Search Direction*

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Abstract

A tractable model of pricing under directed search is proposed and integrated with a position auction for better slots (which rationalizes the consumer search order). Search is always inefficiently low because firms price out further exploration. Equilibrium product prices are such that the marginal consumer's surplus decreases in the order of search. Consumers always find it optimal to follow the order of search that results from whatever allocation rule used to determine firms' positions. optimal rankings that achieve the maximization of joint profit, social welfare or consumer surplus are characterized by means of firm specific scores. In a generalized second price auction, any equilibrium that satisfies a "no-envy" condition always implements the joint profit maximizing order and such an equilibrium is shown to exist for a wide range of parameters characterizing product heterogeneity.

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1 Introduction

Internet search is sequential across options, and surfer search is directed by position placement of ads. Until now, most work on consumer search and firm pricing has involved random search across options (e.g. the work following Stahl's (1989) mixed strategy model with homogenous goods, or the search for match following Wolinsky, 1986, and Anderson-Renault, 1999). Directed search is quite different, and the theory needs to be developed. Research has been stymied so far by lack of tractable frameworks that can accommodate heterogeneous firms, a key ingredient for the analysis of directed search. We propose a tractable framework that reflects a directed search environment suitable for the internet environment as well as other applications and engages Weitzman's (1979) powerful results on search behavior. It enables us to study equilibrium with optimal consumer search, product pricing, and advertiser bidding for positions. It delivers a falling surplus for the marginal consumer in the order of search, with consumers (strictly) wanting to follow the pre set order of directed search. Pricing excessively curtails search. The socially optimal order, joint profit maximizing order and consumer surplus maximizing order may each be characterized by associating a score to each firm and ranking the firms according to that score. The setting involves two positional externalities. We characterize some situations where firm bidding leads nonetheless to joint profit maximization.

An important property of our model is the externality imposed by a firm's position on other firms' profits. This effect is only partially incorporated if at all in the literature on position auctions. In our context, this also means that a firm's willingness-to-pay for a slot depends on which firm is demoted and the distribution of tastes for its product. To see these effects, note that if a firm in the ith position becomes more attractive to search, then the prices and profits of firms before it are reduced. Previous literature only accounted for the negative externality from a firm selling a popular product and being searched early on firms that follow (Chen and He, 2012, and Athey and Ellison, 2012). By accounting for firm pricing we introduce an additional externality imparted by firms that are searched later, which is determined by those firm's search attractiveness. One key to a broader understanding of the

link between equilibrium search and positions is to look at asymmetries in the other variables of the model. Fortunately, it is populated with several parameters that play differently and can be distributed across firms. These we break out in the model. We derive optimal ranking scores in terms of these parameters. Whereas the scores that characterize the joint profit maximizing order and the social welfare maximizing order are qualitatively similar, they both differ strongly from the scores associated with consumer surplus maximization. This is another sharp difference with the analysis in Athey-Ellison (2011), which assumes exogenous prices.

Consider now a position auction for slots, with slots going to firms in the order of their bids, and firms paying the bid of the next highest bidder. Such auctions generally admit multiple equilibria. Following Varian (2007) and Edelman, Ostrovsky, and Schwartz (2007), we therefore consider a "no-envy" refinement. This means that no firm would like the position of another if it had to pay the price the other is paying for its slot. When a firm bids to up its slot, it recognizes that it demotes others and thus changes its equilibrium price. The important result is that we show that in any envy-free equilibrium, there is a unique order of firms induced by the auction. This order maximizes total industry profit. We characterize some form of demand heterogeneity across products for which such an equilibrium exists.

There are two relevant streams of literature. Sequential ordered search has only recently been broached. A step forward is made by Armstrong, Vickers, and Zhou (2009),[1] who show that a firm which is searched first will earn more profit, and will also be more attractive to consumers to search first because it charges a lower price. However, their model has a single "prominent" firm searched first, and then the remaining firms are searched at random (without order). Moreover, they assume an independent and identical distributions of consumer tastes for the various products, which limits introducing heterogeneity in the distribution of tastes across different products. Zhou (2011) addresses some of these concerns with an ordered search model, again with symmetric firms. Song (2012) considers firms that are asymmetric regarding taste heterogeneity, but only looks at the duopoly case. The latter

[1] see Arbatskaya (2007) for an earlier contribution with homogenous products.

two papers find that earlier firms charge lower prices, which is consistent with our result that the marginal consumer's surplus goes down with firm searched later. Finally, Chen and He (2011) introduce some heterogeneity in the probability that a product is suitable for a consumer, although their model delivers monopoly pricing: hence there is no externality through prices.

The position auctions literature has made valuable progress on the auction side of the slate while suppressing the market competition side. Athey and Ellison (2011) use a setting very similar to that of Chen and He (2011) to look at auctions with asymmetric information and then optimal auction design, while assuming that consumers go on searching until a "need" is fulfilled, so they do not allow for competing products on the market-place. Their setting allows for a position externality through demand which depends on how likely are previous products in the queue to fill a consumer's need. Our setting does allow for this type of externality as well as the pricing externalities described above. Furthermore, this paper, as well as Chen and He (2011), establish that it is optimal for consumers to search in the order that emerges from the auction because firms with a higher probability to meet a consumer's need bid more. By contrast, we find that it is optimal for consumers to search in the pre specified order because of the pricing behavior they expect from firms. hence, our result holds independently of the characterization of the auction's outcome. Varian (2007) and Edelman, Ostrovsky, and Schwartz (2007) have no position externalities between firms, and they do not engage the broader consumer search and pricing either.

Section 2 describes our search and competition environment in a simple step demand setting while optimal ranking scores for maximization of total industry profit, social welfare and consumer surplus are derived in section 3. A more general demand specification is introduced in Section 4. Finally we consider how allocation rules such as auctions used on internet platforms might achieve total profit maximization in Section 5.

2 Market equilibrium

2.1 Competition with ordered search

We first describe a basic model of oligopolistic competition with ordered search, where the order of search is exogenous.

Consumers have independent valuations for n competing products. The valuations for product i, $i = 1, .., n$, are either 0, $q_i > 0$, or $q_i + \Delta_i$ where $\Delta_i > 0$,[2] and q_i is taken as "sufficiently large", as explained below. Let the corresponding match probabilities be $\gamma_i = 1 - (\alpha_i + \beta_i)$, α_i, and β_i. This structure begets a two-step demand function, and is the simplest in which we can get to the essence of ordered search.

Search is sequential and ordered, with the search cost $s > 0$ per additional search. As is standard in sequential search settings the consumer may always purchase from any previously searched firm with no additional search cost. There are n firms with firm i selling product i, with zero production costs. The order of search is from the lowest to the highest value of i. A consumer who has searched all firms has a continuation value V_n. For now we treat V_n as exogenous, merely assuming that it is identical for all consumers and positive. Allowing for $V_n > 0$ means that once the consumer is done going through the n firms, she still has additional options to purchase a product. This could be for instance, searching the organic links of a search engine after searching the sponsored link, or purchasing a product off line.

We seek conditions for a particular pricing equilibrium, namely that each firm retains all consumers with a non zero match value. This means that we seek an equilibrium where firms render indifferent any consumer drawing q_i: the constraint therefore is that further search is not desirable for such a consumer. This implies that a consumer has zero willingness to pay for any product encountered before product i, and, as long as prices are strictly positive, never goes back. Let $\omega_i \equiv \beta_i \Delta_i - s$. Henceforth ω_i is assumed to be strictly greater than zero.

[2] Alternatively, Δ_i is the expectation of the surplus increment over the base quality, conditional on it being strictly positive. In particular, if this positive increment has a continuous distribution with a logconcave density, the monopoly price is q_i, if q_i is large enough.

Let V_i denote the minimum value a consumer must hold at firm $i = 1, ..., n-1$ to give up searching on. It is defined by $E_{v_i} \max\{v_i - V_i, 0\} = s$, where v_i is the realization of a random variable measuring the best surplus the consumer can obtain by searching optimally from firm $i+1$ on and we use the free recall assumption. Hence firm i's price must be such that

$$q_i - p_i = V_i. \tag{1}$$

We now derive the equilibrium prices.

2.2 Pricing

Consider first the pricing problem of the last firm in the queue, firm n. Since consumers who reach firm n have a zero valuation of products at previously visited firms, it behaves like a monopolist against the continuation value V_n. For q_n large enough, the firm will choose to price at $p_n = q_n - V_n$. Suppose then that a consumer at the firm in position $n-1$ holding surplus $q_{n-1} - p_{n-1}$ contemplates searching firm n. Given she expects firm n's optimal pricing behavior, and using (1) for $i = n-1$, her continuation value from searching on may be written as

$$V_{n-1} = (\gamma_n + \alpha_n) \max\{V_{n-1}, V_n\} + \beta_n \max\{V_{n-1}, V_n + \Delta_n\} - s, \tag{2}$$

where we use the free recall assumption. If $V_n > V_{n-1}$, then we have $V_{n-1} = V_n + \omega_n$. This cannot be the case since $\omega_n > 0$.[3] Hence we must have $V_{n-1} > V_n$ and from (2), $V_{n-1} = V_n + \frac{\omega_n}{\beta_n}$ (This is because, in order for (2) to hold with $s > 0$, we need $V_n + \Delta_n > V_{n-1}$). From (1) we have $p_{n-1} = q_{n-1} - \frac{\omega_n}{\beta_n} - V_n$. We now use a similar line of argument to establish by induction the following result.

Proposition 1 *If $\omega_i > 0$ and q_i is large enough, then there exists an equilibrium that satisfies*

$$V_i = V_n + \sum_{j=i+1}^{n} \frac{\omega_j}{\beta_j}, \tag{3}$$

[3] If V_n is too small and ω_n is sufficiently negative so that $V_{n-1} < 0$, the consumer prefers dropping out rather than searching on to firm n. Then firm $n-1$ can retain her while charging the monopoly price (which is q_{n-1} if q_{n-1} is large enough), so the Diamond paradox would apply. Previous literature has used settings where the Diamond paradox applies, as in Chen and He, 2013.

for all $i = 1, ..., n-1$*, so that*

$$p_i = q_i - \sum_{j=i+1}^{n} \frac{\omega_j}{\beta_j} - V_n, \tag{4}$$

for all $i = 1, ..., n$*.*

Proof. First, we have already established (3) for $i = n-1$ and (4) for $i = n$. Now, because of (1), if (3) holds for $i = 1, ..., n-1$, then pricing satisfies (4) for $i = 1, ...n-1$. Hence, to prove the result it suffices to show by induction that if (3) is true for some $i = 2, ..., n-1$ then it is true for $i-1$.

Consider a consumer at firm $i-1$, holding surplus $q_{i-1} - p_{i-1}$. Since firm i's price satisfies $q_i - p_i = V_i$, her expected surplus from searching may be written as

$$V_{i-1} = (\gamma_i + \alpha_i) \max\{V_{i-1}, V_i\} + \beta_i \max\{V_{i-1}, V_i + \Delta_i\} - s, \tag{5}$$

The arguments used to derive V_{n-1} can be replicated here to show that, $\omega_i > 0$ implies that $V_{i-1} \geq V_i$ and hence $V_{i-1} = V_i + \frac{\omega_i}{\beta}{}_i$ (again, in order for (5 to hold with $s > 0$, we must have $V_{i-1} < V_i + \Delta_i$). Thus if V_i satisfies (3), so does V_{i-1}. ■

2.3 Directed search.

Our equilibrium analysis thus far has assumed that consumers must search in a set order. We have established that, if $\omega_i > 0$ for all i and firms choose to retain all consumers with a positive valuation with their product, then equilibrium prices are given by (4) and such an equilibrium exists if q_i is large enough for all $i = 1, ..., n$.[4] This characterization of equilibrium does not require that the order of search is optimal for consumers. In particular, it does not rely on the standard myopic reservation value rule of Weitzman (1979). As has been already pointed out, we do want our characterization to be robust to the possibility that a consumer freely selects the order in which she searches. We now show that, for the equilibrium pricing rule derived above, the pre specified search order is always optimal, for

[4]We expect that if q_i is large enough for all i, there cannot be any other equilibrium. Such alternative equilibrium would have some firms price in a way such that only consumers with the highest valuation stop searching when they get to the firm.

all values of q_i and Δ_i, as long as $\omega_i > 0$ for all $i = 1, ..., n$. Zhou (2011) also found that firm pricing makes it optimal for consumers to start searching the firms that expect to be searched early. In the symmetric case he considers, this follows immediately from the higher prices charged by firms later in the queue. We extend this result to asymmetric products. The underlying force here is that the marginal consumer's surplus is lower with firm that are searched later. Indeed, from equation (1) this surplus is V_i at firm i, and from equation (4) V_i is strictly decreasing in i.

As shown by Weitzman (1979), in order to determine the optimal search order, it suffices to compute a reservation value associated with each search alternative: it is then optimal to search the alternatives following the decreasing order of reservation values. in our setting, the consumer's utility with product i is $u_i = 0$ with probability γ_i, $u_i = q_i - p_i = V_n + \sum_{j>i} \frac{\omega_j}{\beta_j}$ with probability α_i and $u_i = q_i + \Delta_i - p_i = \Delta_i + V_n + \sum_{j>i} \frac{\omega_j}{\beta_j}$ with probability β_i. Then the reservation utility associated with searching firm i, $\hat{u}_i$ satisfies

$$E_{u_i} \max\{u_i - \hat{u}_i, 0\} = \gamma_i \max\{-\hat{u}_i, 0\} + \alpha_i \max\{V_n + \sum_{j>i} \frac{\omega_j}{\beta_j} - \hat{u}_i, 0\}$$

$$+\beta_i \max\{\Delta_i + V_n + \sum_{j>i} \frac{\omega_j}{\beta_j} - \hat{u}_i, 0\} = s \tag{6}$$

The left-hand side is zero for $\hat{u}_i = \Delta_i + V_n + \sum_{j>i} \frac{\omega_j}{\beta_j}$. It is continuous and strictly decreasing in $\hat{u}_i$. Hence, for $s > 0$ (6) has a unique solution $\hat{u}_i < \Delta_i + V_n + \sum_{j>i} \frac{\omega_j}{\beta_j}$ It is readily verified that $\hat{u}_i = V_n + \sum_{j \geq i} \frac{\omega_j}{\beta_j}$. This is clearly decreasing in i so it is optimal for the consumer to search earlier firms first. Note that $\hat{u}_i > V_n$ for all $i = 1, ..., n$, so that the consumer finds it optimal to search any of the n firms rather than moving on directly to her best alternative shopping strategy (e.g. organic links or shopping off line). Furthermore, for $i = 2, ..., n$, $\hat{u}_i = V_{i-1}$, which, from (1), is the marginal consumer's surplus at firm $i - 1$. This reflects firm $i - 1$'s strategy to make the consumer drawing q_{i-1} indifferent between buying and searching on.

Results thus far establish that for any order in which the firms are ranked, there is a pricing equilibrium such that each firm retains all consumers who reach it and are willing to pay some positive amount of money for its product, and consumers find it optimal to search

according to the pre specified order. We next investigate on what basis this ranking could be determined.

3 Optimal rankings

In the sequel, we will speak to an equilibrium order of firms that will be determined by the equilibrium to a position auction. The results above indicate that ANY particular order of presentation will be followed by consumers in their optimal search, with the corresponding prices inducing it rational for consumers to follow the order they are expected to follow. When there are asymmetries among firms though, order matters to various measures of market performance. Typically, the optimal order varies by market performance measure. We here determine the various optimal orders.

What is the best order for consumer surplus, social welfare, and total industry profit (gross of any position fees paid)? (For short, call these CS, W, and TIP respectively.) Given asymmetries across firms in the parameters, which order of presentation (given equilibrium search and pricing) maximizes these? A priori, this is a complicated problem because position order affects all prices and search probabilities: with n active firms there are $n!$ positions to check. Nevertheless, our model delivers a structure such that we can simply characterize the optimal order under each criterion, and the optimal order is described by ordering a simple summary statistic (which is different for each criterion).

The idea is as follows. Suppose that we rank firms in some arbitrary way. Then for any neighboring pair of firms, A and B, in the ranking (and for each criterion), we can find a summary statistic Φ_k for firm k such that the maximand (CS, W, or TIP) is higher if $\Phi_A > \Phi_B$. Crucially, whether or not $\Phi_A > \Phi_B$ does not depend on which two slots are flipped (e.g., first and second or fifteenth and sixteenth). Suppose for clarity (and to eliminate ties, which have no consequence anyway – the order is then indifferent between when tied firms are presented – that the Φ_k are all different across firms. Then the claimed result is that there is a unique maximum, and simply follows the order of the Φ_k. Clearly a necessary condition is that in each successive pair the one with the higher Φ_k goes first – otherwise

we can increase the maximand by flipping any pair which violates this. But then, because the flipping rule is independent of the positions i and $i+1$ to be flipped, this criterion just promotes up the order each firm to the positions claimed. Put another way, for any order not satisfying the claimed optimal ranking, there must be at least one pair violating the pairwise flip condition, and so this cannot be an optimum. Thus the ranking of firms by the size of their summary statistics is a necessary and sufficient condition to characterize the optimum rankings.

We now derive the particular summary statistics for the different criteria. We also look at some intuition for the various orders.

3.1 Total Industry Profit

For TIP, we just need to look at the change in (gross) profit from the switch. Thus we have A before B as long as

$$\pi_A^i + \pi_B^{i+1} \geq \pi_B^i + \pi_A^{i+1} \tag{7}$$

where π_k^i denotes the profit of firm k when it is in slot i. For our model, we can write this out to yield:

$$(1-\gamma_A)(1-\gamma_B)(q_A - q_B) + (1-\gamma_B)\frac{\omega_A}{\beta_A} - (1-\gamma_A)\frac{\omega_B}{\beta_B} > 0 \tag{8}$$

(notice the terms in all prices after $i+1$ cancel in the TIP comparison, and we divide through by the total number of consumers that search up to slot i). Dividing through (8) by $(1-\gamma_A)(1-\gamma_B)$ delivers the TIP summary statistics such that A should be before B (in any consecutive pair, and hence in the global maximum) as long as

$$\Phi_A^\pi \equiv q_A + \frac{1}{(1-\gamma_A)}\frac{\omega_A}{\beta_A} > q_B + \frac{1}{(1-\gamma_B)}\frac{\omega_B}{\beta_B} \equiv \Phi_B^\pi.$$

Therefore the TIP summary statistic is as given next

Proposition 2 *The order of firms that maximizes Total Industry Profit follows the ranking of the summary statistics*

$$\Phi_k^\pi \equiv q_k + \frac{1}{(1-\gamma_k)}\frac{\omega_k}{\beta_k} \tag{9}$$

and firms should follow a decreasing order of the Φ_k^π*. Ceteris paribus, higher* q_k*,* $\frac{\omega_k}{\beta_k}$*,* γ_k*, should go earlier in the order.*

The reason why higher q's take precedence is because such firms get more consumers with their high prices. More interestingly, higher $\frac{\omega_k}{\beta_k}$'s should be placed earlier is to clear the decks of those who bring down the price a lot for all if they were late.

Finally, and surprisingly, higher γ_k firms should go earlier even though they have less chance of a successful sale. This is because then consumers are more likely to end up buying from firms with higher prices.

3.2 Social Welfare

We next consider the pairwise ranking condition for Welfare (given equilibrium firm pricing). To find the corresponding summary statistic, first note that transposing any pair does not affect the welfare gained on EITHER earlier or later firms, given that consumers stop when they draw at least the medium valuation. Thus we can look at a pair in isolation. Notice that prices are just a transfer, and so do not enter the calculus.

With these remarks in mind, consider the surplus on searching A then B (conditional on having reached A at some position i), and compare with the converse. The relevant part of surplus for searching A then B is $q_A(1-\gamma_A)+\beta_A\Delta_A$ earned on A plus the chance of not liking A and getting an analogous surplus on B, which also entails a search cost s. Adding this together and using the analogous expression (switching subscripts) for the opposite order yields the condition for the sequence AB (for any pair) to be more profitable in aggregate than BA as:

$$\begin{aligned} & q_A(1-\gamma_A)+\beta_A\Delta_A+\gamma_A(-s+q_B(1-\gamma_B)+\beta_B\Delta_B) \\ > \; & q_B(1-\gamma_B)+\beta_B\Delta_B+\gamma_B(-s+q_A(1-\gamma_A)+\beta_A\Delta_A), \end{aligned}$$

which rearranges to

$$q_A(1-\gamma_A)+\omega_A+\gamma_A(q_B(1-\gamma_B)+\omega_B) > q_B(1-\gamma_B)+\omega_B+\gamma_B(q_A(1-\gamma_A)+\omega_A),$$

or $q_A(1-\gamma_A)(1-\gamma_B)+\omega_A(1-\gamma_B) > q_B(1-\gamma_B)(1-\gamma_A)+\omega_B(1-\gamma_A)$, and hence

$$\Phi_A^W \equiv q_A + \frac{\omega_A}{(1-\gamma_A)} > q_B + \frac{\omega_B}{(1-\gamma_B)} \equiv \Phi_B^W.$$

The summary statistic is thus the one given next:

Proposition 3 *The order of firms that maximizes Social Welfare follows the ranking of the summary statistics*

$$\Phi_k^W \equiv q_k + \frac{\omega_k}{(1-\gamma_k)} \tag{10}$$

and firms should follow a decreasing order of the Φ_k^W. Ceteris paribus, higher q_k, ω_k, γ_k, should go earlier in the order.

To interpret, big q_k's are ranked early, ceteris paribus, because they deliver higher surplus earlier, and likewise for the surpluses on the high matches (the ω_k). Also, high γ_k are preferred earlier to get more shots at the High surplus. (need to embellish with new general version)

We can compare to the order under TIP-maximization, where the summary statistic is $\Phi_k^\pi \equiv q_k + \frac{1}{(1-\gamma_k)}\frac{\omega_k}{\beta_k}$. This puts weight on ω because of its effect on prices. [expand on this!! – Prop on comparison in Comparison section?]

3.3 Consumer Surplus

The consumer surplus case proceeds analogously to the welfare one, except now prices feature explicitly. Another key difference is that the q_k's do not enter because they are priced out. The varying part (the later and earlier surpluses are unaffected by the order switch) of consumer surplus for the AB pair sequence with A in slot i and B in slot $i+1$ is

$$(1-\gamma_A)\left(q_A - p_A^i\right) + \beta_A\Delta_A + \gamma_A\left(-s + (1-\gamma_B)\left(q_B - p_B^{i+1}\right) + \beta_B\Delta_B\right)$$

and the pricing rule gives $p_A^i = q_A - \frac{\omega_B}{\beta_B} - \kappa_{i+1}$ and $p_B^{i+1} = q_B - \kappa_{i+1}$ where $\kappa_{i+1} = \Sigma_{j>i+1}\frac{\omega_j}{\beta_j}$ denotes the sum of later price steps. Hence the consumer surplus difference of AB exceeds that of BA (which is found by transposing subscripts again) if

$$(1-\gamma_A)\frac{\omega_B}{\beta_B} + \beta_A\Delta_A + \gamma_A(-s+\beta_B\Delta_B) > (1-\gamma_B)\frac{\omega_A}{\beta_A} + \beta_B\Delta_B + \gamma_B(-s+\beta_A\Delta_A)$$

where the κ_{i+1} terms all cancel out: hence the same calculus applies regardless of which slot i is the base one.

Rearranging yields

$$\Phi_B^{CS} \equiv \frac{1}{(1-\gamma_B)}\left(\frac{\omega_B}{\beta_B} - \omega_B\right) > \frac{1}{(1-\gamma_A)}\left(\frac{\omega_A}{\beta_A} - \omega_A\right) \equiv \Phi_A^{CS},$$

and the implication for the summary statistic is given next:

Proposition 4 *The order of firms that maximizes Consumer Surplus follows the ranking of the summary statistics*

$$\Phi_k^{CS} \equiv \frac{1}{(1-\gamma_k)}\left(\frac{\omega_k}{\beta_k} - \omega_k\right) > 0 \tag{11}$$

and firms should follow an increasing order of the Φ_k^{CS}. The q_k value is irrelevant, ceteris paribus, while higher ω_k should go earlier in the order, while higher $\frac{\omega_k}{\beta_k}$ and γ_k should go later in the order.

So A before B as $\gamma_B > \gamma_A$ which means more acceptable choices earlier, ceteris paribus.

The other term can be decomposed into two components, corresponding to price and surplus effects. First, a higher $\frac{\omega}{\beta}$ entails a higher price step and so should be placed later to keep consumers happier. Second, a higher ω means a higher surplus from the best match, ceteris paribus, and so should be placed earlier.

4 General match distribution.

The analysis can readily be extended to a much more general setting. Assume now that the valuation for product i, v_i, has support $\{0\} \cup S_i$, where S_i admits a strictly positive minimum and let q_i now be that minimum and $\gamma_i = \Pr\{v_i = 0\}$. The cumulative distribution of v_i is F_i. Further assume that S_i admits a finite maximum. This assumption ensures that, the added revenue that can be obtained by increasing firm i's price above the level that guaranties that it sells to all consumers with match of at least q_i is bounded. As a result, for q_i large enough, firm i wants to price so as to sell to all consumers with a strictly positive match (a more formal argument should be provided here). As before, we are looking for an

equilibrium where all consumers with valuations of at least q_i stop at firm i. The *Weitzman reservation value* for product i is defined by

$$g_i(r_i) \equiv \int_{r_i}^{+\infty} (v - r_i) dF_i(v) = s. \tag{12}$$

The function g_i is strictly decreasing on $(-\infty, \max S_i)$. Throughout we postulate $r_i > q_i$. This condition rules out a Diamond-like equilibrium where all firms would charge their base quality q_i, for $i = 1, ..., n$ (this is the analogue to the assumption that $\omega_i > 0$ in the three point distribution setting).

Once again consider a consumer holding valuation q_i at firm i. If firm $i+1$ charges price p_{i+1}, then the consumer's surplus at that firm is $u_{i+1} = \max\{v_{i+1} - p_{i+1}, 0\}$. Because we seek to characterize an equilibrium where firm $i+1$ sells to all consumers with valuation in S_{i+1}, we have $p_{i+1} \leq q_{i+1}$. Then the reservation value associated with searching firm $i+1$ is $\hat{u}_{i+1}$ solution to

$$\int_{\hat{u}_{i+1}}^{+\infty} (u - \hat{u}_{i+1}) dG_{i+1}(u) = s, \tag{13}$$

where G_{i+1} is the cumulative distribution of u_{i+1}. The left-hand-side of (13) is strictly decreasing in $\hat{u}_{i+1}$ on $(-\infty, \sup S_{i+1} - p_{i+1})$. Since $q_{i+1} - p_{i+1} \geq 0$, for $\hat{u} \geq q_{i+1} - p_{i+1}$ (13) may be written as $\int_{\hat{u}_{i+1}+p_{i+1}}^{+\infty} (v - p_{i+1} - \hat{u}_{i+1}) dF_{i+1}(v) = g_{i+1}(\hat{u}_{i+1} + p_{i+1}) = s$. Because, $r_{i+1} > q_{i+1}$ and g_{i+1} is strictly decreasing, we have $g_{i+1}(q_{i+1}) > s$. It follows that $\hat{u}_{i+1} + p_{i+1} > q_{i+1}$. Hence $\hat{u}_{i+1}$ is defined by $g_{i+1}(\hat{u}_{i+1} + p_{i+1}) = s$ and since g_{i+1} is strictly decreasing, $\hat{u}_{i+1} = r_{i+1} - p_{i+1}$.

As before, assuming large base qualities q_i for all i, firm n charges $p_n = q_n - V_n$ and, if consumers are expected to follow the myopic rule, all other firms $i < n$, charge $p_i = q_i - \hat{u}_{i+1} = q_i - (r_{i+1} - p_{i+1})$. Then, the equilibrium price for firm i, $i = 1, ..., n$ is given by

$$p_i = q_i - V_n - \sum_{j>i} (r_j - q_j). \tag{14}$$

For the step demand considered in previous sections we have $r_i - p_i = \frac{\omega_i}{\beta_i}$. Now let $\beta_{i+1} = 1 - F(r_{i+1})$ and $\omega_{i+1} = [1 - F(r_{i+1})](r_{i+1} - q_{i+1})$ equilibrium pricing may be characterized by (4) in Proposition 1. Furthermore, using the definition of the Weitzman reservation value (12), we have $\omega_{i+1} = \int_{r_{i+1}}^{+\infty} v - q_{i+1} dF_{i+1}(v) - s$.

The arguments in Section 3 can be applied to this more general model to derive firm specific scores that characterize the optimal ranking of firms to achieve the maximization of total industry profit, social welfare or consumer surplus. The corresponding expressions are

$$\Phi_k^{\pi} = q_k + \frac{r_k - q_k}{1 - \gamma_k} \tag{15}$$

$$\Phi_i^{SW} = q_k + \frac{\beta_k(r_k - q_k) + \delta_k}{1 - \gamma_k} \tag{16}$$

$$\Phi_i^{CS} = \frac{(1 - \beta_k)(r_k - q_k) - \delta_k}{1 - \gamma_k}. \tag{17}$$

5 Allocation rules

In view of our equilibrium characterization, it is unclear whether it is more profitable for a firm to be searched earlier. Although a firm that is earlier in the search order gets to sell more, it charges a lower price in equilibrium. In this section we explore how an allocation rule that relies on the firms' private incentives can implement a "desirable" outcome. We are primarily interested in the implementation of the joint profit maximizing outcome. This may be desirable for an internet platform that hopes to attract advertisers and generate large advertising revenue. A simple condition that insures that earlier firms earn more profit is that base qualities q_i are large enough so that the percentage drop in price needed to prevent further search is less than the percentage increase in potential searches afforded by an earlier search slot.

5.1 Incremental values

We now wish to characterize each firm's willingness to pay for being searched earlier. More specifically we consider a firm's willingness to pay for being placed one slot ahead of another firm. We therefore consider two consecutive slots. Because of the externalities involved, this incremental value cannot be independent of the identity of the firms holding the other slots, before or after the two slots under consideration. Furthermore, it depends on which two slots are at stake. However, we now show that the ranking of incremental values between any two firms A and B is independent of which two slots they are competing for.

Consider again firms A and B: *which has the highest willingness to pay for being in slot i rather than in slot $i+1$?*. We have

$$\pi_A^i - \pi_A^{i+1} > \pi_B^i - \pi_B^{i+1} \Leftrightarrow \pi_A^i + \pi_B^{i+1} > \pi_B^i + \pi_A^{i+1}.$$

Thus, A's incremental value of being one slot ahead of B is larger than B's incremental value of being ahead of A if and only if $\Phi_A^\pi > \Phi_B^\pi$ and this is true no matter which two slots are considered.

Next we look at how this property can be used to characterize an equilibrium of a generalized second price auction.

5.2 Generalized second price auction: per impression bidding

Following previous literature we consider an allocation of the slots on an internet platform through an auction that assigns positions according to the ranking of bids (where higher bidders get earlier positions) and where a firm who wins a position is charged the next highest bid. Bids are per position meaning that a firm pays for a position some lump sum amount. We call the latter per impression bidding because in our setting, the number of consumers who see any ad is exogenous and corresponds to the entire consumer population: the number of impressions is therefore 1 for each ad so the bid is a lump sum payment.

We extend the search and competition model to have $n \geq 2$ firms, that bid for $n-1$ positions on a platform. Hence, only the firms with the $n-1$ highest bids get a slot and the remaining firm is assumed to be searched last: to simplify the exposition we however say that it is in slot n. Having only one outside firm avoids having to model search and competition among outside firms.[5] The corresponding complete information auction game typically has multiple equilibria. We follow previous literature and focus primarily on envy=free equilibria (Edelman, *et al.*, 2007) also called symmetric equilibria in Varian (2007) to refine the equilibrium concept. In those papers, the price paid by firms is per click.

[5]There is no obvious way of modeling the search behavior of consumers among outside firms. Chen and He (2012) assume that search costs outside the platform are high enough that a consumer only searches one firm picked at random among the outsiders.

According to the envy free condition, the firm in slot i should not wish to be in slot $j \neq i$ while paying b^{j+1}, which denotes the $j+1$th highest bid, which is paid by firm j (where b^{n+1} is set to zero because a firm can get the outside slot n for free). For $j > i$, it is equivalent to Nash equilibrium. For $j < i$ it is stronger: it allows i to pay b^{j+1} to be in slot j whereas it would have to pay at least b^j so as to outbid the firm in slot j and be able to deviate to slot j. Formally, no envy says that if some firm A is in slot i in equilibrium then

$$\pi_A^i - b^{i+1} \geq \pi_A^j - b^{j+1}, \tag{18}$$

for all $j = 1, ..., n$.

Now consider again two firms A and B in consecutive slots i and $i+1$. No envy for firm A in slot i not moving to slot $i+1$ can be written as

$$\pi_A^i - \pi_A^{i+1} \geq b^{i+1} - b^{i+2} \tag{19}$$

For firm B in slot $i+1$, no envy *vis-a-vis* slot i yields

$$\pi_B^{i+1} - \pi_B^i \geq b^{i+2} - b^{i+1}, \tag{20}$$

or equivalently,

$$b^{i+1} - b^{i+2} \geq \pi_B^i - \pi_B^{i+1}. \tag{21}$$

Hence we must have

$$\pi_A^i - \pi_A^{i+1} \geq \pi_B^i - \pi_B^{i+1}. \tag{22}$$

As before, this ranking of incremental value holds if and only if the ranking of A before B maximizes total industry profit. Hence, if there exists an envy-free equilibrium, then it yields the joint profit maximizing order.

An obvious candidate for equilibrium has each firm bid its incremental value for moving up one slot, on top of what the next firm down in the order is bidding. That is, if we consider again two consecutive firms, A in slot i and B in slot $i+1$, then B's bid is given by $b^{i+1} = b^{i+2} + \pi_B^i - \pi_B^{i+1}$, where b^{n+1} is taken to be zero, π_B^{i+1} is B's equilibrium profit gross of the bid it pays and π_B^i is B's gross profit if it switches position with firm A. Note

that if such an equilibrium exists, firms are necessarily ranked according to the decreasing order of TIP maximization scores Φ_i^π and hence, total profit is maximal. If this were not the case, then there would be two consecutive slots, i and $i+1$, such that the firm in slot i would have a lower incremental value for being in slot i than does the firm in slot $i+1$. Then the firm in slot i would be better off dropping its bid slightly below b^{i+2} in order to be in slot $i+1$, rather than having to pay b^{i+1} and be in slot i: this is because the bid difference reflects the incremental value of the firm in slot $i+1$ for being in slot i, which exceeds that of the firm in slot i. Also note that envy free necessarily holds for any two consecutive firms. Indeed, the additional amount of money the firm in slot i must pay in order to be in slot i rather than in slot $i+1$, exactly reflects the incremental value of the firm in slot $i+1$, so that the latter firm does not wish to be in slot i while having to pay the amount charged for that slot. It is *a priori* less clear whether envy free holds for any two positions, or indeed, whether such incremental value bidding yields an equilibrium. The following result shows that, if all products' consumer bases have the same size, then such an equilibrium exists and is envy-free, provided that base qualities q_i, follow the same order as the scores characterizing the joint profit maximizing order.

Proposition 5 *Suppose $\gamma_i = \gamma$ for all $i = 1, ..., n$, $\Phi_i^\pi > \Phi_{i+1}^\pi$ and $q_i > q_{i+1}$ for $i = 1, ..., n-1$. Then there exists an envy-free equilibrium where firm i is in slot i, $i = 1, ..., n$ and bids are such that:*

- $b^i = b^{i+1} + (1-\gamma)\gamma^{i-2}\left((1-\gamma)\left(q_i - \sum_{j=i+1}^n (r_j - q_j)\right) - (r_{i-1} - q_{i-1})\right)$, *for $i = 2, ...n$, taking $b^{n+1} = 0$.*

Proof. First consider $n = 2$. The two firms are just bidding for one slot and we have a Vickrey auction with only two bidders. Then, despite the externality, it is a dominant strategy for each firm to bid its incremental value for being searched first (this would not be true if there were more than two firms because the incremental value for one firm would depend on the identity of the firm that gets the unique slot if the firm loses). Firm i's incremental value for being searched before firm j, $i, j = 1, 2$, $j \neq i$ is $(1-\gamma)\left((1-\gamma)q_i - (r_j - q_j)\right)$, which is larger for firm 1, because $\Phi_1^\pi \geq \Phi_2^\pi$.

Now assume that $n > 2$. The bid structure is such that no firm would wish to deviate one slot up or one slot down. Indeed, the difference between a firm's bid and what it ends up paying is exactly equal to the firm's incremental value for being one slot up. In order to move up one slot, it would have to be paying the equilibrium bid of the firm it would be demoting which would involve an increase in it payment which strictly exceeds its incremental value. Now if a firm chooses to move down by one slot, it decreases its payment by the incremental value of the next firm down for being in its slot. However, because that firm has a lower TIP maximization score, its incremental value is also lower. Hence, the firm that deviates downward by 1 saves less than its own incremental value for staying where it is and this is not profitable.

Next consider a firm in slot $i < n-1$ deviating downward by t slots, $t > 1$. Instead of paying b_{i+1}, it pays b_{i+t+1}. It therefore saves

$$(1-\gamma)\gamma^{i-2}\sum_{k=1}^{t}\gamma^{k}\left((1-\gamma)\left(q_{i+k}-\sum_{j=i+k+1}^{n}(r_j-q_j)\right)-(r_{i+k-1}-q_{i+k-1})\right). \tag{23}$$

Because $\Phi^{\pi}_{i+k-1} > \Phi^{\pi}_{i+k}$, we have $(1-\gamma)q_{i+k-1}-(r_{i+k}-q_{i+k}) > (1-\gamma)q_{i+k}-(r_{i+k-1}-q_{i+k-1})$. Using $q_i \geq q_{i+k-1}$, it follows that firm i's saving when it deviates is bounded above by

$$(1-\gamma)\gamma^{i-2}\sum_{k=1}^{t}\gamma^{k}\left((1-\gamma)\left(q_{i}-\sum_{j=i+k+1}^{n}(r_j-q_j)\right)-(r_{i+k}-q_{i+k})\right). \tag{24}$$

Now consider firm i's drop in profit gross of the bid payment when going down one slot at a time, with all the other firms remaining in the equilibrium order. When it goes from slot $i+k-1$ to slot $i+k$, it loses a fraction γ of its demand (those who stop at firm $i+k$ which is now in slot $i+k-1$) but it increases its price by $r_{i+k}-q_{i+k}$ because firm $i+k$ is now ahead of firm i. Hence its change in profit is $(1-\gamma)\gamma^{i+k-2}\left(r_{i+k}-q_{i+k}-(1-\gamma)\left(q_i-\sum_{j=i+k+1}^{n}(r_j-q_j)\right)\right)$. Taking the sum over $k = 1, ..., t$ yields minus the expression in (24) which is the total loss in gross profit from moving from slot i to slot $i+t$. The saving in bidding payments is therefore lower than the drop in gross profit so the deviation is not profitable.

Consider now some firm $i = 3, ..., n$ that deviates from slot i to slot $i-t$, $t \geq 2$. In order to encompass the no envy condition, which is stricter than the equilibrium condition

in that case, assume it can do so while having to pay only b_{i-t+1}, the amount that firm $i-t$ is paying in equilibrium. The additional cost for firm i is then $b_{i-t+1} - b_{i+1}$, that is

$$(1-\gamma)\gamma^{i-t-2}\sum_{k=1}^{t}\gamma^{k}\left((1-\gamma)\left(q_{i-t+k} - \sum_{j=i-t+k+1}^{n}(r_j - q_j)\right) - (r_{i-t+k-1} - q_{i-t+k-1})\right). \quad (25)$$

Because $\Phi^{\pi}_{i-t+k} > \Phi^{\pi}_{i}$, we have $(1-\gamma)q_{i-t+k} - (r_i - q_i) > (1-\gamma)q_i - (r_{i-t+k} - q_{i-t+k})$. Now using $q_{i-t+k} \geq q_i$, we have $q_{i-t+k} - (r_i - q_i) > q_i - (r_{i-t+k} - q_{i-t+k})$ so that a lower bound for $b^{i-t+1} - b^{i+1}$ is

$$(1-\gamma)\gamma^{i-t-2}\sum_{k=1}^{t}\gamma^{k}\left((1-\gamma)\left(q_{i} - \sum_{j=i-t+k}^{n}(r_j - q_j) + (r_i - q_i)\right) - (r_{i-t+k-1} - q_{i-t+k-1})\right). \quad (26)$$

Now $(1-\gamma)\gamma^{i-t+k-2}\left((1-\gamma)\left(q_i - \sum_{j=i-t+k}^{n}(r_j - q_j) + (r_i - q_i)\right) - (r_{i-t+k-1} - q_{i-t+k+1})\right)$ is the incremental profit firm i earns by moving from slot $i-t+k$ to $i-t+k-1$ while all other firms are ordered as in the candidate equilibrium, so that the above lower bound is exactly the increase in gross profit for firm i if it moves up from slot i to slot $i-t$. Hence firm i would not gain from such a deviation, even if it had to pay only b_{i-t+1} for being in slot $i-t$, so that no envy is satisfied. This in turn implies that we have an equilibrium. ■

Now consider a situation where firm heterogeneity arises only because they have different probabilities of selling a product that fits a consumer's need, $\gamma_1 > \gamma_2 >, ..., > \gamma_n$. To illustrate, suppose there are 3 firms with $\gamma_1 > \gamma_2 > \gamma_3$. Joint profit maximization requires that firm 1 be ranked before firm 2, which in turn should precede firm 3. In a candidate equilibrium that implements this order with incremental bidding, bids would satisfy:

$$b^3 = \gamma_1(1-\gamma_3)((1-\gamma_2)q - (r-q)) \quad (27)$$

$$b^2 = b^3 + (1-\gamma_2)((1-\gamma_1)(q-(r-q)) - (r-q)). \quad (28)$$

Firm 1 can bid any value that exceeds b_2. Because of incremental bidding, no firm wishes to deviate by only one slot. We now show, however, that this cannot be an envy-free equilibrium, because firm 3 is always better off in slot 1 while paying b^2 than staying in slot 3.

Along the lines of the method used in the proof of Proposition 5, we proceed by writing the incremental profit for firm 3 if it moves from slot 3 to slot 2, $\gamma_1(1-\gamma_3)((1-\gamma_2)q-(r-q))$, and then the incremental profit from moving from slot 2 to slot 1 $(1-\gamma_3)((1-\gamma_1)(q-(r-q))-(r-q))$. Note that the latter term is merely b^3. Hence in order for firm 3 not to be willing to pay b^2 to be in slot 1, the former term must be less than b^2-b^3. Comparing the two expressions it is readily seen that firm 3's incremental profit exceeds the bid difference, so firm 3 is better off in slot 1 paying b^2.

In this three firms example, it is always possible to make firm 3's deviation unprofitable by picking b^1 to be large enough. Still, in order to have an equilibrium, it is necessary to check that 1 does not want to deviate to slot 3. This is the case with the proposed bids as long as γ_2 is close enough to γ_1.[6]

5.3 Per click bidding

As mentioned above, previous literature has considered per click bidding. Then the profit of firm A in slot i is given by $\pi_A^i - \lambda_i b_{i+1}$, where λ_i denotes the number of clicks in slot i. It is easy to see that the argument about no envy equilibria maximizing joint profit maximization remains valid in the case where all products have the same consumer base $\gamma_i = \gamma$ for all $i = 1, ..., n$. This is because λ_i and λ_{i+1} are independent of which firm comes first and we have $\lambda_{i+1} = \gamma\lambda_i$.

When products differ in terms of consumer base, it is no more clear whether, with per click bidding, a no envy equilibrium would actually yield total industry profit maximization. The work by Chen and He (2012) and Athey and Ellison (2012) rather suggest the opposite. They have firms charging essentially exogenous prices and differing only in terms of their consumer base. Both papers characterize an equilibrium where firms with a larger consumer base bid more and are ranked earlier. However, our analysis of joint profit maximization shows that it requires, all other things equal, that firms with a large γ_i (small consumer

[6]The arguments developed for this three firms example show that incremental value bidding cannot sustain an envy-free equilibrium even if there are more than three firms. However, it could still be an equilibrium because, in order for a firm to move up two slots, it is necessary to outbid the firm in that slot and hence, having to pay that firm's bid.

base) are first.

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