much more stringent than the ones in pure currency economies.³

The objective of this paper is to revisit these two key insights—the implementation of constrained-efficient allocations and the existence of endogenous credit cycles—in the context of a pure credit economy with limited commitment. Our main contributions are twofold. On the positive side, we give a complete characterization of the (perfect Bayesian) equilibrium set of a pure credit economy. On the normative side, we characterize constrained-efficient allocations of economies with pairwise meetings and competitive economies with large meetings.

The pure credit economy we consider features random matching—in pairwise meetings or in large

¹In Wicksell's (1936) words, <<a thorough analysis of this purely imaginary case seems to me to be worth while, for it provides a precise antithesis to the equally imaginary case of a pure cash system, in which credit plays no part whatever.>>

²In a related paper Bloise et al. (2013) prove indeterminacy of competitive equilibrium in sequential economies under "not-too-tight" solvency constraints. While they do not focus on endogenous credit cycles they show that for any value of social welfare in between autarchy and constrained optimality, there exists an equilibrium attaining that value.

³For instance, Lagos and Wright (2003) find that monetary economies can generate endogenous cycles under monotone trading mechanisms, such as buyers-take-all bargaining. Under the same trading mechanisms, Gu et al. (2013b) do not find any cycle.

groups—and incorporates intertemporal gains from trade that can be exploited with one-period debt contracts. In the absence of public record keeping, the environment corresponds to the New-Monetarist framework of Lagos and Wright (2005) so that one can easily compare equilibrium allocations in credit and monetary economies. In the presence of a public record-keeping technology the environment is mathematically equivalent to the one in GMMW.⁴

We start with a simple mechanism where the borrower in each bilateral match sets the terms of the loan contract unilaterally, which allows us to analyze the economy as a standard infinitely-repeated game with imperfect monitoring. If we impose the AJ "not-too-tight" solvency constraints exogenously—which amounts to restricting strategies and beliefs such that any form of default is punished with permanent autarky—then there is a unique active steady-state equilibrium and no equilibrium with endogenous cycles. When we look for all perfect Bayesian equilibria, we find a continuum of steady-state equilibria, a continuum of periodic equilibria of any periodicity, and much more. Each equilibrium can be reduced to a sequence of debt limits, where the debt limit in a period specifies the amount that agents can be trusted to repay. Moreover, there is a wide variety of outcomes: in some credit cycle equilibria debt limits are binding in all periods, in other equilibria they are never binding, or they bind periodically. These results are robust to the choice of the mechanism to determine the terms of the loan contract—Nash or proportional bargaining, or even competitive pricing if agents meet in large groups.

Figure 1 plots the set of 2-period cycles in the space of debt limits for a representative example under competitive pricing (example 3A in GMMW). The horizontal axis gives the debt limit (expressed in terms of the good used for repayment) in even periods and the vertical axis specifies the debt limit in odd periods. The blue area represents the continuum of 2-period cycle equilibria. Under "not-too-tight" solvency constraints there is a unique active steady state, marked by a green star, and two strict 2-period cycles marked by red stars.

The multiplicity of credit equilibria captures the basic notion that trust is a self-fulfilling phenomenon. To that extent, and following Mailath and Samuelson (2006, p.9), "we consider multiple equilibria a virtue." But this multiplicity does not imply that everything goes. Fundamentals, including preferences and market structure, do matter for an outcome to be consistent with an equilibrium. We show that the set of credit-cycle equilibria expands (in Figure 1 the blue area expands outwards) as trading frictions are reduced, agents are more patient, and borrowers have more bargaining power (in the version of the model with bargaining).

We also show that for a given trading mechanism the set of equilibrium outcomes of a pure monetary economy (with fiat money but no record keeping) is a strict subset of the outcomes of a pure credit econ-

⁴As we discuss later in details, there are differences regarding the timing of production that are inconsequential.

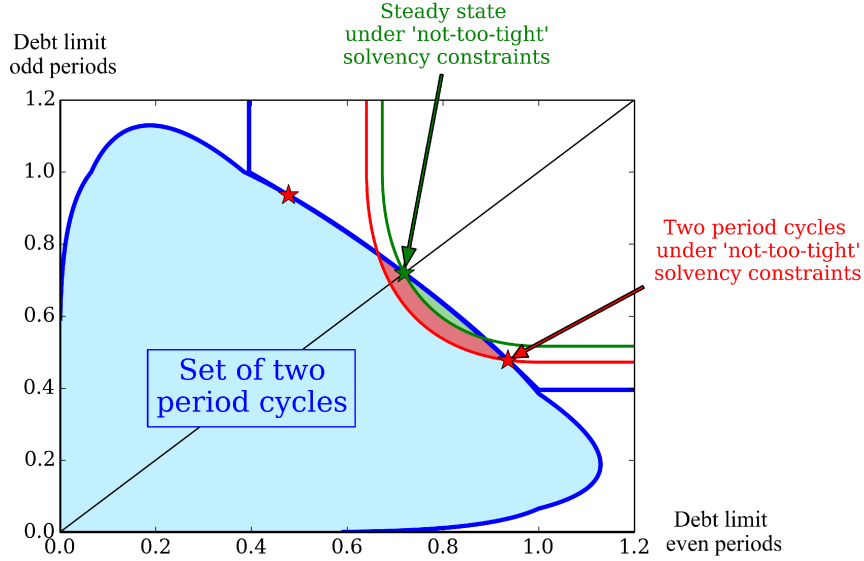


Figure 1: A representative example: Set of 2-period cycles under competitive pricing. Green star: steady state under "not-too-tight" solvency constraints. Red star: cycle under "not-too-tight" constraints. The red and green curves are society's indifference curves.

omy (with record keeping but no fiat money). Hence all dynamic allocations of monetary economies (e.g., cycles, chaos) also exist in credit environments. The reverse is not true. There are equilibrium outcomes of the pure credit economy that cannot be sustained as equilibrium outcomes of the pure currency economy. For instance, there are equilibria where credit and output shut down periodically—which is ruled out by backward-induction in monetary economies.

In order to understand why the equilibrium set for credit economies—the blue area in Figure 1—is so vastly larger than the one found in GMMW—the green and red stars—it is worth recalling that the AJ "not-too-tight" solvency constraints were meant to provide a way to decentralize constrained-efficient allocations in one particular economy with limited commitment. Such constraints are not warranted for positive analysis. We will argue later that they are also restrictive for normative analysis in our environment. We avoid arbitrary restrictions on the set of equilibrium outcomes by working with simple strategies that punish both default and excessive lending (i.e., lending in excess of the amount that is deemed trustworthy along the equilibrium path). We show that such simple strategies implement the full set of outcomes of perfect Bayesian equilibria (subject to mild restrictions).

In terms of normative analysis we determine the constrained-efficient allocations under different assumptions: we consider both economies where agents meet in pairs and economies with large-group meetings

and, following the formulation in GMMW, we vary preferences to affect an agent's temptation to renege on his debt. If agents are matched bilaterally then the constrained-efficient allocation is implemented with take-it-or-leave-it offers by buyers and "not-too-tight" solvency constraints, which generalizes the AJ welfare theorem to economies with pairwise meetings. Under competitive pricing the "not-too-tight" solvency constraints are suboptimal when the temptation to renege is low. In this case the constrained-efficient allocation is non-stationary and it features slack participation constraints (i.e., solvency constraints that are "too-tight", according to the terminology of AJ) for all periods except the initial one. Slack participation constraints are socially optimal due to a pecuniary externality according to which a decrease in debt limits generates lower contemporaneous output, which reduces prices and increases gains from trade for buyers/borrowers, thereby relaxing borrowing constraints in earlier periods. Moreover, there exists a continuum of credit cycles that yield a higher social welfare than those with "not-too-tight" solvency constraints. We illustrate this point in Figure 1 by representing with red and green areas the set of 2-period cycles that dominate the equilibria obtained under "not-too-tight" solvency constraints.

1.1 Related literature

We adopt an environment similar to the pure currency economy of Lagos and Wright (2005) and Rocheteau and Wright (2005), but we replace currency with a public record-keeping technology, as in Sanches and Williamson (2010, Section 4). The first part of the paper, on the characterization of the equilibrium set (Sections 3 and 4), extends the analysis of Sanches-Williamson which focuses on steady states and of GMMW which focuses on cycles. In both cases, the equilibrium notion imposes the "not-too-tight" solvency constraints of AJ. Instead, we present our model as a repeated game with imperfect monitoring with few restrictions on strategies and beliefs (the same restrictions typically imposed on equilibria of pure currency economies).⁵ In addition, we consider both stationary and non-stationary equilibria (including endogenous cycles and sunspots), various trading mechanisms (ultimatum games, axiomatic bargaining solutions, competitive pricing), and we conduct a normative analysis to determine constrained-efficient allocations. Our methods to characterize equilibrium outcomes (Sections 3 and 5) are related but different from the ones used by Abreu (1988) and Abreu et al. (1990) as our stage game has an extensive form and only buyers/borrowers are (imperfectly) monitored.

Kocherlakota (1998) shows that the set of implementable outcomes of monetary economies is a subset of the implementable outcomes of pure credit economies.⁶ We find a similar result, but in contrast to

⁵Repeated games where agents are matched bilaterally and at random and change trading partners over time are studied in Kandori (1992) and Ellison (1994). A thorough review of the literature is provided by Mailath and Samuelson (2006).

⁶Hellwig and Lorenzoni (2009) study an environment similar to Alvarez and Jermann (2000) and show that the set of equilibrium allocations with self-enforcing private debt is equivalent to the allocations that are sustained with money.

Kocherlakota, we take the trading mechanism as given and we do not restrict outcomes to stationary ones. Moreover, we establish in Proposition 9 that the condition to implement the first best allocation in the pure credit economy with record keeping is identical to the one in the pure currency economy with no record keeping (Hu et al., 2009).

Our paper is part of the literature on limited commitment in macroeconomics. Seminal contributions on risk sharing in endowment economies where agents lack commitment include Kehoe and Levine (1993), Kocherlakota (1996), and A.J. Kocherlakota (1996) adopts a mechanism design approach in a two-agent economy with a single good. Our Section 5.1 on constrained-efficient allocations under pairwise meetings is related with some key differences: we study a two-good production economy where a continuum of agents search for new partners every period and we select the allocation that maximizes a social welfare criterion under quasi-linear preferences. Gu et al. (2013a, Section 7) has a similar environment but solves for the contract curve. In our Section 5.2 we study constrained-efficient allocations under large meetings and competitive pricing, as in Kehoe and Levine (1993) or A.J. Kehoe and Levine (1993, Section 7) conjectured that punishments based on partial exclusion might allow the implementation of socially desirable allocations.⁷ This conjecture is verified in our economy with the caveat that the extent of exclusion has to vary over time. Our normative results are also related to the Second Welfare Theorem in AJ according to which constrained-efficient allocations can be implemented with "not-too-tight" solvency constraints. We provide a necessary and sufficient condition under which this theorem applies to our environment.

2 Description of the game

Time is discrete and starts with period 0. Each date has two stages. The first stage will be referred to as DM (decentralized market) while the second stage will be referred to as CM (centralized market). There is a single, perishable good at each stage and the CM good will be taken as the numéraire. There is a continuum of agents of measure two divided evenly into a subset of buyers, $\mathbb{B}$, and a subset of sellers, $\mathbb{S}$.⁸ The labels “buyer” and “seller” refer to agents’ roles in the DM: only the sellers can produce the DM good (and hence will be lenders) and only the buyers desire DM goods (and hence will be borrowers). In the DM, a fraction $\alpha \in (0, 1]$ of buyers meet with sellers in pairs. (We consider a version of the model with large meetings later.) The CM will be the place where agents settle debts.

⁷Azariadis and Kass (2013) relaxed the assumption of permanent autarky and assumed that agents are only temporarily excluded from credit markets. Gu et al. (2013a) and GMMW allow for partial monitoring, which is formally equivalent to partial exclusion, except that the parameter governing the monitoring intensity, π , is time-invariant. Kocherlakota and Wallace (1998) consider the case of an imperfect record-keeping technology where the public record of individual transactions is updated after a probabilistic lag.

⁸The assumption of ex-ante heterogeneity among agents is borrowed from Rocheteau and Wright (2005). Alternatively, one could assume that an agent’s role in the DM is determined at random in every period without affecting any of our results.

Preferences are additively separable over dates and stages. The DM utility of a seller who produces $y \in \mathbb{R}_+$ is $-v(y)$, while that of a buyer who consumes y is $u(y)$, where $v(0) = u(0) = 0$, v and u are strictly increasing and differentiable with v convex and u strictly concave, and $u'(0) = +\infty > v'(0) = 0$. Moreover, there exists $\tilde{y} > 0$ such that $v(\tilde{y}) = u(\tilde{y})$. We denote by $y^* = \arg \max [u(y) - v(y)] > 0$ the quantity that maximizes the match surplus. The utility of consuming $z \in \mathbb{R}$ units of the numéraire good is z , where $z < 0$ is interpreted as production.⁹ Agents' common discount factor across periods is $\beta = 1/(1+r) \in (0,1)$.

With no loss in generality we restrict our attention to intra-period loans issued in the DM and repaid in the subsequent CM.¹⁰ The terms of the loan contracts are determined according to a simple protocol whereby buyers make take-it-or-leave-it offers to sellers. We describe alternative mechanisms later in the paper. Agents cannot commit to future actions. Therefore, the repayment of loans in the CM has to be self-enforcing.

There is a technology allowing loan contracts in the DM and repayments in the CM to be publicly recorded. The entry in the public record for each loan is a triple, (ℓ, x, i) , composed of the size of the loan negotiated in the DM in terms of the numéraire good, $\ell \in \mathbb{R}_+$, the amount repaid in the CM, $x \in \mathbb{R}_+$, and the identity of the buyer, $i \in \mathbb{B}$. If no credit is issued in a pairwise meeting, or if i was unmatched, the entry in the public record is $(0, 0, i)$. The record is updated at the end of each period t as follows:

$$\rho_{t+1}^i = \rho_t^i \circ (\ell_t, x_t, i), \quad (1)$$

where $\rho_0^i = (\ell_0, x_0, i)$. The list of records for all buyers, $\rho_t = \langle \rho_t^i : i \in \mathbb{B} \rangle$, is public information to all agents.¹¹ Agents have private information about their trading histories that are not recorded; in particular, if $\rho_t^i = (0, 0, i)$, then agents other than i do not know whether i was matched but his offer got rejected (in that case, the offer made is not observed either) or was unmatched. However, as discussed later, this private information plays no role in our construction of equilibria.

3 Equilibria

For each buyer $i \in \mathbb{B}$, a strategy, s^i , consists of two functions $s_t^i = (s_{t,1}^i, s_{t,2}^i)$ at each period t conditional on being matched: $s_{t,1}^i$ maps his private trading history, h_t^i , and public records of other buyers, ρ_t^{-i} , to

⁹Kehoe and Levine (1993) and AJ consider pure exchange economies. One could reinterpret our economy as an endowment economy as follows. Suppose that sellers receive an endowment $\bar{y}$ in the DM and $\bar{z}$ in the CM. Buyers have no endowment in the DM but an endowment $\bar{z}$ in the CM. The DM utility of the seller is $w(c)$ where w is a concave function with $w'(\bar{y}) = 0$. Hence, the opportunity cost to the seller of giving up y units of consumption is $v(y) = w(\bar{y}) - w(\bar{y} - y)$.

¹⁰Under linear payoffs in the CM one-period debt contracts are optimal, i.e., agents have no incentives to smooth the repayment of debt across multiple periods. This assumption will facilitate the comparison with pure monetary economies of the type studied in Lagos and Wright (2005).

¹¹We could make alternative assumptions regarding what is recorded in a match. For instance, the technology could also record the output level, y , together with the promises made by the buyer, i.e., $\rho^i = (y, \ell, x, i)$. Not surprisingly, this would expand the set of equilibrium outcomes. Moreover, we could assume that the seller only observes the record of the buyer he is matched with, ρ^i , without affecting our results.

an offer to the seller, (y_t, ℓ_t) ; $s_{t,2}^i$ maps $((h_t^i, \rho_t^{-i}), (y_t, \ell_t))$, together with the seller's response, to his CM repayment, x_t . For each seller $j \in \mathbb{S}$, a strategy, s^j , consists of one function at each period t , conditional on being matched with buyer i : s_t^j maps the seller's private trading history, h_t^j , the buyer's identity and public records, $(i, \rho_t^i, \rho_t^{-i})$, and his current offer, (y_t, ℓ_t) , to a response, *yes* or *no*. We restrict our attention to perfect Bayesian equilibria (see Osborne and Rubinstein, 1994, Definition 232.1) satisfying the following conditions:

(A1) **Public strategies.** In any DM meeting the strategies only depend on histories that are common knowledge in the match, including the buyer's public trading history, his offer and the seller's response in the current match, but not on private histories (nor the public records of other buyers).¹²

(A2) **Symmetry.** All buyers adopt the same strategy, s^b , and all sellers adopt the same strategy, s^s . Moreover, the buyer's offer strategy, $s_{t,1}^b$, is constant over all public trading histories of the buyer that are consistent with equilibrium behavior, in particular, equilibrium offers at date- t are independent of matching histories.

(A3) **Threshold rule for repayments.** For each buyer i and each date t following any history, there exists a number, d_t , such that d_t is weakly larger than the equilibrium loan amount at date t , and $s_{t,2}^b(\rho_t^i, (y_t, \ell_t), \text{yes}) = \ell_t$ if $\ell_t \leq d_t$ and if ρ_t^i is consistent with equilibrium behavior.

We call a perfect Bayesian equilibrium, (s^b, s^s) , satisfying conditions (A1)-(A3) above a *credit equilibrium*. A few remarks are in order about these conditions. Our record-keeping technology does not record all actions taken by the agents. Agents have private information about the number of matches they had, quantities they consumed, or offers that were rejected. Because of this private information using perfect Bayesian equilibrium (PBE) as the solution concept is both standard and necessary. Alternatively, one may assume that all actions are observable, and PBE is reduced to subgame perfection. Although we prefer our environment, which is closer to the existing literature on monetary economics, our multiplicity result does not rely on the presence of private information. In fact, because of our focus on public strategies, (A1), any PBE we construct is also a subgame-perfect equilibrium (SPE) if all actions were observable.¹³ However, agents' belief about how other agents will respond to deviations do matter but they are pinned down by equilibrium strategies.

Conditions (A1) and (A2) imply that, for any credit equilibrium, its outcomes are characterized by $\{(y_t, \ell_t)\}_{t=0}^{+\infty}$, the sequence of equilibrium offers made by buyers. Moreover, (A3) implies that $x_t = \ell_t$ for each t , and hence the sequence $\{(y_t, \ell_t)\}_{t=0}^{+\infty}$ also determines the equilibrium allocation. Without (A1),

¹²For a formal definition of public strategies see Definition 7.1.1 in Mailath and Samuelson (2006).

¹³In such an equilibrium sellers' beliefs about buyers' private information are irrelevant for their decisions to accept or reject offers. Hence, actions that correspond to agents' private trading histories would not matter even if they were publicly observable.

equilibrium offers may depend on the buyer's past matching histories.¹⁴ Condition (A3) is not vacuous either. It restricts sellers to believe that buyers will repay their debt when observing a deviating offer with obligations smaller than those in equilibrium.¹⁵ This restriction will rule out inefficiently large trades. As we will see later, taken together the restrictions (A1)-(A3) will allow us to obtain a simple representation of credit equilibria with solvency constraints added to the bargaining problem.

Let $\{(y_t, \ell_t)\}_{t=0}^{+\infty}$ be a sequence of equilibrium offers. Along the equilibrium path the lifetime expected discounted utility of a buyer at the beginning of period t is

$$V_t^b = \sum_{s=0}^{\infty} \beta^s \alpha [u(y_{t+s}) - \ell_{t+s}]. \quad (2)$$

In each period $t + s$ the buyer is matched with a seller with probability α in which case the buyer asks for y_{t+s} units of DM output in exchange for a repayment of ℓ_{t+s} units of the numéraire in the following CM and the seller agrees. In any equilibrium $-\ell_t + \beta V_{t+1}^b \geq 0$, which simply says that a buyer must be better off repaying his debt and going along with the equilibrium rather than defaulting on his debt and offering no-trade in all future matches, $(y_{t+s}, \ell_{t+s}) = (0, 0)$ for all $s > 0$. By a similar reasoning the lifetime expected utility of a seller along the equilibrium path is

$$V_t^s = \sum_{s=0}^{\infty} \beta^s \alpha [-v(y_{t+s}) - \ell_{t+s}]. \quad (3)$$

The seller's participation constraint in the DM requires $-v(y_t) + \ell_t \geq 0$ since a seller can reject a trade without fear of retribution. (He is not monitored.) Given that buyers set the terms of trade unilaterally, and the output level is not part of the record ρ^i , this participation constraint holds at equality. Our first proposition builds on these observations to characterize outcomes of credit equilibria.

Proposition 1 *A sequence, $\{(y_t, x_t, \ell_t)\}_{t=0}^{\infty}$, is a credit equilibrium outcome if and only if, for each $t = 0, 1, \dots$,*

$$\ell_t \leq \sum_{s=1}^{\infty} \beta^s \alpha [u(y_{t+s}) - \ell_{t+s}] \quad (4)$$

$$\ell_t = x_t = v(y_t) \leq v(y^*). \quad (5)$$

As mentioned earlier, a sequence of equilibrium offers, $\{(y_t, \ell_t)\}_{t=0}^{+\infty}$, also determines the sequence of allocations, $\{(y_t, x_t)\}_{t=0}^{+\infty}$, with $x_t = \ell_t$ for each t , and hence, Proposition 1 also gives a characterization of allocations that can be sustained in a credit equilibrium. Condition (4), which follows directly from (2) and

¹⁴Obviously, when $\alpha = 1$, the matching-history-independence element in (A1) is vacuous. However, when $\alpha < 1$, it would be difficult to fully characterize all equilibrium outcomes without (A1) but it certainly adds many more equilibria.

¹⁵Without this restriction one could sustain equilibria in which $y_t > y^*$ for some t ; to do so, one can adopt a strategy that triggers a permanent autarky for the buyer if his offer ℓ_t is smaller than the equilibrium one.

the incentive constraint $-\ell_t + \beta V_{t+1}^b \geq 0$, is analogous to the participation constraint (IR) in Kehoe and Levine (1993), and the participation constraint in Proposition 2.1 in Kocherlakota (1996). However, while Kehoe and Levine assume the IR constraint from the outset as a primitive condition, (4) is derived as an equilibrium condition in our framework. The condition (5) is the outcome of the buyer take-it-or-leave-if offer and pairwise Pareto efficiency (which follows from the threshold rule A3).¹⁶

Proposition 1 shows that the conditions (4)-(5) are not only necessary but also sufficient for an equilibrium by constructing a simple equilibrium strategy profile. This strategy profile relies on punishments—the "penal code" in Abreu's (1988) terminology—for both default and excessive lending.¹⁷ Specifically, buyers can be in two states at the beginning of period t , $\chi_{i,t} \in \{G, A\}$, where G means "good standing" and A means "autarky", and each buyer's initial state is $\chi_{i,0} = G$. The law of motion of a buyer i 's state following a loan and repayment $(\tilde{\ell}, \tilde{x})$ are given by:

$$\chi_{i,t+1}(\tilde{\ell}, \tilde{x}, \chi_{i,t}) = \begin{cases} A & \text{if } \tilde{x} < \min(\tilde{\ell}, \ell_t) \text{ or } \chi_{i,t} = A \\ G & \text{otherwise} \end{cases}, \quad (6)$$

where $(\tilde{\ell}, \tilde{x})$ might differ from the loan and repayment along the equilibrium path, $\ell_t = x_t$. In order to remain in good standing, or state G , the buyer must repay his loan, $\tilde{x} \geq \tilde{\ell}$, if the size of the loan is no greater than the equilibrium loan size, $\tilde{\ell} \leq \ell_t$, and he must repay the equilibrium loan size, $\tilde{x} \geq \ell_t$, otherwise.¹⁸ The autarky state, A , is absorbing: once a buyer becomes untrustworthy, he stays untrustworthy forever.¹⁹ Sellers cannot be punished in future periods for accepting a loan larger than ℓ_t since their identity is not recorded. However, they are punished in the current period because buyers are allowed to partially default on loans larger than ℓ_t while keeping their good standing with future lenders.

The strategies, (s^b, s^s) , depend on the buyer's state as follows. The seller's strategy, s_t^s , consists of accepting all offers, $(\tilde{y}, \tilde{\ell})$, such that $v(\tilde{y}) \leq \min\{\tilde{\ell}, \ell_t\}$ provided that the buyer's state is $\chi_{i,t} = G$. The buyer repays $s_{t,2}^b = \min\{\ell_t, \tilde{\ell}\}$ if he is in state G , and he does not repay anything otherwise, $s_{t,2}^b = 0$. These strategies are depicted in Figure 2 where (y_t, ℓ_t) is the offer made by a buyer in state G along the equilibrium

¹⁶To derive these conditions formally one has to use the assumption that y_t is not publicly recorded—only the loan contract is—and the threshold property in (A3). See proof of Proposition 1.

¹⁷There are different approaches for finding equilibria of repeated games. Abreu et al. (1990) introduce the idea of self-generating set of equilibrium payoffs while Abreu (1988) introduces the notion of simple strategies. See Mailath and Samuelson (2006, Section 2.5) for a review of these approaches. We use a related but different approach from the one of Abreu (1988) as our stage game has an extensive form and only a subset of the agents (the buyers) are monitored.

¹⁸Note that the buyer can remain in state G even if he does not pay his debt in full, and hence default is with respect to the common belief that buyers repay up to the size of the equilibrium loan. Also, notice that there are alternative strategy profiles that deliver the same equilibrium outcome. For instance, an alternative automaton is such that the transition to state A only occurs if $\tilde{x} < \tilde{\ell} \leq \ell_t$. If a loan such that $\tilde{\ell} > \ell_t$ is accepted, then the buyer can default without fear of retribution.

¹⁹While no player has an incentive to deviate unilaterally along a subgame perfect equilibrium, at some heuristic level it seems that two players may want to "renegotiate" the punishment and coordinate on some preferred outcome in the Pareto sense. In the context of our credit equilibrium, the seller might want to forgive, and trust, a buyer who defaulted in the past instead of punishing him with no trade. This idea was formalized by Farrell and Maskin (1989), among others, with the refinement concept of weakly renegotiation-proof (WRP) equilibrium, defined as one where any two continuation payoffs are not Pareto-rankable. From our viewpoint, WRP eliminates the possibility of interesting coordination failures and the possibility of belief-driven credit cycles.

path and $(\tilde{y}, \tilde{\ell})$ is any offer. By the one-stage-deviation principle it is then straightforward to show that any $\{(y_t, \ell_t)\}_{t=0}^\infty$ that satisfies (4)-(5) is an outcome for the strategy profile (s^b, s^s) .

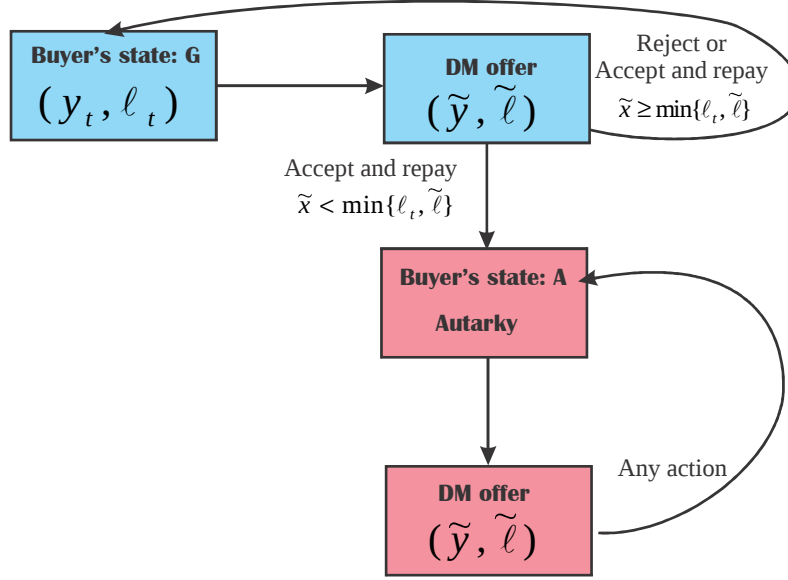


Figure 2: Automaton representation of the buyer's strategy

In the following we propose an alternative formulation of a credit equilibrium in terms of solvency constraints imposed on the bargaining problems in the DM. As in AJ in the context of an economy with competitive trades, a solvency constraint specifies an upper bound—called a *debt limit*—on the quantity of debt an agent can issue, $\ell \leq d_t$. According to this formulation, the buyer in a DM match sets the terms of the loan contract so as to maximize his surplus, $u(y) - \ell$, subject to the seller's participation constraint and the solvency (or borrowing) constraint, $\ell \leq d_t$, i.e.,

$$\max_{y, \ell} \{u(y) - \ell\} \quad \text{s.t.} \quad -v(y) + \ell \geq 0 \quad \text{and} \quad \ell \leq d_t. \quad (7)$$

The solution to (7) is $\ell_t = v(y_t)$ where

$$y_t = z(d_t) \equiv \min\{y^*, v^{-1}(d_t)\}. \quad (8)$$

The solvency constraint is reminiscent to the feasibility constraint in monetary models (e.g., Lagos and Wright, 2005) according to which buyers in bilateral matches cannot spend more than their real balances.

We say that a sequence of debt limits, $\{d_t\}_{t=0}^\infty$, is consistent with a credit equilibrium outcome, $\{(y_t, x_t, \ell_t)\}_{t=0}^\infty$, if (y_t, ℓ_t) is a solution to the bargaining problem, (7), given d_t for all $t \in \mathbb{N}_0$, and the buyer's CM strategy consists of repaying his debt up to d_t provided that his past public histories (up to period $t-1$) are consistent with equilibrium behavior, that is, the sequence $\{d_t\}$ satisfies condition (A3).

It is easy to check from the proof of Proposition 1 that any credit equilibrium outcome, $\{(y_t, x_t, \ell_t)\}_{t=0}^{\infty}$, is consistent with the sequence of debt limits, $\{d_t\}_{t=0}^{\infty}$, such that $d_t = \ell_t$ for all $t \in \mathbb{N}_0$. But the same equilibrium outcome may be implementable by multiple debt limits if (9) is slack and $y_t = y^*$. The following corollary summarizes these results and reduces a credit equilibrium to a sequence of debt limits, $\{d_t\}_{t=0}^{\infty}$, that satisfies a sequence of participation constraints.

Corollary 1 (*Equilibrium representation with debt limits*) *A sequence of debt limits, $\{d_t\}_{t=0}^{\infty}$, is consistent with a credit equilibrium outcome if and only if*

$$d_t \leq \sum_{s=1}^{\infty} \beta^s \alpha [u(y_{t+s}) - v(y_{t+s})] \quad (9)$$

$$v(y_t) = \min\{d_t, v(y^*)\}. \quad (10)$$

Corollary 1 gives a complete characterization of equilibrium outcomes using debt limits. Indeed, by (10), y_t is determined by d_t , and hence (9) can be viewed as an inequality that involves $\{d_t\}_{t=0}^{\infty}$ as the only endogenous variables. Without the danger of confusion, we also call a sequence of debt limits, $\{d_t\}_{t=0}^{\infty}$, a credit equilibrium if it satisfies (9) and (10).

While AJ introduces the solvency constraint as a primitive condition, we derive debt limits endogenously as part of equilibrium strategies. AJ focuses on solvency constraints that are "not-too-tight", meaning that d_t is the largest debt limit that solves the buyer's CM participation constraint, (9), at equality, thereby preventing default while allowing as much trade as possible. A "too-tight" solvency constraint is such that (9) is slack. In contrast to AJ and GMMW, we do not impose buyers' participation constraint to bind, i.e., the solvency constraint to be "not-too-tight", as such restriction would reduce the equilibrium set dramatically and might eliminate equilibria with good welfare properties. The next Corollary provides a sufficient condition for a credit equilibrium in recursive form.

Corollary 2 (*Recursive sufficient condition*) *Any bounded sequence, $\{d_t\}_{t=0}^{+\infty}$, that satisfies*

$$d_t \leq \beta \{ \alpha [u(y_{t+1}) - v(y_{t+1})] + d_{t+1} \}, \quad (11)$$

where $v(y_t) = \min\{d_t, v(y^)\}$, is a credit equilibrium.*

The left side of (11) is the cost of repaying the current debt limit while the right side of (11) is the benefit which has two components: the expected match surplus of a buyer who has access to credit and his continuation value given by the debt limit next-period. In Figure 3 we represent the pairs, (d_t, d_{t+1}) , that satisfy (11) at equality by a red curve. We plot a truncated sequence of debt limits, (d_{T-2}, d_{T-1}, d_T) , that

solves (11), i.e., (d_{T-2}, d_{T-1}) and (d_{T-1}, d_T) are located to the left of the red curve. Under “not-too-tight” solvency constraints $\{d_t\}$ solves (11) where the weak inequality is replaced with an equality and hence any pair, (d_{t-1}, d_t) , is on the red curve. The sequence of inequalities, (11), are sufficient conditions for a credit equilibrium, but they are not necessary. We will provide examples of credit equilibria that do not satisfy (11) in Section 3.3.

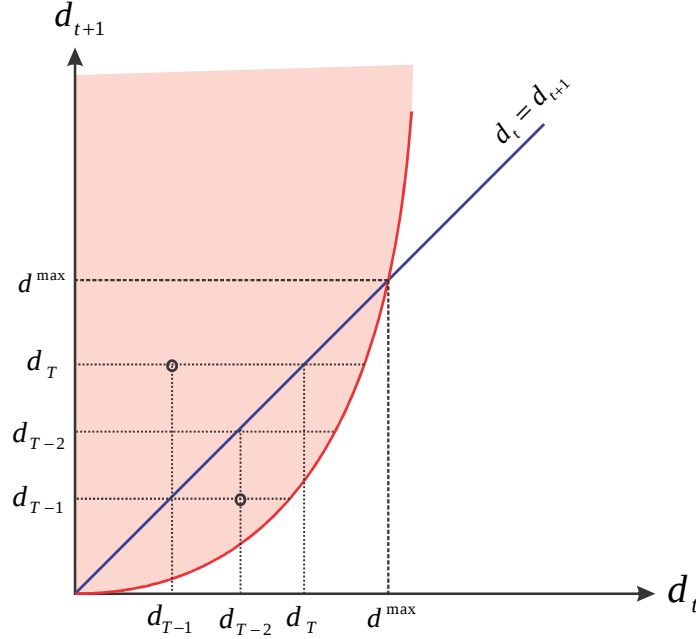


Figure 3: Recursive representation

3.1 Steady-state equilibria

We first characterize steady states where debt limits and DM allocations are constant over time, $(d_t, y_t, \ell_t) = (d, y, \ell)$ for all t . Under such restriction the incentive-compatibility condition, (9), or, equivalently, (11), can be simplified to read:

$$rd \leq \alpha \{u[z(d)] - v[z(d)]\}, \quad (12)$$

where z given by (8) indicates the DM level of output as a function of d . The left side of (12) is the flow cost of repaying debt while the right side is the flow benefit from maintaining access to credit. This benefit is equal to the probability of a trading opportunity, α , times the match surplus, $u(y) - v(y)$, where $y = z(d)$. Let $d^{\max}$ denote the highest value of the debt limit that satisfies (12), i.e., $d^{\max}$ is the unique positive root to $rd^{\max} = \alpha \{u[z(d^{\max})] - v[z(d^{\max})]\}$. It is determined graphically in Figure 4 at the intersection of the left side of (12) that is linear and the right side of (12) that is concave. For all $d < d^{\max}$, the gain from

defaulting is less than the cost associated with permanent autarky. The next Proposition shows that any debt limit between $d = 0$ and $d = d^{\max}$ is part of an equilibrium.

Proposition 2 (*Steady-State Equilibria*) *There exists a continuum of steady-state, credit equilibria indexed by $d \in [0, d^{\max}]$ with $d^{\max} > 0$.*

The two extreme debt limits, $\{0, d^{\max}\}$, correspond to the two steady-state equilibria under the AJ "not-too-tight" solvency constraints where the gain from defaulting is exactly equal to the cost of permanent autarky. Proposition 2 establishes that any debt limit in between these two extreme values is also part of an equilibrium. The intuition is as follows. For any debt d between 0 and $d^{\max}$ the gain from defaulting is strictly smaller than the cost associated with permanent autarky. So the buyer has incentives to repay such a loan. What about a slightly larger loan? The borrower cannot convince his current lender that he could repay more than d as his incentive to repay depends on how his future lenders (who he has not met yet) will "punish" him would he decide to renege on his current obligations, and these punishments are taken as given by the borrower and his lender. As a result, if a buyer offers $\ell > d$ then he only repays d , which is the repayment that keeps him trustworthy to the other sellers he may meet in future periods.

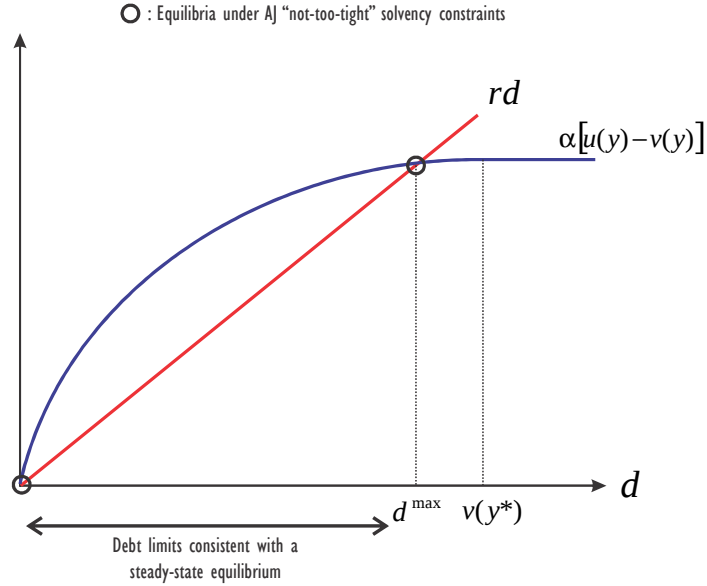


Figure 4: Set of debt limits at steady-state, credit equilibria

3.2 Periodic equilibria

Here we consider deterministic credit cycles where the extent to which buyers are trustworthy to repay their debts changes over time. We start with 2-period cycles, $\{d_0, d_1\}$, where d_0 is the debt limit in even periods

and d_1 is the debt limit in odd periods. The incentive-compatibility condition, (9), becomes:

$$rd_t \leq \frac{\alpha [u(y_{(t+1) \bmod 2}) - v(y_{(t+1) \bmod 2})] + \beta \alpha [u(y_t) - v(y_t)]}{1 + \beta}, \quad t \in \{0, 1\}, \quad (13)$$

where we used that $\ell_t = v(y_t) = \min\{d_t, v(y^*)\}$ from (10). The term on the numerator on the right side of (13) is the buyer's expected discounted utility over the 2-period cycle starting in $t + 1$. Obviously, the steady-state equilibria described in Proposition 2 are special cases of 2-period cycles; indeed, for any $d \in [0, d^{\max}]$, (d, d) satisfy (13). We define, for each $d_0 \in [0, d^{\max}]$,

$$\gamma(d_0) \equiv \max\{d_1 : (d_0, d_1) \text{ satisfies (13) with } t = 1\}, \quad (14)$$

the highest debt limit in odd periods consistent with a debt limit equal to d_0 in even periods. A 2-period-cycle equilibrium, or simply a *2-period cycle*, is a pair (d_0, d_1) that satisfies $d_0 \leq \gamma(d_1)$ and $d_1 \leq \gamma(d_0)$.

Lemma 1 *The function $\gamma(d)$ is positive, non-decreasing, and concave. Moreover, $d^{\min} \equiv \gamma(0) > 0$, $\gamma(d) > d$ for all $d \in (0, d^{\max})$, and $\gamma(d^{\max}) = d^{\max}$. If $v(y^*) < d^{\max}$, then $\gamma(d) = d^{\max}$ for all $d \in [v(y^*), d^{\max}]$.*

The function γ is represented in the two panels of Figure 5. It is non-decreasing because if the debt limit in even periods increases, then the punishment from defaulting gets larger and, as a consequence, higher debt limits can be sustained in odd periods. So there are complementarities between agents' trustworthiness in odd periods and agents' trustworthiness in even periods. The function $\gamma(d)$ is always positive because even if credit shuts down in even periods, it can be sustained in odd periods by the threat of autarky in both odd and even periods. For a given d_0 we define the set of debt limits in odd periods that are consistent with a 2-period cycle by

$$\Omega(d_0) \equiv \{d_1 : d_0 \leq \gamma(d_1), d_1 \leq \gamma(d_0)\}. \quad (15)$$

In Figure 5 the set of credit cycles is the area between γ and its mirror image with respect to the 45° line.

Proposition 3 (2-Period Credit Cycles) *For all $d_0 \in [0, d^{\max})$ the set of 2-period cycles with initial debt limit, d_0 , denoted $\Omega(d_0)$, is a nondegenerate interval.*

GMMW restrict attention to equilibria with "not-too-tight" solvency constraints, i.e., $d_0 = \gamma(d_1)$ and $d_1 = \gamma(d_0)$. Given the monotonicity and concavity of $\gamma(d)$ such equilibria do not occur outside of the 45° line, i.e., there are no (strict) credit cycles with "not-too-tight" solvency constraints. Indeed, if $d_0 \in (0, d^{\max})$ then the maximum debt limit in odd periods is $d_1 = \gamma(d_0) > d_0$. But given d_1 the maximum debt limit in even periods is $d'_0 = \gamma(d_1) > d_1 > d_0$. Following this argument we obtain an increasing sequence, $\{d_0, \gamma(d_0), \gamma(\gamma(d_0)), \dots\}$, that converges to $d^{\max}$. In contrast, by relaxing the restriction of "not-too-tight"

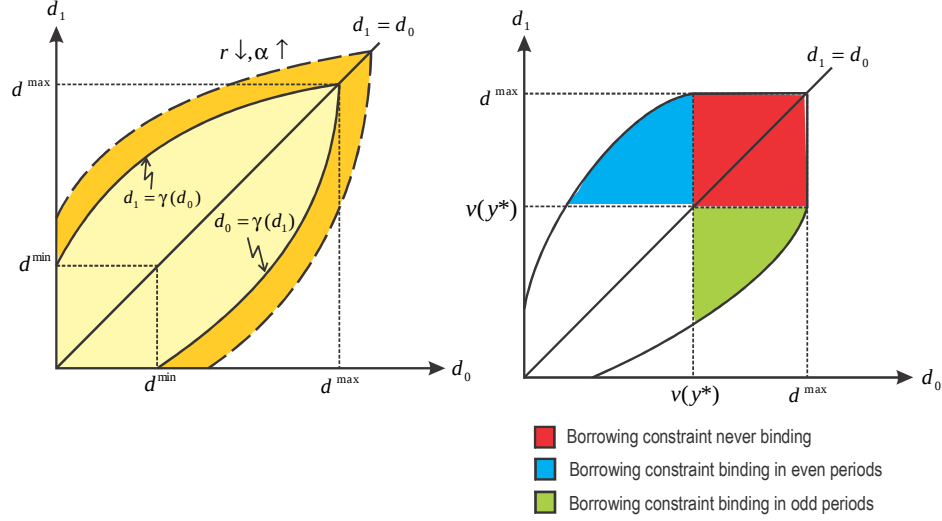


Figure 5: Set of two-period credit cycles

solvency constraints we find a continuum of (strict) two-period cycle equilibria. Moreover, the set of steady-state equilibria is of measure 0 in the set of all 2-period equilibria. Indeed, for any d_0 in the interval $(0, d^{\max})$ there are a continuum of 2-period cycles where d_0 is the debt limit in even periods.

The set of credit equilibria described in Proposition 3 contains equilibria where credit dries up periodically. In the left panel of Figure 5 such equilibria correspond to the case where $d_0 = 0 < d_1 < \gamma(0) = d^{\min}$, i.e., even-period IOUs are believed to be worthless while odd-period IOUs are repaid. If a seller extends a loan in an even period, the buyer defaults, in accordance with equilibrium beliefs, but remains trustworthy in subsequent odd periods. Such outcomes are ruled out by backward induction in pure-currency economies. In contrast a credit economy has IOUs issued at different dates (and by different agents), and hence agents can form different beliefs regarding the terminal value of these different securities.

The result according to which there are a continuum of equilibria does not imply that everything goes. Fundamentals, such as preferences and matching technology, do matter for the outcomes that can emerge. The following corollary investigates how changes in fundamentals affect the equilibrium set.

Corollary 3 (*Comparative statics*) *As r decreases or α increases the set of 2-period cycles expands.*

If agents become more patient, i.e., r decreases, then γ shifts upward, as the discounted sum of future utility flows associated with a given allocation increases, and the set of 2-period cycle equilibria expands. The expansion of the equilibrium set is represented by the dark yellow area in the left panel of Figure 5. Similarly, if the frequency of matches, α , increases, then $d^{\max}$ increases as permanent autarky entails a larger opportunity cost, and the set of credit cycles expands.

Corollary 4 (*Credit tightness over the cycle*)

If $r \geq \alpha [u(y^*) - v(y^*)] / v(y^*)$ then $\ell_t \leq d_t$ binds for both $t \in \{0, 1\}$ in any 2-period cycle.

If $r < \alpha [u(y^*) - v(y^*)] / v(y^*)$, then there are 2-period cycles such that $\ell_t \leq d_t$ is slack for all $t \in \{0, 1\}$, and there are 2-period cycles where $\ell_t \leq d_t$ binds only periodically.

Corollary 4 shows that if agents are sufficiently impatient, as in the left panel of Figure 5, then the debt limit binds and output is inefficiently low in every period for all credit cycles. However, if agents are patient, then there are equilibria where the debt limit binds periodically. Such equilibria are represented by the blue and green areas in the right panel of Figure 5. There are also equilibria where the debt limit fluctuates over time but never binds. These fluctuations, however, are payoff-irrelevant since the allocation is constant and the first best is implemented, $y_0 = y_1 = y^*$. These equilibria are represented by the red square, $[v(y^*), d^{\max}]^2$, in the right panel of Figure 5.

One can generalize the above arguments to T -period cycles, $\{d_j\}_{j=0}^{T-1}$. The debt limits must solve the following inequalities:

$$d_t \leq \frac{\alpha \sum_{j=1}^T \beta^j \{u[y_{(t+j) \bmod T}] - v[y_{(t+j) \bmod T}]\}}{1 - \beta^T}, \quad t = 0, \dots, T-1 \quad (16)$$

The numerator on the right side of (16) is the expected discounted sum of utility flows over the T -period cycle. Following the same reasoning as above:

Proposition 4 (*T-Period Credit Cycles*) For any $T \geq 2$ and for all $d_0 \in [0, d^{\max}]$, the set of T -period credit cycles with initial debt limit, d_0 , denoted $\Omega_T(d_0)$, is a bounded, convex, and closed set in $\mathbb{R}^{T-1}$ with positive Lebesgue measure.

Our environment can lead to cycles of any periodicity, and for a given length of the cycle there are a continuum of equilibria. As an illustration, in Figure 6 we represent the set of 3-period cycles for a given d_2 . The outer edge of this set, which has positive measure in $\mathbb{R}^2$, is represented by a thick black curve. One can also see from the right panel that there is a non-empty set of 3-period cycles (the pink area) where credit shuts down periodically, once ($d_2 = 0$) or twice (e.g., $d_1 = d_2 = 0$) every three periods. Also, for our parametrization the first best is implementable, i.e., there are equilibria in the purple area with $d_t \geq d^* = 1$ for all $t \in \{0, 1, 2\}$.

3.3 Monetary vs credit economies

We now consider the same environment as before but we shut down the record-keeping technology: individual trading histories are private information in a match. Without public memory credit is no longer incentive-feasible as a buyer would find it optimal to renege on his debt. Suppose that all buyers are endowed with

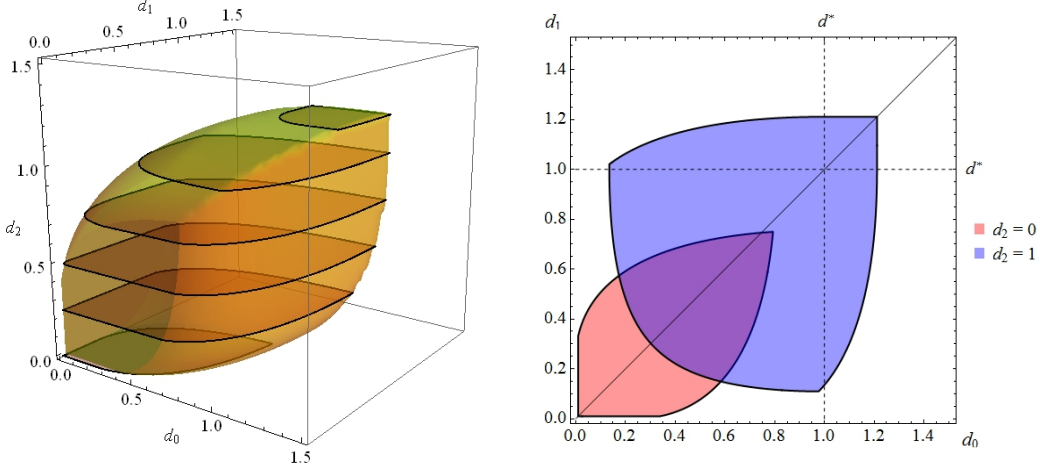


Figure 6: Set of three-period credit cycles: $u(y) = 2\sqrt{y}$, $v(y) = y$, $\beta = 0.9$, $\alpha = 0.25$

$M = 1$ units of fiat money at time $t = 0$. Money is a perfectly divisible, storable, and intrinsically useless object, and its supply is constant over time. The environment is now identical to the one in Lagos and Wright (2003, 2005).²⁰ We show in the following that any allocation, $\{(x_t, y_t)\}_{t=0}^{+\infty}$, of a pure monetary economy, where x_t is CM output and y_t is DM output, is also an allocation of a pure credit economy.

The CM price of money in terms of the numéraire good is denoted ϕ_t . The buyer's choice of money holdings in period t is the solution to the following problem:

$$\max_{m \geq 0} \{-\phi_t m + \beta \alpha [u(y_{t+1}) - v(y_{t+1})] + \beta \phi_{t+1} m\}, \quad (17)$$

where, from buyers' take-it-or-leave-it offers in the DM, sellers are indifferent between trading and not trading, $v(y_t) = \phi_t m$. From (17) it costs $\phi_t m$ to the buyer in the CM of period t to accumulate m units of money. In the following DM the buyer can purchase $y_{t+1} = v^{-1}(\phi_{t+1} m)$ if he happens to be matched with probability α . Otherwise the buyer can resell his units of money at the price ϕ_{t+1} in the CM of period $t+1$. From the first-order condition of (17), $\{\phi_t\}_{t=0}^{+\infty}$ solves the following first-order difference equation,

$$\phi_t = \beta \phi_{t+1} \left[1 + \alpha \frac{u'(y_{t+1}) - v'(y_{t+1})}{v'(y_{t+1})} \right]. \quad (18)$$

According to (18) the value of fiat money in period t is equal to the discounted value of money in period $t+1$ augmented with a liquidity term that captures the expected marginal surplus from holding an additional unit of money in a pairwise meeting in the DM. A monetary equilibrium is a bounded sequence, $\{(y_t, x_t, \phi_t)\}_{t=0}^{+\infty}$, that solves (18), with $v(y_t) = x_t = \min\{\phi_t, v(y^*)\}$.

²⁰This version of the environment with ex-ante heterogeneity between buyers and sellers is due to Rocheteau and Wright (2005).

Proposition 5 (Monetary vs Credit Equilibria) Let $\{(y_t, x_t, \phi_t)\}_{t=0}^{+\infty}$ be a monetary equilibrium of the economy with no record-keeping. Then, $\{(y_t, x_t, \ell_t)\}_{t=0}^{+\infty}$ where $\ell_t = \min\{\phi_t, v(y^*)\}$ is a credit equilibrium of the economy with record-keeping.

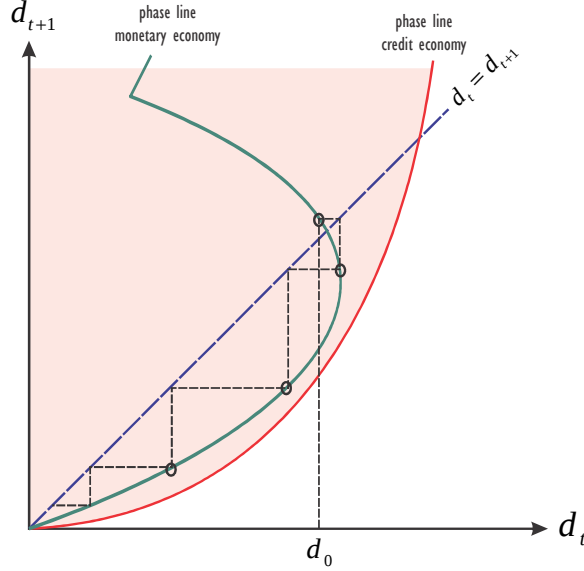


Figure 7: Monetary vs credit outcomes

Proposition 5 establishes that the set of (dynamic) equilibrium allocations in pure credit economies encompasses the set of equilibrium allocations of pure monetary economies taking as given the trading mechanism. This result is related to those in Kocherlakota (1998), but with a key difference: while Kocherlakota (1998) shows that the set of *all* implementable outcomes (allowing for arbitrary trading mechanisms) using money is contained in the set of all implementable outcomes with memory, we compare the equilibrium outcomes for the two economies under a *particular* trading mechanism. Later on we discuss the robustness to other trading mechanisms.

We illustrate this result in Figure 7 where the green, backward-bending line represents the first-order difference equation for a monetary equilibrium, (18), while the red area is the first-order difference inequality for a credit equilibrium, (11). Starting from some initial condition, d_0 , we represent by a dashed line a sequence $\{d_t\}$ that satisfies the conditions for a monetary equilibrium. This sequence also satisfies the conditions for a credit equilibrium, i.e., all pairs (d_t, d_{t+1}) are located in the red area. Therefore, if the equilibrium set of a pure monetary economy contains cycles and chaotic dynamics, the same must be true for the equilibrium set of the same economy with no money but record-keeping.

The inclusion result in Proposition 5 breaks down if one imposes the "not-too-tight" solvency constraints

since the phase line for credit economies differs from the phase line for monetary economies. The reason for this discrepancy is as follows. Under the "not-too-tight" solvency constraints the payment capacity of buyers, d_t , in the pure credit economy is the discounted sum of all future match surpluses,

$$d_t = \beta \{ \alpha [u(y_{t+1}) - v(y_{t+1})] + d_{t+1} \}.$$

In pure monetary economy the payment capacity of buyers, ϕ_t , is the discounted sum of all future marginal surpluses multiplied by the value of money,

$$\phi_t = \beta \left\{ \alpha \frac{\partial [u \circ v^{-1}(\phi_{t+1}) - \phi_{t+1}]}{\partial \phi_{t+1}} \phi_{t+1} + \phi_{t+1} \right\}.$$

From the concavity of the match surplus, if $\phi_t = d_t$, then $\phi_{t+1} > d_{t+1}$.

The reverse of Proposition 5 does not hold; There are equilibria of pure credit economies that are not equilibria of pure monetary economies. As we saw above there are credit equilibria where trades shut down periodically, and such equilibria cannot be captured by Figure 7. (Recall that the recursive condition in Corollary 2 is sufficient but not necessary.) As another example, one can construct equilibria where the debt limit, d_t , increases in a monotonic fashion over time as buyers become more and more trustworthy. Such equilibria would not be sustainable in monetary economies.

3.4 Sunspot equilibria

There is a view that deterministic cycles might not provide a realistic description of actual business cycle fluctuations. Hence we show in the following that one can also construct sunspot equilibria where the DM allocation, $\{(y_\chi, \ell_\chi)\}$, depends on the realization of a sunspot state, $\chi \in \mathbb{X}$, at the beginning of the DM. Suppose that $\mathbb{X}$ is finite and the process driving the sunspot state is i.i.d. with distribution π . We assume that π has a full support, i.e., $\pi(\chi) > 0$ for all $\chi \in \mathbb{X}$. The value of a buyer along the equilibrium path solves

$$V_\chi^b = \alpha [u(y_\chi) - v(y_\chi)] + \beta \bar{V}^b \quad (19)$$

$$\bar{V}^b = \int V_{\chi'}^b d\pi(\chi'), \quad (20)$$

for all $\chi \in \mathbb{X}$. As before, the lifetime utility of a buyer is the expected discounted sum of the surpluses coming from DM trades. It follows that a sunspot credit equilibrium is a vector, $\langle d_\chi; \chi \in \mathbb{X} \rangle$, that satisfies $d_\chi \leq \beta \bar{V}^b$. Hence, $\{d_\chi\}$ satisfies

$$rd_\chi \leq \alpha \int \{u[z(d_{\chi'})] - v[z(d_{\chi'})]\} d\pi(\chi') \quad \forall \chi \in \mathbb{X} \quad (21)$$

Proposition 6 (*Sunspot equilibria*) *Suppose that $\mathbb{X}$ has at least two elements and let π be a distribution over $\mathbb{X}$ with a full support. For a given $(\chi, d_\chi) \in \mathbb{X} \times (0, d^{\max})$, the set of sunspot credit equilibria with debt limit d_χ in state χ , denoted by $\Omega_{\mathbb{X}, \pi}(\chi, d_\chi)$, has a positive Lebesgue measure in $\mathbb{R}^{|\mathbb{X}|-1}$.*

4 Alternative trading mechanisms

In the following we show that our results regarding the equilibrium set of pure credit economies are robust to trading mechanisms other than take-it-or-leave-it offers by buyers. We also extend our model in order to parametrize buyers' temptation to renege on their debt. This extension adds a new parameter that plays a key role for the normative results in Section 5.

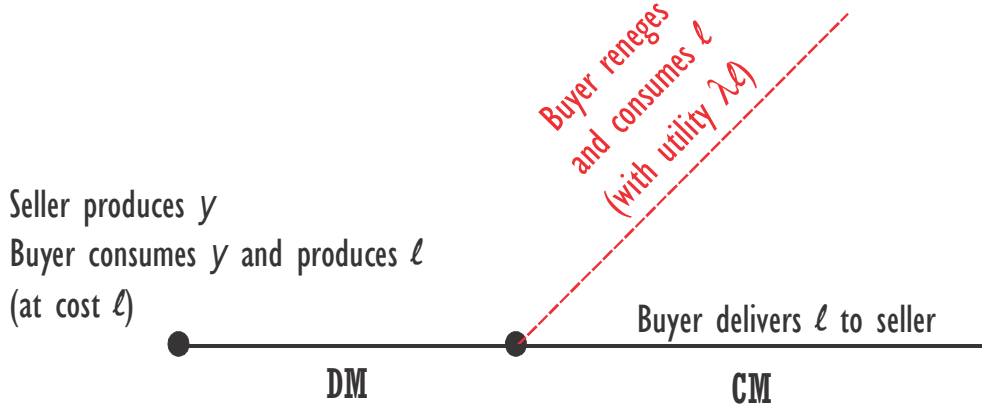


Figure 8: Timing of the extended model with temptation to renege

Suppose from now on that a buyer who promises to deliver ℓ units of goods in the next CM incurs the linear disutility of producing at the time he is matched in the DM. This new timing is illustrated in Figure 8.²¹ The effort exerted by the buyer in the DM, ℓ , is perfectly observable to the seller. At the time of delivery, at the beginning of the CM, the disutility of production has been sunk and the buyer has the option to renege on his promise to deliver the good. The buyer's utility from consuming his own output is $\lambda\ell$ with $\lambda \leq 1$. A buyer has no incentive to produce more good than the amount he promises to repay to the seller since the net utility gain from producing x units of the good for oneself is $(\lambda - 1)x \leq 0$. Although the physical environment is different, mathematically speaking, the model of the previous section can be regarded as a special case with $\lambda = 1$. As before we will focus on symmetric perfect Bayesian equilibria that satisfy (A1)-(A3).²²

Let $\{(d_t, y_t, \ell_t)\}_{t=0}^{\infty}$ be the sequence of equilibrium debt limits and trades. A necessary condition for the repayment of d_t to be incentive feasible is $\beta V_{t+1}^b \geq \lambda d_t$, where the left side is a buyer's continuation value from delivering the promised output and the right side of the inequality is the utility of a buyer if he keeps

²¹The description of the buyer's incentive problem is taken from Gu et al. (2013a) and GMMW.

²²GMMW also introduce an imperfect record-keeping technology as follows. At the end of the CM of period t the repayments are recorded for a subset of buyers, $\mathbb{B}_t^r \subset \mathbb{B}$, chosen at random among all buyers. The set, $\mathbb{B}_t^r$, of monitored buyers is of measure π , and the draws from $\mathbb{B}$ are independent across periods. So in every period, while his promise is always recorded, a buyer has a probability π of having his repayment decision being recorded. Any equilibrium of our model with $\pi < 1$ is also an equilibrium with $\pi = 1$. Hence, setting $\pi = 1$ is with no loss in generality.

the output for himself, in which case he enjoys a utility flow λd_t , and goes to autarky. Following the same reasoning as before, a credit equilibrium is reduced to a sequence, $\{d_t\}_{t=0}^{\infty}$, that satisfies

$$\lambda d_t \leq \beta V_{t+1}^b = \alpha \sum_{s=1}^{+\infty} \beta^s [u(y_{t+s}) - \ell_{t+s}], \quad t \in \mathbb{N}_0, \quad (22)$$

where the relationship between y_t , ℓ_t , and d_t will depend on the assumed trading mechanism.

4.1 Bargaining

It is standard in the literature on markets with pairwise meetings to determine the outcome of a meeting by an axiomatic bargaining solution. In this section we consider two well-known solutions: (i) the Kalai proportional bargaining solution and (ii) the generalized Nash solution. We adopt the representation of the equilibrium with solvency constraints, $\ell_t \leq d_t$, in order to obtain a convex bargaining set.²³ For a given sequence of debt limits, $\{d_t\}_{t=0}^{+\infty}$, the buyer repays $\min\{\ell_t, d_t\}$ if his date- t obligation from his DM trade is ℓ_t . Due to the linearity of the CM value functions, the buyer's surplus from a DM trade, (y_t, ℓ_t) with $\ell_t \leq d_t$, is $u(y_t) - \ell_t$ and the seller's surplus is $-v(y_t) + \ell_t$.

Kalai proportional bargaining We amend the take-it-or-leave-it offer game by restricting the set of buyers' feasible offers: a buyer can only make offers such that the fraction of the match surplus he receives is no greater than a given $\theta \in [0, 1]$, i.e.,

$$u(y) - \ell \leq \theta[u(y) - v(y)]. \quad (23)$$

Thus, the buyer's offer in the DM, assuming he is in state G , solves

$$(y_t, \ell_t) = \arg \max_{y, \ell} [u(y) - \ell] \text{ s.t. (23) and } \ell \leq d_t. \quad (24)$$

According to (24) the buyer maximizes his utility of consumption net of the cost of repaying his debt subject to the feasibility constraint, (23), and the repayment constraint, $\ell \leq d_t$. The solution to (24) is

$$y_t = z(d_t) \equiv \min\{y^*, \eta^{-1}(d_t)\} \text{ and } \ell_t = \eta[z(d_t)]. \quad (25)$$

where $\eta(y) = (1 - \theta)u(y) + \theta v(y)$. In equilibrium, the buyer offers (y_t, ℓ_t) given by (25) and the seller accepts it. The seller rejects any offer from a buyer with state A .

²³Even though the bargaining solution is axiomatic we could consider a simple game where upon being matched the buyer and the seller receive a proposal that they can either accept or reject. The focus here, however, is not on strategic foundations for axiomatic bargaining solutions.

Proposition 7 (*Credit equilibrium under proportional bargaining*) *A sequence, $\{d_t\}$, is a credit equilibrium under proportional bargaining if and only if*

$$\lambda d_t \leq \alpha \theta \sum_{i=1}^{+\infty} \beta^i [u(y_{t+i}) - v(y_{t+i})], \quad \forall t \in \mathbb{N}_0, \quad (26)$$

where (y_t, ℓ_t) is given by (25).

Proposition 7 describes the set of all debt limits, $\{d_t\}_{t=0}^{+\infty}$, and associated allocations, $\{(y_t, \ell_t)\}_{t=0}^{+\infty}$, that are generated by credit equilibria under bargaining weight θ . The right side of (26) takes into account that buyers only receive a fraction θ of the match surplus. Note that Corollary 1 is a special case of Proposition 7 by taking $\theta = \lambda = 1$.

We can generalize Proposition 2 by showing that the set of steady-state equilibria is the interval $[0, d^{\max}]$, where $d^{\max}$ is the largest nonnegative root to $r\lambda d = \alpha\theta\{u[z(d)] - v[z(d)]\}$, and $d^{\max} > 0$ if and only if $\lambda r < \alpha\theta/(1 - \theta)$. If buyers do not have all the bargaining power, then an active steady-state credit equilibrium exists only if buyers are sufficiently patient. The lower the value of θ the lower the rate of time preference that is required for credit to emerge. Indeed, if θ decreases buyers receive a lower share in current and future match surpluses and, for a given d , the amount of DM consumption they can purchase is lower. Both effects reduce the gains from participating in the DM and hence reduce the maximum sustainable debt limit. It can also be checked that a higher λ reduces $d^{\max}$. As a result any (equilibrium) allocation under $\lambda = 1$ is also an (equilibrium) allocation under $\lambda < 1$. We now move to equilibria with endogenous fluctuations.

Proposition 8 (*2-Period Credit Cycles under proportional bargaining*) *If $\lambda r < \alpha\theta/(1 - \theta)$, then there exists a continuum of strict, 2-period, credit cycle equilibria. Moreover, if $r < \sqrt{1 + \alpha\theta/[\lambda(1 - \theta)]} - 1$, then there exist equilibria where credit shuts down periodically.*

Proposition 8 establishes a condition for the existence of a continuum of credit-cycle equilibria under proportional bargaining. The set of equilibria can also be represented by Figure 5, and, under proportional bargaining, the outer envelope shifts outward as θ increases. Moreover, if agents are sufficiently patient then there are equilibria where credit shuts down periodically, i.e., $\gamma(0) > 0$. In contrast, if we impose the "not-too-tight" solvency constraints, then there are no periodic equilibrium under proportional bargaining, irrespective of the buyer's bargaining share.²⁴

²⁴Propositions 4 and 5 regarding the existence of N -period credit cycles and the relationship between monetary and credit equilibria can be generalized to proportional bargaining in a similar fashion.

Generalized Nash bargaining Under generalized Nash bargaining the terms of the loan contract are

$$(y_t, \ell_t) = \arg \max [u(y) - \ell]^\theta [\ell - v(y)]^{1-\theta} \quad \text{s.t.} \quad \ell \leq d_t.$$

The solution is given by (25) where

$$\eta(y) = \Theta(y)v(y) + [1 - \Theta(y)]u(y) \text{ and } \Theta(y) = \theta u'(y) / [\theta u'(y) + (1 - \theta)v'(y)]. \quad (27)$$

A sequence, $\{d_t\}_{t=0}^{+\infty}$, is a credit equilibrium under generalized Nash bargaining if and only if

$$\lambda d_t \leq \alpha \sum_{i=1}^{+\infty} \beta^i [u(y_{t+i}) - \eta(y_{t+i})], \quad \forall t \in \mathbb{N}_0, \quad (28)$$

where y_t is the solution to (25).

We denote $\hat{y} = \arg \max \{u(y) - \eta(y)\}$ the output level that maximizes the buyer's surplus. Unlike the proportional solution $\hat{y} < y^*$ for all $\theta < 1$. As a result the buyer's surplus, $u(y) - \eta(y)$, in the right side of the participation constraint, (28), is non-monotonic with the debt limit provided that $\theta < 1$.²⁵ It follows that the function $\gamma(d)$ is hump-shaped, reaching a maximum at $d = \hat{d} \equiv \eta(\hat{y})$ and it is constant for $d > \eta(y^*)$. In Figure 9 we represent the function γ and the set of pairs, (d_0, d_1) , consistent with a 2-period credit cycle equilibrium. One can see that the results are qualitatively unchanged except for the fact that the credit limits at a periodic equilibrium can be greater than the highest debt limit at a stationary equilibrium.²⁶ This result will have important normative implications.

The two red stars in the left panel of Figure 9 are the strict two-period cycles under "not-too-tight" solvency constraints that GMMW focuses on. Such cycles are located at the intersection of γ and its mirror image with respect to the line $d_1 = d_0$. It should be clear that the non-monotonicity of the trading mechanism is necessary to obtain such cycles. It can also be checked that cycles under "not-too-tight" solvency constraints do not exist when $\lambda = 1$ (see GMMW).

In the top panels of Figure 10 we plot the numerical examples in Gu and Wright (2011) under generalized Nash bargaining for the following functional forms and parameter values: $u(y) = [(x+b)^{1-a} - b^{1-a}]/(1-a)$ with $a = 2$ and $b = 0.082$, $v(y) = Ay$, $\beta = 0.6$, $\alpha = 1$, $\theta = 0.01$, and $\lambda = 3/40$. In the top left panel, $A = 1.1$, the two 2-period cycles under "not-too-tight" solvency constraints are such that borrowing constraints bind periodically. In the top right panel, $A = 1.5$, the borrowing constraint binds in all periods. For both examples there exists a continuum of PBE 2-period cycles, a fraction of which feature borrowing constraints that bind periodically and a fraction of which have borrowing constraints that bind in all periods.

²⁵This non-monotonicity property of the Nash bargaining solution and its implications for monetary equilibria is discussed at length in Aruoba et al. (2007).

²⁶In the Appendix we prove that any 2-period cycle under proportional bargaining is also a 2-period cycle under Nash bargaining.

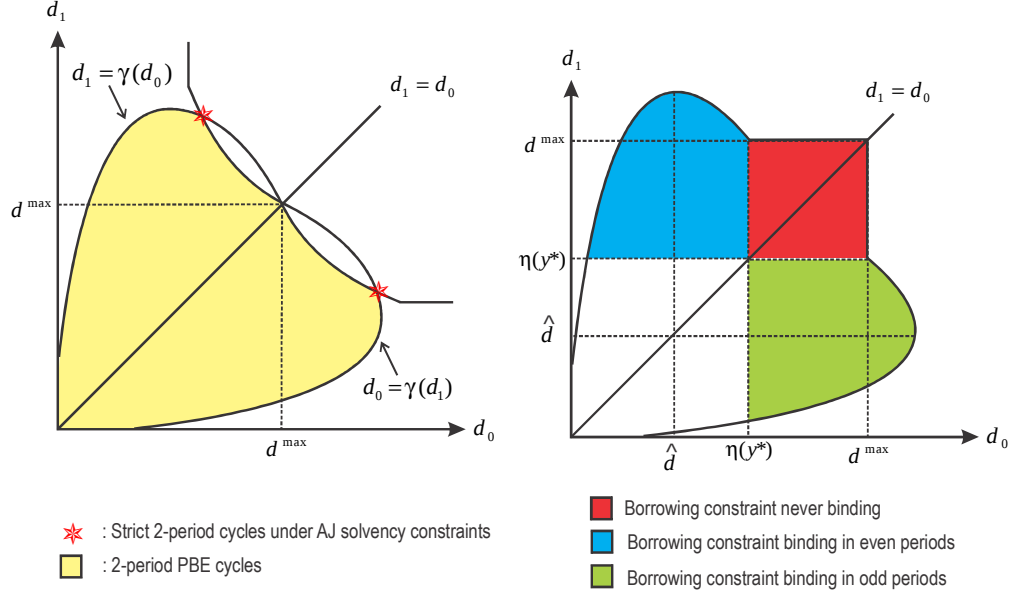


Figure 9: 2-period cycles under Nash bargaining or competitive pricing

4.2 Competitive pricing

Here we follow Kehoe and Levine (1993) and AJ and assume that the terms of the loan contract in the DM are determined by competitive pricing. We reinterpret matching shocks as preference and productivity shocks, i.e., only α buyers want to consume and only α sellers can produce. As in the previous sections, buyers' repayment strategy follows a threshold rule: for a given sequence of debt limits, $\{d_t\}_{t=0}^{+\infty}$, the buyer repays $\min\{\ell_t, d_t\}$ if his date- t obligation from his DM trade is ℓ_t .²⁷ Moreover, the overall amount of debt issued by a buyer in the DM of period t , ℓ_t , is known to all agents. Hence, if p_t denotes the price of DM output in terms of the numéraire, a buyer's demand is subject to the borrowing constraint, $p_t y \leq d_t$. For a given $\{d_t\}_{t=0}^{+\infty}$ the market-clearing price is given by $p_t = v'(y_t)$, where

$$y_t = z(d_t) \equiv \min\{y^*, \eta^{-1}(d_t)\} \text{ and } \ell_t = \eta[z(d_t)], \quad (29)$$

with $\eta(y) = v'(y)y$.²⁸ The buyer's surplus is $u(y) - py = u(y) - v'(y)y$. For a given p , the buyer's surplus is non-decreasing in his borrowing capacity, d_t . However, once one takes into account the fact that $p = v'(y)$ then the buyer's surplus is non-monotone in his capacity to borrow, d_t . Provided that v is strictly convex, the buyer's surplus reaches a maximum for $y = \hat{y} < y^*$. A sequence, $\{d_t\}_{t=0}^{+\infty}$, is a credit equilibrium under

²⁷If a buyer repays $x_t \neq \ell_t$ in the CM, then each unit of IOU issued by that buyer has a payoff equal to x_t/ℓ_t units of numéraire to its owner.

²⁸The buyer's problem is $\max_y \{u(y) - p_t y\}$ s.t. $p_t y \leq d_t$. The solution is $y_t = \min\{u'^{-1}(p_t), d_t/p_t\}$. Using that there is the same measure, α , of buyers and sellers participating in the market, market clearing implies $p_t = v'(y_t)$. As a result $y_t = y^*$ if $y^* v'(y^*) \geq d_t$ and $y_t v'(y_t) = d_t$ otherwise. For a detailed description of this problem in the context of a pure monetary economy, see Rocheteau and Wright (2005, Section 4).

competitive pricing if and only if (28) holds for all $t \in \mathbb{N}_0$, where y_t is given by (29). A steady state is a d such that

$$r\lambda d \leq \alpha \{u[z(d)] - v'[z(d)]z(d)\}. \quad (30)$$

Under some weak assumptions on v (for example, $\eta(y) = v'(y)y$ is convex), $d^{\max} > 0$, i.e., there exists a continuum of steady-state equilibria. This also implies that there exist a continuum of strict, 2-period, credit cycle equilibria.²⁹ This result can be contrasted with the ones in GMMW (Corollary 1-3) where conditions on parameter values are needed to generate a finite number (typically, two) of cycles. The right panel of Figure 9 illustrates these differences. Under "not-too-tight" solvency constraints credit cycles are determined at the intersection between $\gamma(d)$ and its mirror image with respect to the 45° line. These cycles are marked by a red star. If we allow for slack buyers' participation constraints, cycles are at the intersection of the area underneath $\gamma(d)$ and its mirror image with respect to the 45° line—the blue area in the figure. Finally, Proposition 5 on the equivalence result between monetary equilibria and credit equilibria holds for Walrasian pricing as well. (See the Supplementary Appendix S1 for a formal proof).

We now review the numerical examples in GMMW in the case where the DM market is assumed to be competitive. The functional forms are $u(y) = y$, $v(y) = y^{1+\gamma}/(1+\gamma)$, and there are no idiosyncratic shocks, $\alpha = 1$. The first example in the bottom left panel of Figure 10 is obtained with the following parameter values: $\gamma = 2.1$, $\beta = 0.4$, $\lambda = 1/6$. GMMW identify two (strict) two-period cycles under "not-too-tight" solvency constraints, $(d_0, d_1) = (0.477, 0.936)$ and its converse, marked by red dots in the figure. The second example in the bottom right panel is obtained with the following parameter values: $\gamma = 0.5$, $\beta = 0.9$, $\lambda = 1/10$. The credit cycles under "not-too-tight" solvency constraints, $(d_0, d_1) = (0.933, 1.037)$ and its converse, are such that period allocations fluctuate between being debt-constrained and unconstrained. We find a much bigger set of PBE credit cycles represented by the blue colored region. There are a continuum of cycles such that the allocations fluctuate between being debt-constrained and unconstrained and a continuum of cycles such that agents are debt-constrained in all periods. In the second example, the credit cycle under "not-too-tight" solvency constraints is such that $(y_0, y_1) = (0.96, 1.00)$ while the most volatile PBE is $(y_0, y_1) = (0.96, 0.00)$.

5 Normative analysis

We now turn to the normative implications of our model. We will characterize constrained-efficient allocations under two alternative market structures: pairwise meetings and large-group meetings. We will show that the

²⁹Under competitive pricing, the function γ (analogous to (14)) may not be monotone or concave, but the logic for Proposition 3 does not depend on those properties. See also the supplementary appendix S2 for a formal proof of the existence of 2-period cycles.

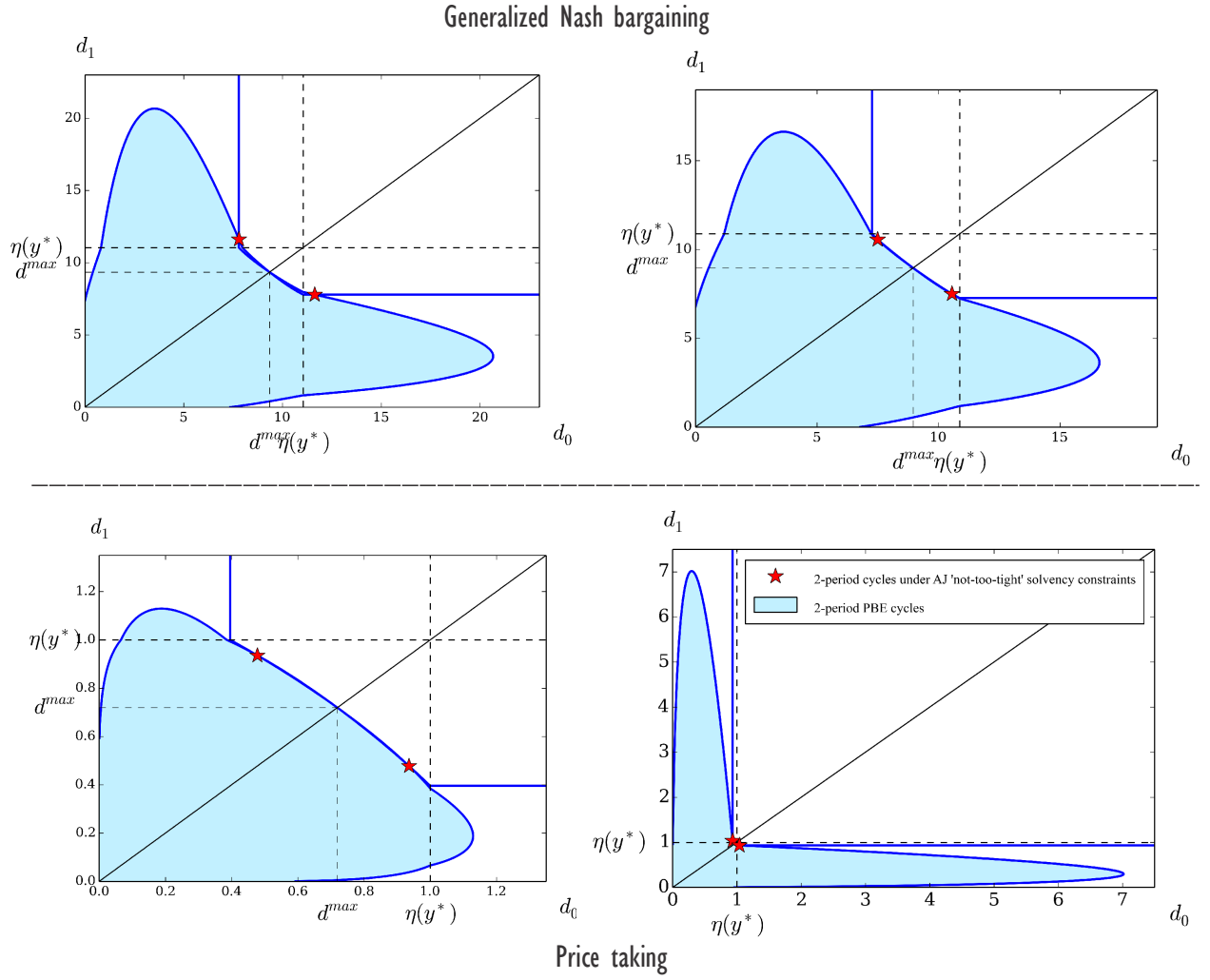


Figure 10: The blue area is the set of all PBE credit cycles. The red dots are credit cycles under AJ "not-too-tight" solvency constraints. The top panels are obtained under generalized Nash bargaining while the bottom panels are obtained under price taking.

optimal mechanism for the economy with pairwise meetings is the one studied in Section 3 where buyers have all the bargaining power, and "not-too-tight" solvency constraints are socially optimal. Under large-group meetings the optimality of "not-too-tight" solvency constraints depends on λ that parameterizes buyers' temptation to renege on their obligations.

5.1 Optimal mechanism with pairwise meetings

We study the problem of a planner who chooses the allocation, $\{(y_t, \ell_t)\}_{t=0}^{+\infty}$, in order to maximize the discounted sum of all match surpluses subject to incentive-feasibility conditions:³⁰

$$\max_{\{(y_t, \ell_t)\}} \sum_{t=0}^{+\infty} \beta^t \alpha [u(y_t) - v(y_t)] \quad (31)$$

$$\text{s.t. } \lambda \ell_t \leq \sum_{s=1}^{+\infty} \beta^s \alpha [u(y_{t+s}) - \ell_{t+s}] \quad (32)$$

$$v(y_t) \leq \ell_t \leq u(y_t). \quad (33)$$

The inequality, (32), is the participation constraint guaranteeing that buyers prefer to repay their debt rather than going to permanent autarky. The conditions in (33) make sure that both buyers and sellers receive a positive surplus from their DM trades. Coalition-proofness in pairwise meetings requires that $y_t \leq y^*$, which is satisfied endogenously (and hence ignored thereafter). We call a solution to (31)-(33) a constrained-efficient allocation (c.e.a.). In the following we use $y^{\max}$ to denote the highest, stationary level of output consistent with both the seller's and buyer's participation constraints. It is the positive solution to $\lambda r v(y^{\max}) = \alpha [u(y^{\max}) - v(y^{\max})]$.

Proposition 9 (c.e.a. under pairwise meetings)

1. If $y^* \leq y^{\max}$, then any c.e.a. is such that $y_t = y^*$ and $\ell_t \in [v(y^*), \bar{\ell}]$ for all $t \in \mathbb{N}_0$, where $\bar{\ell} = \alpha [u(y^*) - v(y^*)] / \lambda r$.
2. If $y^* > y^{\max}$, then the c.e.a. is such that $y_t = y^{\max}$ and $\ell_t = v(y_t)$ for all $t \in \mathbb{N}_0$.

If agents are sufficiently patient (r low) and if the temptation to renege is not too large (λ low), then the first-best allocation is implementable.³¹ In contrast, if $\lambda r > \alpha [u(y^*)/v(y^*) - 1]$, then the c.e.a. is $y_t = y^{\max} < y^*$, which corresponds to the highest steady state. The c.e.a. can be implemented by having

³⁰Kocherlakota (1996) and Gu et al. (2013a, Section 7) study a Pareto problem to determine a contract curve linking the expected discounted utilities of buyers and sellers. In contrast the planner's objective in our model is a social welfare function that aggregates the buyers' and sellers' utilities. One can interpret this social welfare function as the ex ante expected utility of a representative agent in a version of the model where the role of an agent in the DM is determined at random in each period.

³¹Hu, Kennan, and Wallace (2009) derive the same condition for pure monetary economies in the case where $\lambda = 1$. A difference, however, is that the game where buyers make take-it-or-leave-it offers is not the optimal mechanism in monetary economies.

buyers set the terms of the loan contract unilaterally, in which case $\ell_t = v(y_t)$ for all t .³² By giving all the bargaining power to buyers the planner relaxes participation constraints in the CM, which allows for higher levels of output. Moreover, the solvency constraint in the buyer's bargaining problem must be "not-too-tight," in accordance with AJ's Second Welfare Theorem. We summarize this implementation result in the following Corollary.

Corollary 5 (*Second Welfare Theorem for economies with pairwise meetings*) *The c.e.a. is implemented with take-it-or-leave-it offers by buyers under "not-too-tight" solvency constraints.*

5.2 Optimal mechanism with large-group meetings

Suppose next that agents meet in a centralized location in the DM. If we do not allow for defections with coalitions, then the planner's problem is subject to the same incentive constraints as before, (32) and (33), and Proposition 9 holds. However, the restriction according to which no coalition of agents can defect from the proposed allocation is binding when $v'' > 0$.³³ In order to prevent such defections we impose the core requirement in the DM or, equivalently, the competitive equilibrium outcome.³⁴ Hence, from (29) the terms of the loan contract are given by $\ell = \eta(y) = v'(y)y$.

The planner's problem, which is analogous to (31)-(33), is easier to solve when written recursively with the buyer's "promised utility," ω_t , as a new state variable.³⁵ Society's welfare, denoted $V(\omega)$, solves the following Bellman equation,

$$V(\omega) = \max_{y, \omega'} \{ \alpha [u(y) - v(y)] + \beta V(\omega') \} \quad (34)$$

$$\text{s.t.} \quad -\eta(y) + \beta \frac{\omega'}{\lambda} \geq 0 \quad (35)$$

$$\omega' \geq (1+r) \{ \omega - \alpha [u(y) - \eta(y)] \} \quad (36)$$

$$y \in [0, y^*], \quad \omega' \in [0, \bar{\omega}], \quad (37)$$

where $\bar{\omega} = \max_{y \in [0, y^*]} [u(y) - \eta(y)] / (1 - \beta)$ is an upper bound for the lifetime expected utility of a buyer.

Equation (35) is the buyer's participation constraint in the CM that replaces (32) taking into account the competitive pricing mechanism, $\ell = \eta(y) = v'(y)y$. The novelty is the promise-keeping constraint,

³²Notice, however, that the c.e.a. is not uniquely implemented by the optimal mechanism. There are a continuum of equilibria under "not-too-tight" solvency constraints converging to the autarky steady state. The c.e.a. is the only bounded sequence that does not converge to the autarky equilibrium. For related results in the context of the AJ model see Bloise et al. (2013).

³³Indeed, a buyer and two sellers can form a deviating coalition in which each seller produces $y_t/2$ at a total cost of $2v(y_t/2) < v(y_t)$ and the buyer compensates the sellers by offering them a positive surplus, $\ell_t/2 - v(y_t/2) > 0$.

³⁴See Hu et al. (2009) and Wallace (2013) for a related assumption in the context of monetary economies. The equivalence result between the core and competitive equilibrium allocations for economies with a continuum of agents was first shown by Aumann (1964). See supplementary appendix S3 for a proof of this equivalence result in the context of our model.

³⁵Our recursive formulation is similar to the self-generation technique in Abreu et al. (1990), which characterizes the set of payoffs generated by Perfect Public Equilibria. We go beyond their characterization by providing the set of allocations and debt limits.

(36), according to which the lifetime expected utility promised to the buyer along the equilibrium path, ω , is implemented by generating an expected surplus in the current period equal to $\alpha[u(y) - \eta(y)]$ and by promising $\beta\omega'$ for the future. In the Supplementary Appendix S4 we show that there is a unique V solution to (34)-(37) in the space of continuous, bounded and concave functions, and this solution is non-increasing. As a result, the maximum value for society's welfare is $V(0) = \max_{\omega \in [0, \bar{\omega}]} V(\omega)$, as the initial promised utility to the buyer is a choice variable.

We define two critical values for DM output:

$$\hat{y} = \arg \max_{y \in [0, y^*]} [u(y) - \eta(y)] \quad (38)$$

$$y^{\max} = \max\{y > 0 : \alpha[u(y) - \eta(y)] \geq r\lambda\eta(y)\}. \quad (39)$$

The quantity $\hat{y}$ is the output level that maximizes the buyer's surplus in the DM. The quantity $y^{\max}$ is the highest, stationary level of output that is consistent with the buyer's participation constraint in the CM. We assume that both $\hat{y}$ and $y^{\max}$ are well-defined and, for all $0 \leq y \leq y^{\max}$, $\alpha[u(y) - \eta(y)] \geq r\lambda\eta(y)$.

Proposition 10 (c.e.a. under centralized meetings) *Assume η is a convex function.*

1. If $y^* \leq y^{\max}$, then the c.e.a. is such that $y_t = y^*$ for all $t \in \mathbb{N}_0$.
2. If $y^{\max} \leq \hat{y} \leq y^*$, then the c.e.a. is such that $y_t = y^{\max}$ for all $t \in \mathbb{N}_0$.
3. If $\hat{y} < y^{\max} < y^*$ then there are two cases:
 - (a) If $\lambda \geq \alpha[1 - u'(y^{\max})/\eta'(y^{\max})]$, then the c.e.a. is such that $y_t = y^{\max}$ for all $t \in \mathbb{N}_0$.
 - (b) If $\lambda < \alpha[1 - u'(y^{\max})/\eta'(y^{\max})]$, then the c.e.a. is such that $y_0 \in (y^{\max}, y^*)$ and $y_t = y_1 \in (\hat{y}, y^{\max})$ for all $t \geq 1$, where (y_0, y_1) is the unique solution to

$$\max_{y_0, y_1} \left\{ u(y_0) - v(y_0) + \frac{u(y_1) - v(y_1)}{r} \right\} \quad (40)$$

$$s.t. \quad \eta(y_0) = \frac{\alpha[u(y_1) - \eta(y_1)]}{\lambda r}. \quad (41)$$

In accordance with "Folk theorems" for repeated games, provided that agents are sufficiently patient, $r \leq \alpha[u(y^*) - \eta(y^*)]/\lambda\eta(y^*)$, the first-best allocation is an equilibrium outcome. If $y^{\max} < y^*$ then the first best violates the buyers' participation constraint. In this case, the characterization of the c.e.a. depends on the ordering of $y^{\max}$ and $\hat{y}$. As shown in the left panel of Figure 11, if $y^{\max} \leq \hat{y}$ then a buyer's welfare and society's welfare are both increasing with y over $(0, y^{\max})$ and the highest steady state maximizes social welfare.

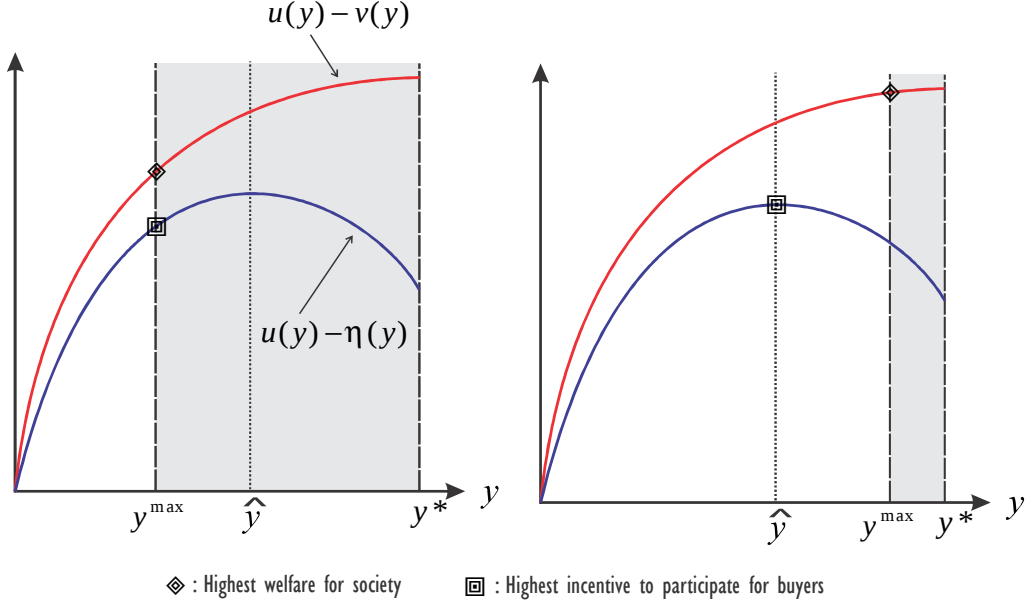


Figure 11: Left panel: No trade-off between efficiency and incentives over $[0, y^{\max}]$; Right panel: A trade-off between efficiency and incentives over $[\hat{y}, y^*]$.

We now turn to the case where $\hat{y} < y^{\max} < y^*$. For all $y \in (\hat{y}, y^*)$ the buyer's surplus, $u(y) - \eta(y)$, and society's surplus, $u(y) - v(y)$, covary negatively with y , as shown in the right panel of Figure 11. This negative relationship gives rise to a trade-off between social efficiency and incentives for debt repayment. As a result of this trade-off the highest steady state, $y^{\max}$, might no longer be the PBE outcome that maximizes social welfare.

It is shown in the proof of Proposition 10 that it is always socially optimal to keep future output constant, $y_t = y_1$ for all $t \geq 1$. Moreover, if the first best cannot be achieved, then the buyer's participation constraint at $t = 0$ must be binding since otherwise y_0 could be raised without affecting any future incentive constraints. As a result of these two properties the buyer's participation constraint at $t = 0$ is given by (41). If $y_1 > \hat{y}$, (41) gives a trade-off between current and future output. The magnitude of this trade-off in the neighborhood of the highest steady state is:

$$\left. \frac{dy_1}{dy_0} \right|_{y^{\max}} = \frac{\lambda r \eta'(y^{\max})}{\alpha [u'(y^{\max}) - \eta'(y^{\max})]} < 0.$$

When $\lambda \geq \alpha [1 - u'(y^{\max})/\eta'(y^{\max})]$ exploiting this trade-off is harmful since one would have to implement a large drop in future output below $y^{\max}$ in order to raise current output by a small amount above $y^{\max}$ while maintaining the buyer's incentive to repay his debt.

In contrast, when λ is small, it is optimal to exploit the trade-off between current and future output arising from (41). The optimal allocation is such that y_0 is larger than $y^{\max}$ while y_1 is lower than $y^{\max}$. Even though,

in future periods, society would be better-off at the highest steady state, $u(y_1) - v(y_1) < u(y^{\max}) - v(y^{\max})$, buyers enjoy a higher surplus, $u(y_1) - \eta(y_1) > u(y^{\max}) - \eta(y^{\max})$, which relaxes their incentive constraint for repayment at $t = 0$. As a result, output and society's welfare in the initial period are higher than the highest steady-state levels, $u(y_0) - v(y_0) > u(y^{\max}) - v(y^{\max})$.³⁶

In Figure 12 we illustrate the determination of (y_0, y_1) . The red curve labelled IR corresponds to (41). It slopes downward because of the trade-off between current and future output described above. By definition the IR curve intersects the 45°-line at $y^{\max}$. The blue curve labelled FOC corresponds to the first-order condition of the problem (40)-(41). Given the strict concavity of the surplus function it is optimal to smooth consumption by increasing y_0 when y_1 increases. When λ is low the FOC curve is located above the IR curve at $y_1 = y^{\max}$. Hence, the optimal solution, denoted (y_0^{**}, y_1^{**}) , is such that $y_0^{**} > y^{\max}$ and $y_1^{**} < y^{\max}$. The next Corollary reviews the role of "not-too-tight" solvency constraints to implement a constrained-efficient allocation.

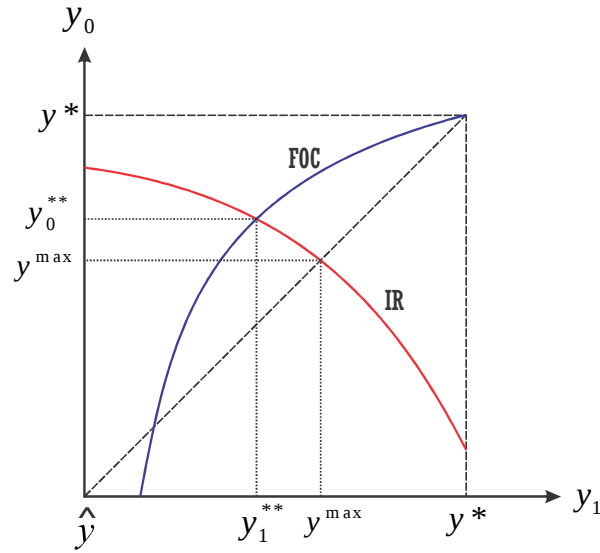


Figure 12: Determination of the constrained-efficient allocation, (y_0, y_1)

Corollary 6 (Second Welfare Theorem under large-group meetings) *Assume that η is a convex function.*

1. *If either $y^{\max} \leq \hat{y} \leq y^*$ or $\hat{y} < y^{\max} < y^*$ and $\lambda \geq \alpha [1 - u'(y^{\max})/\eta'(y^{\max})]$, then the c.e.a. is implemented with "not-too-tight" solvency constraints.*

³⁶Kehoe and Levine (1993) provide an example where partial exclusion leads to a welfare-improving outcome. See their Example 2 on p. 875. In the Supplementary Appendix S5 we conduct a similar analysis for different trading mechanisms, η . We show that the results obtained under Nash bargaining are qualitatively similar to the ones obtained in this section under competitive pricing.

2. If $\hat{y} < y^{\max} < y^*$ and $\lambda < \alpha [1 - u'(y^{\max})/\eta'(y^{\max})]$, then the c.e.a. is implemented with slack repayment constraints (i.e., "too-tight" solvency constraints) in all future periods, $t \geq 1$.

The failure of the AJ Welfare Theorem in the second part of Corollary 6 is surprising as one would conjecture that higher debt limits allow society to generate larger gains from trade. This reasoning is valid in a static sense. If d_t increases, the sum of all surpluses in period t , $\alpha [u(y_t) - v(y_t)]$, increases. However, there is a general equilibrium effect according to which more IOUs are competing for DM goods, which raises the price of DM goods, $p_t = v'(y_t)$. If the economy is close enough to the first best, this pecuniary externality lowers the buyers' welfare (even though society as a whole is better off) and worsens their incentive to repay their debt in earlier periods.

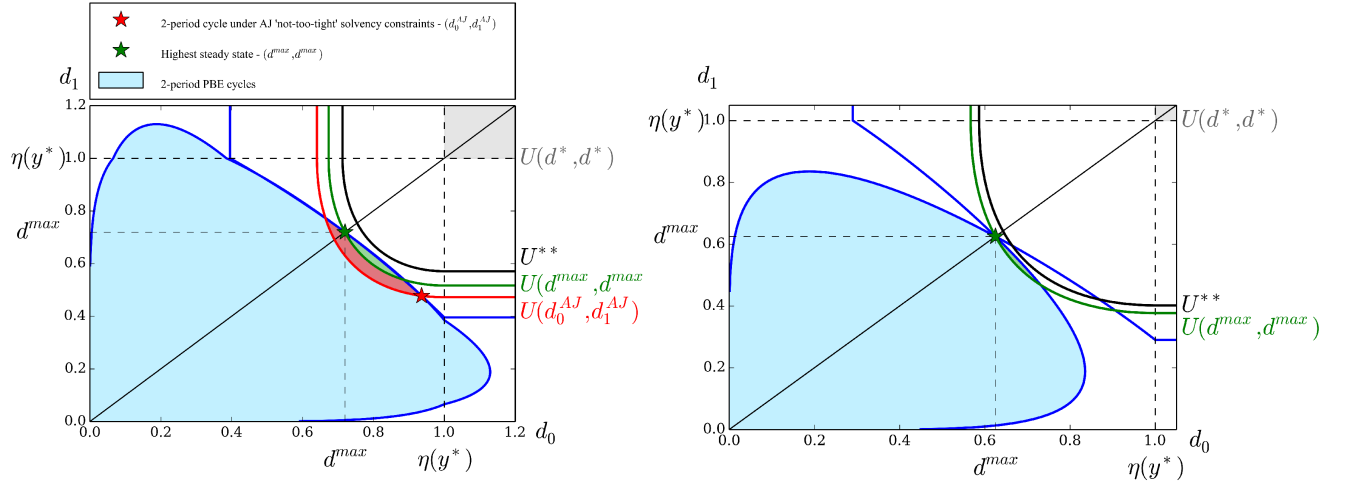


Figure 13: The blue area is set of all 2-period cycles. The red star is the 2-period cycle in GMMW and the green star is the highest steady state. Left panel: $\lambda = 1/6$; Right panel: $\lambda = 1/4$.

The results in Proposition 10 are robust if we restrict the equilibrium set to 2-period cycles. To see this we adopt the numerical example from the left panel of Figure 10, $\gamma = 2.1$, $\beta = 0.4$, $\lambda = 1/6$. For these parameter values $\lambda < \alpha [1 - u'(y^{\max})/\eta'(y^{\max})]$. Society's welfare over a 2-period cycle is measured by $u[y(d_0)] - v[y(d_0)] + \beta \{u[y(d_1)] - v[y(d_1)]\}$. In the left panel of Figure 13 we highlight in red and green the set of 2-period cycles, (d_0, d_1) , that dominate the equilibria under "not-too-tight" solvency constraints. There exist a continuum of such cycles that feature slack participation constraints. Hence, the imposition of "not-too-tight" solvency constraints eliminates good equilibria. Moreover, we represent society's welfare at the c.e.a. with a black indifference curve. This curve lies outside of the set of 2-period cycles (the blue area), which confirms Part 3(b) of Proposition 10, i.e., the c.e.a. is not a 2-period cycle. The right panel of Figure 13 reduces λ from $\lambda = 1/6$ to $\lambda = 1/4$. The condition in Part 3(b) of Proposition 10 holds so that the

highest steady state is not constrained efficient. There is no credit cycle under the "not-too-tight" solvency constraints, but there are a continuum of cycles under "too-tight" constraints, a fraction of which dominate the highest steady state.

6 Conclusion

We have characterized the set of equilibrium outcomes of a pure credit economy and their welfare properties. The economy features intertemporal gains from trade that can be exploited with simple one-period loan contracts. Such contracts and their execution are publicly recorded. Agents interact either through random, pairwise meetings under various trading mechanisms, as in the New-Monetarist literature, or in competitive spot markets, as in AJ. In contrast with the existing literature we have shown that such economies exhibit a continuum of steady states and a continuum of endogenous cycles of any periodicity. Moreover, any equilibrium outcome of the pure monetary economy with no record-keeping but fiat money is an outcome of the pure credit economy, but the reverse is not true.

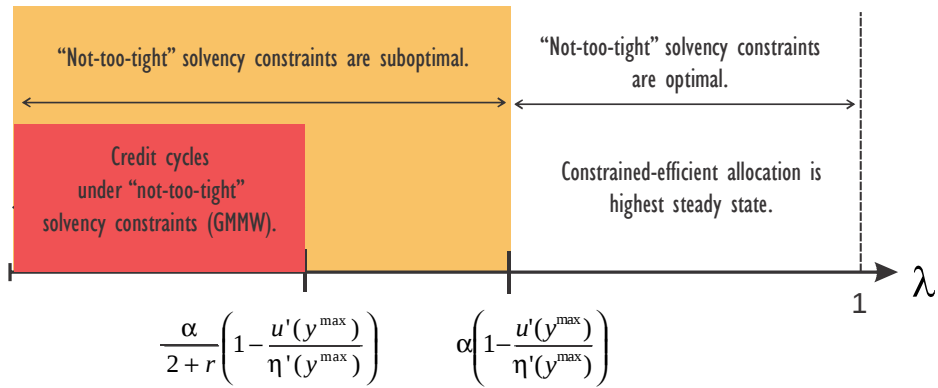


Figure 14: Optimality of "not-too-tight" solvency constraints and credit cycles under competitive pricing.

Finally, we have characterized the constrained-efficient allocations for economies with pairwise and large-group meetings. We generalized the AJ Second Welfare Theorem to economies with pairwise meetings by showing that constrained-efficient allocations are implemented with take-it-or-leave-it offers by buyers and "not-too-tight" solvency constraints. In contrast, under large group meetings the AJ Second Welfare Theorem fails when the temptation to renege (λ) is small. As shown in Figure 14, cycles under "not-too-tight" solvency constraints emerge for low values of λ (see GMMW), but for such values constrained efficiency requires slack participation constraints or, equivalently, "too-tight" solvency constraints. Hence, imposing the "not-too-tight" solvency constraint entails a loss in generality for both positive and normative analysis.

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APPENDIX: Proofs of lemmas and propositions

Proof of Proposition 1 ($\Rightarrow$) Here we prove necessity. Suppose that $\{(y_t, x_t, \ell_t)\}_{t=0}^\infty$ is an equilibrium outcome in a credit equilibrium, (s^b, s^s) .

(i) Here we show condition (4). Because the worst payoff to buyers at each period is 0 (autarky) while the equilibrium payoff at period t is $u(y_t) - x_t$, condition (4) is necessary for buyers to repay their promises at each period.

(ii) To show condition (5), we first show that $x_t = v(y_t)$ for all t . Note that (A3) implies that $x_t = \ell_t$ for all t . If $x_t < v(y_t)$, then the seller would not accept the offer. Suppose, by contradiction, that $x_t > v(y_t)$. Then, the buyer may deviate and offer (y', ℓ_t) with $v(y') \in (v(y_t), \ell_t)$. Because this deviation does not affect the buyer's public record and the buyer has the same incentive to repay his debt, it is dominant for the seller to accept it. It then is a profitable deviation because $y' > y_t$.

Next, to show that $y_t \leq y^*$ for all t , suppose, by contradiction, that $y_t > y^*$ and hence $u(y_t) \geq x_t \geq v(y_t) > v(y^*)$. Then there exists an alternative offer, $(y', \ell') = (y', x')$, such that $u(y') - x' > u(y_t) - x_t$ and $-v(y') + x' > -v(y_t) + x_t$ and $\ell' \leq \ell_t$. It is dominant for the seller to accept this alternative offer. The seller's payoff at the current period is 0 if he rejects. However, if he accepts, then by (A3), the threshold rule for repayment, the buyer will repay his promise $\ell' = x'$. Then, by accepting the offer the seller obtains $-v(y') + x' > 0$. Thus, (y', ℓ') is a profitable deviation for the buyer.

($\Leftarrow$) Here we show sufficiency. Let $\{(y_t, x_t, \ell_t)\}_{t=0}^\infty$ be a sequence satisfying (4) and (5). Consider (s^b, s^s) given as follows. Buyers can be in two states, $\chi_{i,t} \in \{G, A\}$, and each buyer's initial state is $\chi_{i,0} = G$. The law of motion of the buyer i 's state are given by:

$$\chi_{i,t+1} [(\ell', x', i), \chi_{i,t}] = \begin{cases} A & \text{if } x' < \min(x_t, \ell') \text{ or } \chi_{i,t} = A \\ G & \text{otherwise} \end{cases} \quad (42)$$

The strategies are such that $s_{t,1}^b(\rho_t^i) = (y_t, \ell_t)$ if the state for ρ_t^i is G and $s_{t,1}^b(\rho_t^i) = (0, 0)$ otherwise; $s_{t,2}^b(\rho_t^i, (y', \ell'), yes) = \min\{\ell', \ell_t\}$ if the state for ρ_t^i is G and $s_{t,2}^b(\rho_t^i, (y', \ell'), yes) = 0$ otherwise; $s_t^s(\rho_t^i, (y', \ell')) = yes$ if the state for ρ_t^i is G and $v(y') \leq \min\{\ell', \ell_t\}$, and $s_t^s(\rho_t^i, (y', \ell')) = no$ otherwise. We show that (s^b, s^s) is a credit equilibrium.

Given s^b, s^s is optimal: the seller expects a buyer in state G to repay up to ℓ_t at period t and hence he accepts an offer, (y', ℓ') , if $v(y') \leq \min\{\ell', \ell_t\}$; with buyers in state A he expects no repayment at all and hence rejects any offer. Next, we show that s^b is optimal given s^s . Consider a buyer with state A at the beginning of period t . Any offer to the seller is rejected and therefore it is optimal for the buyer to offer $(0, 0)$. Similarly, for such a buyer at the CM stage at period t with a promise ℓ' , his state will remain in A ,

independent of his repayment decision and hence it is optimal to repay nothing.

Now consider a buyer with state G at the CM stage of period t , with a promise ℓ' made to the seller. The buyer has to pay $\min\{\ell_t, \ell'\}$ to maintain state G . By (4), paying this amount is better than becoming an A person, whose continuation value is 0. Finally, consider a buyer with state G at the beginning of period t . Note that under s^b , his continuation value from period $t + 1$ onward is independent of his offer at period t . Moreover, for any offer (y, ℓ) , the seller accepts the offer if and only if $v(y) \leq \min\{\ell, \ell_t\}$. Thus, a buyer's problem is

$$\max_{(y, \ell)} u(y) - \min\{\ell, \ell_t\} \text{ s.t. } v(y) \leq \min\{\ell, \ell_t\}.$$

Because $\ell_t = v(y_t) \leq v(y^*)$, (y_t, ℓ_t) is a solution to the problem. $\square$

Proof of Corollary 1 ($\Leftarrow$) Here we show sufficiency. Let $\{d_t\}_{t=0}^\infty$ be a sequence satisfying (9) and (10). Then, we can determine the outcome, $\{(y_t, x_t, \ell_t)\}_{t=0}^\infty$, consistent with $\{d_t\}_{t=0}^\infty$ by the solution to the bargaining problem, (8), that is, $x_t = \ell_t = v(y_t) = \min\{v(y^*), d_t\}$ for each t . It remains to show that $\{(y_t, x_t, \ell_t)\}_{t=0}^\infty$ is the outcome of a credit equilibrium, (s^b, s^s) , with buyers' repayment strategy consistent with $\{d_t\}_{t=0}^\infty$. As in the proof of Proposition 1, the strategy follows a simple finite automaton with two states, $\chi_{i,t} \in \{G, A\}$, and each buyer's initial state is $\chi_{i,0} = G$. The law of motion of the buyer i 's state are given by:

$$\chi_{i,t+1} [(\ell', x', i), \chi_{i,t}] = \begin{cases} A & \text{if } x' < \min(d_t, \ell') \text{ or } \chi_{i,t} = A \\ G & \text{otherwise} \end{cases}. \quad (43)$$

This law of motion is the same as (42), where d_t replaces x_t . The strategies are analogous to those constructed in the proof of Proposition 1, but with d_t as the maximum amount of debt the buyer repays: at date t , the buyer offers (y_t, ℓ_t) in state G , the seller accepts the offer (y', ℓ') iff $v(y') \leq \ell' \leq d_t$ and the buyer's state is G , and the buyer repays $\min(\ell', d_t)$ in the CM in state G if ℓ' is the loan issued in DM. Following exactly the same logic as in the proof of Proposition 1, (9) and (10) ensure that (s^b, s^s) is a credit equilibrium.

($\Rightarrow$) Here we show necessity. Let $\{d_t\}_{t=0}^\infty$ be a sequence consistent with a credit equilibrium outcome, $\{(y_t, x_t, \ell_t)\}_{t=0}^\infty$. By definition, $\{d_t\}_{t=0}^\infty$ satisfies (10). To show (9), consider a buyer at period- t CM with a loan size $\ell' = d_t$ (perhaps on an off-equilibrium path). For repayment of d_t to be optimal in state G , (9) must hold, i.e., the buyer prefers repaying d_t to permanent autarky.

Proof of Corollary 2 Rewrite the incentive-compatibility constraint (11) at time $t + 1$ and multiply it by β to obtain:

$$\beta d_{t+1} \leq \beta^2 \{ \alpha [u(y_{t+2}) - v(y_{t+2})] + d_{t+2} \}. \quad (44)$$

Combining (11) and (44) we get:

$$d_t \leq \beta \{ \alpha [u(y_{t+1}) - v(y_{t+1})] \} + \beta^2 \{ \alpha [u(y_{t+2}) - v(y_{t+2})] \} + \beta^2 d_{t+2}.$$

By successive iterations we generalize the inequality above as follows:

$$d_t \leq \sum_{s=1}^T \beta^s \{ \alpha [u(y_{t+s}) - v(y_{t+s})] \} + \beta^{t+T} d_{t+T}. \quad (45)$$

By assumption, $\{d_t\}$ is bounded, $\lim_{T \rightarrow \infty} \beta^{t+T} d_{t+T} = 0$. Hence, by taking T to infinity, it follows from (45) that $\{d_t\}$ satisfies (9).

Proof of Proposition 2 Define the right side of (12) as a function

$$\Psi(d) = \alpha \{ u[z(d)] - v[z(d)] \}. \quad (46)$$

Ψ is continuous in d with $\Psi(0) = 0$ and $\Psi(d) = \alpha [u(y^*) - v(y^*)]$ for all $d \geq v(y^*)$. Moreover, it is differentiable with

$$\Psi'(d) = \alpha \left\{ \frac{u'[z(d)] - v'[z(d)]}{v'[z(d)]} \right\} \text{ if } d \in (0, v(y^*)), \text{ and } \Psi'(d) = 0 \text{ if } d > v(y^*).$$

This derivative is decreasing in d for all $d \in (0, v(y^*))$. Hence, Ψ is a concave function of d , and the set of values for d that satisfies (12) is an interval $[0, d^{\max}]$, where $d^{\max} \geq 0$ is the largest number that satisfies $\Psi(d^{\max}) = r d^{\max}$. Moreover, $d^{\max} > 0$ if and only if $\Psi'(0) > r$, which is always satisfied since $\Psi'(0) = \infty$ by assumption on preferences.

Proof of Lemma 1 Define the correspondence $\Gamma : \mathbb{R}_+ \supset \mathbb{R}_+$ as follows:

$$\Gamma(d) = \{x \in \mathbb{R}_+ : r(1 + \beta)x \leq \alpha \{u[z(d)] - v[z(d)]\} + \beta \alpha \{u[z(x)] - v[z(x)]\}\}. \quad (47)$$

Then, $\gamma(d) = \max \Gamma(d)$. First we show that $\Gamma(d)$ is a closed interval and γ is well-defined. By definition, $x \in \Gamma(d)$ if and only if

$$r(1 + \beta)x \leq \Psi(d) + \beta \Psi(x),$$

where $\Psi(d) = \alpha \{u[z(d)] - v[z(d)]\}$. Using a similar argument to that in Proposition 2, $\Gamma(d)$ is a closed interval with zero as the lower end point. Thus, γ is well-defined, and $\gamma(d)$ is the largest x that satisfies

$$r(1 + \beta)x = \Psi(d) + \beta \Psi(x). \quad (48)$$

Moreover, if $d > d'$, then $\Gamma(d') \subseteq \Gamma(d)$, and hence γ is a non-decreasing function: Because $\Psi(d)$ is constant for all $d \geq v(y^*)$, γ is constant for all $d \geq v(y^*)$, but it is strictly increasing for $d < v(y^*)$. Now we show that

$$\gamma(0) > 0, \quad \gamma(d^{\max}) = d^{\max},$$

where $d^{\max}$ is given in Proposition 2. First, as $\Psi(0) = 0$, and $\Psi(x)$ is a concave function, $\gamma(0) > 0$ if and only if $r(1 + \beta) < \Psi'(0) = \infty$, which holds by Inada conditions. Moreover, as the two curves $r(1 + \beta)x$ and $\beta\Psi(x)$ intersect at $\gamma(0) \equiv d^{\min} > 0$, by concavity of Ψ we have $\beta\Psi'(d^{\min}) < r(1 + \beta)$. Second, by Proposition 2, $d^{\max} > 0$ and $rd^{\max} = \Psi(d^{\max})$. Therefore, $r(1 + \beta)d^{\max} = \Psi(d^{\max}) + \beta\Psi(d^{\max})$ and hence $\gamma(d^{\max}) = d^{\max}$.

Finally, we show that γ is a concave function. Applying the implicit function theorem to (48), for all $0 < d < v(y^*)$,

$$\gamma'(d) = \frac{\Psi'(d)}{(1 + \beta)r - \beta\Psi'[\gamma(d)]}.$$

Note that $(1 + \beta)r - \beta\Psi'[\gamma(0)] = (1 + \beta)r - \beta\Psi'(d^{\min}) > 0$ and hence $(1 + \beta)r - \beta\Psi'[\gamma(d)] > 0$ for all d . By concavity of Ψ , $\gamma'(d)$ is decreasing in d . Hence, γ is a concave function.

Proof of Proposition 3 Notice that, by definition, any pair (d_0, d_1) that satisfies $d_0 \leq \gamma(d_1)$ and $d_1 \leq \gamma(d_0)$ also satisfies (13) with $y_0 = z(d_0)$ and $y_1 = z(d_1)$, and hence (d_0, d_1) is a 2-period credit cycle. By Lemma 1, γ is a concave function with $\gamma(0) > 0$ and $\gamma(d^{\max}) = d^{\max}$, and hence, $\gamma(d) > d$ for all $d \in [0, d^{\max})$, where $d^{\max}$ is given in Proposition 2. Thus, for each $d_0 \in [0, d^{\max})$, the interval $[d_0, \gamma(d_0)]$ is nondegenerate and $\gamma(d_0) < d^{\max}$. Hence, for each $d_1 \in [d_0, \gamma(d_0)]$, $d_0 \leq d_1 < \gamma(d_1)$, where we used that $\gamma(d) > d$ for all $d \leq \gamma(d_0) < d^{\max}$, so (d_0, d_1) is a 2-period credit cycle. This gives a full characterization of the set of 2-period cycles with $d_0 \leq d_1$, and the set of cycles with $d_1 \leq d_0$ is its mirror image with respect to the 45° line. Thus, for each $d_0 \in [0, d^{\max})$, the set $\Omega(d_0)$ is a nondegenerate interval.

Proof of Corollary 3 As shown earlier, a pair (d_0, d_1) is a 2-period cycle if and only if $d_0 \leq \gamma(d_1; \alpha, r)$ and $d_1 \leq \gamma(d_0; \alpha, r)$, where γ is given by Lemma 1. Note that here we make the parameters (α, r) explicit. By Proposition 3, for all $d_0 \in [0, d^{\max})$, there exists a continuum of d_1 such that (d_0, d_1) is a 2-period cycle. Now, $d_1 \leq \gamma(d_0; \alpha, r)$ if and only if

$$\frac{r(2 + r)}{1 + r}d_1 \leq \Psi(d_0; \alpha) + \frac{1}{1 + r}\Psi(d_1; \alpha), \quad (49)$$

where $\Psi(d; \alpha) = \alpha\{u[z(d)] - v[z(d)]\}$. By the proof of Proposition 3, for each $d_0 \in [0, d^{\max})$, (d_0, d_1) is a 2-period cycle with $d_0 \leq d_1$ if and only if d_1 satisfies (49). Let $\bar{\Omega}(d_0; \alpha, r)$ be the set of such d_1 . Because of symmetry between d_0 and d_1 in a 2-period cycle, it suffices to show that $\bar{\Omega}(d_0; \alpha, r)$ expands as α increases and as r decreases. Because $\Psi(d; \alpha)$ is strictly increasing in α , it follows that for any $\alpha' > \alpha''$, $d_1 \in \bar{\Omega}(d_0; \alpha'', r)$ implies that $d_1 \in \bar{\Omega}(d_0; \alpha', r)$, but there exists $d_1 \in \bar{\Omega}(d_0; \alpha', r)$ that is not in $\bar{\Omega}(d_0; \alpha'', r)$, that is, $\bar{\Omega}(d_0; \alpha'', r) \subsetneq \bar{\Omega}(d_0; \alpha', r)$. Similarly, because the left-side of (49) is increasing in r but the right-

side is decreasing in r , for any $r' > r''$, $d_1 \in \bar{\Omega}(d_0; \alpha, r')$ implies that $d_1 \in \bar{\Omega}(d_0; \alpha, r'')$, but there exists $d_1 \in \bar{\Omega}(d_0; \alpha, r'')$ that is not in $\bar{\Omega}(d_0; \alpha, r')$, that is, $\bar{\Omega}(d_0; \alpha, r') \subsetneq \bar{\Omega}(d_0; \alpha, r'')$.

Proof of Corollary 4 Note that $d^{\max} \leq v(y^*)$ if and only if $r \geq \alpha[u(y^*) - v(y^*)]/v(y^*)$. We prove the two cases separately.

Case 1: $d^{\max} \leq v(y^*)$. Consider a 2-period cycle, (d_0, d_1) , with $d_0 \leq d_1$. By Proposition 3, $d_0 \leq d_1 \leq \gamma(d_0) \leq d^{\max} \leq v(y^*)$. Thus, by (8), the loan contract is given by $\ell_t = d_t$ for $t = 0, 1$. The case $d_0 > d_1$ is completely symmetric.

Case 2: $d^{\max} > v(y^*)$. Consider a 2-period cycle, (d_0, d_1) , with $d_0 \leq d_1$. If $d_0 \leq d_1 \leq v(y^*)$, then using identical arguments as case (1) above, we can show that the borrowing constraints always bind. Because, as shown in Lemma 1, $\gamma[v(y^*)] = d^{\max} > v(y^*)$, there exists a unique $\hat{d}_0 < v(y^*)$ such that $\gamma(\hat{d}_0) = v(y^*)$. Then, for any $d_0 \in (\hat{d}_0, d^{\max}]$ and for any $d_1 \in [d_0, \gamma(d_0)]$, $d_1 > v(y^*)$, and hence, by (8), $\ell_1 = v(y^*) < d_1$. Thus, for any 2-period cycle, (d_0, d_1) , with $\hat{d}_0 < d_0 \leq v(y^*)$ and $d_1 \in [d_0, \gamma(d_0)]$, the borrowing constraint is slack in odd periods but binds in even periods. For any 2-period cycle, (d_0, d_1) , with $v(y^*) < d_0$ and $d_1 \in [d_0, \gamma(d_0)]$, the borrowing constraints are slack in all periods. The case $d_0 > d_1$ is symmetric.

Proof of Proposition 4 By a similar argument to Proposition 3, a T -tuple, $(d_0, \dots, d_{T-1}) \in \mathbb{R}_+^T$, is a T -period credit cycle if and only if $d_t \leq \gamma_T(d_{t+1}, \dots, d_{t+T-1})$, where $\gamma_T(d_0, \dots, d_{T-2})$ is the largest x that satisfies

$$r \frac{\beta}{1 - \beta^T} x = \sum_{t=0}^{T-2} \beta^t \Psi(d_t) + \beta^{T-1} \Psi(x), \quad (50)$$

and $\Psi(x) = \alpha\{u[z(x)] - v[z(x)]\}$. The function $\gamma_T(d_0, \dots, d_{T-2})$ is a non-decreasing function in all its arguments. By similar arguments to Lemma 1, we can also show that $\gamma_T(d^{\max}, \dots, d^{\max}) = d^{\max}$ and $\gamma_T(d, \dots, d) > d$ for all $d \in [0, d^{\max}]$. Moreover, γ_T is a concave function. For each $d_0 \in [0, d^{\max}]$, define

$$\Omega_T(d_0) = \{(d_1, \dots, d_{T-1}) \in \mathbb{R}_+^{T-1} : d_t \leq \gamma_T(d_{t+1}, \dots, d_{t+T-1}) \text{ for all } t = 0, \dots, T\}.$$

The set $\Omega_T(d_0)$ is closed and bounded. Moreover, $(d_0, \dots, d_0) \in \Omega_T(d_0)$ where all inequalities in the definition above are strict inequalities. Hence, there exists an open ball with a positive radius centered at $(d_0, \dots, d_0)$ that is contained in $\Omega_T(d_0)$. Thus, $\Omega_T(d_0)$ has positive Lebesgue measure in $\mathbb{R}^{T-1}$. Finally, because γ_T is concave, $\Omega_T(d_0)$ is a convex set.

Proof of Proposition 5 Replace $d_t = \phi_t$ into the buyer's optimality condition in a monetary economy, (18), to get

$$d_t = \beta d_{t+1} \left[1 + \alpha \frac{u'(y_{t+1}) - v'(y_{t+1})}{v'(y_{t+1})} \right]. \quad (51)$$

The right side of (51), $[u'(y_{t+1}) - v'(y_{t+1})]/v'(y_{t+1})$, is the derivative of the function, $u[v^{-1}(d_{t+1})] - d_{t+1}$, with respect to d_{t+1} . From the strict concavity of the function and the fact that it is equal to 0 when evaluated at $d_{t+1} = 0$,

$$\frac{u'(y_{t+1}) - v'(y_{t+1})}{v'(y_{t+1})} d_{t+1} < u(y_{t+1}) - v(y_{t+1}). \quad (52)$$

From (51) and (52),

$$d_t < \beta \alpha [u(y_{t+1}) - v(y_{t+1})] + \beta d_{t+1}. \quad (53)$$

Iterating (53),

$$d_t < \sum_{j=1}^J \beta^j \alpha [u(y_{t+j}) - v(y_{t+j})] + \beta^J d_{t+J}. \quad (54)$$

Applying the transversality condition, $\lim_{J \rightarrow \infty} \beta^J d_{t+J} = 0$ to (54), we prove that the sequence, $\{d_t\}$, is a solution to (51) satisfies (9), and hence it is part of a credit equilibrium.

Proof of Proposition 6 Here we show that for any distribution over $\mathbb{X}$ with a full support, denoted by π , we have a continuum of sunspot equilibria indexed by $d \in (0, d^{\max})$. For any $d \in (0, d^{\max})$, we have

$$rd < \alpha \{u[z(d)] - v[z(d)]\}. \quad (55)$$

Fix an element $\chi_0 \in \mathbb{X}$ and let $\mathbb{X}_{-0} = \mathbb{X} - \{\chi_0\}$. Define the set

$$\Omega_{(\mathbb{X}, \pi)}(d_{\chi_0}) = \left\{ \langle d_{\chi}; \chi \in \mathbb{X}_{-0} \rangle : rd_{\chi} \leq \pi(\chi_0) \alpha \{u[z(d_{\chi_0})] - v[z(d_{\chi_0})]\} + \sum_{\chi \in \mathbb{X}_{-0}} \pi(\chi) \alpha \{u[z(d_{\chi})] - v[z(d_{\chi})]\} \text{ for all } \chi \in \mathbb{X} \right\}.$$

By (55), the sequence $\langle d_{\chi}; \chi \in \mathbb{X}_{-0} \rangle$ with $d_{\chi} = d_{\chi_0}$ for all $\chi \in \mathbb{X}_{-0}$ is in $\Omega_{(\mathbb{X}, \pi)}(d_{\chi_0})$ where all inequalities in the definition above are strict inequalities. Thus, the set $\Omega_{(\mathbb{X}, \pi)}(d_{\chi_0})$ contains an open ball with a positive radius centered at $\langle d_{\chi}; \chi \in \mathbb{X}_{-0} \rangle$ with $d_{\chi} = d_{\chi_0}$ for all $\chi \in \mathbb{X}_{-0}$. Hence, it has a positive Lebesgue measure in $\mathbb{R}^{|\mathbb{X}_{-0}|}$ and almost all points in it satisfy $d_{\chi} \neq d_{\chi'}$ for all $\chi \neq \chi'$. Note that for any $\langle d_{\chi}; \chi \in \mathbb{X}_{-0} \rangle \in \Phi(d)$, $\langle d_{\chi}; \chi \in \mathbb{X} \rangle$ with $d_{\chi_0} = d$ is a sunspot credit equilibrium by (21).

Proof of Proposition 7 In the main text we have shown that (22) is necessary for the buyer to deliver their promised output in the DM, and, by (25), for a given sequence of debt limits, $\{d_t\}$, the equilibrium amount of loan is given by $\ell_t = \eta[z(d_t)]$ and hence $u(y_t) - \ell_t = \theta[u(y_t) - v(y_t)]$. Thus, (22) becomes (26). This proves the necessity. For sufficiency, let $\{(y_t, \ell_t, d_t)\}$ be a sequence satisfying (25) and (26). We use the same

strategies as in the proof of Corollary 1, but we have to modify s_2^b in accordance with the new environment. As the buyer makes the production decision in the DM, the buyer strategy s_2^b becomes a delivery strategy that specifies the amount of the output that the buyer delivers to the seller in the CM, the difference being what the buyer consumes. Analogous to the strategies in Corollary 1, s_2^b satisfies the following threshold property. If ℓ' is the amount of output that the buyer promises to deliver in the period- t DM, then his actual delivery is $x' = \min\{\ell', d_t\}$. As in the proof of Corollary 1, (26) ensures that this delivery strategy is optimal.

Proof of Proposition 8 From (26), a pair (d_0, d_1) is a 2-period credit cycle equilibrium outcome if and only if $d_0 \leq \gamma(d_1)$ and $d_1 \leq \gamma(d_0)$, where $\gamma(d)$ is the largest x that satisfies

$$r\lambda(1 + \beta)x = \Psi(d) + \beta\Psi(x), \quad (56)$$

and

$$\Psi(d) = \alpha\theta \{u[z(d)] - v[z(d)]\}.$$

The left side of the equation in (56), $r\lambda(1 + \beta)x$, is linear and increasing while the right side, $\beta\alpha\theta \{u[z(x)] - v[z(x)]\}$, is non-decreasing and concave. Given that the first term on the right side is non-negative, $\gamma(d)$ is well-defined. Note that $\gamma(d^{\max}) = d^{\max}$, where $d^{\max}$ is defined as the highest solution to $r\lambda d = \Psi(d)$, and, by similar arguments to Lemma 1, we can show γ is concave.

Note that $d^{\max} > 0$ if and only if $\Psi'(0) > r\lambda$. Now,

$$\Psi'(d) = \alpha\theta \left[\frac{u'[z(d)] - v'[z(d)]}{(1 - \theta)u'[z(d)] + \theta v'[z(d)]} \right]$$

and hence $\Psi'(0) = \alpha\theta/(1 - \theta)$, which shows that $d^{\max} > 0$ if and only if $r\lambda < \alpha\theta/(1 - \theta)$.

When $d^{\max} > 0$, i.e., when $r\lambda < \alpha\theta/(1 - \theta)$, for each $0 < d_0 < d^{\max}$, $d_0 < \gamma(d_0)$, and, for each $d_1 \in (d_0, \gamma(d_0)]$, $d_0 < \gamma(d_0) \leq \gamma(d_1)$. So any such (d_0, d_1) is a 2-period cycle and there are continuum of them.

To show the existence of 2-period cycles with periodic credit shutdowns, we need to show that $\gamma(0) > 0$. From (56), $\gamma(0) > 0$ if and only if $r\lambda(1 + \beta) < \beta\Psi'(0) = \beta\alpha\theta/(1 - \theta)$, which corresponds to the condition $r < \sqrt{1 + \alpha\theta/[\lambda(1 - \theta)]} - 1$. Given that $\gamma(0) > 0$, any $(d_0, d_1) \in \{0\} \times (0, \gamma(0))$ is a credit equilibrium where credit shuts down in even periods.

Proof of Proposition 9 (1) Suppose that $y^* \leq y^{\max}$. Then, the outcome $\{(y_t, \ell_t)\}_{t=0}^{\infty}$ with $y_t = y^*$ and $\ell_t = v(y_t)$ for all t is implementable.

(2) Suppose that $y^* > y^{\max}$. We show that the optimal sequence that has $y_t = y^{\max}$ and $\ell_t = v(y_t)$ for all t . Suppose, by contradiction, that there is another sequence $\{y'_t, \ell'_t\}_{t=0}^{\infty}$ satisfying (32) and (33) with

a strictly higher welfare. It then follows that $y^* \geq y'_t > y^{\max}$ for some t . Let t_0 be the first t such that $u(y'_t) - v(y'_t) > u(y^{\max}) - v(y^{\max})$. Now we show that for some $t_1 > t_0$, $y'_{t_1} > y'_{t_0}$. Suppose, by contradiction, that $y'_t \leq y'_{t_0}$ for all $t > t_0$. We have the following inequality,

$$v(y'_{t_0}) \leq \ell'_{t_0} \leq \lambda^{-1} \sum_{s=1}^{+\infty} \beta^s \alpha [u(y'_{t_0+s}) - \ell'_{t_0+s}] \leq \lambda^{-1} \sum_{s=1}^{+\infty} \beta^s \alpha [u(y'_{t_0}) - v(y'_{t_0})],$$

where the first inequality follows from the seller's participation constraint, (33), at $t = t_0$, the second follows from the buyer's participation constraint, (32), and the third follows from $u(y'_{t_0+s}) - \ell'_{t_0+s} \leq u(y'_{t_0+s}) - v(y'_{t_0+s}) \leq u(y'_{t_0}) - v(y'_{t_0})$ since $u - v$ is increasing for $y < y^*$ and $\ell'_{t_0+s} \leq v(y'_{t_0+s})$ for all s . Because $y^{\max}$ is the maximal value of y'_{t_0} that equalizes the left side and the right side of this series of inequalities, it follows that $y'_{t_0} \leq y^{\max}$, a contradiction. So $y^* \geq y'_{t_1} > y'_{t_0}$ for some t_1 (and we choose $t_1 > t_0$ to be the first index for this to happen). By induction, we can then find a subsequence $\{y'_{t_i}\}$ that is strictly increasing and is bounded from above. So there exists a limit $\tilde{y} = \lim_{i \rightarrow \infty} y'_{t_i} > y^{\max}$. Hence, by monotonicity, we have for all i ,

$$rv(y'_{t_i}) \leq r\ell_{t_i} \leq \frac{\alpha[u(\tilde{y}) - v(\tilde{y})]}{\lambda},$$

and, by taking i to infinity, we have

$$rv(\tilde{y}) \leq \frac{\alpha[u(\tilde{y}) - v(\tilde{y})]}{\lambda}.$$

However, as explained above, this implies that $\tilde{y} \leq y^{\max}$, and this leads to a contradiction.

Proof of Proposition 10 The program that selects the best PBE is

$$\max_{\{y_t\}_{t=0}^{\infty}} \sum_{t=0}^{+\infty} \beta^t \alpha [u(y_t) - v(y_t)] \quad (57)$$

$$\text{s.t.} \quad \lambda \eta(y_t) \leq \alpha \sum_{s=1}^{+\infty} \beta^s [u(y_{t+s}) - \eta(y_{t+s})] \quad (58)$$

$$y_t \leq y^* \text{ for all } t = 0, 1, 2, \dots \quad (59)$$

(1) Suppose that $y^* \leq y^{\max}$. In this case, the outcome $\{y_t\}_{t=0}^{\infty}$ with $y_t = y^*$ for all t is implementable and hence is the c.e.a.

(2) Suppose that $y^* > y^{\max}$ but $y^{\max} \leq \hat{y}$. We show that the outcome $\{y_t\}_{t=0}^{\infty}$ with $y_t = y^{\max}$ for all t is the optimum. Suppose, by contradiction, that there is another outcome $\{y'_t\}_{t=0}^{\infty}$ satisfying (58) and (59) with a strictly higher welfare. First we show that $y'_t \leq \hat{y}$ for all t . Suppose, by contradiction, that there is a t such that $y'_t > \hat{y}$. Then, because $\hat{y} \geq y^{\max}$,

$$\lambda \eta(y'_t) > \lambda \eta(\hat{y}) \geq \sum_{s=1}^{\infty} \beta^s \alpha [u(\hat{y}) - \eta(\hat{y})] \geq \sum_{s=1}^{\infty} \beta^s \alpha [u(y'_{t+s}) - \eta(y'_{t+s})],$$

a contradiction to (58). Given that this alternative outcome can only lie in the range $[0, \hat{y}]$ and hence the trade surplus is increasing in the output, the rest of the arguments are exactly the same as those in the proof of Proposition 9.

(3) Suppose that $\hat{y} < y^{\max} < y^*$. In the supplementary Appendix S4 we show that the constrained-efficient allocation, $\{x_t, y_t\}$, can be determined recursively as follows:

$$V(\omega) = \max_{y, \omega'} \{ \alpha [u(y) - v(y)] + \beta V(\omega') \} \quad (60)$$

$$\text{s.t.} \quad -\eta(y) + \beta \frac{\omega'}{\lambda} \geq 0 \quad (61)$$

$$\beta \omega' \geq \{ \omega - \alpha [u(y) - \eta(y)] \} \quad (62)$$

$$y \in [0, y^*], \quad \omega' \in [0, \bar{\omega}], \quad (63)$$

with $\omega_0 = 0$, $\omega_{t+1} = \omega'(\omega_t)$, $y_t = y(\omega_t)$, and $x_t = \eta(y_t)$. Moreover, the value function V is unique, and it is nonincreasing and concave.

The Lagrangian associated with the above Bellman equation is

$$\begin{aligned} \mathcal{L} = & \alpha [u(y) - v(y)] + \beta V(\omega') + \xi \left(\beta \frac{\omega'}{\lambda} - \eta(y) \right) \\ & + \nu \{ \alpha [u(y) - \eta(y)] + \beta \omega' - \omega \}, \end{aligned} \quad (64)$$

where the Lagrange multipliers, ξ and ν , are non-negative. In general V may not be differentiable everywhere. However, because V is concave, the following first-order conditions are still necessary and sufficient for (y, ω') to be optimal (Clarke (1976), Theorems 1 and 2):

$$\alpha [u'(y) - v'(y)] - \xi \eta'(y) + \nu \alpha [u'(y) - \eta'(y)] = 0 \quad (65)$$

$$\beta V'_+(\omega') + \beta \frac{\xi}{\lambda} + \beta \nu \leq 0 \leq \beta V'_-(\omega') + \beta \frac{\xi}{\lambda} + \beta \nu, \quad (66)$$

where $V'_+(\omega') = \lim_{\omega \downarrow \omega'} V'(\omega)$ and $V'_-(\omega') = \lim_{\omega \uparrow \omega'} V'(\omega)$. Both $V'_+(\omega')$ and $V'_-(\omega')$ exist because of concavity. The envelope condition, provided that $V'(\omega)$ exists, is

$$V'(\omega) = -\nu. \quad (67)$$

We define two critical values for the buyer's promised utility:

$$\omega^{\max} = \frac{\alpha [u(y^{\max}) - \eta(y^{\max})]}{1 - \beta} \quad \text{and} \quad \bar{\omega} = \frac{\alpha [u(\hat{y}) - \eta(\hat{y})]}{1 - \beta}.$$

The first threshold is the buyer's life-time expected utility at the highest steady state, while the second is the maximum life-time expected utility achieved by the buyers across all steady states. Note that by the definition of $y^{\max}$, $\eta(y^{\max}) = \beta \omega^{\max} / \lambda$.

(a) $\lambda \geq \alpha [1 - u'(y^{\max})/\eta'(y^{\max})]$.

The following claim provides conditions under which the constrained-efficient allocation corresponds to the highest steady state. In order to establish this claim, we show that, for $\omega = 0$ and $\omega = \omega^{\max}$, the optimal solution to the maximization problem in (60)-(63) is $(\omega^{\max}, y^{\max})$.

Claim 1 *If $\hat{y} < y^{\max} < y^*$ and $\lambda \geq \alpha [1 - u'(y^{\max})/\eta'(y^{\max})]$, then the unique solution to (60)-(63) is*

$$V(\omega) = \frac{\alpha [u(y^{\max}) - v(y^{\max})]}{1 - \beta} \text{ if } \omega \in [0, \omega^{\max}], \quad (68)$$

$$= \frac{\alpha \{u[g(\omega)] - v[g(\omega)]\}}{1 - \beta} \text{ if } \omega \in (\omega^{\max}, \bar{\omega}), \quad (69)$$

where $g(\omega)$ is the unique solution to $\alpha[u(y) - \eta(y)] = (1 - \beta)\omega$ for all $\omega \in (\omega^{\max}, \bar{\omega})$.

The function V given by (68)-(69) is flat in the interval $[0, \omega^{\max}]$ and is strictly concave for all $\omega \in (\omega^{\max}, \bar{\omega})$, and hence is concave overall. To show the strict concavity, we compute $V''(\omega)$ for all $\omega \in (\omega^{\max}, \bar{\omega})$. By the Implicit Function Theorem, we have

$$g'(\omega) = \frac{1 - \beta}{\alpha \{u'[g(\omega)] - \eta'[g(\omega)]\}} < 0,$$

and hence

$$V'(\omega) = \frac{u'[g(\omega)] - v'[g(\omega)]}{u'[g(\omega)] - \eta'[g(\omega)]} \quad (70)$$

for all $\omega \in (\omega^{\max}, \bar{\omega})$. Thus,

$$\begin{aligned} V''(\omega) &= \frac{\{u''[g(\omega)] - v''[g(\omega)]\} \{u'[g(\omega)] - \eta'[g(\omega)]\}}{\{u'[g(\omega)] - \eta'[g(\omega)]\}^2} g'(\omega) \\ &+ \frac{-\{u'[g(\omega)] - v'[g(\omega)]\} \{u''[g(\omega)] - \eta''[g(\omega)]\}}{\{u'[g(\omega)] - \eta'[g(\omega)]\}^2} g'(\omega) < 0. \end{aligned}$$

Note that, for all $\omega \in (\omega^{\max}, \bar{\omega})$, $u'[g(\omega)] - \eta'[g(\omega)] < 0$ as $g(\omega) > \hat{y}$ and that $u'[g(\omega)] - v'[g(\omega)] > 0$ as $g(\omega) \leq y^{\max} < y^*$.

To prove that V satisfies (68) and (69), we consider two cases.

(i) Suppose that $\omega \in [0, \omega^{\max}]$. The solution to the maximization problem in (60)-(63) is given by $(\omega', y) = (\omega^{\max}, y^{\max})$. This solution is feasible because $(\omega^{\max}, y^{\max})$ satisfies (62) for all $\omega \leq \omega^{\max}$ and it satisfies (61) at equality. Next, we show that it satisfies (65)-(66) with $\nu = 0$ and

$$\xi = \frac{\alpha [u'(y^{\max}) - v'(y^{\max})]}{\eta'(y^{\max})} > 0.$$

The condition (65) holds by the definition of ξ . To establish (66), first note that $V'_-(\omega^{\max}) = 0$ and

$$V'_+(\omega^{\max}) \equiv \lim_{\omega \downarrow \omega^{\max}} V'(\omega) = \frac{u'(y^{\max}) - v'(y^{\max})}{u'(y^{\max}) - \eta'(y^{\max})}.$$

Thus, $V'_-(\omega^{\max}) + \xi/\lambda > 0$ and the first inequality in (66) holds if and only if

$$\begin{aligned} & V'_+(\omega^{\max}) + \frac{\xi}{\lambda} \\ &= \frac{1}{\lambda} \frac{[u'(y^{\max}) - v'(y^{\max})]}{\eta'(y^{\max})} \left\{ \alpha + \lambda \frac{\eta'(y^{\max})}{u'(y^{\max}) - \eta'(y^{\max})} \right\} \leq 0, \end{aligned}$$

and, because $\hat{y} < y^{\max} < y^*$ and hence $u'(y^{\max}) - v'(y^{\max}) > 0$ and $u'(y^{\max}) - \eta'(y^{\max}) < 0$, the last inequality holds if and only if

$$\alpha \left[\frac{u'(y^{\max})}{\eta'(y^{\max})} - 1 \right] \geq -\lambda,$$

that is, $\lambda \geq \alpha [1 - u'(y^{\max})/\eta'(y^{\max})]$. This implies V satisfies (68).

(ii) Suppose that $\omega \in (\omega^{\max}, \bar{\omega})$. Here we show that $(\omega', y) = (\omega, g(\omega))$ is the solution to the maximization problem in (60)-(63). This solution is feasible: (62) holds by construction; because $\omega' = \omega = \alpha[u(y) - \eta(y)]/(1 - \beta)$ and because $y = g(\omega) \leq y^{\max}$,

$$\lambda \eta(y) \leq \beta \alpha [u(y) - \eta(y)]/(1 - \beta),$$

(61) holds. Next, we show that the FOC's (65) and (66) are satisfied by $(\omega', y) = (\omega, g(\omega))$ with $\xi = 0$ and

$$\nu = -\frac{u'[g(\omega)] - v'[g(\omega)]}{u'[g(\omega)] - \eta'[g(\omega)]} > 0.$$

The FOC for y , (65), holds by the definition of ν . The FOC for ω' , (66), holds if and only if

$$\nu + V'(\omega) = 0,$$

which holds by (70). Thus, if $\omega_0 = \omega \in (\omega^{\max}, \bar{\omega})$, then the optimal sequence is $(\omega_t, y_t) = (\omega, g(\omega))$ for all t . Hence, $V(\omega)$ satisfies (69) for all $\omega \in (\omega^{\max}, \bar{\omega})$. Finally, V satisfies (69) at $\omega = \bar{\omega}$ by continuity.

(b) $\lambda < \alpha [1 - u'(y^{\max})/\eta'(y^{\max})]$.

We will show that $V(\omega)$ has the same closed-form solution as derived in claim 1 when $\omega > \omega^{\max}$. Given this observation, we will establish that if $\omega = 0$ then $\omega' > \omega^{\max}$ and y can be solved in closed form.

Claim 2 Suppose that $\hat{y} < y^{\max} < y^*$ and $\lambda < \alpha [1 - u'(y^{\max})/\eta'(y^{\max})]$. Then, there exists a unique (y_0, y_1) with $\hat{y} < y_1 < y^{\max} < y_0 < y^*$ that solves (40)-(41), and the unique V that solves (60)-(63) satisfies

$$V(\omega) = \alpha[u(y_0) - v(y_0)] + \frac{\beta}{1 - \beta} \alpha[u(y_1) - v(y_1)] \quad \text{if } \omega = 0, \quad (71)$$

$$= \frac{\alpha}{1 - \beta} \{u[g(\omega)] - v[g(\omega)]\} \quad \text{if } \omega \in [\omega^{\max}, \bar{\omega}], \quad (72)$$

where $g(\omega)$ is given in Part 1.

The fact that V satisfies (72) follows the proof of the second case in the claim in the proof of Part 1 and the Contraction Mapping Theorem. Note that by (72), $V'(\omega)$ is given by (70) for $\omega > \omega^{\max}$ and hence the proof there applies exactly.

Here we show (71). First we rewrite the problem in (60)-(63) at $\omega = 0$ as follows:

$$\max_{y, \omega'} \{ \alpha [u(y) - v(y)] + \beta V(\omega') \} \quad (73)$$

$$\text{s.t.} \quad -\eta(y) + \beta \frac{\omega'}{\lambda} \geq 0 \quad (74)$$

$$y \in [0, y^*], \quad \omega' \in [0, \bar{\omega}]. \quad (75)$$

Note that (62) is trivially satisfied when $\omega = 0$. Now, conjecturing that $\omega' \geq \omega^{\max}$, we can replace $V(\omega')$ by the expression given by (72), y by y_0 and $g(\omega')$ by y_1 , and transform the above problem to

$$\max_{(y_0, y_1) \in [0, y^*] \times [\hat{y}, y^{\max}]} \left\{ \alpha [u(y_0) - v(y_0)] + \alpha \frac{u(y_1) - v(y_1)}{r} \right\} \quad (76)$$

$$\text{s.t.} \quad -\eta(y_0) + \alpha \frac{u(y_1) - \eta(y_1)}{\lambda r} \geq 0, \quad (77)$$

which is exactly the same as (40)-(41). By the Kuhn-Tucker conditions, a pair (y_0, y_1) solves the above problem if it satisfies the following FOC and feasibility condition:

$$\frac{u'(y_0) - v'(y_0)}{\eta'(y_0)} = -\frac{\lambda}{\alpha} \left[\frac{u'(y_1) - v'(y_1)}{u'(y_1) - \eta'(y_1)} \right] \quad (78)$$

$$\alpha [u(y_1) - \eta(y_1)] = r \lambda \eta(y_0). \quad (79)$$

In order to show that the solution (y_0, y_1) is also a solution to the problem in (60)-(63) at $\omega = 0$ we only need to verify our conjecture,

$$\omega_1 = \frac{1}{1 - \beta} \alpha [u(y_1) - \eta(y_1)] > \omega^{\max},$$

because the necessary conditions, (78)-(79), are also sufficient by the concavity of V over its entire domain.

Now we show that there exists a unique pair (y_0, y_1) with $\hat{y} < y_1 < y^{\max} < y_0 < y^*$ that satisfies (78)-(79). For each $y_1 \in (\hat{y}, y^{\max}]$, define

$$h(y_1) = \eta^{-1} \left[\frac{\alpha}{r \lambda} [u(y_1) - \eta(y_1)] \right].$$

as the unique solution of y_0 to (79) for a given y_1 . Note that $h(y^{\max}) = y^{\max}$. For any $y_1 \in (\hat{y}, y^{\max}]$,

$$h'(y_1) = \frac{\alpha}{r \lambda} \frac{[u'(y_1) - \eta'(y_1)]}{\eta'[h(y_1)]} < 0.$$

Substituting y_0 by its expression given by $h(y_1)$ in the left side of (78), we rewrite (78) as $H(y_1) = 0$ where

$$H(y_1) = \frac{u'[h(y_1)] - v'[h(y_1)]}{\eta'[h(y_1)]} + \frac{\lambda}{\alpha} \left[\frac{u'(y_1) - v'(y_1)}{u'(y_1) - \eta'(y_1)} \right].$$

The function $H(y_1)$ is continuous and strictly increasing in $(\hat{y}, y^{\max}]$ with

$$\lim_{y_1 \downarrow \hat{y}} H(y_1) = -\infty,$$

and, at $y_1 = y^{\max}$, we have

$$\begin{aligned} H(y^{\max}) &= \frac{u'(y^{\max}) - v'(y^{\max})}{\eta'(y^{\max})} + \frac{\lambda}{\alpha} \left[\frac{u'(y^{\max}) - v'(y^{\max})}{u'(y^{\max}) - \eta'(y^{\max})} \right] \\ &= [u'(y^{\max}) - v'(y^{\max})] \left\{ \frac{1}{\eta'(y^{\max})} + \frac{\lambda}{\alpha} \left[\frac{1}{u'(y^{\max}) - \eta'(y^{\max})} \right] \right\} > 0 \end{aligned}$$

because $\lambda < \alpha [1 - u'(y^{\max})/\eta'(y^{\max})]$. Thus, by Intermediate Value Theorem, there exists a unique $y_1 \in (\hat{y}, y^{\max})$ such that $H(y_1) = 0$ and hence (78) holds for $(h(y_1), y_1)$, and $h(y_1) > y^{\max}$ as h is strictly decreasing with $h(y^{\max}) = y^{\max}$. This proves that there exists a unique pair (y_0, y_1) with $\hat{y} < y_1 < y^{\max} < y_0 < y^*$ that satisfies (78) and (79).

Finally, because $\hat{y} < y_1 < y^{\max} < y_0 < y^*$ and because $(\omega', y) = (\omega_1, y_0)$ with $\omega_1 = \alpha[u(y_1) - \eta(y_1)]/(1 - \beta)$ is the solution to the maximization problem in (60)-(63) for $\omega = 0$, V satisfies (71). $\square$

SUPPLEMENTARY APPENDICES

S1. Equivalence between monetary and credit equilibria

Here we extend the equivalence result, Proposition 5, to other trading mechanisms. We first consider bargaining in the pairwise meetings and then consider Walrasian pricing for large group meetings. We adopt the environment introduced in Section 4 without record-keeping. The monetary trades follow a similar pattern to that in Section 3.3: buyers who cannot commit to deliver goods in the CM use money to buy DM goods from sellers in the DM. They produce CM goods in the first stage of each period in order to sell them for money in the CM. Notice that the timing of producing CM goods (whether it takes place in the first or second stage of each period) is irrelevant for buyers' behavior because it is only incentive-feasible to sell these goods in the CM for money. Sellers use money obtained from DM sales to buy CM goods. Because $\lambda \leq 1$, buyers never produce CM goods for self-consumption. As a result, the parameter λ plays no role in monetary equilibria. So with no loss of generality we set $\lambda = 1$.

Bargaining Under a general bargaining solution represented by the function $\eta(y)$, the sequence for the values of money, $\{\phi_t\}$, solves

$$\max_{m \geq 0} \{\phi_t m + \beta \alpha [u(y_{t+1}) - \eta(y_{t+1})]\}$$

where $\phi_{t+1} m = \eta(y_{t+1})$ for all t . Replace $d_t = \phi_t$ for all t in the above problem and take the FOC, we obtain

$$d_t = \beta d_{t+1} \left\{ \alpha \left[\frac{u'(y_{t+1})}{\eta'(y_{t+1})} - 1 \right] + 1 \right\}, \quad (80)$$

where $\eta(y_t) = d_t$ for all t . In the credit economy, the debt limits, $\{d_t\}$, solves

$$d_t \leq \beta \{ \alpha [u(y_{t+1}) - \eta(y_{t+1})] + d_{t+1} \}. \quad (81)$$

Because η is concave, $u \circ \eta^{-1}(d_t) - d_t$ is concave in terms of the value of money. The right side of (80), $[u'(y_{t+1}) - \eta'(y_{t+1})] / \eta'(y_{t+1})$, is the derivative of the function, $u[\eta^{-1}(d_{t+1})] - d_{t+1}$, with respect to d_{t+1} .

From the strict concavity of the function and the fact that it is equal to 0 when evaluated at $d_{t+1} = 0$,

$$\frac{u'(y_{t+1}) - \eta'(y_{t+1})}{\eta'(y_{t+1})} d_{t+1} < u(y_{t+1}) - \eta(y_{t+1}). \quad (82)$$

From (80) and (82),

$$d_t < \beta \alpha [u(y_{t+1}) - \eta(y_{t+1})] + \beta d_{t+1}. \quad (83)$$

Iterating (83),

$$d_t < \sum_{j=1}^J \beta^j \alpha [u(y_{t+j}) - \eta(y_{t+j})] + \beta^J d_{t+J}. \quad (84)$$

Applying the transversality condition, $\lim_{J \rightarrow \infty} \beta^J d_{t+J} = 0$ to (84), we prove that the sequence, $\{d_t\}$, solution to (80) satisfies (81), and hence it is part of a credit equilibrium.

This concavity of η is satisfied for the proportional bargaining solution and for the general Nash bargaining solution under the functional forms for u and v that guarantee the concavity of the buyer's surplus.

Walrasian pricing Suppose the DM is competitive and p_t denotes the price of DM goods in terms of CM goods. In a monetary economy the buyer chooses money holdings as the solution to:

$$\max_{m, y_{t+1} \geq 0} \left\{ -\phi_t m + \beta \alpha [u(y_{t+1}) - p_{t+1} y_{t+1}] + \beta \phi_{t+1} m \right\}, \quad (85)$$

where, $\phi_{t+1} m \geq p_{t+1} y_{t+1}$. The first-order condition for (85) is

$$\phi_t = \beta \phi_{t+1} \left\{ \alpha \left[\frac{u'(y_{t+1})}{p_{t+1}} - 1 \right] + 1 \right\}.$$

From the seller's maximization problem, $p_{t+1} = v'(y_{t+1})$ so that $\{\phi_t\}$ solves

$$\phi_t = \beta \phi_{t+1} \left\{ \alpha \left[\frac{u'(y_{t+1})}{v'(y_{t+1})} - 1 \right] + 1 \right\}. \quad (86)$$

It should be noticed that it is the same first-order difference equation as the one obtained under buyers' take-it-or-leave-it offers. Notice, using $\phi_{t+1} = v'(y_{t+1}) y_{t+1}$ by market-clearing (i.e., $m = 1$), that

$$\phi_{t+1} \left[\frac{u'(y_{t+1})}{v'(y_{t+1})} - 1 \right] = u'(y_{t+1}) y_{t+1} - v'(y_{t+1}) y_{t+1} < u(y_{t+1}) - v(y_{t+1}) y_{t+1},$$

from the concavity of u . Recall that a sufficient condition for the sequence of debt limits to be a credit equilibrium is

$$d_t \leq \beta \{ \alpha [u(y_{t+1}) - v'(y_{t+1}) y_{t+1}] + d_{t+1} \}.$$

This proves that the phase of the monetary equilibrium is located to the left of the phase line of the credit equilibrium. Hence, by the same reasoning as before, any outcome of the monetary economy is an outcome of the credit economy.

S2. Existence of 2-period cycles under alternative mechanisms

Walrasian pricing Under Walrasian pricing, $\eta(y) = v'(y)y$. Here we show existence of a continuum of 2-period cycles when $\eta(y)$ is convex. Recall that $z(d) = \min\{\eta^{-1}(d), y^*\}$. Let $d^{\max}$ be the unique positive solution to

$$r\lambda d = \alpha \{u[z(d)] - \eta[z(d)]\}. \quad (87)$$

Lemma 2 *Suppose that $\eta(y)$ is convex. For each $d_0 \in [0, d^{\max})$, there is a nondegenerate interval, $\Omega(d_0)$, such that for any $d_1 \in \Omega(d_0)$, (d_0, d_1) is a (strict) 2-period cycle.*

Proof. Because $\eta(y)$ is convex, there is a unique positive number, denoted $y^{\max}$, such that

$$r\lambda\eta(y^{\max}) = \alpha[u(y^{\max}) - \eta(y^{\max})].$$

It can be verified that that $d^{\max}$ is given by

$$d^{\max} = \begin{cases} \eta(y^{\max}) & \text{if } y^* \geq y^{\max} \\ \frac{\alpha\{u(y^*) - \eta(y^*)\}}{r\lambda} & \text{otherwise.} \end{cases}$$

Note that any $d \in [0, d^{\max}]$ corresponds to a steady-state equilibrium. Let us turn to 2-period cycles. A pair, (d_0, d_1) , is a 2-period cycle if for $t = 0, 1$,

$$r\lambda d_t \leq \frac{\alpha \{u[z(d_{t+1})] - \eta[z(d_{t+1})]\} + \beta \alpha \{u[z(d_t)] - \eta[z(d_t)]\}}{1 + \beta}. \quad (88)$$

Hence,

$$\Omega(d_0) = \{d_1 \geq 0 : (d_0, d_1) \text{ satisfies (88)}\}.$$

For all $d \in [0, d^{\max})$, because $r\lambda d < \alpha \{u[z(d)] - \eta[z(d)]\}$, $(d_0, d_1) = (d, d)$ satisfies (88) with a strict inequality. Hence, by continuity, there is a nonempty open set contained in $\Omega(d)$. Moreover, because η is concave, the set $\Omega(d)$ is convex and hence is a nondegenerate interval. ■

Nash bargaining For all $y \leq y^*$, $u(y) - \eta(y) \geq \theta[u(y) - v(y)]$ and hence $\eta(y) \leq (1 - \theta)u(y) + \theta v(y)$. Under proportional bargaining a 2-period cycle solves

$$r\lambda [(1 - \theta)u(y_t) + \theta v(y_t)] \leq \frac{\{\alpha\theta [u(y_{t+1}) - v(y_{t+1})] + \beta\alpha\theta [u(y_t) - v(y_t)]\}}{1 + \beta}.$$

It implies

$$r\lambda\eta(y_t) \leq \frac{\{\alpha [u(y_{t+1}) - \eta(y_{t+1})] + \beta\alpha [u(y_t) - \eta(y_t)]\}}{1 + \beta}.$$

Hence (y_t, y_{t+1}) , and the associated $(d_t, d_{t+1}) = (\eta(y_t), \eta(y_{t+1}))$, is a credit cycle under generalized Nash bargaining.

S3. Core and competitive equilibrium

Recall that an allocation, $\mathcal{L} = \{(y(i), x(i)), (y(j), x(j)) : i \in \mathbb{B}, j \in \mathbb{S}\}$, where $(y(i), x(i))$ denotes buyer i 's DM and CM consumptions and $(y(j), x(j))$ denotes seller j 's DM and CM consumptions, is in the core if there is no blocking (finite) coalition, $\mathcal{I} \subset \mathbb{B} \cup \mathbb{S}$, such that each agent in $\mathcal{I}$ enjoys at least the same utility as his allocation in $\mathcal{L}$, but at least one of them is strictly better off. Now we show that the only core allocation is the competitive outcome, with debt limit, d , is given by the symmetric allocation, (y, ℓ) , such that $\ell = \eta(y) \equiv v'(y)y$ and $y = \min\{y^*, \eta^{-1}(d)\}$.

First notice that, by standard arguments, the competitive outcome is in the core. For necessity, we restrict ourselves to symmetric allocations. For a justification of such assumption, see Mas-Colell et al. (1995). Note that to be in the core, $u(y) \geq \ell \geq v(y)$. First we show that $\ell = v'(y)y$. Suppose, by contradiction, $\ell \neq v'(y)y$. Assume that $\ell < v'(y)y$. The other direction has a similar proof. Let ε be so small that

$$[v'(y) - \varepsilon]y > \ell. \quad (89)$$

Consider a coalition with m buyers and n sellers such that with $\delta = m/n < 1$, we have

$$\frac{v(y) - v(\delta y)}{(1 - \delta)y} > v'(y) - \varepsilon. \quad (90)$$

Consider the following allocation: each buyer consumes y and issues an IOU with face value ℓ , and each seller produces δy and receives an IOU with face value $\delta \ell$. Note that such allocation is feasible:

$$my = n\delta y \text{ and } m\ell = n\delta \ell.$$

Now, each buyer enjoys the same utility as before, but each seller has a higher utility: combining (89) and (90),

$$v(y) - v(\delta y) > [v'(y) - \varepsilon](1 - \delta)y > (1 - \delta)\ell,$$

and hence

$$\delta \ell - v(\delta y) > \ell - v(y).$$

This proves $\ell = v'(y)y = \eta(y)$. Finally, if $y < \min\{y^*, \eta^{-1}(d)\}$, then a buyer and a seller can form a coalition to increase surplus.

S4. Recursive formulation of the mechanism design problem

Here we show that we can solve the problem (57)-(59) recursively. First we show that recursive formulation with promised utility as a state variable is equivalent to the original sequence problem.

Lemma 3 *A sequence $\{y_t\}_{t=0}^\infty$ satisfies (58) and (59) if and only if there is a sequence $\{\omega_t\}_{t=0}^\infty$ such that, for all $t = 0, 1, 2, \dots$,*

$$\omega_t \leq \alpha[u(y_t) - \eta(y_t)] + \beta\omega_{t+1}, \quad (91)$$

$$\eta(y_t) \leq \beta\omega_{t+1}/\lambda, \quad (92)$$

$$y_t \in [0, y^*], \quad (93)$$

$$\omega_t \in [0, \bar{\omega}]. \quad (94)$$

Proof. Suppose that $\{y_t\}_{t=0}^\infty$ satisfies (58) and (59). Then, define, for each $t = 0, 1, 2, \dots$,

$$\omega_t = \sum_{s=0}^{\infty} \beta^s \alpha[u(y_{t+s}) - \eta(y_{t+s})]. \quad (95)$$

The right side of (58) is equal to $\beta\omega_{t+1}/\lambda$ for each t . Hence, $\{\omega_t, y_t\}_{t=0}^\infty$ satisfies (92). By definition of $\hat{y}$,

$$u(y_t) - \eta(y_t) \leq u(\hat{y}) - \eta(\hat{y}) \text{ for all } t \in \mathbb{N}_0.$$

It follows from (59) that $\{\omega_t\}_{t=0}^\infty$ satisfies (94). Finally, by (95),

$$\omega_t = \alpha[u(y_t) - \eta(y_t)] + \beta \sum_{s=0}^{\infty} \beta^s \alpha[u(y_{t+s+1}) - \eta(y_{t+s+1})] = \alpha[u(y_t) - \eta(y_t)] + \beta\omega_{t+1} \text{ for all } t \in \mathbb{N}_0.$$

Hence, $\{\omega_t, y_t\}_{t=0}^\infty$ satisfies (91).

Conversely, suppose that $\{\omega_t, y_t\}_{t=0}^\infty$ satisfies (91)-(94). Then, $\{y_t\}_{t=0}^\infty$ satisfies (59) by (93). To show (58), define, for each $t \in \mathbb{N}_0$,

$$\omega'_t = \sum_{s=0}^{\infty} \beta^s \alpha[u(y_{t+s}) - \eta(y_{t+s})]. \quad (96)$$

By (92), it suffices to show that $\omega_t \leq \omega'_t$ for all $t \geq 0$. Let t be given. We show by induction on T that

$$\omega_t \leq \sum_{s=0}^T \beta^s \alpha[u(y_{t+s}) - \eta(y_{t+s})] + \beta^{T+1} \omega_{t+T+1}. \quad (97)$$

When $T = 0$, this follows from (91). Suppose that it holds for T . Then,

$$\begin{aligned} \omega_t &\leq \sum_{s=0}^T \beta^s \alpha[u(y_{t+s}) - \eta(y_{t+s})] + \beta^{T+1} \omega_{t+T+1} \\ &= \sum_{s=0}^T \beta^s \alpha[u(y_{t+s}) - \eta(y_{t+s})] + \beta^{T+1} \{ \alpha[u(y_{t+T+1}) - \eta(y_{t+T+1})] + \beta\omega_{t+T+2} \} \\ &= \sum_{s=0}^{T+1} \beta^s \alpha[u(y_{t+s}) - \eta(y_{t+s})] + \beta^{T+2} \omega_{t+T+2}. \end{aligned}$$

This proves (97). Now, because, by (94), $\omega_{T+1} \leq \bar{\omega}$ for all T , it follows from the limit by taking T to infinity in (97) that $\omega_t \leq \omega'_t$. ■

Because of Lemma 3, we may replace the constraints (58) and (59) by (91)-(94). Note that the initial condition for the promised utility, ω_0 , is also a choice variable.

Define the planner's value function, $V(\omega)$, as follows:

$$V(\omega) = \max_{\{y_t\}_{t=0}^{\infty}} \sum_{t=0}^{+\infty} \beta^t \alpha [u(y_t) - v(y_t)]$$

subject to (91)-(94) with $\omega_0 = \omega$. From the Principle of Optimality V satisfies the following Bellman equations,

$$V(\omega) = \max_{y, \omega'} \{ \alpha [u(y) - v(y)] + \beta V(\omega') \} \quad (98)$$

$$\text{s.t.} \quad -\eta(y) + \beta \frac{\omega'}{\lambda} \geq 0 \quad (99)$$

$$\beta \omega' \geq \{ \omega - \alpha [u(y) - \eta(y)] \} \quad (100)$$

$$y \in [0, y^*], \omega' \in [0, \bar{\omega}]. \quad (101)$$

The proposition below shows that the above Bellman equation is well-defined and that V is uniquely determined. As a result, the maximization problem (57)-(59) is reduced to

$$\max_{\omega_0 \in [0, \bar{\omega}]} V(\omega_0).$$

Proposition 11 *Suppose that $y^* > y^{\max} > \hat{y}$.*

(1) *The value function V is the unique solution to (98)-(101), and is continuous and weakly decreasing in ω .*

(2) *The function V is concave in ω if η is convex.*

Proof. (1) First we show that, for any $\omega \in [0, \bar{\omega}]$, the set of elements $(y, \omega') \in [0, y^*] \times [0, \bar{\omega}]$ satisfying (99)-(101) is nonempty and hence the maximization problem is well-defined. For all $\omega \in [0, \bar{\omega}]$, define $y_\omega \leq \hat{y} \leq y^*$ as the unique solution to

$$\omega = \frac{\alpha}{1-\beta} [u(y_\omega) - \eta(y_\omega)]. \quad (102)$$

As $u(0) - \eta(0) = 0$ and $\frac{\alpha}{1-\beta} [u(\hat{y}) - \eta(\hat{y})] = \bar{\omega}$, such $y_\omega \in [0, \hat{y}]$ exists by the Intermediate Value Theorem. We claim that (y_ω, ω') satisfies (99)-(101) for any $\omega' \in [\omega, \bar{\omega}]$. First (101) holds by construction. Moreover, rearranging (102), we have

$$\beta \omega = \omega - \alpha [u(y_\omega) - \eta(y_\omega)]$$

which implies (100) for any $\omega' \geq \omega$. Finally, by (102) and the fact that $y \leq \hat{y} \leq y^{\max}$,

$$\eta(y_\omega) \leq \beta \frac{\omega}{\lambda} \leq \beta \frac{\omega'}{\lambda}$$

for any $\omega' \geq \omega$.

We now show that the Bellman equation (99)-(101) has a unique solution. Let $\mathcal{C}[0, \bar{\omega}]$ be the complete metric space of continuous functions over $[0, \bar{\omega}]$ equipped with the sup norm. Define $T : \mathcal{C}[0, \bar{\omega}] \rightarrow \mathcal{C}[0, \bar{\omega}]$ by

$$T(W)(\omega) = \max_{y, \omega'} \{ \alpha [u(y) - v(y)] + \beta W(\omega') \},$$

subject to (99)-(101). Note that $T(W) \in \mathcal{C}[0, \bar{\omega}]$ by the Theorem of Maximum. The mapping T satisfies the Blackwell sufficient condition (Lucas and Stokey, 1989, Theorem 3.3), and hence T is a contraction mapping, which admits a unique fixed point by the Banach Fixed-Point Theorem. Hence, V is the unique solution to the Bellman equation and is continuous.

Notice that by decreasing ω we increase the set of (y, ω') that satisfies (99)-(101), but without affecting the objective function. Hence, V is weakly decreasing.

(2) Assume now that η is convex. To show that V is concave, we show that T preserves concavity. Let $\omega_0, \omega_1 \in [0, \bar{\omega}]$ be given. Let (y_0, ω_0) and (y_1, ω_1) solves (99)-(101) for ω_0 and ω_1 , respectively. let $\epsilon \in (0, 1)$ be given. Then,

$$\begin{aligned} & T(W)(\epsilon\omega_0 + (1-\epsilon)\omega_1) \\ & \geq \alpha [u(\epsilon y_0 + (1-\epsilon)y_1) - v(\epsilon y_0 + (1-\epsilon)y_1)] + \beta W(\epsilon\omega'_0 + (1-\epsilon)\omega'_1) \\ & \geq \alpha \epsilon [u(y_0) - v(y_0)] + \alpha (1-\epsilon) [u(y_1) - v(y_1)] + \beta [\epsilon W(\omega'_0) + (1-\epsilon)W(\omega'_1)] \\ & = \epsilon T(W)(\omega_0) + (1-\epsilon)T(W)(\omega_1). \end{aligned}$$

The first inequality follows from the fact that $(\epsilon y_0 + (1-\epsilon)y_1, \epsilon\omega'_0 + (1-\epsilon)\omega'_1)$ also satisfies (99)-(101) for $\omega = \epsilon\omega_0 + (1-\epsilon)\omega_1$ because η is convex. The second inequality follows from the concavity of $u - v$ and the assumed concavity of W . ■

S5. Optima under arbitrary trading mechanism

We characterize the optimal credit equilibrium allocation taking the mechanism to set the terms of the loan contract, η , as given. Although Propositions 10 are obtained under competitive pricing, they hold for any arbitrary trading mechanism, η . For example, if η is determined by proportional bargaining, then $\hat{y} = y^*$, and Parts 1-2 of Proposition 10 imply that the best PBE corresponds to the highest steady state, $y_t = y^{\max}$ for all t . Proposition 10 also applies to generalized Nash bargaining: if $y^{\max} \leq \hat{y} \leq y^*$ or $y^* \leq y^{\max}$, then the best PBE is the highest steady state (in the proof of the proposition we only use the fact that $\hat{y}$ is the unique maximizer). However, under Nash bargaining, the loan contract η may not be convex in general, and hence Proposition 10 may not apply. Nevertheless, we showed that (34)-(37) defines a contraction mapping so that we can easily solve for the best PBE allocation numerically.

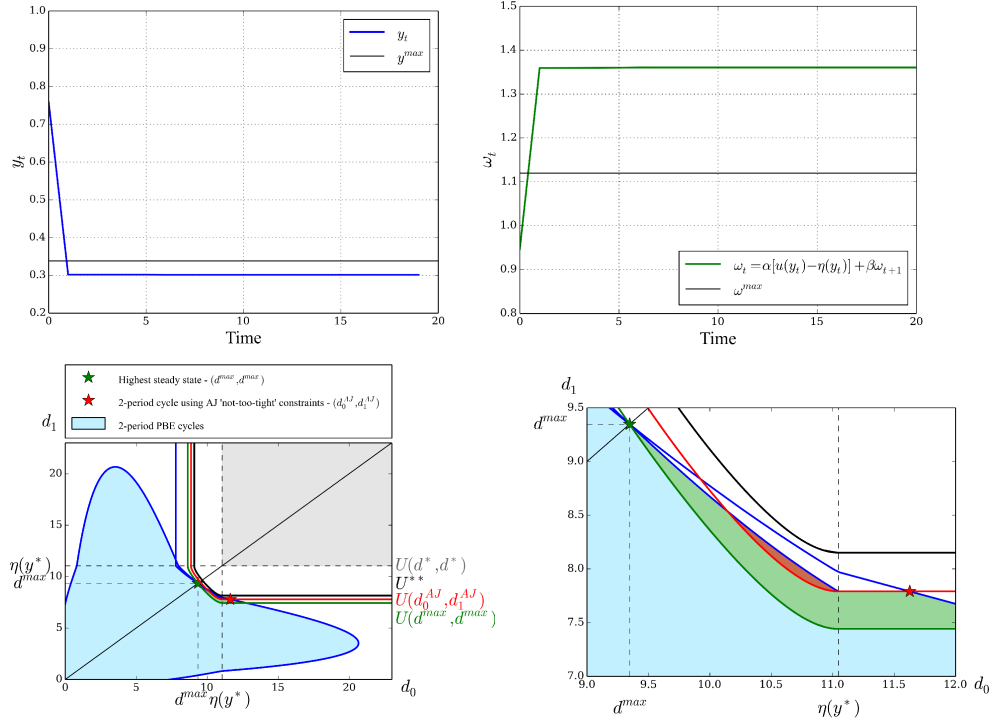


Figure 15: Top panels: Best PBE under Nash bargaining; Bottom panels: 2-period cycles and their welfare properties under Nash bargaining

In Figure 15 we adopt the same functional forms and parameter values as the ones in the bottom left panel of Figure 10. The top left panel plots y_t while the right panel plots V_t^b . It can be seen that the allocation that maximizes social welfare is non-stationary: $y_0 > y_1 = y_t$ for all $t \geq 1$, in accordance with Part 3(b) of Proposition 10. The logic for why the solution is non-stationary is similar to the one described

in the case of price taking. Given that the buyer's surplus is hump-shaped, one can implement a high level of output in the initial period by promising a high utility to buyers in the future, which is achieved by lowering future output. In the bottom panels of Figure 15 we represent the set of 2-period cycles under the same parametrization. There are a continuum of cycles that dominate the periodic equilibria obtained under "not-too-tight" solvency constraints (the red area) and the highest steady state (the green area).