

A new characterization of perfect public equilibrium payoffs in repeated games with imperfect public monitoring in continuous time^{*}

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Abstract

This paper continues the study of a new class of repeated games with imperfect public monitoring launched by Sannikov (2007). I provide a new characterization of self-generating sets for a class of games in continuous time and Brownian information. This new characterization relies on partial differential equation techniques. Our approach gives a geometric characterization of the set of perfect public equilibrium payoffs, similar to the 2-player characterization obtained by Sannikov (2007) who obtains a curvature relation through a direct argument. Our characterization via partial differential equations is obtained by first identifying self-generating sets as stochastically viable under the dynamic determining the continuation payoff process induced by the players' strategies. Based on this formal identification we use viscosity solution techniques to derive a geometric characterization of the boundary of self-generating sets. In case of two players my characterization reduces to the result reported by Sannikov (2007), relating the curvature parameters of a set to incentives.

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1. Introduction

The seminal paper Sannikov (2007) introduced a new class of repeated games in continuous time that are analogous to repeated games with imperfect public monitoring in

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discrete time. In these games players don't see each others' actions. Instead they just observe a public signal which is distorted by some underlying noise. As in Sannikov (2007) we assume that the noisy distortions come from an exogenous Brownian motion. The players' strategies influence the distribution of public signals. Like most of the papers on games with imperfect public monitoring this paper studies a subset of the set of sequential equilibrium payoffs, called *perfect-public equilibrium* (PPE). The two main results of this paper provide new analytic characterizations of the set of perfect public equilibrium payoffs (abbreviated as PPE-p) for N -player games.¹ These two results extend and generalize the main findings of Sannikov (2007) to N -players, as we explain in the following.

The first main result of this paper, Theorem 4.3, provides a characterization of subsets of the set of perfect-public equilibrium payoffs $\mathcal{E}(r)$. We tend to call this an "inner approximation" of $\mathcal{E}(r)$.² This inner approximation result is obtained by using the dynamic properties of equilibrium continuation payoff processes. To understand the content of the theorem, let us recall the well-known recursive approach to repeated games due to Abreu et al. (1990).³ The recursive characterization of $\mathcal{E}(r)$ derived in Abreu et al. (1990) rests on the notion of *self-generation*. Self-generation is the set-valued generalization of Bellman's principle of optimality. A set K of payoff vectors is self-generating if every payoff vector from K can be *generated* by successively constructing continuation payoffs from K together with *enforcing* action profiles. By construction, the induced continuation value process must leave the set K forward invariant. This simple, but crucial, observation translates in continuous-time to stochastic viability (or weak-invariance) of a controlled stochastic process representing the continuation payoff process induced by a public strategy profile. In order to find stochastically viable subsets of payoff vectors analytically, we follow a significant literature in stochastic control theory and PDEs, and propose a stochastic optimal control framework to detect self-generating sets. Using the dynamic programming principle (i.e. the "recursive approach") we obtain from this stochastic optimal control problem a Hamilton-Jacobi-Bellman PDE, which reduces to a *geometric equation* at the boundary of the self-generating set. We should emphasize that this equation pins down self-generating sets under very weak conditions, as no a-priori convexity or smoothness of the self-generating set are assumed. However, in the case of smooth self-generating sets in the sense of Fudenberg et al. (1994), the geometric equation gives

¹Perfect public equilibrium is, in some sense, a natural solution concept in games with public signals. Nevertheless, it is a restrictive concept which can be justified on grounds of tractability. These arguments are well known from the perceived discrete-time literature and all of these arguments in favor of public equilibrium apply equally well to continuous-time games. See Mailath and Samuelson (2006) for an excellent summary of the discrete-time literature.

²The terminology is influenced by the algorithmic approach of Judd et al. (2003).

³See also Sorin (2000) and Mailath and Samuelson (2006).

us a curvature characterization of the boundary of the self-generating set which recovers one of the central insights provided by Sannikov (2007) in the two-player case. Unfortunately, this characterization is much more involved, because with more than two players curvature of a manifold is not a scalar quantity. Instead, for N -players and under a standard full dimensionality condition on the space of the feasible payoffs, there will be $N - 1$ curvature parameters, which are combined in our multi-player version of the *optimality equation*.

The second main result of this paper, which we present in two parts as Theorem 4.10 and Theorem 4.11, complements the inner characterization of $\mathcal{E}(r)$ with an "outer characterization" using a different stochastic optimization problem. Theorem 4.10 shows that finding the set $\mathcal{E}(r)$ is equivalent to solving a Dirichlet-type problem for a semi-linear elliptic PDE. This elegant characterization is completely new in the game-theoretic literature. The idea behind this problem is the following; We know from the first part of the paper that self-generating sets are those subsets of payoff vectors which are forward invariant under some equilibrium trajectory of continuation payoffs. In addition, the continuation payoff trajectory must remain in the set of weakly individual rational and feasible payoffs. This allows us to construct the following test: Let v be a feasible and individually rational payoff vector (a point in a set which is known a-priori). If $v \in \mathcal{E}(r)$, then we must be able to construct a stochastic process $\{W_t\}_{t \geq 0}$, corresponding to the continuation payoff process of the players in the repeated game, which stays in the feasible and individually rational set for all future times, and in addition satisfies the incentive compatibility constraint characterizing equilibrium continuation flows. With other words, the *exit time* from the set $\mathcal{V}^{IR}$, the set of feasible and weakly individually rational payoffs, is infinite. Using this observation, we set up a stochastic optimal control problem which aims to maximize the time a continuation payoff process, constructed to meet the above mentioned constraints, can stay inside the set $\mathcal{V}^{IR}$. The dynamic programming equation corresponding to this stochastic optimal control problem yields the same Hamilton-Jacobi-Bellman PDE as the one characterizing self-generating sets, but it holds on the whole domain $\mathcal{V}^{IR}$, not just on subsets. That's why we think of this procedure as an outer approximation of the set of PPE-p's.

1.1 Related Literature

Recently there has been a great effort of studying versions of repeated games in continuous time. A common tenet of these papers is that a continuous-time analysis sheds new light on important features of dynamic strategic interactions. Beside the growing literature on principal-agent problems, noteworthy contributions are Sannikov (2007); San-

nikov and Skrzypacz (2010) and Bernard and Frei (2014). In particular, Bernard and Frei (2014) prove a folk theorem for the class of games studied in this paper. They show in detail how continuous-time arguments differ (and in some cases simplify!) arguments used in the discrete-time theory. It shows that under the familiar full-dimensionality and identifiability conditions (suitably adapted to the Brownian information setting), the set of perfect public equilibrium payoffs $\mathcal{E}(r)$ approaches the set of feasible and weakly individually rational payoffs. The present paper gives a characterization of $\mathcal{E}(r)$ for *every* discount factor r using “recursive methods” in the spirit of Abreu et al. (1990). Finally, we would like to mention the work by Faingold and Sannikov (2011), who study reputation games in continuous-time. Their paper is also the first which highlights the role of viscosity solutions in games with more than 2 players, and implicitly uses our notion of stochastic viability to characterize the set of sequential equilibria in the reputation game between a single large player and a continuum of small players. Beside these important contributions this is the first paper which provides an exact characterization of self-generating sets for any level of discounting and in continuous-time.

The rest of this paper is organized as follows. Section 2 describes the strategic and stochastic environment in quite some detail. Section 3 introduces the class of equilibria which we will characterize in this work. In that section we also introduce the fundamental concept of self-generation and formally demonstrate its equivalence with the control-theoretic notion of stochastic viability. Section 4 commences our characterization of the set of PPE-p’s by developing the inner approximation procedure. In that section the first main result, Theorem 4.3 is stated and informally discussed. The outer approximation is given in Section 4.2 and Section 5 concludes. All technical proofs are collected in several appendices.

2. The Setting

We consider an N -player repeated game in continuous time with imperfect public monitoring. Every player $i \in I \triangleq \{1, 2, \dots, N\}$ has finitely many actions, collected in the set A^i , at her disposal. A stage game action profile is an N -tuple $a = (a^1, \dots, a^N) = (a^i, a^{-i}) \in \prod_{1 \leq i \leq N} A^i \triangleq A$. As usual one can think of the stage-game payoff function as representing the ex-post payoffs, i.e. an expected value of flow utilities computed before the signal is realized. Each stage game action profile a results in an instantaneous payoff vector $g(a) = (g^i(a))_{i \in I}$ (see Example 2.3 for an illustration). The set of feasible payoffs in the game is defined as the convex hull of stage game payoffs, i.e. $\mathcal{V} \triangleq \text{conv}(g(A))$. The

pure action minimax payoff of player i is defined as

$$\underline{v}^i \triangleq \min_{a^{-i} \in A^{-i}} \max_{a^i \in A^i} g^i(a^i, a^{-i}).$$

Without loss of generality we assume that $\underline{v} = \mathbf{0} \in \mathbb{R}^N$. The set of feasible and individually rational payoff vectors is thus $\mathcal{V}^{IR} \triangleq \mathcal{V} \cap \mathbb{R}_+^N$. A pure action profile $\bar{a} \in A$ is a (pure action stage-game) Nash equilibrium (NE) if $g^i(a^i, \bar{a}^{-i}) \leq g^i(\bar{a})$ for all $i \in I$ and $a^i \in A^i$. The set of pure action NEs is denoted by $\text{NE}(g)$, and the set of payoff vectors is denoted by $\mathcal{NE}$. Note that $\mathcal{NE} \subset \mathcal{V}^{IR} \subset \mathcal{V}$. Standing hypothesis of this paper is the following

Assumption 2.1. $\mathcal{NE} \neq \emptyset$ and $\dim(\mathcal{V}) = N$.

Let us briefly make a remark on the notation employed in this paper. If not mentioned differently, vectors are always column vectors. The transpose of a vector $x \in \mathbb{R}^d$ is the row vector x^* . The scalar product of two vectors $x, y \in \mathbb{R}^d$ is $\langle x, y \rangle \triangleq x^* y$, and the length of a vector x is $\|x\| \triangleq \langle x, x \rangle$. For a matrix $Y \in \mathbb{R}^{n \times d}$ we define its length as $\|Y\| \triangleq \text{tr}(Y Y^*)^{1/2}$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a stochastic basis, i.e. a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ together with a filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions. If τ is a $\mathbb{F}$ stopping time and E is some set, we let $L_{\mathbb{F}}^{2, \beta}(0, \tau; E)$, $\beta \in \mathbb{R}$, denote the space of $\mathbb{F}$ -progressively measurable processes taking values in E such that

$$\mathbb{E} \left(\int_0^\tau e^{-2\beta t} \|X_t\|^2 dt \right) < \infty.$$

For $\beta = 0$ this reduces to the Hilbert space of square integrable processes $L_{\mathbb{F}}^{2, 0}(0, \tau; E) = L_{\mathbb{F}}^2(0, \tau; E)$.

2.1 The repeated game model

We consider a situation in which players play the stage game g in continuous-time. The information sets of the players are generated by an exogenous Brownian motion process B , which serves as the public signal. As in discrete-time, the strategies of the players will influence the distribution of the public signal, and we formalize this via a change of measure approach.⁴ The stochastic basis of our repeated game model is the canonical Wiener space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega = \mathbf{C}(\mathbb{R}_+; \mathbb{R}^d)$ is the space of continuous functions $t \mapsto \omega(t) \in \mathbb{R}^d$. We endow this space with the topology of uniform convergence on

⁴This weak-solution approach is the most natural one in the present situation of imperfect public monitoring. See also Staudigl and Steg (2014), and the related analysis of Cvitanic and Zhang (2013) restricted to principal-agent problems.

compact intervals and let $\mathcal{F}$ denote the resulting Borel σ -algebra. The measure $\mathbf{P}$ is the Wiener measure on Ω , putting mass 1 on functions starting at $\mathbf{0} \in \mathbb{R}^d$. On this set-up the coordinate process $B_t(\omega) \triangleq \omega(t)$ is a standard d -dimensional Brownian motion. $\mathcal{F}_t^\circ$ denotes the sigma-algebra generated by the sample paths of the process B up to time t , i.e. $\mathcal{F}_t^\circ = \sigma(B_s; 0 \leq s \leq t)$ and denote by $\mathcal{F}_t$ its augmented version. The resulting filtrations are given by $\mathbb{F}^\circ \triangleq \{\mathcal{F}_t^\circ\}_{t \geq 0}$ and $\mathbb{F} \triangleq \{\mathcal{F}_t\}_{t \geq 0}$, respectively (see Appendix A for a formal definition). Now that the information structure of the game is defined, we can introduce our class of continuous-time strategies. Following most of the literature on dynamic games with imperfect public monitoring, we restrict the analysis to public strategies. Even more so, we particularize the setting to public strategies which are pure, in the following sense.

Definition 2.2. *A pure public strategy for player $i \in I$ is a process $\alpha^i : \Omega \times \mathbb{R}_+ \rightarrow A^i$ which is $\mathbb{F}$ -progressively measurable. The set of pure public strategies is denoted by $\mathcal{A}^i$. A pure public strategy profile is an N -tuple of public strategies $\alpha \in \mathcal{A} = \prod_{1 \leq i \leq N} \mathcal{A}^i$.*

On the basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$ we define the *signal process* $X = \{(X_t, \mathcal{F}_t); t \geq 0\}$ by $X_t = B_t$. Let

$$(1) \quad M_t^\alpha \triangleq \exp \left(\int_0^t \langle \mu(\alpha_s), dB_s \rangle - \frac{1}{2} \int_0^t \|\mu(\alpha_s)\|^2 ds \right).$$

For every finite time t , we define the probability measure $\mathbf{P}_t^\alpha$ on $(\Omega, \mathcal{F}_t^\circ)$ via the Radon-Nikodym derivative

$$\frac{d\mathbf{P}_t^\alpha}{d\mathbf{P}|_{\mathcal{F}_t}} = M_t^\alpha.$$

For every fixed t the measure $\mathbf{P}_t^\alpha$ is a probability measure on $(\Omega, \mathcal{F}_t^\circ)$. The family $\{\mathbf{P}_t^\alpha\}_{t \geq 0}$ is consistent, and can therefore be extended to a unique probability measure $\mathbf{P}^\alpha$ on $(\Omega, \mathcal{F}_\infty^\circ)$, where $\mathcal{F}_\infty^\circ \triangleq \sigma(\bigcup_{t \geq 0} \mathcal{F}_t^\circ)$ (see Karatzas and Shreve (2000), section 3.5, and Appendix A). It is well known that $\mathbf{P}^\alpha$ is equivalent to $\mathbf{P}$ on $\mathcal{F}_t$ for every finite $t \geq 0$. By Girsanov's theorem the process $B^\alpha = \{(B_t^\alpha, \mathcal{F}_{t+}^\circ); t \geq 0\}$, defined by

$$B_t^\alpha = B_t - \int_0^t \mu(\alpha_s) ds,$$

is a standard d -dimensional Brownian motion on $(\Omega, \mathcal{F}_\infty^\circ, \mathbb{F}^\circ, \mathbf{P}^\alpha)$. Using the new Brownian motion as the driving continuous martingale noise process, we see that the signal

process solves the stochastic differential equation

$$X_t = \int_0^t \mu(\alpha_s) ds + B_t^\alpha.$$

Hence, the tuple $(\Omega, \mathcal{F}_\infty^\circ, \mathbb{F}^\circ, \mathbb{P}^\alpha), (X, B^\alpha)$ is a weak solution to the SDE

$$y_t = \int_0^t \mu(\alpha_s) ds + w_t,$$

where $w = \{(w_t, \mathcal{F}_{t+}^\circ); t \geq 0\}$ is a Wiener process.

Given the public strategy profile α , we define the payoff process $U_t(\alpha)$ as⁵

$$(2) \quad U_t(\alpha) = \mathbb{E}_t^\alpha \left[\int_0^\infty r e^{-rs} g(\alpha_s) ds \right],$$

where $\mathbb{E}_t^\alpha[\cdot] \triangleq \mathbb{E}^{\mathbb{P}^\alpha}[\cdot | \mathcal{F}_t]$. The process $U(\alpha) = \{(U_t(\alpha), \mathcal{F}_t); t \geq 0\}$ is a bounded martingale under $\mathbb{P}^\alpha$. $U_0(\alpha)$ corresponds to the expected discounted payoff of the infinite horizon game.

Example 2.3. A large class of games with imperfect public monitoring fitting into the above framework is the one studied in Staudigl and Steg (2014). Suppose that the flow payoff of each player is given by the process

$$R_t^i(\alpha^i) \triangleq \int_0^t \left\langle \phi^i(\alpha_s^i), dX_s \right\rangle + \int_0^t \psi^i(\alpha_s) ds,$$

where ϕ^i, ψ^i are deterministic maps depending only on the action of player i of appropriate dimension. The discounted expected payoff in the infinite-horizon game is then given by

$$U^i(\alpha) = \mathbb{E}^\alpha \left[\int_0^\infty r e^{-rt} dR_t^i(\alpha^i) \right].$$

By the Law of iterated expectations this reduces to the specification given above, where

$$g^i(a^i, a^{-i}) = \left\langle \phi^i(a^i), \mu(a^i, a^{-i}) \right\rangle + \psi^i(a^i).$$

⁵Observe the change of the filtration in the definition of the stochastic process $U(\alpha)$. We performed the change of measure on the uncompleted set-up $(\Omega, \mathcal{F}_\infty^\circ, \mathbb{F}^\circ)$, but now define $U(\alpha)$ on the completed set-up $(\Omega, \mathcal{F}, \mathbb{F})$, satisfying the usual hypothesis. This change in the filtered space is permissible as explained in Appendix A.

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Using a suitable version of a martingale representation theorem⁶ there exists a $\mathbb{F}$ -progressively measurable process $\Phi^\alpha = \{(\Phi_t^\alpha, \mathcal{F}_t); t \geq 0\} \in L_{\mathbb{F}}^2(0, \infty; \mathbb{R}^{N \times d})$ such that

$$(3) \quad U_t(\alpha) = \mathbb{E}^\alpha[U_\infty(\alpha)] + \int_0^t \Phi_s^\alpha dB_s^\alpha = U(\alpha) + \int_0^t \Phi_s^\alpha dB_s^\alpha$$

for all $t \geq 0$ $\mathbf{P}^\alpha$ -a.s. The continuation payoff following time t is the random variable

$$W_t^\alpha = \mathbb{E}_t^\alpha \left[\int_t^\infty re^{-r(s-t)} g(\alpha_s) ds \right].$$

Note that $W_0^\alpha = U(\alpha)$, and

$$U_t(\alpha) = \int_0^t re^{-rs} g(\alpha_s) ds + e^{-rt} W_t^\alpha \quad \forall t \geq 0, \mathbf{P} - \text{a.s.}$$

When combined with (3) this gives

$$(4) \quad e^{-rt} W_t^\alpha = W_0^\alpha + \int_0^t \Phi_s^\alpha dB_s^\alpha - \int_0^t re^{-rs} g(\alpha_s) ds.$$

Finally, using the definition of the process B^α , and setting

$$Z_t^\alpha \triangleq \frac{1}{r} e^{rt} \Phi_t^\alpha,$$

we can express (4) in differential form as

$$(5) \quad dW_t^\alpha = r[W_t^\alpha - g(\alpha_t) - Z_t^\alpha \mu(\alpha_t)] dt + rZ_t^\alpha dB_t.$$

Our derivation shows that every public strategy α induces a progressively measurable process W_t of the diffusion type whose differential is given by eq. (5). Conversely, if we are given the stochastic differential equation as primitive, we can ask the question whether equation (5) has a solution over any *finite* time interval $[0, T]$ with given terminal condition $W_T = \xi \in L_{\mathcal{F}_T}^2(\Omega, \mathbf{P}; \mathcal{V})$. In this context a solution is a pair (W, Z) of adapted processes such that

(i) $\{(W_t, \mathcal{F}_t); t \geq 0\}$ is *viable* in $\mathcal{V}$, meaning that

$$(6) \quad \mathbf{P}(W_t \in \mathcal{V}, \forall t \in [0, T]) = 1,$$

⁶See Appendix A and Staudigl (2014).

and

- (ii) the martingale integrand process $Z = \{(Z_t, \mathcal{F}_t); t \in [0, T]\}$ is an element of $L_{\mathbb{F}}^{2,r}(0, T; \mathbb{R}^{N \times d})$, so that

$$\mathbb{E} \left[\int_0^T e^{-2rt} \|Z_t\|^2 dt \right] < \infty.$$

With this interpretation in mind, the problem of identifying the continuation payoff process over a finite time interval $[0, T]$ induced by a public strategy profile reduces to finding a solution (in the above sense) of the backward stochastic differential equation (BSDE)

$$(7) \quad \begin{cases} dW_t = r[W_t - g(\alpha_t) - Z_t \mu(\alpha_t)]dt + rZ_t dB_t, \\ W_T = \xi \in L^2(\Omega, \mathcal{F}_T, \mathbf{P}; \mathcal{V}). \end{cases}$$

This is a BSDE with linear driver, and hence can be solved explicitly as

$$(8) \quad e^{-rt} W_t = e^{-rT} W_T + \int_t^T r e^{-rs} [g(\alpha_s) + Z_s \mu(\alpha_s)] ds - \int_t^T r e^{-rs} Z_s dB_s.$$

The BSDE is standard in the literature, and existence and uniqueness of solutions follows from arguments summarized in Pardoux and Rascanu (2014). It is also not difficult to construct solutions on an infinite time horizon, modulo changes in the boundary condition. To see the necessary changes, let

$$\eta \triangleq \int_0^\infty r e^{-rs} g(\alpha_s) ds.$$

Then $\eta \in L_{\mathcal{F}_\infty}^2(\Omega, \mathbf{P}^\alpha; \mathbb{R}^N)$. By the same martingale representation theorem used before, there exists an almost surely unique pair $(Y^\alpha, \Phi^\alpha) \in L_{\mathbb{F}}^2(0, \infty; \mathbb{R}^N) \times L_{\mathbb{F}}^2(0, \infty; \mathbb{R}^{N \times d})$ such that

$$\begin{aligned} \eta &= \mathbb{E}^\alpha(\eta) + \int_0^\infty \Phi_s^\alpha dB_s^\alpha, \\ Y_t^\alpha &= \mathbb{E}^\alpha(\eta | \mathcal{F}_t). \end{aligned}$$

Hence,

$$Y_t^\alpha = \eta - \int_t^\infty \Phi_s^\alpha dB_s^\alpha.$$

From Doob's martingale convergence theorem (e.g. Rogers and Williams, 2000, Theorem

II.69.1) it follows that

$$\lim_{t \rightarrow \infty} Y_t^\alpha = Y_\infty^\alpha = \eta \quad \mathbf{P}^\alpha - \text{a.s.}$$

For every $t \geq 0$, let us define

$$e^{-rt} W_t \triangleq Y_t^\alpha - \int_0^t r e^{-rs} g(\alpha_s) ds.$$

It is easy to see that W defined in that way has the differential from condition (7), after setting $Z_t^\alpha \triangleq \frac{1}{r} e^{rt} \Phi_t^\alpha$. Additionally, by dominated convergence,

$$\begin{aligned} \lim_{t \rightarrow \infty} \mathbf{E}^\alpha \left(\|e^{-rt} W_t\|^2 \right) &= \lim_{t \rightarrow \infty} \mathbf{E}^\alpha \left(\left\| Y_t^\alpha - \int_0^t r e^{-rs} g(\alpha_s) ds \right\|^2 \right) \\ &= \lim_{t \rightarrow \infty} \mathbf{E}^\alpha \left\{ \left\| \mathbf{E}^\alpha \left(\int_t^\infty r e^{-rs} g(\alpha_s) ds \middle| \mathcal{F}_t \right) \right\|^2 \right\} \\ &= 0. \end{aligned}$$

Hence, W_t satisfies the *transversality condition*

$$(9) \quad \lim_{t \rightarrow \infty} e^{-2rt} \mathbf{E} \|W_t\|^2 = 0.$$

Finally, we observe that

$$W_0 = Y_0^\alpha = \mathbf{E}^\alpha(\eta) = U(\alpha) \in \mathcal{V}.$$

Summarizing these steps, we deduce that W_t is a version of the continuation payoff process under strategy α . This leads us to the following characterization theorem, generalizing Proposition 1 of Sannikov (2007) to the N -player case (see also the recent work of Bernard and Frei (2014)).

Proposition 2.4. *The continuation value process corresponding to a public strategy profile $\alpha \in \mathcal{A}$ is the a.s. unique progressively measurable process $W = \{(W_t, \mathcal{F}_t); t \geq 0\}$ taking values in $\mathcal{V}$, such that there exists a progressively measurable process $Z \in L_{\mathbb{F}}^{2,r}(0, \infty; \mathbb{R}^N \times d)$ so that the pair (W, Z) solves the BSDE (7) for every $T > 0$, together with the transversality condition (9).*

Proof. We already know that the continuation payoff process W^α satisfies the conditions mentioned in the text of the proposition. It remains to show uniqueness. Recall that any solution to the BSDE (7) on the finite time interval $[0, T]$ is given by the expression (8).

Switching to the probability space $(\Omega, \mathcal{F}, \mathbf{P}^\alpha)$, this equation has the representation

$$e^{-rt}W_t = e^{-rT}W_T + \int_t^T re^{-rs}g(\alpha_s) - \int_t^T re^{-rs}Z_s dB_s^\alpha.$$

Taking conditional expectations leaves us with

$$e^{-rt}W_t = \mathbb{E}_t^\alpha \left[e^{-rT}W_T + \int_t^T re^{-rs}g(\alpha_s)ds \right].$$

Using the transversality condition (9), we can use dominated convergence to arrive at

$$e^{-rt}W_t = \mathbb{E}_t^\alpha \left[\int_t^\infty re^{-rs}g(\alpha_s)ds \right] = e^{-rt}W_t^\alpha.$$

Hence, $W_t = W_t^\alpha$ almost surely. □

Remark 2.5. (i) Both Sannikov (2007) and Staudigl and Steg (2014) model the continuous-time game including a public correlation device. In the latter reference it is shown that this is needed when one wants to interpret the continuous time game as a suitable limit of a sequence of approximating discrete time games. For the purpose of this paper the introduction of additional exogenous noise is not needed.

(ii) The quadratic cross variation of the the continuation payoff processes of players i and j is given by

$$[W^{i,\alpha}, W^{j,\alpha}]_t = \int_0^t r^2 \langle Z_s^i, Z_s^j \rangle ds,$$

or $[W]_t = \int_0^t r^2 Z_s Z_s^* ds$. The $N \times d$ matrix-valued process of cross variations between the continuation payoff process and the driving Brownian motion is

$$[W, B]_t = \int_0^t r Z_s ds.$$

In this sense, it is natural to call the martingale integrand Z the "sensitivity" of continuations with respect to the underlying information structure.

◆

3. Perfect public equilibrium payoffs in continuous time

Proposition 2.4 shows that the continuation payoff process induced by a public strategy profile α is the unique progressive process $W = \{(W_t, \mathcal{F}_t); t \geq 0\}$ with differential

$$(10) \quad dW_t = r[W_t - g(\alpha_t) - Z_t \mu(\alpha_t)]dt + rZ_t dB_t,$$

subject to the initial condition $W_0 \in \mathcal{V}$ and boundary condition (9). In terms of continuation payoff process, a perfect public equilibrium is defined as follows.

Definition 3.1. A public strategy profile $\alpha^* \in \mathcal{A}$ is a perfect public equilibrium if for all $\mathbb{F}$ -stopping times τ we have

$$W_\tau^{i, \alpha^*} \geq W_\tau^{i, (\alpha^i, \alpha^{-i,*})} \quad \mathbf{P} - a.s.$$

for all players $i = 1, 2, \dots, N$, and alternative strategies $\alpha^i \in \mathcal{A}^i$.

We follow the recursive approach to repeated games by deriving convenient necessary and sufficient conditions characterizing perfect public equilibria in the continuous-time game. In discrete-time, a recursive formulation of public equilibrium is possible in terms of the Shapley-operator, which can be interpreted as an auxiliary one-shot game representing the link between current and future payoffs.⁷ In the continuous-time game we have a stochastic maximum principle which leads us to the following auxiliary game

$$G(a, Z) \triangleq g(a) + Z\mu(a) \quad \forall (a, Z) \in A \times \mathbb{R}^{N \times d}.$$

Definition 3.2. An action profile $a \in A$ is said to be enforceable if there exists a matrix $Z \in \mathbb{R}^{N \times d}$ of sensitivity coefficients to the public signal such that $a^i \in \mathbf{NE}(G(\cdot, Z))$ for all $i \in I$.

Theorem 3.3. A progressive process $W = \{W_t, \mathcal{F}_t; t \geq 0\}$ taking values in $\mathcal{V}$ is the continuation value process giving the PPE-p $v \in \mathcal{V}$ if and only if there exists a pure strategy profile $\alpha \in \mathcal{A}$ together with a $\mathbb{F}$ -progressively measurable process $Z_t \in L_{\mathbb{F}}^{2,r}(0, \infty; \mathbb{R}^{N \times d})$ such that $\alpha_t \in \mathbf{NE}(G(\cdot, Z_t))$ for all $t \geq 0$ $\mathbf{P}$ -a.e and the process W solves the differential equation (10) with initial condition $W_0 = v$.

Proof. The proof is a special case of the general analysis in Staudigl (2014) and therefore omitted. $\square$

⁷See Sorin (2000) for an exposition on the recursive approach in zero-sum games and Mailath and Samuelson (2006) for the general case.

3.1 Self-generation and stochastic viability

Theorem 3.3 gives us a full characterization of stochastic processes which correspond to perfect public equilibrium payoffs in continuous time. In this section we report a new characterization of the whole set of perfect public equilibrium payoffs. Let us briefly recall some terminology familiar from the discrete-time literature.

In discrete-time a (closed nonempty) subset of payoff vectors $K \subset \mathcal{V}$ is called *self generating* if every point in the set corresponds to a Nash equilibrium payoff of an auxiliary one-shot game constructed from the Shapley operator. The requirement that the continuation payoffs are inside the set K ensures that the sequence of thus generated continuation payoffs remains inside K for all future times. Hence, a self-generating set is a set which is *weakly invariant* under the dynamics defining a continuation payoff process. Building on this insight, we propose the following definitions.

Definition 3.4. A pair of processes $u = (\alpha, Z) \in \mathcal{A} \times L_{\mathbb{F}}^{2,r}(0, \infty; \mathbb{R}^{N \times d}) \triangleq \mathcal{U}$ is called a control.

Controls are the basic inputs of our stochastic differential equation (10). Since we want to construct continuation payoff processes which are public equilibria, we have to add some restrictions on the space of controls.

Definition 3.5. A control $u = (\alpha, Z) \in \mathcal{U}$ is called admissible if $\alpha_t \in \text{NE}(G(\cdot, Z_t))$ for all $t \geq 0$ and $\mathbf{P}$ -a.s. Equivalently, if we denote by

$$U \triangleq \{(a, Z) \in A \times \mathbb{R}^{N \times d} \mid a \in \text{NE}(G(\cdot, Z))\},$$

then the space of admissible controls is defined as

$$\mathcal{U}_0 \triangleq \{u \in \mathcal{U} \mid u_t \in U \text{ a.e.}\} \subseteq \mathcal{U}.$$

Remark 3.6. Note that $U \neq \emptyset$ by Assumption 2.1. Indeed, playing a stage-game NE with the enforcing matrix $Z = \mathbf{0}$ defines an element of the set U . Observe also that the set U is independent of the current position v of the payoff vector.

If at a given payoff vector $v \in \mathcal{V}$ there exists a process $u \in \mathcal{U}_0$ such that the SDE (10) has a unique strong solution, then we denote the resulting flow by $W_t^{v,u}$. Note that under Assumption 2.1 the set of controls is non-empty.⁸ The next definition has been proposed, but stated differently, by Sannikov (2007).

⁸Indeed, one possible control is the constant strategy $\alpha_t(\omega) = a \in \text{NE}(g)$, together with $Z_t = \mathbf{0}_{N \times d}$ the null matrix. Here we need the existence of a pure-action stage game Nash equilibrium, which is why Assumption 2.1 is needed.

Definition 3.7. A closed set $K \subset \mathcal{V}$ is self-generating if for every $v \in K$ there exists a process $u \in \mathcal{U}_0$ such that $W_t^{v,u} \in K$ for all $t \geq 0$ and $\mathbf{P}$ -a.s.

If K is self-generating then, by definition, every point $v \in K$ is a PPE-p. Hence, we have the following analogue of Theorem 1 and 2 of APS, stated as Corollary 1 in Sannikov (2007) for the two player case.

Theorem 3.8. The set $\mathcal{E}(r)$ of public perfect equilibrium payoffs is the largest self-generating set.

The beauty of this result lies in the fact that everything can be formulated in terms of solutions to controlled stochastic differential equations, where the set of admissible controls have to satisfy the incentive constraints. The test of whether a payoff vector $v \in \mathcal{V}$ is a PPE-payoff can therefore be reformulated as follows: For arbitrary payoff vector $v \in \mathcal{V}$, let

$$\mathcal{U}_0(v) \triangleq \{u \in \mathcal{U}_0 \mid W_t^{v,u} \in K, \forall t \geq 0, \mathbf{P} - \text{a.s.}\}.$$

If $\mathcal{U}_0(v) \neq \emptyset$ for all $v \in K \subset \mathcal{V}$, then the set K is called viable with respect to (10). Given this definition, the following observation is immediate:

Observation 3.9. Let K be a no-empty closed subset of the set of feasible payoffs. Then

$$K \subseteq \mathcal{E}(r) \Leftrightarrow (\forall v \in K) : \mathcal{U}_0(v) \neq \emptyset.$$

In particular, $\mathcal{E}(r)$ is the largest set (with respect to set-inclusion) with this property.

Remark 3.10. Using the same argument as in discrete time, it is easy to see that $\mathcal{E}(r) \subseteq \mathcal{V}^{IR}$. To wit, a player can choose the constant strategy $\alpha^i \equiv \underline{a}^i$ giving her at least the pure-action stage game min-max payoff in every "subgame". Hence, $W_t \in \mathcal{V}^{IR}$ for all $t \geq 0$ and $\mathbf{P}$ -a.s, so that $\mathcal{V}^{IR}$ is viable under the set of controls $\mathcal{U}$. ♦

4. Geometric characterization of perfect public equilibrium payoffs

This section uses stochastic control theory to give an inner description of the set PPE-p's. The viability approach allows us to characterize smooth subsets in the interior of $\mathcal{E}(r)$. Later on, in Section 4.2 we provide a complementary description of the set of PPE-p's from the outside. For conceptual clarity, it is however good to start with the inner

approximation procedure.

Fix a closed nonempty set $K \subset \mathcal{V}$ which we suspect to be a subset of $\mathcal{E}(r)$. Let $c : \mathbb{R}^N \rightarrow \mathbb{R}$ be a bounded and uniformly continuous function, for short $c \in \text{BUC}(\mathbb{R}^N)$. Additionally, this function should have similar properties like the distance function to the set K . By this we mean that it additionally has the properties that $c \geq 0$ and $c(v) = 0$ only if $v \in K$.⁹ We interpret c as the cost function of an optimal control problem whose objective is to minimize the functional

$$J(v; u) \triangleq \mathbb{E} \left[\int_0^\infty r e^{-rt} c(W_t^{v,u}) dt \right].$$

The value function is defined as

$$(11) \quad F(v) \triangleq \inf_{u \in \mathcal{U}_0} J(v; u).$$

This defines an infinite horizon stochastic optimal control problem on the fixed stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with state dynamics

$$(12) \quad \begin{cases} dW_t^{v,u} &= r[W_t^{v,u} - G(u_t)]dt + r\sigma(u_t)dB_t, \\ W_0^{v,u} &= v, \end{cases}$$

where $G(u) = g(a) + z\mu(a)$ and $\sigma(u) = z$ for $u = (a, z)$. If $\mathcal{U}_0(v) = \emptyset$, then we set $F(v) = \infty$. Therefore, by definition we have $F(v) \in [0, \infty]$ for all $v \in \mathcal{V}$ and, if F is finite, then $F(v) \leq \sup_{\mathbb{R}^N} |c| \triangleq \bar{c} < \infty$.

By standard dynamic programming arguments, the function F is a viscosity solution of the second-order elliptic partial differential equation

$$(13) \quad F(v) - c(v) - H(v, DF(v), D^2F(v)) = 0 \quad \forall v \in \mathcal{V},$$

where

$$(14) \quad H(v, p, Y) \triangleq \inf_{(a,z) \in \mathcal{U}} \left\{ \langle p, v - G(a, z) \rangle + \frac{r}{2} \text{tr}(Yzz^*) \right\}.$$

Note that the set of controls already includes the incentive compatibility constraint. This gives the optimization problem actually its bite.

⁹Indeed, the squared distance function $c(x) = \text{dist}(x, K)^2$ satisfies our needs, when localized to a sufficiently large hypercube containing $\mathcal{V}$.

In order to use the full power of viscosity solution methods we make the following (standard) assumption.

Assumption 4.1. *The HJB equation (13) satisfies the comparison principle and the Hamiltonian H is continuous on the interior of its domain.*

Beside the infinite horizon optimization problem (11), we are able to formulate our characterization of self-generating sets in terms of finite-horizon problems. The appropriate notion of viability is given in the following definition.

Definition 4.2. *A closed set $K \subset \mathcal{V}$ is called approximately-viable if for all $c \in \text{BUC}(\mathbb{R}^N)$ with $c \geq 0$ and $c = 0$ on K and all $T \geq 0$, we have*

$$\inf_{u \in \mathcal{U}_0} \mathbb{E} \int_0^T c(W_t^{v,u}) dt = 0$$

for all $v \in K$.

Equivalently, a set $K \subset \mathcal{V}$ is approximately-viable if for every cost function c , positive on $\mathbb{R}^N \setminus K$ and zero on K , every $\varepsilon > 0$ and every $T > 0$ we can find a control $u \in \mathcal{U}_0$ such that the total accumulated costs of the flow $W^{v,u}$ on the time interval $[0, T]$ is smaller than ε . Trivially, if K is a self-generating set (and therefore viable) it is approximately viable.

We come now to the first result of this paper. The proof is given in Appendix C.

Theorem 4.3. *Given a closed set $K \subset \mathbb{R}^N$. The following properties are equivalent:*

- (a) *The set K is approximately-viable;*
- (b) *For all $v \in K$ and all $c \in \text{BUC}(\mathbb{R}^N)$ with $c \geq 0$ and $c = 0$ on K , we have $F(v) = 0$;*
- (c) *The geometric condition*

$$(15) \quad \forall v \in \text{bd}(K), \forall (p, Y) \in \mathcal{N}_K^2(v), \exists (a, z) \in U : \\ \langle p, v - G(a, z) \rangle + \frac{r}{2} \text{tr}[Yzz^*] \leq 0$$

holds, where

$$\mathcal{N}_K^2(v) \triangleq \left\{ (p, Y) \in \mathbb{R}^N \times \mathcal{S}^N \left| \begin{array}{l} \text{for } K \ni w \rightarrow v : \\ \langle p, w - v \rangle + \frac{1}{2} \langle Y(w - v), w - v \rangle \leq o(\|w - v\|^2) \end{array} \right. \right\}$$

is the second-order normal cone of K at v .

This result tells us two things. First, approximate-viability and exact viability are, under an additional existence result, equivalent.¹⁰ Second, condition (15) shows that the (approximate) viability property of a subset of payoff vectors depends only on a geometric condition which has to be satisfied at the boundary of a self-generating set.

Remark 4.4. Note that the geometric condition only involves points from the second-order normal cone. Hence, at the boundary it suffices that the value function is a viscosity super-solution (see Appendix B.1 for an explanation of the terminology and the references mentioned there). ♦

Now we use Theorem 4.3 to obtain necessary and sufficient conditions for a closed subset K to be self-generating. Suppose that K is a self-generating set. Then, by definition, for every point $v \in K$ we can find an admissible control $u \in \mathcal{U}_0$ such that the induced continuation payoff process stays inside the set K . Theorem 4.3 shows that, under this control, whenever the process $W^{v,u}$ hits the boundary of K the controls must satisfy the geometric condition (15) so that the trajectory is stabilized to stay inside K . Conversely, if the geometric condition (15) holds, then Theorem 4.3 tells us that K is approximately viable. If the optimal control problem (11) has a solution, we know that K is even self-generating. Corollary 4.5 summarizes this finding.

Corollary 4.5. *Suppose that for all $v \in K$ the optimal control problem (11) has a solution. Then*

- (a) *K is self-generating and hence $K \subseteq \mathcal{E}(r)$, and*
- (b) *the geometric condition (15) holds on $\text{bd}(K)$.*

4.1 Smooth domains

We now use the geometric condition (15) to get more information on the elements from the second-order normal cone in the special case where the “test set” K is a smooth manifold with compact boundary.¹¹ In this scenario we obtain an interesting curvature condition which holds at the boundary of the target set. This curvature condition reduces in the two player case to the optimality equation provided by Sannikov (2007). Therefore, our viability analysis generalizes his result, and in fact provides a new proof. Before

¹⁰Note that we do not assume in the statement of the theorem that K is viable. If K would be viable then, trivially, assertions (a) and (b) are true. Once we *assume* that K is viable, we have additionally the geometric condition (15). See Corollary 4.5, and the discussion preceding it.

¹¹This is the typical class of sets used in proofs of folk theorems. See Fudenberg et al. (1994) and Mailath and Samuelson (2006) for an analysis in discrete-time and Bernard and Frei (2014) for the continuous-time theory.

jumping into details, let us however introduce some terms, well known from the discrete-time literature (and adapted to continuous-time models by Bernard and Frei (2014).)

Definition 4.6.

- Let $\mathbf{T}$ be a $N \times (N - 1)$ matrix whose columns $\vec{t}_1, \dots, \vec{t}_{N-1}$ span the hyperplane $H \subset \mathbb{R}^N$. An action profile $a \in A$ is enforceable on the hyperplane H if there exists a matrix $\beta \in \mathbb{R}^{(N-1) \times d}$ such that $a \in \text{NE}(G(\cdot, \mathbf{T}\beta))$.
- Let $p \in \mathbb{R}^N$. A matrix $Z \in \mathbb{R}^{N \times d}$ enforces $a \in A$ orthogonally to p , if $a \in \text{NE}(G(\cdot, Z))$ and $p^*Z = \mathbf{0}$.

It is easy to see that these two conditions are actually equivalent. The next Lemma establishes a precise connection between orthogonal enforceability and our viability characterization in terms of the geometric equation (15).

Lemma 4.7. *Let the set of (regular) normal vectors at $v \in K$ be*

$$\mathcal{N}_K^1(v) \triangleq \left\{ p \in \mathbb{R}^N \mid \limsup_{w \rightarrow v, w \in K \setminus \{v\}} \frac{\langle p, w - v \rangle}{\|w - v\|} \leq 0 \right\}.$$

Then

- If $(p, Y) \in \mathcal{N}_K^2(v)$ it follows that $p \in \mathcal{N}_K^1(v)$.*
- If the geometric condition (15) is satisfied at a payoff vector $v \in \text{bd}(K)$, then*

$$p^*Z = \mathbf{0} \quad \forall p \in \mathcal{N}_K^1(v).$$

Proof. See Appendix B.2. □

This lemma states that whenever the geometric condition (15) applies, then the “enforcing” martingale integrand Z must enforce the action profile a orthogonally. This is precisely the “tangential span” condition, first reported in Sannikov (2007). Its discrete-time analogue is the Fudenberg et al. (1994) concept of enforceability on tangent hyperplanes. Note that this lemma restricts the coefficient Z for general target sets K . It is easy to see that if K is smooth, then it reduces to the one-dimensional subspace spanned by the unit outward normal at the boundary point v .

With the help of this tangential span condition we obtain some control over the choice of the sensitivity coefficient Z in the representation of the continuation value process. Indeed, if the continuation value process is in the interior of some closed subset $K \subseteq \mathcal{V}$, then

Lemma 4.7 does not restrict the choice of the enforcing matrix Z in any way. But once the continuation payoff process hits the boundary, then the choice of possible enforcing sensitivities is significantly limited to be orthogonally to the normal cone.

Let $K \subset \mathcal{V}$ be a nonempty closed convex self-generating set. Assume that $\text{bd}(K)$ is $\mathbf{C}^2$. For $v \in \text{bd}(K)$, let $\vec{n}$ and $T(v)$ denote the unit outer normal vector and tangent hyperplane to $\text{bd}(K)$ at v , respectively. The linearly independent set of vectors $\{\vec{t}_1, \dots, \vec{t}_{N-1}\}$ defines an orthonormal basis of $T(v)$, so that $\mathcal{B} \triangleq \{\vec{t}_1, \dots, \vec{t}_{N-1}, \vec{n}\}$ is an orthonormal basis of $\mathbb{R}^N$. In this coordinate system the v^N -axis points along the direction defined by the unit outward normal vector $\vec{n}$.

By the assumed smoothness of the boundary of K , there exists a function $\varphi \in \mathbf{C}^2(\mathcal{O}' \cap T(v))$ such that $\text{bd}(K)$ can be represented in a small neighborhood $\mathcal{O}$ around the point v as the solution of the equation $\varphi(w') = w^N$, where $w' = (w^1, \dots, w^{N-1}) \in \mathcal{O}'$. Additionally, one can see that $D\varphi(v')^* \triangleq \left[\frac{\partial \varphi(v')}{\partial v^i} \right]_{1 \leq i \leq N-1} = \mathbf{0}$.¹² In terms of this parametrization, the outward unit vector has coordinates

$$\vec{n}(w) = \frac{(-D\varphi(w')^*, 1)}{\sqrt{1 + \|D\varphi(w')\|^2}} \quad \forall w = (w', w^N) \in \mathcal{O}.$$

On $\mathcal{O} \cap K$ this defines the Gauss-map $w \mapsto \vec{n}(w) \in \mathbb{S} \triangleq \{x \in \mathbb{R}^N \mid \|x\| = 1\}$, i.e. a vector field associating to each point $w \in \mathcal{O} \cap K$ its unit outward normal vector $\vec{n}(w)$. Since the boundary of K is smooth, we have $\vec{n}(\cdot) \in \mathbf{C}^1(\mathcal{O} \cap K)$, and the choice of the parametrization gives $\vec{n}(v) = (0, 0, \dots, 1)^* \triangleq e_N$. The rate of change of the normal vector along the tangent directions is captured by the second-fundamental form of $\text{bd}(K)$ at v . To get an analytic expression for this geometric operation, we define the *Weingarten map* as the endomorphism

$$L_w \triangleq -D\vec{n}(w) : \mathcal{T}_K(w) \rightarrow \mathcal{T}_{\mathbb{S}}(\vec{n}(w)) = \{\vec{n}(w)\}^\perp = \mathcal{T}_K(w),$$

assigning to every tangent vector $\eta \in \mathcal{T}_K(w)$ a new tangent vector $L_w(\eta) = -D\vec{n}|_w \eta$. The second fundamental form at the point $w \in \text{bd}(K)$ is defined as the bilinear form acting on

¹²The coordinates of the vector $w = (w', w^N) \in \text{bd}(K) \cap \mathcal{O}$ correspond to the coordinates relative to the tangent plane at v and the one-dimensional subspace spanned by the outward normal $\vec{n}(v)$. Then $K \cap \mathcal{O} = \{(w', w^N) \in \mathcal{O}' \times I \mid w^N \leq \varphi(w')\}$ for some interval $I \subset \mathbb{R}_+$. The inequality becomes an equality on $\text{bd}(K) \cap \mathcal{O}$. For the details, see Appendix B.3.

the tangent space given by ¹³

$$II_w(\eta, \xi) \triangleq L_w(\eta)(\xi) \quad \forall \eta, \xi \in \mathcal{T}_K(w).$$

It can be shown that the operator L_v is symmetric, and thus has $N - 1$ real eigenvalues, called the *principal curvatures* at $v \in \text{bd}(K)$. In fact, a simple computation given in Appendix B.3 shows that under the tangent-normal coordinates defined by the basis $\mathcal{B}$ the Weingarten map admits the Matrix representation

$$(16) \quad [L_v]_{\mathcal{B}} = \begin{pmatrix} D^2\varphi(v') & \mathbf{0}_{N-1} \\ \mathbf{0}_{N-1}^* & 0 \end{pmatrix}.$$

Switching back to the original coordinates, we see that

$$L_v = \mathbf{T} D^2\varphi(\mathbf{T}^* v) \mathbf{T}^*,$$

where $\mathbf{T} = [\vec{t}_1, \dots, \vec{t}_{N-1}]$ is the $N \times (N - 1)$ matrix formed by the spanning tangent vectors at $v \in \text{bd}(K)$.

We now use this convenient tangent-normal coordinate system in order to characterize the points in the second-order normal cone in some detail. To focus on interesting cases, pick a pair $(p, Y) \in \mathcal{N}_K^2(v)$ such that $p \neq \mathbf{0}$. In our choice of coordinates, recall that $v' = (v_1, \dots, v_{N-1})$ are the tangent directions and v_N points in the direction of the normal vector. Additionally, the curvature matrix L_v is given by the matrix (16). From Lemma 4.7, we deduce that $p = \lambda \vec{n}(v)$ for some $\lambda > 0$. Let $\{v_n\}_{n \in \mathbb{N}}$ be a sequence of payoff vectors in K converging to $v \in \text{bd}(K)$. Choosing n sufficiently large, we can assume that $\{v_n\}_n \subset \mathcal{O} \cap K$. Then, we see that (cf. footnote 12)

$$\begin{aligned} 0 &\leq \varphi(v'_n) - \varphi(v') - (v'_n - v')^N \\ &= \left\langle \begin{pmatrix} D\varphi|_{v'}^* \\ -1 \end{pmatrix}, v_n - v \right\rangle + \frac{1}{2} \left\langle D^2\varphi|_{v'}(v'_n - v'), v'_n - v' \right\rangle + o(\|v_n - v\|^2) \\ &= -\langle \vec{n}(v), v_n - v \rangle + \frac{1}{2} \langle L_v(v_n - v), v_n - v \rangle + o(\|v_n - v\|^2). \end{aligned}$$

This implies that for every sequence $w \rightarrow v$ in K we have

$$\langle \vec{n}(v), w - v \rangle - \frac{1}{2} \langle L_v(w - v), w - v \rangle \leq o(\|w - v\|^2),$$

¹³See Lee (1997) for references to these concepts from differential geometry.

so that $(\vec{n}(v), -L_v) \in \mathcal{N}_K^2(v)$. Additionally, from the same line of equations (using the rotated coordinates under which we have $\vec{n}(v) = e_N$) we deduce that

$$\begin{aligned}\langle \vec{n}(v), v_n - v \rangle &\leq \frac{1}{2} \left\langle D^2\varphi|_{v'}(v'_n - v'), v'_n - v' \right\rangle + o(\|v_n - v\|^2) \\ &= \frac{1}{2} \langle L_v(v_n - v), v_n - v \rangle + o(\|v_n - v\|^2).\end{aligned}$$

Plugging this into the defining inequality for elements from the second-order normal cone, we see that when $p = \lambda \vec{n}(v), \lambda > 0$, it must be the case that

$$-\frac{\lambda}{2} \langle L_v(v_n - v), v_n - v \rangle + \frac{1}{2} \langle Y(v_n - v), v_n - v \rangle \leq o(\|v_n - v\|^2).$$

Hence, the elements of the second-order normal cone are characterized as follows:

- (a) If $p = \mathbf{0}$ then $\langle Y\eta, \eta \rangle \leq 0$ for all $\eta \in \mathcal{T}_K(v)$;
- (b) If $p = \lambda \vec{n}(v), \lambda > 0$, then

$$\langle Y\eta, \eta \rangle \leq \lambda \langle L_v\eta, \eta \rangle \quad \forall \eta \in \mathcal{T}_K(v).$$

In particular, it follows from these bounds that, if K is convex, then Y is negative semi-definite as an operator on the tangent space, as then the map φ is concave.

Combining these calculations with lemma 4.7, we get the following characterization of smooth self-generating sets.

Proposition 4.8. *Let K be a subset in $\mathcal{V}$ with $\mathbf{C}^2$ boundary $\text{bd}(K)$ of dimension $N - 1$. If K is self-generating, then for every $v \in \text{bd}(K)$ there exists a pair $(a, Z) \in U$ such that*

$$(17) \quad \vec{n}(v)^* Z = \mathbf{0}, \text{ and}$$

$$(18) \quad \langle \vec{n}(v), v - g(a) \rangle - \frac{r}{2} \text{tr}(L_v Z Z^*) \leq 0.$$

Proof. Eq. (17) is an immediate consequence of Lemma 4.7. Eq. (18) follows by combining the geometric condition (15) with the just derived fact that $(\vec{n}(v), -L_v) \in \mathcal{N}_K^2(v)$. $\square$

Condition (17) is the “tangential span” condition on the sensitivity coefficients first reported by Sannikov (2007) in the two player case. This condition means that if $(a, Z) \in U$ is an enforcing pair at the payoff vector v , then there exists a matrix $\beta = \beta_a \in \mathbb{R}^{(N-1) \times d}$

such that $Z = \mathbf{T}\beta$. From this observation we now deduce a curvature relation which must hold at enforced payoff vectors.

We can diagonalize the matrix $D^2\varphi(\mathbf{T}^*v)$ so that $D^2\varphi(\mathbf{T}^*v) = \mathbf{S}_v^* \Lambda_v \mathbf{S}_v$. The matrix

$$\Lambda_v \triangleq \text{diag}[\kappa_1(v), \dots, \kappa_{N-1}(v)]$$

is the $(N-1) \times (N-1)$ matrix whose diagonal elements $\kappa_i(v), 1 \leq i \leq N-1$ are called the *principal curvature parameters* of $\text{bd}(K)$ at v . The columns of the matrix $\mathbf{S}_v$ are the eigenvectors corresponding to the principal curvatures and are called the *principal directions*. If we combine this fact with eq. (18), we arrive at the expression

$$\begin{aligned} \text{tr}(L_v Z Z^*) &= \text{tr}(Z^* L_v Z) = \text{tr}(\beta_a^* \mathbf{T}^* \mathbf{T} D^2\varphi(\mathbf{T}^*v) \mathbf{T}^* \mathbf{T} \beta_a) \\ &= \text{tr}(\beta_a^* \mathbf{S}_v^* \Lambda_v \mathbf{S}_v \beta_a) \\ &= \sum_{i=1}^{N-1} \kappa_i(v) \left\| \Phi^i(a, \mathbf{T}_v) \right\|^2, \end{aligned}$$

where $\Phi(a, \mathbf{T}_v) \triangleq \mathbf{S}_v \beta_a$ is the $(N-1) \times d$ matrix formed by the matrix of principal directions at the point $v \in \text{bd}(K)$, and the (orthogonally) enforcing matrix β_a , and $\Phi^i(a, \mathbf{T}_v)$ is the i -th row of this matrix. We are thus left with the *optimality condition*

$$(19) \quad \langle \vec{n}(v), v - g(a) \rangle - \frac{r}{2} \sum_{i=1}^{N-1} \kappa_i(v) \left\| \Phi^i(a, \mathbf{T}_v) \right\|^2 \leq 0.$$

The second term in this condition can be interpreted as a weighted mean curvature of the boundary of the self-generating set K .¹⁴ The optimality equation shows that any smooth convex subset of PPE-p must have a weighted mean curvature bounded from below by $\frac{2}{r} \langle \vec{n}(v), g(a) - v \rangle$. Figure 1 illustrates this result. There we have drawn a smooth convex set $K \subset \mathcal{V}$ and a point $v \in \text{bd}(K)$. The red-dashed line indicates one possible trajectory of continuation payoffs passing through v . The curvature of the flow must be at least as large as the curvature of the boundary of K at v , because otherwise the trajectory would immediately escape K , contradicting its viability. Figure 2 shows what happens when the curvature condition is violated. In this case the solution trajectory must instantaneously escape K since the curvature is lower than the one of $\text{bd}(K)$ at v .

Example 4.9. Let us investigate the optimality equation (19) in the two-player case. This is the setting studied by Sannikov (2007). In this case, the enforcing matrix projected to

¹⁴Mean curvature is defined as the arithmetic mean of the principal curvatures. In eq. (19) we would obtain the mean curvature if each vector $\Phi^i(a, \mathbf{T}_v)$ would have unit length.

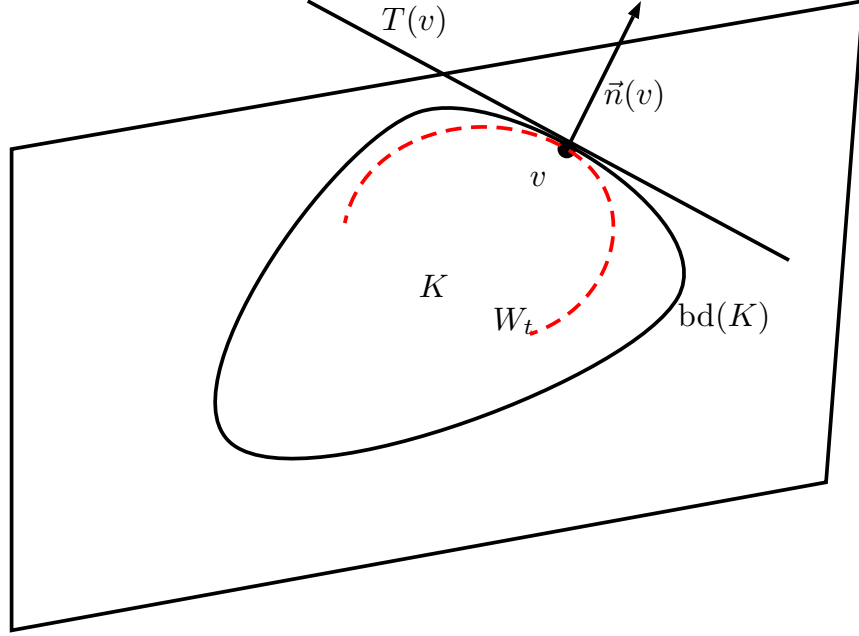


Figure 1: Illustration of the curvature condition on viable solutions. Here the solution trajectory W_t touching $v \in \text{bd}(K)$ stays in K since the flow has a larger curvature at v than $\text{bd}(K)$.

the tangent coordinates is a $1 \times d$ vector $\Phi(a, \mathbf{T}_v)$, and equation (19) reduces to

$$\langle \vec{n}(v), v - g(a) \rangle - \frac{r}{2} \kappa(v) \|\Phi(a, \mathbf{T}_v)\|^2 \leq 0 \Leftrightarrow \kappa(v) \geq \frac{2}{r} \frac{\langle \vec{n}(v), v - g(a) \rangle}{\|\Phi(a, \mathbf{T}_v)\|^2},$$

provided that $\Phi(a, \mathbf{T}_v) \neq \mathbf{0}$ (which is the case whenever a is not a stage-game NE). While this condition guarantees viability of the set K , Sannikov (2007) has shown that actually the inequality holds with equality for $K = \mathcal{E}(r)$.¹⁵ This means that $\text{bd}(\mathcal{E}(r))$ is an invariant set under equilibrium continuation payoffs, i.e. once the payoff process reaches the boundary of the set of PPE-p's it must stay there. This is in line with results reported in Cvitanic et al. (2012), and with classical results due to Friedman (1976), chapter 9-12, derived in the case where K is the closure of an open set with smooth boundary. ♦

4.2 Computation and equivalent characterization of $\mathcal{E}(r)$

Theorem 4.3 provides us with a general geometric characterization of self-generating subsets. This characterization was achieved by exploiting the recursive nature of public

¹⁵We use a different signing convention than Sannikov (2007), as becomes clear when comparing his definition of curvature on p. 1305, and ours.

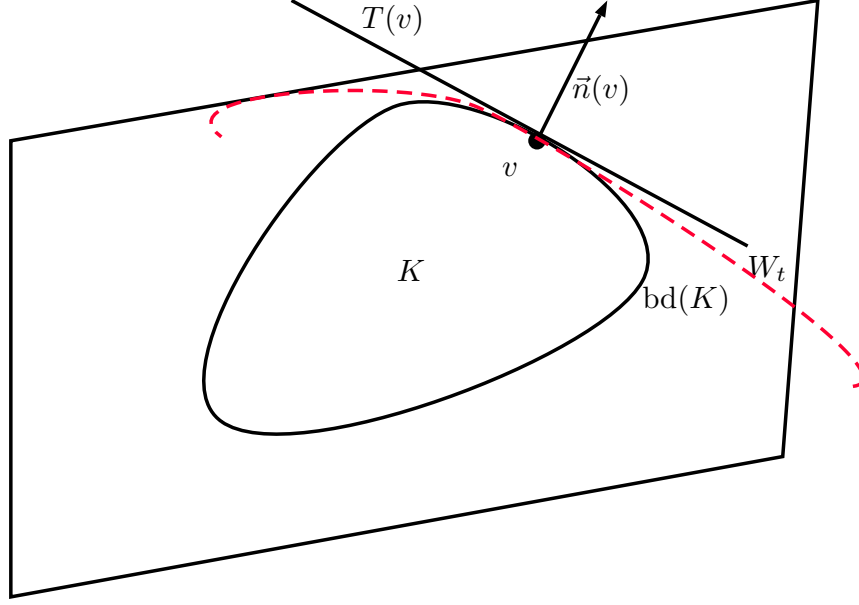


Figure 2: Illustration of the curvature condition on viable solutions. Here the solution trajectory W_t touching $v \in \text{bd}(K)$ escapes K instantaneously since the flow has a lower curvature at v than $\text{bd}(K)$.

equilibrium. Building on this recursive approach we now present a computational procedure which yields in general as an output the set $\mathcal{E}(r)$.

Fix a payoff vector $v \in \mathcal{V}^{IR}$ and a control process $u \in \mathcal{U}$. If this admissible control process gives rise to a well-defined continuation payoff process $\{W_t^{v,u}\}_{t \geq 0}$, then we can define the stopping time

$$\tau_v(u) \triangleq \inf\{t \geq 0 \mid W_t^{v,u} \notin \mathcal{V}^{IR}\}.$$

Define

$$(20) \quad f(v) \triangleq \sup_{u \in \mathcal{U}_0} \mathbb{E} \left[1 - e^{-r\tau_v(u)} \right] = \sup_{u \in \mathcal{U}_0} \mathcal{J}(v, u).$$

The idea behind the optimization problem (20) is the following. For a vector $v \in \mathbb{R}^N$ to be a PPE two conditions have to be satisfied. First, we need that $v \in \mathcal{V}^{IR}$. Second, at v there must exist an admissible control process $u \in \mathcal{U}_0$ such that the continuation payoff process starting from v is the process $\{W_t^{v,u}\}_{t \geq 0}$ having differential (10) and $W_t^{v,u} \in \mathcal{V}^{IR}$ for all $t \geq 0$ P-a.s. Hence, if $f(v) = 1$ and there exists a $u \in \mathcal{U}_0(v)$ such that $\mathcal{J}(v, u) = 1$, then $\mathbb{P}(\tau_v(u) = \infty) = 1$, meaning that both of these conditions hold. It follows that

$v \in \mathcal{E}(r)$. Motivated by this observation, let us define the set $D \triangleq \{v \in \mathcal{V}^{IR} | f(v) < 1\}$ and its "boundary" $\partial D \triangleq \mathcal{V}^{IR} \setminus D = \{v \in \mathcal{V}^{IR} | f(v) = 1\}$. By the above said, the following characterization of the set of PPE-p's is obvious.

Theorem 4.10. $\mathcal{E}(r) = \{v \in \mathcal{V}^{IR} | (\exists u \in \mathcal{U}_0(v)) : \mathcal{J}(v, u) = f(v) = 1\}$.

Since ∂D is not a-priori known, it is like a "free-boundary" of the PDE corresponding to the optimal control problem. This PDE is given by

$$(21) \quad 1 - f(v) + \mathcal{H}(v, Df(v), D^2f(v)) = 0,$$

where

$$\mathcal{H}(v, p, Y) \triangleq \sup_{(a, z) \in U} \{ \langle p, v - g(a) - Z\mu(a) \rangle + \frac{r}{2} \text{tr}(Yzz^*) \} \quad v \in \mathcal{V}^{IR}, (p, Y) \in \mathbb{R}^N \times \mathcal{S}^N,$$

together with the Dirichlet type boundary condition

$$(22) \quad \min\{f(v) - 1, \mathcal{H}(v, Df(v), D^2f(v))\} = 0 \quad \forall v \in \partial D.$$

Since $\mathcal{E}(r)$ must be stochastically viable and the characterization of a stochastically viable set only involved geometric properties of the set, the present Dirichlet problem should give us the same geometric characterizations as the previous one. Indeed, the next theorem shows that this true.

Theorem 4.11. *The free-boundary ∂D of the HJB-PDE (21) with the boundary condition (22) characterizes the set of perfect public equilibrium payoffs if and only if the geometric condition (15) holds on ∂D .*

Proof. By the above said, for ∂D to coincide with $\mathcal{E}(r)$ we need that $f \equiv 1$ on ∂D . Since $f \in [0, 1]$ by definition, we must have that $\psi = 1$ is a subsolution of the PDE (21) with boundary condition (22). This means that for every element (p, Y) of the "positive" second-order normal cone

$$\mathcal{N}_K^{2,+}(v) \triangleq \left\{ (p, Y) \in \mathbb{R}^N \times \mathcal{S}^N \mid \begin{array}{l} \text{for } K \ni w \rightarrow v : \\ \langle p, w - v \rangle + \frac{1}{2} \langle Y(w - v), w - v \rangle \geq o(\|w - v\|^2) \end{array} \right\}.$$

we have

$$\sup_{(a, z) \in U} \{ \langle p, v - g(a) - Z\mu(a) \rangle + \frac{r}{2} \text{tr}(Yzz^*) \} \geq 0.$$

Since $\mathcal{N}_{\partial D}^{2,+}(v) = -\mathcal{N}_{\partial D}^2(v)$, this gives

$$\inf_{(a,z) \in U} \{ \langle p, v - g(a) - Z\mu(a) \rangle + \frac{r}{2} \text{tr}(Yzz^*) \} \leq 0,$$

which is seen to be just the geometric condition (15). □

5. Conclusion

This paper gives a novel geometric characterization of self-generating sets in a class of repeated games with imperfect public monitoring with Brownian signals. Our analysis relied on stochastic viability theory, viscosity solutions and geometry. The combination of all these methods gives a new geometric characterization of compact subsets of perfect public equilibrium payoffs with smooth boundaries in terms of the principal curvatures, and a new computational procedure to detect stochastically viable sets (and *a-fortiori* subsets of PPE-p's). With the same methods one obtains also a novel characterization of the whole set of perfect public equilibrium payoffs, showing that this set is the solution to a Dirichlet boundary value problem. We hope that this identification allows us to come up with more efficient methods to compute the set of perfect public equilibrium payoffs in continuous-time repeated games. Moreover, it would be of some importance to extend the present characterization to include processes with jumps.

It is clear that none of these characterizations could be applied in the same powerful way in models where players have private information. Indeed, the reduction of games with imperfect monitoring to optimal control problems is only possible because players condition their behavior on the same information filtration. This is exactly the reason why the concept of perfect public equilibrium is analytically tractable (an observation which also applies to the discrete-time case). In order to analyze games with private information, a new mathematical framework has to be developed, and we regard this a very challenging and important question for future research. Additionally, it will be interesting to extend the present framework to a class of stochastic games with imperfect monitoring, as studied in Hörner et al. (2011).

Appendix

A. On the derivation of the continuation payoff process

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$ be the given stochastic basis on which we have defined the Brownian motion B . The measure $\mathbf{P}$ is the Wiener measure and fixes the sets of measure 0 in our repeated game model. In the main text we constructed the repeated game model in presence of a public strategy α via a change of measure argument, using Girsanov's theorem. Since the game is defined on an infinite horizon we have to add some technical details to make our approach precise. This appendix provides the details omitted from the main text.

The filtration $\mathbb{F}$ consists of sub-sigma-algebras $\mathcal{F}_t$ generated by the Brownian motion B , completed by the sets of $\mathbf{P}$ -measure 0. The uncompleted version of this sub-sigma algebra is denoted by $\mathcal{F}_t^\circ$. Formally,

$$\mathcal{F}_t^\circ \triangleq \sigma(B_s; 0 \leq s \leq t), \quad \mathcal{F}_{t+}^\circ \triangleq \bigcup_{s>0} \mathcal{F}_{t+s}^\circ, \quad \mathcal{F}_\infty^\circ \triangleq \sigma\left(\bigcup_{t \geq 0} \mathcal{F}_t^\circ\right),$$

and $\mathcal{F}_t \triangleq \sigma(\mathcal{F}_t^\circ \cup \mathcal{N})$,

where

$$\mathcal{N} \triangleq \{N \subset \Omega \mid \exists B \subset \mathcal{F}_\infty^\circ \text{ s.t. } N \subset B \text{ and } \mathbf{P}(B) = 0\},$$

is the collection of $\mathbf{P}$ -null sets. The so generated filtration is denoted by $\mathbb{F}^\circ \triangleq \{\mathcal{F}_t^\circ\}_{t \geq 0}$.

Let α be a pure public strategy. In Section 2.1 we defined for every $T \in [0, \infty)$ a probability measure $\mathbf{P}_T^\alpha$ on $(\Omega, \mathcal{F}_{T+}^\circ)$ by

$$\mathbf{P}_T^\alpha(A) = \mathbf{E}[\mathbf{1}_A M_T^\alpha] \quad \forall A \in \mathcal{F}_{T+}^\circ.$$

We also defined a process

$$B_t^\alpha \triangleq B_t - \int_0^t \mu(\alpha_s) ds.$$

By Girsanov's theorem (Karatzas and Shreve, 2000, Section 3.5) the restricted process $\{B_t^\alpha; t \in [0, T]\}$ is a Brownian motion under $\mathbf{P}_T^\alpha$. Moreover, for every $t \in [0, T]$, the probability measure $\mathbf{P}_T^\alpha$ is equivalent to $\mathbf{P}$ on $\mathcal{F}_t^{(T)} \triangleq \sigma(\mathcal{F}_t^\circ \cup \mathcal{N}^T)$, where

$$\mathcal{N}^T \triangleq \{N \subset \Omega \mid \exists B \subset \mathcal{F}_T^\circ \text{ s.t. } N \subseteq B \text{ and } \mathbf{P}(B) = 0\},$$

is the collection of $\mathbf{P}|_{\mathcal{F}_T^\circ}$ -measure 0 sets. Note that $\{\mathcal{F}_t^{(T)}\}_{t \in [0, T]}$ is continuous, thanks to completion, and $\mathcal{F}_T^{(T)} = \mathcal{F}_T$ for every $T \geq 0$.

The family of probability measure $\{\mathbf{P}_t^\alpha\}_{t \geq 0}$ is consistent, hence can be extended to a unique probability measure on $(\Omega, \mathcal{F}_\infty)$ (Karatzas and Shreve, 2000, Corollary 3.5.2). The important thing to note is that we have not defined a probability measure on $(\Omega, \mathcal{F})$. In fact, it is well known that $\mathbf{P}^\alpha$ cannot be equivalent to $\mathbf{P}$ on $(\Omega, \mathcal{F}_\infty)$ in general. Nevertheless, $\{B_t^\alpha; t \geq 0\}$ is a Brownian motion under $\mathbf{P}^\alpha$. We have to be careful in our derivation of the representation formula for the continuation value process, because there we used a martingale representation theorem, and martingales have to come up together with a filtration.

For a fixed public strategy, let $\eta \triangleq \int_0^\infty r e^{-rt} g(\alpha_t) dt$. Then $|\eta(\omega)| \leq \bar{g} = \max_{a \in A} |g(a)| < \infty$, hence is a uniformly bounded random variable on $(\Omega, \mathcal{F}_\infty, \mathbf{P})$. The total expected payoff of player i under strategy α is given by $U(\alpha) = \mathbf{E}^\alpha(\eta)$. For fixed $T \in [0, \infty)$, let

$$Y_T \triangleq \mathbf{E}^\alpha[\eta | \mathcal{F}_T^{(T)}] = \mathbf{E}^\alpha[\eta | \mathcal{F}_T],$$

and

$$Y_t \triangleq \mathbf{E}^\alpha[\eta | \mathcal{F}_t^{(T)}] \quad \forall t \in [0, T].$$

Remark A.1. An important observation for the derivation to come is that we have the following almost sure equalities:

$$Y_t = \mathbf{E}^\alpha[\eta | \mathcal{F}_t^\circ] = \mathbf{E}^\alpha[\eta | \mathcal{F}_t] = U_t(\alpha),$$

almost surely, where $U_t(\alpha)$ is the process defined in eq. (2). (see Bain and Crisan, 2000, Exercise 1.2.4). ♦

The adapted process $t \mapsto Y_t$ is a continuous martingale under $\mathbf{P}_T^\alpha$ with respect to the filtration $\mathbb{F}^{(T)} \triangleq \{\mathcal{F}_t^{(T)}\}_{t \in [0, T]}$. Hence, by the martingale representation theorem of Fujisaki et al. (1972) there exists a progressively measurable process $\{\Phi_t^{\alpha, T}; t \in [0, T]\} \in L_{\mathbb{F}^{(T)}}^2(\Omega, \mathbf{P}^\alpha; \mathbb{R}^{N \times d})$ such that

$$Y_t = \mathbf{E}^\alpha[Y_0] + \int_0^t \Phi_s^{\alpha, T} dB_s^\alpha = U(\alpha) + \int_0^t \Phi_s^{\alpha, T} dB_s^\alpha, \quad t \in [0, T].$$

If $\tilde{T} \leq T$ is chosen and we repeat the above computation for the process $\tilde{Y}_t \triangleq \mathbf{E}^\alpha[\eta | \mathcal{F}_t^{(\tilde{T})}]$, $t \in$

$[0, \tilde{T}]$, then there exists a process $\{\Phi_t^{\alpha, \tilde{T}}; t \in [0, \tilde{T}]\}$ such that

$$\tilde{Y}_t = U(\alpha) + \int_0^t \Phi_s^{\alpha, \tilde{T}} dB_s^\alpha \quad t \in [0, \tilde{T}].$$

Since Y_t is $\mathcal{F}_t^{(T)}$ measurable, we have the almost sure identity (cf. Remark A.1)

$$Y_t = \mathbb{E}^\alpha[\eta | \mathcal{F}_t^\circ] = \mathbb{E}^\alpha[\eta | \mathcal{F}_t^{(\tilde{T})}] = \tilde{Y}_t.$$

Hence,

$$\int_0^t (\Phi_s^{\alpha, \tilde{T}} - \Phi_s^{\alpha, T}) dB_s^\alpha = 0$$

almost surely for every $t \in [0, \tilde{T}]$, which implies that $\Phi_t^{\alpha, \tilde{T}} = \Phi_t^{\alpha, T}$ almost surely for Lebesgue almost all $t \in [0, \tilde{T}]$ and $\tilde{T} \leq T$. Now set

$$\Phi_t^\alpha \triangleq \sum_{n \geq 1} \mathbf{1}_{[n-1, n)}(t) \Phi_t^{\alpha, n},$$

where $\Phi^{\alpha, n}$ is the progressively measurable process obtained from the martingale representation theorem of the bounded martingale $\{\mathbb{E}^\alpha[\eta | \mathcal{F}_t^{(n)}]; t \in [0, n]\}$; I.e.

$$Y_t^{(n)} \triangleq \mathbb{E}^\alpha(\eta | \mathcal{F}_t^{(n)}) = U(\alpha) + \int_0^t \Phi_s^{\alpha, n} dB_s^\alpha \quad t \in [0, n].$$

Finally, let

$$Y_t = \sum_{n \geq 1} \mathbf{1}_{[n-1, n)}(t) Y_t^{(n)},$$

to see that

$$Y_t = U(\alpha) + \int_0^t \Phi_s^\alpha dB_s^\alpha,$$

for all $t \geq 0$. For every $t \geq 0$ pick $n \in \mathbb{N}$ such that $t \in [n-1, n)$, so that

$$\int_0^t \Phi_s^\alpha dB_s^\alpha = \int_0^t \Phi_s^{\alpha, n} dB_s^\alpha.$$

Then, we have the following string of equalities in the almost sure sense

$$Y_t = Y_t^n = \mathbb{E}^\alpha(\eta | \mathcal{F}_t^{(n)}) = \mathbb{E}^\alpha(\eta | \mathcal{F}_t) = U_t(\alpha).$$

Setting

$$W_t^\alpha \triangleq \mathbb{E}^\alpha \left[\int_t^\infty r e^{-r(s-t)} g(\alpha_s) ds | \mathcal{F}_t \right],$$

we see that

$$U_t(\alpha) = \int_0^t r e^{-rs} g(\alpha_s) ds + e^{-rt} W_t^\alpha,$$

from which it is easy to deduce that W_t^α is an Itô process with differential

$$dW_t = r[W_t - g(\alpha_t)]dt + rZ_t^\alpha dB_t^\alpha, \quad Z_t^\alpha \triangleq \frac{1}{r} e^{-rt} \Phi_t^\alpha.$$

B. Auxiliary Facts

B.1 Subjets and Viscosity solutions

The proof of Theorem 4.3 relies on viscosity solution techniques. This short section summarizes all the concepts which are used in the proof of Theorem 4.3. All definitions and results are standard in this literature, and can be found in Fleming and Soner (2006).

Definition B.1. Let $f : \mathbb{R}^N \rightarrow [0, \infty]$ be a lower semi-continuous function. The (second-order) subjet for the map $f|_K$ is defined as

$$(23) \quad \mathcal{J}_K^2 f(v) \triangleq \left\{ (p, Y) \in \mathbb{R}^N \times \mathcal{S}^N \mid \begin{array}{l} \text{for } K \ni w \rightarrow v : f(w) - f(v) \geq \\ \langle p, w - v \rangle + \frac{1}{2} \langle Y(w - v), w - v \rangle + o(\|w - v\|_2^2) \end{array} \right\}$$

It is easy to see that if K is a closed domain and f is constant on K we have $\mathcal{J}_K^2 f(v) = \mathcal{N}_K^2(v)$, where we recall the definition of the second-order normal cone as

$$\mathcal{N}_K^2(v) \triangleq \left\{ (p, Y) \in \mathbb{R}^N \times \mathcal{S}^N \mid \begin{array}{l} \text{for } K \ni w \rightarrow v : \\ \langle p, w - v \rangle + \frac{1}{2} \langle Y(w - v), w - v \rangle \leq o(\|w - v\|^2) \end{array} \right\}.$$

Another useful fact about subjets is that they are closely related to comparison principles. The following fact is needed in the proof of Theorem 4.3.

Lemma B.2. $(p, Y) \in \mathcal{J}_K^2 f(v)$ if and only if there exists $\phi \in \mathbf{C}^2(\mathbb{R}^N; \mathbb{R})$ such that $f - \phi$ achieves its minimum at $v \in K$ satisfying $(\phi'(v), \phi''(v)) = (p, Y)$.

Proof. Fleming and Soner (2006), Lemma V.4.1. $\square$

Definition B.3. Let $\mathcal{O}$ be an open set in $\mathbb{R}^N$ containing K . A lower semi-continuous function $F : \mathcal{O} \rightarrow \mathbb{R}$ is a viscosity super-solution to (13) if

$$F(v) - c(v) - H(v, D\phi(v), D^2\phi(v)) \geq 0$$

for all $\phi \in \mathbf{C}(\mathcal{O}, \mathbb{R})$ such that

$$F(v) - \phi(v) = (\text{strict}) \min_{\mathcal{O}} (F - \phi).$$

In the proof of Theorem 4.3 we need the following characterization of viscosity super-solutions to the Hamilton-Jacobi-Bellman PDE (13).

Lemma B.4. The function $F : \mathbb{R}^N \rightarrow [0, \infty]$ is a viscosity supersolution to (13) if and only if

$$F(v) - c(v) - H(v, p, Y) \geq 0 \quad \forall (p, Y) \in \mathcal{J}_K^2 F(v).$$

Proof. Fleming and Soner (2006), Proposition V.4.1. $\square$

Remark B.5. The subset of a function f is usually denoted by $\mathcal{J}_{\mathcal{O}}^{2,-} f$ to notationally distinguish it from the superjet $\mathcal{J}_{\mathcal{O}}^{2,+} f(v) = -\mathcal{J}_{\mathcal{O}}^{2,-}(-f)(v)$. Since only subsets are important for this paper no notational distinction is necessary. $\blacklozenge$

B.2 Geometric tools

In this appendix we collect some geometric facts, which are needed in the study of enforceable payoffs vectors with respect to smooth sets. The results are not new, and in fact can be found in Bardi and Jensen (2002). To understand the next lemma, recall the definition of the *Bouligand tangent cone* for a closed set $K \subset \mathbb{R}^N$ as¹⁶

$$\mathcal{T}_K(v) \triangleq \left\{ q \in \mathbb{R}^N \mid \exists \{t_n\}_n \rightarrow 0^+ \text{ and } \{v_n\}_n \subset K \text{ s.t. } \lim_{n \rightarrow \infty} \frac{v_n - v}{t_n} = q \right\}.$$

If K is a closed convex set the Bouligand tangent cone coincides with the tangent cone in the sense of convex analysis, i.e. $\text{cl}(\mathbb{R}_+(K - v))$. The Bouligand tangent cone is related to

¹⁶The interested reader may find Vinter (2000) accessible.

the set of strict normals via polarity, i.e.

$$\mathcal{N}_K^1(v) = \{p \in \mathbb{R}^N \mid \langle p, q \rangle \leq 0 \ \forall q \in \mathcal{T}_K(v)\}.$$

The following lemma relates points in the second-order normal cone to (strict) normal vectors.

Lemma B.6. *If $(p, Y) \in \mathcal{N}_K^2(v)$ then $p \in \mathcal{N}_K^1(v) = \left\{ p \in \mathbb{R}^N \mid \limsup_{w \rightarrow v, w \in K \setminus \{v\}} \frac{\langle p, w-v \rangle}{\|w-v\|} \leq 0 \right\}$.*

Proof. Observe that an equivalent definition of the second-order normal cone is

$$\mathcal{N}_K^2(v) = \left\{ (p, Y) \in \mathbb{R}^N \times \mathcal{S}^N \mid \limsup_{\|\eta\| \rightarrow 0, v+\eta \in K} \frac{\langle p, \eta \rangle + \frac{1}{2} \langle Y\eta, \eta \rangle}{\|\eta\|^2} \leq 0 \right\}.$$

If $v \in \text{int}(K)$ then it is easy to see that for every $(p, Y) \in \mathcal{N}_K^2(v)$ it must be the case that $p = \mathbf{0}$. This is also the only choice for a vector to be in $\mathcal{N}_K^1(v)$. If $v \in \text{bd}(K)$, choose a sequence $\{v_n\}_n$ in K with $\|v_n - v\| \rightarrow 0$ and $\frac{v_n - v}{\|v_n - v\|} \rightarrow q$, so that $q \in \mathcal{T}_K(v)$. Since $(p, Y) \in \mathcal{N}_K^2(v)$ along this sequence we have

$$0 \geq \left\langle p, \frac{v_n - v}{\|v_n - v\|} \right\rangle + \frac{1}{2} \left\langle Y \frac{v_n - v}{\|v_n - v\|}, v_n - v \right\rangle + o(\|v_n - v\|)$$

Let $n \rightarrow \infty$ to get

$$0 \geq \langle p, q \rangle.$$

Hence, $p \in \mathcal{N}_K^1(v)$, by polarity. □

Proposition B.7. *Let $z \in \mathbb{R}^{N \times d}$ and $f \in \mathbb{R}^N$ a vector such that*

$$\langle p, f \rangle + \frac{r}{2} \text{tr}(Yzz^\top) \leq 0 \quad \forall (p, Y) \in \mathcal{N}_K^2(v).$$

Then

$$p^* z = 0 \quad \forall (p, Y) \in \mathcal{N}_K^2(v).$$

Proof. A simple adaptation of the argument given in Lemma 2 of Bardi and Jensen (2002) shows that if $(p, Y) \in \mathcal{N}_K^2(v)$ then $(p, Y + \mu p p^*) \in \mathcal{N}_K^2(v)$ for every $\mu \in \mathbb{R}$. If $p^* z \neq 0$ for some $(p, Y) \in \mathcal{N}_K^2(v)$ then $\text{tr}(zz^* p p^*) > 0$. Hence,

$$0 \geq \langle f, p \rangle + \frac{r}{2} \text{tr}(zz^* Y) + \frac{\mu r}{2} \text{tr}(zz^* p p^*)$$

holds for every $\mu \in \mathbb{R}$ whenever $(p, Y) \in \mathcal{N}_K^2(v)$. If $p^*z \neq 0$ then we can let $\mu \rightarrow -\infty$ in the above display to obtain a contradiction. $\square$

B.3 The Weingarten map

In Section 4.1 we have explicitly characterized pairs (p, Y) which are elements of the second-order normal cone. The explicit representation used a coordinate transform under which the Weingarten map has a nice matrix representation in terms of the Hessian of a $\mathbf{C}^2$ map which locally represents payoff vectors in a sufficiently small neighborhood centered at $v \in \text{bd}(K)$. The purpose of this Appendix is to provide the computations leading to this representation. The computation is organized in two steps. First, we perform a rotation of coordinates so that the v^N -axis points into the direction of the normal vector $\vec{n}$. Then we use these rectangular coordinates to perform the curvature computations.

Fix a point $v \in \text{bd}(K)$. At this point choose a basis $\mathcal{B} = \{\vec{t}_1, \dots, \vec{t}_{N-1}, \vec{n}\}$ consisting of the vectors of length one spanning the tangent plane at v (denoted by $T(v)$), and its orthogonal complement. Without loss of generality we can assume that the vectors $\vec{t}_i$ are orthonormal, so that $\mathcal{B}$ is an orthonormal basis of $\mathbb{R}^N$. Let $\mathbf{T} = [\vec{t}_1, \dots, \vec{t}_{N-1}]$ denote the $N \times (N-1)$ whose columns is made of the spanning tangent vectors at v , and $B = [\mathbf{T}, \vec{n}]$ the $N \times N$ matrix determining the change-of-coordinates, i.e. the coordinates of a vector $w \in \mathbb{R}^N$ in the basis $\mathcal{B}$ is given by

$$w = \sum_{i=1}^{N-1} w_i^{\text{Tang}} e_i + w^{\text{Norm}} e_N,$$

where $w^{\text{Tang}} = \mathbf{T}^*w$ and $w^{\text{Norm}} = \langle \vec{n}, w \rangle$. Additionally, there exists an open set $\mathcal{O}' \subset \mathbb{R}^{N-1}$ and a mapping $\varphi \in \mathbf{C}^2(\mathcal{O}' \cap T(v))$ such that¹⁷

$$w^{\text{Norm}} \leq \varphi(\mathbf{T}^*w), \quad \forall w \in K \cap \mathcal{O}, \text{ and}$$

¹⁷Choose $w = v \pm \varepsilon \vec{t}_i$ where $\varepsilon > 0$ is small. Then, denoting by $\tilde{e}_i$ the i -th unit vector in $\mathbb{R}^{N-1}$, we see that

$$c(\varepsilon) \triangleq \varphi(\mathbf{T}^*(v + \varepsilon \vec{t}_i)) = \varphi(\mathbf{T}^*v + \varepsilon \tilde{e}_i) \quad \forall \varepsilon > 0 \text{ small.}$$

Since $c(\varepsilon) \leq v^{\text{Norm}} = c(0)$ for all $\varepsilon > 0$ small, it follows

$$c'(0) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} [c(\varepsilon) - c(0)] = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} [\varphi(\mathbf{T}^*v + \varepsilon \tilde{e}_i) - \varphi(\mathbf{T}^*v)] = \nabla \varphi(\mathbf{T}^*v)^* \tilde{e}_i \leq 0.$$

Doing the same computation with the point $v - \varepsilon \vec{t}_i$, we similarly conclude that $-\nabla \varphi(\mathbf{T}^*v)^* \tilde{e}_i \leq 0$. This shows that $\langle \nabla \varphi(\mathbf{T}^*v), \tilde{e}_i \rangle = 0$ for all $i = 1, 2, \dots, N-1$, and hence $\nabla \varphi(\mathbf{T}^*v) = \mathbf{0}_{N-1}$.

$$v^{Norm} = \varphi(\mathbf{T}^*v), \quad \nabla \varphi(\mathbf{T}^*v) = \mathbf{0}.$$

In general, $K \cap \mathcal{O} = \{w \in \mathcal{O} | \varphi(\mathbf{T}^*w) \leq \langle \vec{n}, w \rangle\}$ and $\text{bd}(K) \cap \mathcal{O} = \{w \in \mathcal{O} | \varphi(\mathbf{T}^*w) = \langle \vec{n}, w \rangle\}$. For the curvature computations it is useful to work with standard coordinates. Therefore we introduce the parametrization $f : \mathbb{R}^{N-1} \rightarrow \mathbb{R}^N$ given by

$$f(x) = \begin{pmatrix} x \\ \varphi(x) \end{pmatrix}.$$

Localized to the subspace $\mathbf{T}^*(\text{bd}(K) \cap \mathcal{O}) \subset \mathbb{R}^{N-1}$, the coordinates of any vector $w \in \text{bd}(K) \cap \mathcal{O}$ in the basis $\mathcal{B}$ are given by $f(\mathbf{T}^*w)$. For $x \in \mathbb{R}^{N-1}$ we compute

$$(24) \quad f'_i(x) \triangleq \frac{\partial f}{\partial x_i}(x) = e_i + e_N D_i \varphi(x), \quad i = 1, 2, \dots, N-1, \text{ and}$$

$$(25) \quad f''_{ij}(x) \triangleq \frac{\partial^2 f}{\partial x_j \partial x_i}(x) = e_N D_{ij} \varphi(x).$$

The linear independent set of vectors $\{f'_i(\mathbf{T}^*w), \dots, f'_{N-1}(\mathbf{T}^*w)\}$ spans the tangent plane at $w \in \text{bd}(K) \cap \mathcal{O}$. In particular $f'_i(\mathbf{T}^*v) = e_i$ for $i = 1, 2, \dots, N-1$.

The *Gauss-map* defined in terms of the parametrization f is the mapping $\nu : \mathbb{R}^{N-1} \rightarrow \mathbb{R}^N$ given by

$$\nu(x) \triangleq \frac{(-D\varphi(x), 1)}{\sqrt{1 + \|D\varphi(x)\|^2}}.$$

The vector $\nu(\mathbf{T}^*w)$ spans the normal space at the point $w \in \text{bd}(K) \cap \mathcal{O}$. Observe that $\nu(\mathbf{T}^*v) = e_N$. Hence, the basis $\{f'_1(\mathbf{T}^*v), \dots, f'_{N-1}(\mathbf{T}^*v), \nu(\mathbf{T}^*v)\}$ reduces to the standard basis $\{e_1, \dots, e_{N-1}, e_N\}$. To lift the Gauss map to the outward normals at points $w \in \text{bd}(K) \cap \mathcal{O}$ set

$$\vec{n}(w) \triangleq \nu(\mathbf{T}^*w) \quad w \in \text{bd}(K) \cap \mathcal{O}.$$

Then $\vec{n}(f(\mathbf{T}^*w)) = \nu(\mathbf{T}^*w)$ since $w = f(\mathbf{T}^*w)$ in the coordinates corresponding to $\mathcal{B}$. By definition, we have

$$\langle \vec{n}(w), f'_i(\mathbf{T}^*w) \rangle = 0 \quad \forall i = 1, 2, \dots, N-1, w \in \text{bd}(K) \cap \mathcal{O}.$$

Denoting the standard basis vectors of $\mathbb{R}^{N-1}$ by $\tilde{e}_1, \dots, \tilde{e}_{N-1}$, this gives us for $x = \mathbf{T}^*v$,

$$0 = \langle f'_i(x + \varepsilon_2 \tilde{e}_j), \vec{n}(f(x + \varepsilon_2 \tilde{e}_j)) \rangle = \left\langle \frac{d}{d\varepsilon_1} \Big|_{\varepsilon_1=0} f(x + \varepsilon_1 \tilde{e}_i + \varepsilon_2 \tilde{e}_j), \vec{n}(f(x + \varepsilon_2 \tilde{e}_j)) \right\rangle,$$

for all $\varepsilon_2 > 0$ sufficiently small. Hence, for $x = \mathbf{T}^*v$, we obtain the following string of equalities:

$$\begin{aligned} 0 &= \frac{d}{d\varepsilon_2} \Big|_{\varepsilon_2=0} \left\langle \frac{d}{d\varepsilon_1} \Big|_{\varepsilon_1=0} f(x + \varepsilon_1 \tilde{e}_i + \varepsilon_2 \tilde{e}_j), \vec{n}(f(x + \varepsilon_2 \tilde{e}_j)) \right\rangle \\ &= \frac{d}{d\varepsilon_2} \Big|_{\varepsilon_2=0} \langle f'_i(x + \varepsilon_2 \tilde{e}_j), \vec{n}(f(x + \varepsilon_2 \tilde{e}_j)) \rangle \\ &= \langle f''_{ij}(x), \vec{n}(f(x)) \rangle + \langle f'_i(x), D\vec{n}|_v f'_j(x) \rangle, \end{aligned}$$

where the last equality follows from the Chain rule. Hence, for $x = \mathbf{T}^*v$, we arrive at the expression

$$(26) \quad - \langle f'_i(x), D\vec{n}|_v f'_j(x) \rangle = \langle f''_{ij}(x), \vec{n}(v) \rangle.$$

Let us define the Weingarten map by

$$L_v \triangleq -D\vec{n}|_v,$$

so that $L_v f'_i(\mathbf{T}^*v) = -D\vec{n}|_v f'_i(\mathbf{T}^*v)$ is the rate of change of the normal vector at $v = f(\mathbf{T}^*v)$ when moving to the point $v + \varepsilon \vec{t}_i$ for ε small, expressed in the tangent-normal coordinates at v . From equation (26), we deduce that

$$\langle f'_i(x), L_v f'_j(x) \rangle = \langle f''_{ij}(x), \vec{n}(v) \rangle \quad i, j = 1, \dots, N-1.$$

Since $f'_i(\mathbf{T}^*v) = e_i, 1 \leq i \leq N-1$, as well as $f''_{ij}(\mathbf{T}^*v) = e_N D_{ij} \varphi(\mathbf{T}^*v)$ and $\vec{n}(v) = e_N$, we see that

$$\langle e_j, L_v e_i \rangle = D_{ij} \varphi(\mathbf{T}^*v) \quad i, j = 1, \dots, N-1.$$

Extending the map L_v by orthogonality on the normal component, we can define the mapping $L_v : \mathbb{R}^N \rightarrow \mathbb{R}^N$ by

$$\begin{aligned} \langle e_i L_v, e_j \rangle &= D_{ij} \varphi(\mathbf{T}^*v) \quad 1 \leq i, j \leq N-1, \text{ and} \\ \langle e_i L_v, e_N \rangle &= 0 \quad \forall i = 1, 2, \dots, N. \end{aligned}$$

Hence, we get the $N \times N$ matrix

$$L_v \triangleq \begin{pmatrix} D^2\varphi(\mathbf{T}^*v) & \mathbf{0}_{N-1} \\ \mathbf{0}_{N-1}^* & 0 \end{pmatrix}.$$

C. Proof of Theorem 4.3

(a) $\Rightarrow$ (b): By the dynamic programming principle,

$$F(v) = \inf_{u \in \mathcal{U}_0} \mathbb{E} \left[\int_0^T r e^{-rt} c(W_t^{v,u}) dt + e^{-rT} F(W_T^{v,u}) \right].$$

Let $\bar{c} \triangleq \sup_{\mathbb{R}^N} c$ and choose T such that $e^{-rT} \bar{c} \leq \varepsilon$. Then $F \in [0, \bar{c}]$ and consequently

$$\begin{aligned} F(v) &\leq \inf_{u \in \mathcal{U}_0} \mathbb{E} \left[\int_0^T r e^{-rt} c(W_t^{v,u}) dt + \varepsilon \right] \\ &\leq r \inf_{u \in \mathcal{U}_0} \mathbb{E} \left[\int_0^T c(W_t^{v,u}) dt + \varepsilon/r \right] \\ &= \varepsilon \end{aligned}$$

where in the last equality we have used the approximate-viability of the set K . As ε is arbitrary, the result follows.

(b) $\Rightarrow$ (a): Assume that there are $T, \varepsilon > 0$ and $v \in K$ such that

$$\mathbb{E} \left[\int_0^T c(W_t^{v,u}) dt \right] \geq \varepsilon \quad \forall u \in \mathcal{U}_0.$$

Then, for all $u \in \mathcal{U}_0$,

$$\begin{aligned} \mathbb{E} \left[\int_0^\infty r e^{-rt} c(W_t^{v,u}) dt \right] &\geq \mathbb{E} \left[\int_0^T r e^{-rt} c(W_t^{v,u}) dt \right] \\ &\geq r \mathbb{E} \left[\int_0^T c(W_t^{v,u}) dt \right] e^{-rT} \\ &\geq r \varepsilon e^{-rT} \\ &> 0. \end{aligned}$$

Hence, $F(v) > 0$. A contradiction!

(c) $\Rightarrow$ (b): Fix $c \in \text{BUC}(\mathbb{R}^N)$ such that $c \geq 0$ and $c = 0$ on K . Denote by $\bar{c} \triangleq \sup_{\mathbb{R}^N} c$. The function F is the viscosity solution to the Hamilton-Jacobi-Bellman equation (13). We

claim that the function

$$(27) \quad \hat{F}(v) \triangleq \begin{cases} 0 & \text{if } v \in K, \\ \bar{c} & \text{if } v \notin K, \end{cases}$$

is a viscosity supersolution to (13). Before proving this claim, let us observe that this proves the assertion. Indeed, by the comparison principle (which applies thanks to Assumption 4.1), we have $F \leq \hat{F}$, and thus $F = 0$ on K . Let us now prove the claim. We use the subset definition of a viscosity supersolution (Lemma B.4). For $v \in \text{int}(K)$, we have $\hat{F}(v) = 0$ and $\mathcal{J}_K^2 \hat{F}(v) = \mathcal{N}_K^2(v)$. Hence, the geometric condition (15) implies that $\hat{F}$ is a supersolution on $\text{int}(K)$. The same argument applies on $\mathbb{R}^N \setminus K$. For $v \in \text{bd}(K)$, a computation shows that $\mathcal{J}_K^2 \hat{F}(v) = \mathcal{N}_K^2(v)$, and the geometric condition (15) again implies the supersolution property.

(b) $\Rightarrow$ (c): Assume that $F = 0$ on K . Set $c(v) \triangleq \min(1, \text{dist}(v, K))$ and for all $n \geq 1$ let $c_n(v) \triangleq nc(v)$. Then each c_n is a bounded and continuous function on $\mathbb{R}^N$ with $c_n = 0$ on K and $c_n \geq 0$. Moreover,

$$\lim_{n \rightarrow \infty} c_n(v) = \begin{cases} 0 & \text{if } v \in K \\ +\infty & \text{if } v \notin K. \end{cases}$$

Let

$$\hat{F}_n(v) \triangleq \inf_{u \in \mathcal{U}_0} \mathbb{E} \left[\int_0^\infty re^{-rt} c_n(W_t^{v,u}) dt \right].$$

Whenever the function is finite, we have $\hat{F}_n(v) \in [0, n]$. Set

$$\hat{F}(v) \triangleq \liminf_{(w,n) \rightarrow (v,\infty), w \in K} \hat{F}_n(w)$$

denote the lower weak limit of the sequence $\{\hat{F}_n\}_{n \geq 1}$. By Bardi and Capuzzo-Dolcetta (1997), Lemma V.1.5, the function $\hat{F}$ is lower semi-continuous. Since $\hat{F}_n = 0$ on K , it follows that $\hat{F} = 0$ on K as well. We claim that $\hat{F}$ is a global discontinuous viscosity super-solution of (13). Particularizing to the domain K , this means that

$$H(v, p, Y) \leq 0 \quad \forall (p, Y) \in \mathcal{J}_K^2 \hat{F}(v), \forall v \in K.$$

This implies the geometric condition (15), upon noting that $\mathcal{J}_K^2 \hat{F}(v) = \mathcal{N}_K^2(v)$ for all $v \in K$. To prove this claim, we need the following Lemma, whose proof is given at the end of this

section.

Lemma C.1. $\hat{F} = \infty$ on $\mathbb{R}^N \setminus K$.

Let $v \in \text{bd}(K)$ and let $\psi \in \mathbf{C}^2(\mathbb{R}^N)$ such that

$$0 = (\hat{F} - \psi)(v) = (\text{strict}) \min_{w \in K} (\hat{F} - \psi)(w).$$

Since $\hat{F} = \infty$ on $\mathbb{R}^N \setminus K$, we can alternatively formulate the above minimization problem over the unconstrained domain $\mathbb{R}^N$. By the same procedure as used in the proof of Lemma C.1, we can construct a sequence $\{v_n\}_{n \geq 1}$ such that $v_n \rightarrow v$ and $\hat{F}(v_n) \rightarrow \hat{F}(v)$. By the viscosity super-solution property of the mappings $\hat{F}_n$, we get

$$\hat{F}_n(v_n) - c_n(v_n) - H(v_n, D\psi(v_n), D^2\psi(v_n)) \geq 0.$$

Taking the $\liminf$ in the above display, and assuming that $\{(v_n, D\psi(v_n), D^2\psi(v_n))\}_n \subset \text{int}(\text{dom}(H))$, we see that

$$H(v, D\psi(v), D^2\psi(v)) \leq 0.$$

Since $\mathcal{J}_K^2 \hat{F}(v) = \mathcal{N}_K^2(v)$, the proof is complete. ■

We now provide the proof of the intermediate result needed in the direction (c) $\Rightarrow$ (b).

Proof of Lemma C.1. Suppose there exists $v \notin K$ such that $\hat{F}(v) < \infty$. Since K is compact, there exists $\delta > 0$ such that $c(v) = \delta$. Fix $\varepsilon > 0$ and set $r = r(\varepsilon) \triangleq \sqrt{2\varepsilon \hat{F}(v)}$. Additionally, let $\phi(w) = \Phi_\varepsilon(v, w) \triangleq \frac{-1}{2\varepsilon} \|v - w\|^2$. Recall that $\hat{F}$ is lower semi-continuous, and $\phi(\cdot)$ is continuous on $\mathbb{R}^N$. For each $\varepsilon > 0$ let v_ε be a strict minimizer of the function $(\hat{F} - \phi)$, which is attained in the set $\mathbb{B}_r(v) \triangleq \{w \in \mathbb{R}^N \mid \|v - w\| \leq r\}$. Then,

$$(28) \quad \hat{F}(v_\varepsilon) \leq \hat{F}(v) + \phi(v_\varepsilon) \leq \hat{F}(v).$$

We now use the Barles-Perthame procedure to construct a sequence $\{v_n^{(\varepsilon)}\}_{n \geq 1} \rightarrow v_\varepsilon$ as $n \rightarrow \infty$ along which $\hat{F}(v_n^{(\varepsilon)}) \rightarrow \hat{F}(v_\varepsilon)$. The construction is as follows (we omit the ε -label to simplify the notation):

For all $n \geq 1$ let $v_n \in \text{argmin}_{w \in \mathbb{B}_r(v)} (\hat{F}_n - \phi)(w)$. Then $\{v_n\}_{n \geq 1} \subset \mathbb{B}_r(v)$, and we can assume that $v_n \rightarrow \bar{v}$ after eventually passing to a subsequence. For all $n \geq 1$ we have

$$(\hat{F} - \phi)(v_n) \leq (\hat{F} - \phi)(v_\varepsilon)$$

by definition of the points v_n . Hence,

$$\begin{aligned} (\hat{F} - \phi)(v_\varepsilon) &= \liminf_{n \rightarrow \infty} (\hat{F}_n - \phi)(v_\varepsilon) \\ &\geq \liminf_{n \rightarrow \infty} (\hat{F}_n - \phi)(v_n) = (\hat{F} - \phi)(\bar{v}). \end{aligned}$$

Since v_ε is a strict minimum on $\mathbb{B}_r(v)$, it follows that $\bar{v} = v_\varepsilon$.

Now choose ε small enough so that v_ε must be in a sufficiently small neighborhood of v . Then choose n sufficiently large so that $c(v_n) \geq \delta/4$. The viscosity super-solution property of $\hat{F}$ implies that

$$-\frac{n\delta}{4} + \hat{F}_n(v_n) - H(v_n, D\phi(v_n), D^2\phi(v_n)) \geq 0$$

Assume that $\{(v_n, D\phi(v_n), D^2\phi(v_n))\}_n \subset \text{int}(\text{dom}(H))$ we can let $n \rightarrow \infty$ to conclude that $\hat{F}(v_\varepsilon) = \infty$. But this contradicts eq. (28). We conclude that $\hat{F}(v) = \infty$ for all $v \notin K$. $\square$

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