The Curse of Long Horizons

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Abstract

We study a model of dynamic moral hazard with symmetric ex ante uncertainty about the difficulty of the job. Over time, both the principal and agent update their beliefs about the difficulty of the job as they observe output. Effort is private and so incentives must be provided for the agent to exert effort, and the principal can only make within period commitments. In consequence, the agent may have an additional incentive to shirk when the principal induces effort, because by shirking, the agent causes the principal to have incorrect beliefs. We show that this possibility can result in the contract that induces effort in every period needing incentives that become increasingly high powered as the length of the relationship increases. Thus it is never optimal to always induce effort in very long relationships.

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1 Introduction

This paper analyzes the long-run implications of the ratchet effect, arising from the introduction of new technology, in a context where both firm and worker are learning about its efficacy. Milgrom and Roberts (1990) provide a lucid statement of the problem;. when a firm installs new equipment, firms and workers have to learn what is the appropriate work standard. It is efficient to use future information to adjust the standard, but this reduces work incentives today.[1] Thus the ratchet effect arises from the combination of learning, moral hazard and lack of long term commitment by the employer.

Theoretical work on the ratchet effect usually assumes ex ante differential information. The agent has private information on the nature of the job, and the principal is unable to make long term commitments. Thus the problem is formulated as one of dynamic mechanism design without commitment, and the question is, how does the principal induce the agent to reveal her private information. Lazear (1986) argues that high powered incentives are able to overcome the ratchet effect, without any efficiency loss, assuming that the worker is risk neutral. Gibbons (1987) shows that Lazear's result depends upon an implicit assumption of long term commitment; in its absence, one cannot induce efficient effort provision by the more productive type.[2] Laffont and Tirole (1988) prove a general result, that one cannot induce full separation given a continuum of types.[3] Laffont and Tirole (1993) have a comprehensive discussion, and consider both the case of binary types and of a continuum of types. Gerardi and Maestri (2015) analyze an infinite horizon model with binary types.

The present paper differs from this literature since we formulate the ratchet as arising from the learning problem in a context with moral hazard, where the worker's effort is unobserved by the principal. The principal and the agent are symmetrically uncertain about the how difficult it is for the worker to succeed on the job. We assume that the principal cannot commit to long term contracts, but chooses short-term contracts optimally. Since there is no limited liability and the principal has all the bargaining power, the agent will not be paid any more than his outside option. Furthermore,

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[1]In the sociological literature, Mathewson (1931), Roy (1952), and Edwards (1979) are workplace studies that document the importance of output restriction in order to influence the firm's beliefs.

[2]See also Freixas, Guesnerie, and Tirole (1985) and Carmichael and MacLeod (2000).

[3]Malcomson (2014) shows that the no full-separation result also obtains in a relational contracting setting, where the principal need not have all the bargaining power, as long as continuation play following full separation is efficient.

since uncertainty pertains to the nature of the job, the outside option does not depend upon what is learnt.

The ratchet effect arises since the agent can manipulate the beliefs of the principal by shirking. In a pure strategy equilibrium where high effort is chosen, the principal correct anticipates the agent's effort choices, and the beliefs of the two parties about the nature of the job are identical. However, when the agent deviates and shirks, the beliefs of the two parties diverge, at least temporarily. We show that such a deviation increases the expected continuation value of the agent. In consequence, any incentive compatible contract that induces high effort must be high powered, in order to offset this possible deviation gain. Thus the ratchet effect gives rise to a dynamic incentive cost, since the agent must be exposed to additional risk in order to overcome the incentive problem. Since the principal must compensate the agent for increased risk, his wage costs increase. This finding generalizes Milgrom and Roberts (1990), who show this in a model with a linear technology and normal model signals, since we find that this applies under a more general information structure and production technology.

We study the behavior of the dynamic incentive cost as the time horizon T increases. Our focus is on dynamically sequentially incentive efficient contracts, where the principal induces high effort in every period. Not surprisingly, the cost of incentivizing effort in any period increases with the time horizon. However, one might conjecture that this effect tapers off — since the both principal and agent learn the state of the world, there is very little uncertainty remaining towards the end of the game. Our main result is that this conjecture is false – the cost of inducing effort in any period increases at least linearly with T. Despite the fact that there is little uncertainty at the end, an additional period does have a small effect in the penultimate period of interaction. This has an effect on the period before, and the cascading effect over periods offsets the learning effect.

This paper is related to a growing literature on dynamic moral hazard with learning/experimentation. Holmström's (1982) career concerns model is a pioneering example.[4] A crucial difference is that in the present paper, learning relates to the nature of the job rather than the agent's talent, and does not affect the outside option of the agent. More recently, there has been increased interest in agency models with learning, where the uncertainty also pertains to the nature of the project. Bergemann and Hege (1998, 2005), Manso (2011), Hörner and Samuelson (2015), and Kwon (2011) and analyze

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[4]Extensions of the career concerns model include Gibbons and Murphy (1992) and Dewatripont, Jewitt, and Tirole (1999).

agency models with binary effort, binary signals and limited liability. There is also recent work on learning in agency models with private actions in continuous time and continuum action spaces including DeMarzo and Sannikov (2011), Cisternas (2014), and Pratt and Jovanovic (2014), that examines the agent's incentives for belief manipulation. Bhaskar (2014) studies a two-period model that makes the same informational and contracting assumptions as in the present paper, but allows for continuum effort choices (rather than binary). The main finding is that the principal cannot implement interior effort choices in the first period. Since the agent can increase his continuation value by shirking, this must be dissuaded by high powered incentives. However, this implies that the agent can deviate upwards, and increase his current payoff, without any loss in continuation value since he can always quit the job tomorrow.

2 The model

We study a risk neutral principal (whom we treat as female) who repeatedly hires a risk averse agent (whom we treat as male) to undertake some task. In each period, the principal offers a spot contract to the agent, who decides whether to accept or reject it. If the agent rejects the contract, the relationship is dissolved and the game ends. If the agent accepts the contract, the agent then decides whether to exert effort e (incurring a disutility of $c > 0$) or shirk s (which is costless). As usual, there is moral hazard, with this choice not observed by the principal. Moreover, there is uncertainty about the "difficulty" of the task. Specifically, there are two states of the world $\omega \in \{B, G\}$, with the task being easy in G, and hard in B. The uncertainty concerns how difficult it is to succeed on *this* job. Importantly, it does *not* affect outside option of the agent, which we normalize to 0.

The choice $a \in \{e, s\}$ by the agent determines, with the state of the world, the probability distribution over signals $y \in Y$, where $Y := \{y^1, y^2, \ldots, y^K\}$ is a finite set of signals. The spot contract specifies the wage payment as a function of the realized signal.

The agent will update his beliefs about the state knowing his own effort choice and the realized public signal. The principal updates her beliefs knowing only the signal, since the agent's effort is not public (i.e., it is not observed by the principal).

The agent's flow utility from a wage payment $w \in \mathbb{R}$ is $u(w)$, where u is strictly increasing and concave. We also assume unlimited liability, so that there are no constraints on the size and sign of utility payments.

We find it more convenient to work with utility schedules, so we write a spot contract as a utility schedule $u := (u^1, \ldots, u^K)$, where u^k is the utility the agent will receive after signal y^k. The wage cost of providing utility level u^k is written $w(u^k) := u^{-1}(u^k)$.

We do not specify the principal's preferences. While solving for the equilibrium of the game does require specifying the principal's preferences, that is not our focus. Our focus, rather, is on the important preliminary step of characterizing the expected cost minimizing sequence of spot contracts that induce effort in every period. This step is independent of the principal's preferences (other than time preference).

There are a finite number of signals with the probability of signal y^k at action $a \in \{s, e\}$ and state $\omega \in \{B, G\}$ denoted by $p^k_{a\omega}$. Our interest is in settings where a a signal that the state is good is also a signal of high effort (and conversely), so that it is impossible to disentangle the two. We capture this by the following assumption.

Assumption 1.

1. *There exists an informative signal, i.e., there exists $y^k \in Y$ such that $\left|\{p^k_{sB}, p^k_{eB}, p^k_{sG}, p^k_{eG}\}\right| \neq 1$.*

2. *For any informative signal $y^k \in Y$,*
$$\min\left\{p^k_{sB}, p^k_{eG}\right\} < p^k_{sG}, p^k_{eB} < \max\left\{p^k_{sB}, p^k_{eG}\right\}.$$

3. *Signals have full support: $p^k_{a\omega} > 0$ for all k, a, ω.*

We partition the set of signals into a set of "high" signals Y^H, "low" signals Y^L, and neutral $Y \setminus (Y^H \cup Y^L)$ by setting
$$y^k \in Y^H \text{ if } p^k_{eG} > p^k_{sB}$$
and
$$y^k \in Y^L \text{ if } p^k_{eG} < p^k_{sB}.$$

A player with belief μ that the task is easy ($\omega = G$) assigns a probability to signal y^k of $p^k_{a\mu} := \mu p^k_{aG} + (1-\mu)p^k_{aB}$. Assumption 1 immediately implies
$$y^k \in Y^H \iff p^k_{eG} > p^k_{eB}, p^k_{sG} > p^k_{sB} \iff p^k_{e\mu} > p^k_{s\mu}$$
and
$$y^k \in Y^L \iff p^k_{eG} < p^k_{eB}, p^k_{sG} < p^k_{sB} \iff p^k_{e\mu} < p^k_{s\mu}.$$

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In other words, high signals arise with higher probability when either the agent exerts effort *or* the state is good. An important implication of this property is that if the principal believes that the agent is exerting effort, but the agent is in fact shirking, then on average, the principal is more pessimistic than the agent.

Lemma 1. *Suppose the signals satisfy Assumption 1. Then,*

$$\mu = \sum_k p_{s\mu}^k \frac{\mu p_{sG}^k}{p_{s\mu}^k} > \sum_k p_{s\mu}^k \frac{\mu p_{eG}^k}{p_{e\mu}^k}.$$

Proof. Assumption 1 implies

$$y^k \in Y^H \iff p_{e\mu}^k > p_{s\mu}^k \iff p_{eG}^k > p_{e\mu}^k$$

and

$$y^k \in Y^L \iff p_{e\mu}^k < p_{s\mu}^k \iff p_{eG}^k < p_{e\mu}^k.$$

Thus,

$$\mu - \sum_k p_{s\mu}^k \frac{\mu p_{eG}^k}{p_{e\mu}^k} = \mu \sum_k (p_{e\mu}^k - p_{s\mu}^k) \frac{p_{eG}^k}{p_{e\mu}^k}$$
$$> \mu \sum_k (p_{e\mu}^k - p_{s\mu}^k) = 0. \qquad \square$$

Suppose the principal and agent both assign probability μ to the task being easy. The *statically optimal spot contract* offered by the principal is a contract $u \in \mathbb{R}^K$ minimizing its expected cost of provision

$$p_{e\mu} \cdot w(u),$$

where $p_{a\mu} := (p_{a\mu}^1, \ldots, p_{a\mu}^K)$, subject to incentive compatibility

$$p_{e\mu} \cdot u - c \geq p_{s\mu} \cdot u \tag{IC}$$

and individual rationality

$$p_{e\mu} \cdot u - c = 0. \tag{IR}$$

Since the principal is risk neutral and the agent is risk averse, the statically optimal contract is unique, which we denote u_μ.

Another useful implication of Assumption 1 is the following lemma.

Lemma 2. *Suppose the signals satisfy Assumption 1. Then,*

$$(p_{eG} - p_{eB}) \cdot u_\mu > 0.$$

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Proof. From (IC), we have

$$(p_{e\mu} - p_{s\mu}) \cdot u_\mu > 0.$$

Observe that $u_\mu^k \geq u_\mu^{k'}$ if $p_{e\mu}^k > p_{s\mu}^k$ and $p_{e\mu}^{k'} \leq p_{s\mu}^{k'}$. [If not, there exists k and k' such that $u_\mu^k < u_\mu^{k'}$ with $p_{e\mu}^k > p_{s\mu}^k$ and $p_{e\mu}^{k'} \leq p_{s\mu}^{k'}$. The contract that equals the old contract except at signals y^k and $y^{k'}$, where the utility promises are replaced by the constant value $(p_{e\mu}^k u_\mu^k + p_{e\mu}^{k'} u_\mu^{k'})/(p_{e\mu}^k + p_{e\mu}^{k'})$, satisfies (IC) and (IR), at lower cost.]

Assumption 1 then implies that $u_\mu^k \geq u_\mu^{k'}$ for $y^k \in Y^H$ and $y^{k'} \in Y^L$, proving the lemma. $\square$

We conclude this section with an implication of Lemma 2 and the principal and agent having different beliefs. Suppose the principal assigns probability μ to G, and so offers a static contract u at which the (IR) binds given $p_{e\mu}$. If the agent has belief π and exerts effort, the agent's payoff from exerting effort is

$$
\begin{aligned}
V^*(\pi, \mu) :&= p_{e\pi} \cdot u - c \\
&= p_{e\mu} \cdot u - c + (\pi - \mu)(p_{eG} - p_{eB}) \cdot u \\
&= (\pi - \mu)(p_{eG} - p_{eB}) \cdot u.
\end{aligned}
\tag{1}
$$

Hence, from Lemma 2, when the principal is less optimistic than the agent, the contract u_μ gives the agent a strictly positive payoff.

3 Two time periods

To illustrate the issues we consider first two periods. The principal minimizes the total wage costs. Neither the principal nor the agent can commit in period 1 to wages or effort in period 2, so each period's spot contract satisfies incentive compatibility (IC) and individual rationality (IR) in that period.

We are interested in the most efficient sequence of spot contracts inducing e in every period. Since there is incomplete information, we require that both the principal and the agent's behavior be sequentially rational after every history, and that both actors update using Bayes' rule whenever possible. The common prior probability on G of the principal and agent is denoted $\mu^\dagger$. Let $\mu_a^k := \psi_a^k(\mu^\dagger)$ be the posterior probability on G after y^k under action a. While the principal does not observe effort, under the sequence of incentive efficient contracts, she assigns probability one to the agent choosing e.

Denote the first period spot contract by $u(1) := (u^1(1), \ldots, u^K(1))$, and the second period spot contract offered by the principal after signal y^k by $u(y^k) := (u^1(y^k), \ldots, u^K(y^k))$.

Definition 1. *A two period sequence of contracts $(u(1), (u(y^k))_{y^k \in Y})$ is sequentially effort incentive efficient if*

1. *for every first period signal realization $y^k \in Y$, $u(y^k)$ minimizes*

$$p_{e\mu_e^k} \cdot w(u) = \sum_{k'} p_{e\mu_e^k}^{k'} \, w(u^{k'})$$

   subject to the agent finding it optimal to participate and exert effort in the second period after exerting effort in the first period, and

2. *$u(1)$ minimizes $\sum p_{e\mu^\dagger}^k w(u^k)$ subject to the agent finding it optimal to participate and exert effort in the first period.*

Under a sequentially effort incentive efficient sequence of contracts, the agent exerts effort in every period, and the second period beliefs of the agent and principal agree. In particular, after y^k, in the second period, the effort incentive efficient contract solves the static problem with public beliefs μ_e^k.

The first period is more complicated, since a deviation by the agent to shirking in the first period results in the principal and agent having different beliefs. Given signal y^k, the agent updates his belief on G to μ_s^k after the signal y^k, which differs from the principal's update of μ_e^k. After the signal y^k, not only does the principal have the (incorrect) belief μ_e^k about G, she is also mistaken in her conviction that the agent has the belief μ_e^k.

Since the agent and principal have different beliefs about the state of nature, the incentive compatibility and participation constraints as viewed by the principal are not, in general, valid for the agent. The difference in beliefs can benefit the agent, and provide an additional incentive for the agent to shirk. The principal designs the second period contract so that the second period IR constraint binds at the beliefs μ_e^k, so that the agent earns only his reservation utility from the static contract under beliefs μ_e^k.

Since the principal wants to induce e at $t = 2$, high signals have to be rewarded in the second period. As a consequence, the agent's second period utility *strictly increases* from shirking:

1. Lemma 1 implies there is a signal y^k such that $\mu_s^k > \mu_e^k$, with a resulting second period gain from deviation.

2. For any signal y^k satisfying $\mu_s^k < \mu_e^k$, the IR constraint is violated, and the agent walks away, obtaining his reservation utility.

Thus, the first period spot contract must satisfying the constraint

$$p_{e\mu^\dagger} \cdot u(1) - c_e \geq p_{s\mu^\dagger} \cdot u(1) - c_s + W(\mu^\dagger), \tag{2}$$

where $W(\mu^\dagger)$ is the expected payoff in the second period from shirking rather than exerting effort in the first period. We have just seen that

$$W(\mu^\dagger) \geq \sum_{y^k} p_{s\mu^\dagger}^k \max\{V^*(\mu_s^k, \mu_e^k), 0\} > 0,$$

and so the statically optimal contract $u_{\mu^\dagger}$ does not satisfy (2). The first period spot contract must be more high powered than the statically optimally contract in order to deter shirking.

4   Finite Horizon

We consider next the finite horizon setting, with T periods in the relationship. We index periods backwards, so in period t, there are $t-1$ periods remaining after the current one. In period $\tau = T, \ldots, 1$, the principal has observed the history $h^\tau \in Y^{T-\tau}$, and offers a spot contract $u(h^\tau)$. In the following definition, note that $\hat{h}^t$ is the common $T-t$ initial segment of each h^τ.

Definition 2. *A sequence of contracts $((u(h^\tau))_{h^\tau \in Y^{T-\tau}})_{\tau=1,\ldots,T}$ is* sequentially effort incentive efficient *(SEIE) if for every $t \in \{T, \ldots, 2, 1\}$ and every $\hat{h}^t \in Y^{T-t}$, the sequence minimizes*

$$\sum_{\tau=1}^t E_{h^\tau, y^k}\{w(u^k(h^\tau, y^k)) \mid \hat{h}^t, a^\tau = e, a^T = \cdots = a^{\tau-1} = e]$$

subject to the agent finding it optimal to participate and exert effort in period t and in every subsequent period after every public history, conditional on the agent having exerted effort in every previous period.

Since the behavior of the principal in any period is completely determined by her beliefs about the state updated from the public history, we can solve for SEIE recursively, beginning in the last period (period 1; recall we index periods backwards).

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We need to consider situations in which the agent and principal have different beliefs. Let $V(\pi, \mu, 1)$ denote the agent's value function in period 1 when his belief is π and the principal's belief is μ (for our purposes, these beliefs are the result of updating using $h^1 \in Y^{T-1}$, the period 1 public history). The principal, given his updated beliefs μ, offers the contract $u_\mu(1)$. The agent's value from this contract is

$$V(\pi, \mu, 1) = \max \{ p_{e\pi} \cdot u_\mu(1) - c, \ p_{s\pi} \cdot u_\mu(1), \ 0 \}.$$

If beliefs agree the value is zero, i.e., $V(\mu, \mu, 1) = 1$. Let $V(\pi, \mu, t)$ denote the agent's value function in period t when his belief is π and the principal's belief is μ. Denote the effort incentive efficient contract offered in period t by the principal by $u_\mu(t)$. Then,

$$
\begin{aligned}
V(\pi, \mu, t) = \max \{ &p_{e\pi} \cdot u_\mu(t) - c + \sum_k p_{e\pi}^k V(\psi_e^k(\pi), \psi_e^k(\mu), t - 1), \\
&p_{s\pi} \cdot u_\mu(t) + \sum_k p_{s\pi}^k V(\psi_s^k(\pi), \psi_e^k(\mu), t - 1), \ 0 \},
\end{aligned}
$$

where $\psi_a^k(\beta)$ is the posterior probability on G after y^k under action a, given a prior β.

On the equilibrium path, the agent has always been exerting effort, so that in period t, at belief μ, the contract $u_\mu(t)$ satisfies the incentive constraint

$$
\begin{aligned}
p_{e\mu} \cdot u_\mu(t) - c + \sum_k p_{e\mu}^k V(\psi_e^k(\mu), \psi_e^k(\mu), t - 1) & \\
\geq p_{s\mu} \cdot u_\mu(t) &+ \sum_k p_{s\mu}^k V(\psi_s^k(\mu), \psi_e^k(\mu), t - 1)
\end{aligned}
$$

and the participation constraint

$$p_{e\mu} \cdot u_\mu(t) - c + \sum_k p_{e\mu}^k V(\psi_e^k(\mu), \psi_e^k(\mu), t - 1) = 0.$$

Since $V(\mu', \mu', 1) = 1$ for all μ', induction immediately implies $V(\mu', \mu', t) = 1$ for all μ'.

Defining
$$W(\mu, t) := \sum_k p_{s\mu}^k V(\psi_s^k(\mu), \psi_e^k(\mu), t - 1), \tag{3}$$

as the *future information rent from shirking* (FIRS) in period t, the period-t incentive constraint can then be written as

$$p_{e\mu} \cdot u_\mu(t) - c \geq p_{s\mu} \cdot u_\mu(t) + W(\mu, t).$$

Summarizing this discussion, we have:

Proposition 1. *A sequence of contracts* $((u(h^\tau))_{h^\tau \in Y^{T-\tau}})_{\tau=1,\ldots,T}$ *is sequentially effort incentive efficient (SEIE) if and only if* $u = u(h^\tau)$ *minimizes*

$$p_{e\mu} \cdot w(u)$$

subject to

1. $\mu = \Pr[G \mid h^\tau, a^T = \cdots a^{\tau-1} = e]$,

2. $p_{e\mu} \cdot u - c \geq p_{s\mu} \cdot u + W(\mu, t)$, *and*

3. $p_{e\mu} \cdot u - c \geq 0$.

4. *Furthermore, the constraints (2) and (3) bind.*

From Section 3, we know $W(\mu, 2) > 0$. Is $W(\mu, t)$ increasing in t, and if it is increasing, does it increase without bound.

Intuitively, $W(\mu, 3)$ should be larger than $W(\mu, 2)$, because the latter reflects the value of different beliefs induced by shirking under a statically optimal contract for a less demanding incentive compatibility constraint. This is essentially a question of comparative statics on static contracts with respect to the opportunity cost of shirking, which turns out to be a lot harder than comparative statics with respect to the disutility of effort. The next section outlines the problem.

5 Comparative Statics of Optimal Contracts

The contract $u_\mu(t)$ described in Proposition 1 solves a static incentive problem that is a case of the following. The principal solves (where $w(u^k) = u^{-1}(u^k)$ is the wage necessary for the agent to receive utility u^k)

$$\min_{\{u^k\}} \ \sum_k p^k_{e\mu} w(u^k)$$

subject to

$$\sum_k p^k_{e\mu} u^k - c \geq \sum_k p^k_{s\mu} u^k + W \qquad\qquad \text{(IC)}$$

$$\text{and} \quad \sum_k p^k_{e\mu} u^k - c = 0. \qquad\qquad \text{(IR)}$$

The first order conditions imply

$$w'(u^k) = \lambda + \xi\left(1 - \frac{p^k_{s\mu}}{p^k_{e\mu}}\right), \qquad k = 1, \ldots, K, \qquad\qquad (4)$$

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where λ is the multiplier on the IR constraint and ξ is the multiplier on the IC constraint.

Order the signals so that $p_{s\mu}^k / p_{e\mu}^k$ is decreasing in k. The right side of (4) is then increasing in k, and so $w'(u^k)$ is increasing in k, implying u^k is increasing in k.

Suppose W and $\widetilde{W}$ are two distinct opportunity costs of shirking, with $W > \widetilde{W}$. Let u and $\tilde{u}$ denote the vectors of utilities in the corresponding optimal contracts. Since IC holds with equality we have

$$(p_{e\mu} - p_{s\mu}) \cdot u = c + W$$

and

$$(p_{e\mu} - p_{s\mu}) \cdot \tilde{u} = c + \widetilde{W}.$$

We are interested in the properties of the vector $\tilde{u} - u$. In particular, from (1), we would like to conclude

$$(p_{eG} - p_{eB}) \cdot (u - \tilde{u}) > 0. \tag{5}$$

While we know

$$(p_{e\mu} - p_{s\mu}) \cdot (u - \tilde{u}) \cdot = W - \widetilde{W}, \tag{6}$$

without further assumptions, this does not imply (5).

There is one setting with general probabilities where we can deduce (5), and so the monotonicity of $W(\mu, t)$ in t, and that is where the agent has a particular form of CRRA preferences,

$$u(w) = \sqrt{A + w},$$

where $A > 0$ is a positive constant sufficiently large that IR binds for the following discussion. Then $w'(u^k) = 2u^k$, and (4) can be written as

$$u^k = \frac{\lambda}{2} + \frac{\xi}{2}\left(1 - \frac{p_{s\mu}^k}{p_{e\mu}^k}\right), \qquad k = 1, \dots, K. \tag{7}$$

The incentive constraint $u \cdot (p_{e\mu} - p_{s\mu}) = c + W$ can then be rewritten as

$$c + W = \sum_k \left[\frac{\lambda}{2} + \frac{\xi}{2}\left(1 - \frac{p_{s\mu}^k}{p_{e\mu}^k}\right)\right](p_{e\mu}^k - p_{s\mu}^k)$$

$$= \sum_k \frac{\xi}{2}\left(1 - \frac{p_{s\mu}^k}{p_{e\mu}^k}\right)(p_{e\mu}^k - p_{s\mu}^k)$$

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$$=: \frac{\xi X(\mu)}{2}.$$

This implies

$$\xi = \frac{2(c+W)}{X(\mu)},$$

and so

$$(p_{eG} - p_{eB}) \cdot (u - \tilde{u}) = \sum_k (p_{eG}^k - p_{eB}^k) \cdot \frac{(W - \widetilde{W})}{X(\mu)} \left(1 - \frac{p_{s\mu}^k}{p_{e\mu}^k}\right) > 0,$$

where the inequality is an implication of Assumption 1.

6 A Restriction on Signals

We now pursue a direct path to link (5) and (6) by assuming the vectors $(p_{eG} - p_{eB})$ and $(p_{e\mu} - p_{s\mu})$ are collinear. Our goal is to bound $W(\mu, t)$ as a function of t, since larger information rents require more high powered incentives. We bound $W(\mu, t)$ from below by bounding $V(\pi, \mu, t)$.

Obtaining tight bounds for the value function is in general difficult. However, in some cases, we are able to obtain useful bounds by considering a particular specification of continuation play of the agent, namely, always exert effort. Denote by $V^*(\pi, \mu, t)$ the agent's value function in period t when his belief is π and the principal's belief is μ, and the agent always chooses effort. Since

$$V(\pi, \mu, t) \geq V^*(\pi, \mu, t), \tag{8}$$

it is enough to bound $V^*(\pi, \mu, t)$. The value recursion for V^* is

$$V^*(\pi, \mu, t) = p_{e\pi} \cdot u_\mu(t) - c + \sum_k p_{e\pi}^k V^*(\psi_e^k(\pi), \psi_e^k(\mu), t - 1). \tag{9}$$

As we saw from Section 3, if $\pi > \mu$, the first flow term is positive, with subsequent flows reflecting additional rents from updated differences in beliefs. However, beliefs *merge* (Blackwell and Dubins, 1962): the difference between the agent's and the principal's posteriors vanishes. Consequently, in a long relationship, the impact of a difference in beliefs after a deviation in the initial period on the expected information rent in the last period is small.

Nonetheless, in the last period, any small information rent leads to an increase (albeit small) in the power of the required incentives in the penultimate period. This implies that the information rents in period 2 generated

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from a difference in beliefs are greater than they would have been in the last period.

Proposition 2. *Suppose there exists a vector* $\gamma \in \mathbb{R}^K$, $\gamma \cdot \mathbf{1} = 0$, *and constants* $\alpha > 0$ *and* β *satisfying* $\beta > \max\{\alpha, 1\} > 0$ *such that*

$$p_{sG} = p_{sB} + \alpha\gamma,$$
$$p_{eB} = p_{sB} + \gamma,$$
$$\text{and} \quad p_{eG} = p_{sB} + \beta\gamma.$$

Let

$$K := \min_{\mu} \frac{(\beta - 1)}{[\mu(\beta - \alpha) + (1 - \mu)]} > 0.$$

For any integer t,

$$V(\pi, \mu, t) \geq V^*(\pi, \mu, t) \geq (\pi - \mu)Kct. \tag{10}$$

Remark 1. With binary signals, the collinearity assumption is automatically satisfied, since the space of probabilities is one-dimensional.

We now prove the proposition. Assumption 1 holds without further restrictions on the parameters, with $y^k \in Y^H$ if $\gamma^k > 0$ and $y^k \in Y^L$ if $\gamma^k < 0$. Note that $p_{eG} - p_{eB} = (\beta - 1)\gamma$ and $p_{e\mu} - p_{s\mu} = [\mu(\beta - \alpha) + (1 - \mu)]\gamma$.

We first state two implications of the assumed structure on signals. The optimal spot contract in period t satisfies

$$c + W(\mu, t) = (p_{e\mu} - p_{s\mu}) \cdot u_\mu(t) = [\mu(\beta - \alpha) + (1 - \mu)]\gamma \cdot u_\mu(t), \tag{11}$$

(where $W(\mu, 1) = 0$), and so (since (IR) binds on $u_\mu(t)$ at belief μ, recalling (1))

$$p_{e\pi} \cdot u_\mu(t) - c = (\pi - \mu)(\beta - 1)\gamma \cdot u_\mu(t),$$
$$= (\pi - \mu)\frac{(\beta - 1)}{[\mu(\beta - \alpha) + (1 - \mu)]}(c + W(\mu, t)). \tag{12}$$

The first inequality in (10) is simply (8).

From the value recursion for V^* given in (9), we have

$$V^*(\pi, \mu, t) = p_{e\pi} \cdot u_\mu(t) - c + \sum_k p_{e\pi}^k V^*(\psi_e^k(\pi), \psi_e^k(\mu), t - 1)$$
$$= (\pi - \mu)\frac{(\beta - 1)}{[\mu(\beta - \alpha) + (1 - \mu)]}(c + W(\mu, t))$$

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$$+ \sum\nolimits_k p_{e\pi}^k V^*(\psi_e^k(\pi), \psi_e^k(\mu), t-1). \qquad (13)$$

A natural way to proceed is by induction. Suppose $t = 1$. Then,

$$\begin{aligned}
V(\pi, \mu, 1) &\geq V^*(\pi, \mu, 1) \\
&= (\pi - \mu)(p_{eG} - p_{eB}) \cdot u_\mu(1) \\
&= (\pi - \mu)(\beta - 1)\gamma \cdot u_\mu(1) \\
&\geq (\pi - \mu)Kc.
\end{aligned}$$

The inductive hypothesis is

$$V^*(\pi, \mu, t-1) \geq (\pi - \mu)Kc(t-1).$$

If this implied

$$\sum\nolimits_k p_{e\pi}^k V^*(\psi_e^k(\pi), \psi_e^k(\mu), t-1) \geq (\pi - \mu)Kc(t-1), \qquad (14)$$

then we would be done, since $W(\mu, t) \geq 0$ and so

$$(\pi - \mu)K(c + W(\mu, t)) \geq (\pi - \mu)Kc.$$

However, (14) fails because beliefs merge. From the inductive hypothesis we have

$$\sum\nolimits_k p_{e\pi}^k V^*(\psi_e^k(\pi), \psi_e^k(\mu), t-1) \geq Kc(t-1) \sum\nolimits_k p_{e\pi}^k(\psi_e^k(\pi) - \psi_e^k(\mu)).$$

Using the equality $p_{e\pi}^k = p_{e\mu}^k + (\pi - \mu)(p_{eG}^k - p_{eB}^k)$, we have

$$\begin{aligned}
\sum\nolimits_k p_{e\pi}^k(\psi_e^k(\pi) - \psi_e^k(\mu)) &= \pi - \sum\nolimits_k p_{e\pi}^k \frac{\mu p_{eG}^k}{p_{e\mu}^k} \\
&= \pi - \mu - (\pi - \mu)\mu \sum\nolimits_k (p_{eG}^k - p_{eB}^k)\frac{p_{eG}^k}{p_{e\mu}^k} \\
&= (\pi - \mu)(1 - \xi(\mu)), \qquad (15)
\end{aligned}$$

where

$$\xi(\mu) := \mu \sum\nolimits_k (p_{eG}^k - p_{eB}^k)\frac{p_{eG}^k}{p_{e\mu}^k} > 0$$

is the *merging deficit*.[5] Therefore, all we can conclude from the inductive hypothesis with respect to the second term of (13) is

$$\sum\nolimits_k p_{e\pi}^k V^*(\psi_e^k(\pi), \psi_e^k(\mu), t-1) \geq (\pi - \mu)Kc(t-1)(1 - \xi(\mu)). \qquad (16)$$

---

[5]The strict positivity of $\xi(\mu)$ is an immediate implication of Assumption 1. As one would expect, $\xi(\mu) \to 0$ as $\mu \to 0$ or 1 (recall that $\sum_k \gamma^k = 0$).

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For future reference, a straightforward calculation shows that under the collinear parameterization,

$$\xi(\mu) = \mu(\beta - 1)\sum_k \gamma^k \frac{p_{eG}^k}{p_{e\mu}^k}. \tag{17}$$

But the inductive hypothesis also bounds the future information rents from shirking,

$$W(\mu, t) = \sum_k p_{s\mu}^k V(\psi_s^k(\mu), \psi_e^k(\mu), t - 1)$$
$$\geq K(t - 1)c\sum_k p_{s\mu}^k(\psi_s^k(\mu) - \psi_e^k(\mu)).$$

Now,

$$\sum_k p_{s\mu}^k(\psi_s^k(\mu) - \psi_e^k(\mu)) = \mu - \sum_k p_{s\mu}^k \frac{\mu p_{eG}^k}{p_{e\mu}^k}$$
$$= \mu \sum_k \left\{p_{e\mu}^k - p_{s\mu}^k\right\} \frac{p_{eG}^k}{p_{e\mu}^k}$$
$$= \mu \sum_k [\mu(\beta - \alpha) + (1 - \mu)]\gamma^k \frac{p_{eG}^k}{p_{e\mu}^k}$$
$$= \frac{[\mu(\beta - \alpha) + (1 - \mu)]}{(\beta - 1)}\xi(\mu). \tag{18}$$

Hence,

$$\frac{(\beta - 1)}{[\mu(\beta - \alpha) + (1 - \mu)]}W(\mu, t) \geq K(t - 1)c\xi(\mu). \tag{19}$$

Substituting (16) and (19) into (13) yields

$$V^*(\pi, \mu, t) \geq (\pi - \mu)Kc[1 + (t - 1)\xi(\mu) + (t - 1)(1 - \xi(\mu))] = (\pi - \mu)Kct,$$

completing the proof.

These calculations also give via(3), a lower bound on W:

Corollary 1. *The future information rent from shirking is bounded below by a linear function of time:*

$$W(\mu, t) \geq Kc\frac{[\mu(\beta - \alpha) + 1 - \mu]}{(\beta - 1)}\xi(\mu)(t - 1).$$

The assumption on the structure of signals plays two roles in the analysis. The first is to provide a relationship between $p_{e\pi} \cdot u_\mu(t) - c$ and $W(\mu, t)$. The second is connect the merging deficit with the bound on $W(\mu, t)$. While it is possible to provide a relationship between $p_{e\pi} \cdot u_\mu(t) - c$ and $W(\mu, t)$ under weaker assumptions, the connection of the merging deficit with the bound on $W(\mu, t)$ is more subtle, and we have not found a nice more general condition.

We can precisely characterize the future information rents from shirking under one simple additional restriction.

Proposition 3. *Suppose the probability distribution on signals satisfies the conditions of Proposition 2, and that $p_{eG} + p_{sB} = p_{eB} + p_{sG}$ (i.e., $\beta = \alpha + 1$). Then,*

1. *for all informative signals, y^k, $\psi_s^k(\mu) > \psi_e^k(\mu)$, and so after shirking, the agent is always more optimistic than the principal, and so never takes the outside option.*

2. *if u satisfies IR with equality at $p_{e\mu}$, then $p_{s\pi} \cdot u = p_{e\pi} \cdot u - \gamma \cdot u$, and*

3. *the agent is always indifferent between exerting effort and shirking, and so an optimal continuation play for the agent after shirking at a common belief μ is to accept every future contract and always exert effort, so that*

$$V(\pi, \mu, t) = V^*(\pi, \mu, t) = (\pi - \mu)\alpha c t$$

and

$$W(\mu, t) = c\xi(\mu)(t - 1).$$

Proof.   1. A few lines of algebra shows that $\psi_s^k(\mu) - \psi_e^k(\mu)$ has the same sign as $\alpha(\gamma^k)^2$, which is strictly positive if and only if $\gamma^k \neq 0$, that is, if y^k is informative.

2. Consider a contract u that satisfies IR with equality at $p_{e\mu}$. Then,

$$
\begin{aligned}
p_{s\pi} \cdot u &= (p_{s\pi} - p_{e\mu}) \cdot u + c + p_{e\mu} \cdot u - c \\
&= [\pi(p_{sB} + \alpha\gamma) + (1 - \pi)p_{sB} \\
&\qquad - \mu(p_{sB} + \beta\gamma) - (1 - \mu)(p_{sB} + \gamma)] \cdot u + c \\
&= (\pi\alpha\gamma - \mu\beta\gamma - (1 - \mu)\gamma) \cdot u + c \\
&= (\pi\alpha\gamma - \mu\alpha\gamma - \mu\gamma - (1 - \mu)\gamma) \cdot u + c
\end{aligned}
$$

$$= (\pi - \mu)\alpha\gamma \cdot u - \gamma \cdot u + c,$$

while

$$p_{e\pi} \cdot u = (\pi - \mu)(p_{eG} - p_{eB}) \cdot u + c$$
$$= (\pi - \mu)(\beta - 1)\gamma \cdot u + c$$
$$= (\pi - \mu)\alpha\gamma \cdot u + c.$$

3. We prove by induction. The agent is clearly indifferent between effort and shirt for $t = 1$, and $V^*(\pi, \mu, 1) = (\pi - \mu)\alpha c$. Suppose the agent is indifferent between shirk and effort for $\tau = 1, \ldots, t - 1$. Then,

$$V(\pi, \mu, \tau) = V^*(\pi, \mu, \tau) = (\pi - \mu)\alpha c\tau, \quad \tau = 1, \ldots, t - 1$$

and so
$$W(\mu, t) = c\xi(\mu)(t - 1).$$

The difference between the value from effort and shirk in period t is

$$p_{e\pi} \cdot u_\mu(t) - c + \sum_k p_{e\pi}^k V(\psi_e^k(\pi), \psi_e^k(\mu), t - 1)$$
$$- p_{s\pi} \cdot u_\mu(t) - \sum_k p_{s\pi}^k V(\psi_s^k(\pi), \psi_e^k(\mu), t - 1)$$
$$= p_{e\pi} \cdot u_\mu(t) - c - p_{s\pi} \cdot u_\mu(t) + \alpha c(t - 1) \sum_k (p_{s\pi}^k - p_{e\pi}^k)\psi_e^k(\mu)$$
$$= W(\mu, t) + \alpha c(t - 1) \sum_k (p_{s\pi}^k - p_{e\pi}^k)\psi_e^k(\mu)$$
$$= c\xi(\mu)(t - 1) - \alpha c(t - 1)\xi(\mu)/\alpha = 0,$$

where the first equality comes from substituting for $V = V^*$ evaluated at $t - 1$ and simplifying, the second equality uses part 2 of the proposition (and (11)), and the third equality applies the inductive hypothesis again and (17)).

$\square$

7 Merging with Binary Signals

We have already seen in the two period case that the initial period contract must be more high powered than the one period contract in order to compensate for the one period FIRS. But this means that in the three period contract, the FIRS reflects the increased value of different beliefs in period 2 from the more high powered period 2 contract, in addition to the value of different beliefs in period 1.

$p_{a\omega}^{H}$	$a = e$	$a = s$
$\omega = G$	r	$q + (2r - 1)$
$\omega = B$	$1 - r$	q

Figure 1: The probability of the high signal y^H as a function of the state ω and action a, with $0 < q < r < 1$ and $2r - 1 > 0$ and $q + r < 1$. Under these assumptions Assumption 1 is satisfied. The conditions of Proposition 3 hold.

How much of the lower bound on future information rents from shirking is due to the value from having different beliefs in all future periods, and how much is due to the positive feedback from one period's increase in the required power of the incentives to the previous period?

To shed light on this issue, we consider a symmetric binary signal environment in which we have an exact expression for the FIRS: There are two signals y^H and y^L, with the probability of y^H given in Figure 1 (note that $p_{eG} + p_{sB} = p_{eB} + p_{sG}$ as required in Proposition 3).

By construction, beginning from a common prior, if the principal expects effort, but the agent shirks, then the agents is more optimistic than the principal after both y^H and y^L. We are interested in the value to the agent of shirking in the initial period (and so being optimistic in every future period), when there are *no* expected information rents after the initial period.

So, suppose that in each period (after the initial period), the principal offers the statically optimal contract $u_\mu^\dagger(t)$, where μ is the posterior update assuming the agent has exerted effort previously. The principal has belief $\psi_e(\mu, h^\tau) =: \mu^\tau$. This contract solves

$$u^H - u^L = \frac{c}{p_{e\mu^\tau}^H - p_{s\mu^\tau}^H}$$

and

$$p_{e\mu^\tau}^H u^H + p_{e\mu^\tau}^L u^L = 0.$$

The flow benefit to the agent from exerting effort is then (from (1))

$$[\psi_e(\pi, h^\tau) - \mu^\tau](p_{eG}^H - p_{eB}^H)(u^H - u^L) = [\psi_e(\pi, h^\tau) - \mu^\tau]\frac{(2r - 1)c}{p_{e\mu^\tau}^H - p_{s\mu^\tau}^H}.$$

The value to the agent of having belief $\pi > \mu$ at the end of the initial period

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with t periods remaining is then

$$V^\dagger(\pi,\mu,t) = E_{e\pi} \sum_{\tau=1}^{t} [\psi_e(\pi,h^\tau) - \mu^\tau] \frac{(2r-1)c}{p_{e\mu^\tau}^H - p_{s\mu^\tau}^H},$$

where, as before, $h^\tau \in Y^{t-\tau}$. At the risk of emphasizing the obvious, observe that because $\pi > \mu$, for all h^τ we have that $\psi_e(\pi,h^\tau) - \mu^\tau = \psi_e(\pi,h^\tau) - \psi_e(\mu,h^\tau) > 0$.

The appendix proves the following:

Proposition 4. *Suppose there are two signals with distributions given in Figure 1 and $16r^3(1-r) < 1$. There exists $\bar{V} \in \mathbb{R}$ such that for all t, and $\pi > \mu$,*

$$V^\dagger(\pi,\mu,t) < \bar{V}.$$

While we have not been able to bound $V^\dagger$ for other parameterizations, we conjecture the result holds more generally. We interpret this result as confirming our intuition that the incentive costs are unbounded in t due to the positive feedback from the power of the incentives.

8 Infinite Horizon

In this section, we maintain the hypotheses on the probability distributions of Proposition 2 and show that a similar phenomenon arises with an infinite horizon. We assume both the principal and agent discount with possibly different discount factors δ_A and $\delta_P < 1$. We focus on stationary high effort incentive efficient contracts, and prove that the following result.

Proposition 5. *Suppose the probability distributions satisfy the conditions in Proposition 2, and K is the constant defined in that proposition. Suppose a stationary high effort incentive efficient contract exists and V is the agent's value when his belief is π and the principal's belief is μ. Then, V satisfies*

$$V(\pi,\mu) \geq \frac{Kc(\pi-\mu)}{1-\delta_A}. \tag{20}$$

The denominator $1 - \delta_A$ replaces the horizon, and analogously to the finite horizon, as the agent becomes patient, future information rents from shirking become arbitrarily large.

Let $\mathcal{Y}$ be the set of all functions mapping $[0,1]^2$ to $\mathbb{R}$ equalling zero on the diagonal (i.e., $V(\mu,\mu) = 0$ for all $\mu \in [0,1]$ and all $V \in \mathcal{Y}$),[6] and let $\Psi : \mathcal{Y} \to \mathcal{Y}$ be the mapping defined by $V' = \Psi(V)$ given by

$$V'(\pi,\mu) := \max \big\{ p_{e\pi} \cdot u_\mu^V - c + \delta_A \sum\nolimits_k p_{e\pi}^k V(\psi_e^k(\pi), \psi_e^k(\mu)),$$
$$p_{s\pi} \cdot u_\mu^V + \delta_A \sum\nolimits_k p_{s\pi}^k V(\psi_s^k(\pi), \psi_e^k(\mu)), 0 \big\}, \quad (21)$$

where u_μ^V is the unique cost minimizing vector of utilities satisfying

$$p_{e\mu} \cdot u_\mu^V - c = p_{s\mu} \cdot u_\mu^V + \delta_A \sum\nolimits_k p_{s\pi}^k V(\psi_s^k(\pi), \psi_e^k(\mu)) \quad (22)$$

$$\text{and} \quad p_{e\mu} \cdot u_\mu^V - c = 0. \quad (23)$$

For any stationary high effort incentive efficient contract, the value function V describing the agent's value when his belief is π and the principal's belief is μ is a fixed point of Ψ.

We proceed as in the finite horizon case, bounding V by the value function when the agent exerts effort. Consequently, as for the finite horizon case, we do not need to know the precise details of the spot contracts, here μ^V. It is enough to know that

$$p_{e\pi} \cdot u_\mu^V - c = (\pi - \mu) \frac{(\beta - 1)}{[\mu(\beta - \alpha) + (1 - \mu)]} \left(c + \delta_A \sum\nolimits_k p_{s\pi}^k V(\psi_s^k(\pi), \psi_e^k(\mu)) \right),$$

which follows from familiar arguments (see (11) and (12)).

Lemma 3. *Denote by $\mathcal{V}$ the subset of $\mathcal{Y}$ satisfying the inequality in (20). The mapping $\Psi^e : \mathcal{Y} \to \mathcal{Y}$ defined by $V^* = \Psi^e(V)$, where*

$$V^*(\pi,\mu) := p_{e\pi} \cdot u_\mu^V - c + \delta_A \sum\nolimits_k p_{e\pi}^k V(\psi_e^k(\pi), \psi_e^k(\mu)) \quad (24)$$

is a self-map on $\mathcal{V}$, i.e.,

$$\Psi^e : \mathcal{V} \to \mathcal{V}.$$

Proof. For $V \in \mathcal{V}$,

$$\sum\nolimits_k p_{s\mu}^k V(\psi_s^k(\mu), \psi_e^k(\mu)) \geq \sum\nolimits_k p_{s\mu}^k \frac{K c(\psi_s^k(\mu) - \psi_e^k(\mu))}{1 - \delta_A}$$

---

[6]We have already seen in the finite horizon setting that this property holds, and it could be deduced here as well. Assuming it directly is without loss of generality and simplifies our analysis.

$$\geq \frac{[\mu(\beta - \alpha) + (1 - \mu)]}{(\beta - 1)} \frac{Kc\xi(\mu)}{(1 - \delta_A)}$$

(where the last inequality follows from (18)). This gives

$$V^*(\pi, \mu) \geq (\pi - \mu)K \left\{ c + \delta_A \frac{c\xi(\mu)}{(1 - \delta_A)} \right\} + \delta_A \sum_k p_{e\pi}^k V(\psi_e^k(\pi), \psi_e^k(\mu)).$$

Turning to the second term and applying (15) to obtain the equality gives

$$\sum_k p_{e\pi}^k V(\psi_e^k(\pi), \psi_e^k(\mu)) \geq \frac{Kc \sum_k p_{e\pi}^k (\psi_e^k(\pi) - \psi_e^k(\mu))}{1 - \delta_A}$$
$$= (\pi - \mu) \frac{Kc}{(1 - \delta_A)} (1 - \xi(\mu)),$$

so that

$$V^*(\pi, \mu) \geq (\pi - \mu) \frac{Kc}{(1 - \delta_A)} \{ 1 - \delta_A + \delta_A \xi(\mu) + \delta_A(1 - \xi(\mu)) \}$$
$$= (\pi - \mu) \frac{Kc}{(1 - \delta_A)},$$

and so $V^* \in \mathcal{V}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Since
$$\Psi(V) \geq \Psi^e(V)$$
pointwise (i.e., for all (π, μ), $\Psi(V)(\pi, \mu) \geq \Psi^e(V)(\pi, \mu)$) and $\Psi^e : \mathcal{V} \to \mathcal{V}$, we have $\Psi : \mathcal{V} \to \mathcal{V}$.

We now argue that any fixed point of Ψ must lie in $\mathcal{V}$. Since Ψ need not be a contraction, we argue indirectly.

Let $\mathcal{Y}^0 := \{ V \in \mathcal{Y} \mid V(\pi, \mu) \geq 0 \ \forall (\pi, \mu) \}$. Clearly, $\Psi : \mathcal{Y} \to \mathcal{Y}^0$. For all $V \in \mathcal{Y}^0$,
$$\Psi(V)(\pi, \mu) \geq \Psi^e(V)(\pi, \mu) \geq (\pi - \mu)Kc.$$

Lemma 4. *Defining*

$$\mathcal{Y}^\kappa := \{ V \in \mathcal{Y}^{\kappa-1} \mid V(\pi, \mu) \geq (\pi - \mu)Kc(1 - \delta_A^\kappa)/(1 - \delta_A), \ \forall (\pi, \mu) \},$$

we have
$$\Psi : \mathcal{Y}^\kappa \to \mathcal{Y}^{\kappa+1}, \qquad \forall \kappa \geq 0.$$

Proof. For $V \in \mathcal{Y}^\kappa$, applying (18),

$$\sum_k p_{s\mu}^k V(\psi_s^k(\mu), \psi_e^k(\mu)) \geq \sum_k p_{s\mu}^k \frac{Kc(1-\delta_A^\kappa)(\psi_s^k(\mu) - \psi_e^k(\mu))}{(1-\delta_A)}$$

$$\geq \frac{c(1-\delta_A^\kappa)\xi(\mu)}{(1-\delta_A)}(p_{e\mu}^H - p_{s\mu}^H).$$

Then, as in the beginning of the proof of Lemma 3,

$$\Psi^e(V)(\pi, \mu) \geq (\pi - \mu)K \left\{ c + \delta_A \frac{(1-\delta_A^\kappa)c\xi(\mu)}{(1-\delta_A)} \right\}$$

$$+ \delta_A \sum_k p_{e\pi}^k V(\psi_e^k(\pi), \psi_e^k(\mu)).$$

But

$$\sum_k p_{e\pi}^k V(\psi_e^k(\pi), \psi_e^k(\mu)) \geq \frac{Kc(1-\delta_A^\kappa) \sum_k p_{e\pi}^k (\psi_e^k(\pi) - \psi_e^k(\mu))}{1-\delta_A}$$

$$= (\pi - \mu)\frac{K(1-\delta_A^\kappa)c}{(1-\delta_A)}(1-\xi(\mu)),$$

so that

$$\Psi(V)(\pi, \mu) \geq \Psi^e(V)(\pi, \mu)$$

$$\geq (\pi - \mu)\frac{Kc}{(1-\delta_A)}\{1 - \delta_A + \delta_A(1-\delta_A^\kappa)\xi(\mu) + \delta_A(1-\delta_A^\kappa)(1-\xi(\mu))\}$$

$$= (\pi - \mu)\frac{Kc(1-\delta_A^{\kappa+1})}{(1-\delta_A)},$$

and so $V^* \in \mathcal{Y}^{\kappa+1}$. $\qquad\square$

Lemma 5. *Every fixed point of Ψ is in $\mathcal{V}$.*

Proof. Each fixed point of Ψ must be in every $\mathcal{Y}^\kappa$, so that

$$V(\pi, \mu) \geq \frac{(\pi - \mu)Kc(1-\delta_A^\kappa)}{(1-\delta_A)}, \qquad \forall(\pi, \mu),$$

for all κ, implying $V \in \mathcal{V}$. $\qquad\square$

Lemma 5 implies a similar lower on the future information rent from shirking to that in Corollary 1.

Corollary 2. *The future information rent from shirking becomes unbounded as the agent becomes arbitrarily patient:*

$$W(\mu) \geq \frac{Kc}{(1-\delta_A)}\frac{[\mu(\beta - \alpha) + 1 - \mu]}{(\beta - 1)}\xi(\mu).$$

8.1 Existence of stationary high effort incentive efficient contracts

A natural approach to obtaining existence of a well-defined value function is to find conditions under which Ψ is a contraction. Since Ψ is the point-wise maximum of Ψ^e (defined in (24)), Ψ^s (the analogous operator in which the agent shirks in the current period, corresponding to the second term in (21)), and the zero function, Ψ will be a contraction (under the sup norm) if Ψ^e and Ψ^s are (again, under the sup norm).

Suppose $V, \hat{V} \in \mathcal{V}$. Then,

$$|\Psi^e(V) - \Psi^e(\hat{V})|$$

$$\leq \sup_{\pi,\mu} \left| \frac{(\pi - \mu)(\beta - 1)}{\mu(\beta - \alpha) + (1 - \mu)} \right| \delta_A \left| \sum_k p_{s\pi}^k V(\psi_s^k(\pi), \psi_e^k(\mu)) - \hat{V}(\psi_s^k(\pi), \psi_e^k(\mu)) \right|$$

$$+ \delta_A \left| \sum_k p_{e\pi}^k V(\psi_e^k(\pi), \psi_e^k(\mu)) - \hat{V}(\psi_e^k(\pi), \psi_e^k(\mu)) \right|$$

$$\leq |V - \hat{V}| \times \left\{ \sup_{\pi,\mu} \left| \frac{(\pi - \mu)(\beta - 1)}{\mu(\beta - \alpha) + (1 - \mu)} \right| + 1 \right\} \delta_A.$$

This simple calculation shows that if Ψ^e is not a contraction, the failure arises from the future information rent from shirking (which contributes the sup term in the last expression. We also see that Ψ^e is a contraction if that sup term is sufficiently small (relative to $(1 - \delta_A)/\delta_A$). A similar calculation shows that Ψ^s is also a contraction if a similar sup term is sufficiently small (also relative to $(1 - \delta_A)/\delta_A$).[7]

A second approach to obtaining existence is to impose the same parameter restriction as in Proposition 3. In this case, we again have an exact expression for the value function (for essentially the same reason).

Lemma 6. *Suppose $\beta = \alpha + 1$ (as in Proposition 3). The mapping Ψ has as a fixed point the function*

$$V(\pi, \mu) = \frac{\alpha c(\pi - \mu)}{1 - \delta_A}, \tag{25}$$

and the associated stationary high effort incentive efficient contract is the unique cost minimizing vector of utilities satisfying (22) and (23).

Proof. We need only show that the function specified in (25) is a fixed point of Ψ. It is straightforward to verify that (25)) is a fixed point of

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[7]The sup in Ψ^e is being taken over $|(p_{e\pi}^H - p_{e\mu}^H)/(p_{e\pi}^H - p_{s\mu}^H)|$, while the sup in Ψ^s is being taken over $|(p_{s\pi}^H - p_{e\mu}^H)/(p_{e\pi}^H - p_{s\mu}^H)|$.

Ψ^e. Analogous calculations to those in Proposition 3 shows that $\Psi^e(V) - \Psi^s(V) = 0$ for V given by (25), and so (25) does indeed describe a fixed point of Ψ. □

9 The Cost of Inducing Effort

We have shown that the incentive cost of inducing effort increases at least linearly in the length of the relationship. If the principal is short-lived, this enough to show that eventually the first period principal will not induce effort in the first period. The case of a long-lived principal is more complicated, since such a principal benefits directly from a longer horizon, and may benefit from more informative signals.

We content ourselves with two simple observations: On one hand, if the agent's utility function is log, then the expected cost of inducing effort in the initial period is exponential in the length of the relationship: The expected wage cost from the spot contract $u_\mu(t)$ can be bounded below as

$$\begin{aligned}
p_{e\mu} \cdot w(u_\mu(t)) = \sum_k (p^k_{sB} + \mu(\beta - 1)\gamma^k) \exp(u^k_\mu(t)) \\
\geq \mu(\beta - 1) \sum_k \gamma^k \exp(u^k_\mu(t)) \\
\geq \mu(\beta - 1) \exp(\gamma \cdot u_\mu(t)) \\
\geq \mu(\beta - 1) \exp(c + W(\mu, t)).
\end{aligned}$$

On the other hand, there are utility functions for the agent for which the expected cost of inducing effort in the initial period is linear in the cost of length of the relationship. As an illustration, consider the utility function $\tilde{u}$ given by

$$\tilde{u}(w) = \begin{cases} aw, & w \geq 0, \\ bw, & w < 0, \end{cases}$$

with $0 < a < b$, and binary signals as in Figure 1. (Because of binary signals, $a < b$ is sufficient for our earlier analysis; for more signals, it should be enough to consider utility functions "close" to $\tilde{u}$, in particular, which converge to $\tilde{u}$ as $w \to \pm\infty$.) Under binary signals, $u_\mu(t) = (u^L_\mu(t), u^H_\mu(t))$, and with our parametric assumption,

$$\Delta u_\mu(t) := u^H_\mu(t) - u^L_\mu(t) = \frac{c + W(\mu, t)}{1 - r - q}.$$

The optimal contract is the pair $(u^L_\mu(t), u^H_\mu(t))$ solving

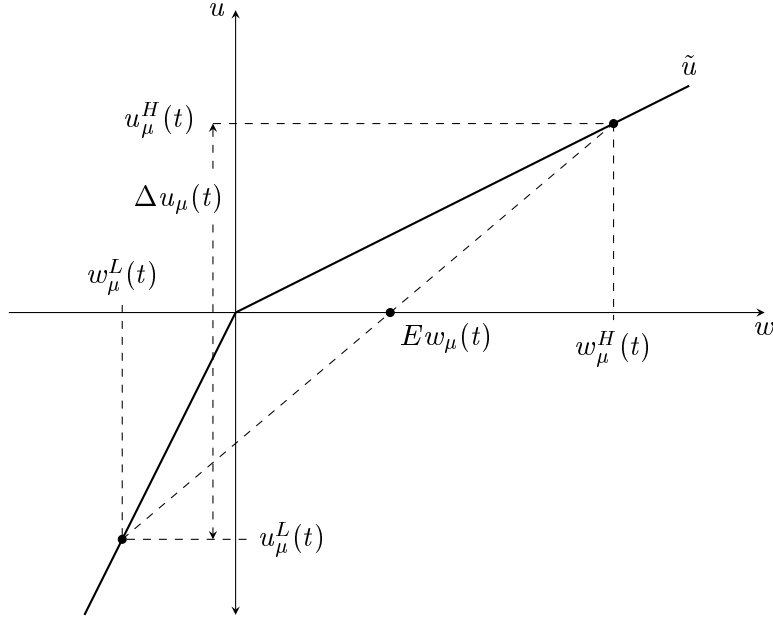$$u^H_\mu(t) = u^L_\mu(t) + \Delta u_\mu(t) \qquad \text{and}$$

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Figure 2: The optimal contract $u_\mu(t)$ is determined by $\Delta u_\mu(t)$ and the requirement that expected utility (under $p_{e\mu}$) is zero. The expected cost of the contract is then the corresponding value on the w-axis.

$$0 = p_{e\mu}^H u_\mu^H(t) + (1 - p_{e\mu}^H)u_\mu^L(t).$$

It is straightforward to verify that the expected cost of this contract is of the same order as $W(\mu, t)$, and so linear in t (see Figure 2; while $p_{e\mu}$ does vary as beliefs are updated, it is bounded by p_{eB} and p_{eG}).

A   Appendix

Lemma A.1. *Suppose μ, $\pi > \frac{1}{2}$. Then, there exists $\sigma \in (0,1)$ such that for all $\mu, \pi \geq \frac{1}{2}$ and for all $y^k \in Y^H$,*

$$\left| \psi_e^k(\pi) - \psi_e^k(\mu) \right| \leq \sigma \left| \pi - \mu \right|.$$

Proof. From some straightforward calculations, we have

$$\psi_e^k(\pi) - \psi_e^k(\mu) = \frac{\pi p_{eG}^k}{p_{e\pi}^k} - \frac{\mu p_{eG}^k}{p_{e\mu}^k}$$

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$$= \frac{(\pi - \mu)p_{eG}^k p_{eB}^k}{p_{e\pi}^k p_{e\mu}^k},$$

and so it remains to bound the ratio of probabilities.

Now, consider

$$f^k(\pi, \mu) := p_{e\pi}^k p_{e\mu}^k - p_{eG}^k p_{eB}^k$$
$$= \pi\mu(p_{eG}^k)^2 + (1 - \pi)(1 - \mu)(p_{eB}^k)^2$$
$$\quad - [\pi\mu + (1 - \pi)(1 - \mu)]p_{eG}^k p_{eB}^k.$$

This function is increasing in π and μ (since $y^k \in Y^H$), and so is minimized at $\pi = \mu = \frac{1}{2}$ over $\pi, \mu \geq \frac{1}{2}$. That is,

$$f^k(\pi, \mu) \geq \frac{1}{4}(p_{eG}^k - p_{eB}^k)^2 \quad \forall \pi, \mu \geq \frac{1}{2}.$$

Define

$$X := \min_{y^k \in Y^H} \frac{(p_{eG}^k - p_{eB}^k)^2}{4p_{eG}^k p_{eB}^k}$$

and set

$$\sigma = \frac{1}{1 + X} \in (0, 1). \tag{A.1}$$

Then,

$$p_{e\pi}^k p_{e\mu}^k - p_{eG}^k p_{eB}^k = f^k(\pi, \mu)$$
$$\geq X p_{eG}^k p_{eB}^k$$
$$= \left(\frac{1}{\sigma} - 1\right) p_{eG}^k p_{eB}^k,$$

and so

$$\frac{p_{eG}^k p_{eB}^k}{p_{e\pi}^k p_{e\mu}^k} \leq \sigma. \qquad \square$$

Proof of Proposition 4. For the purposes of this proof, it is more convenient to index periods forward rather than backward, so that h^τ is the τ length history leading to period τ, with $T - \tau$ periods remaining.

Given h^τ, let $n(h^\tau)$ denote the difference between the number of y^H and y^L realizations in h^τ. Then, since $p_{eB}^H = p_{eG}^L$, histories of different lengths lead to the same posterior as long as they agree in $n(h^\tau)$, i.e., for all h^τ and $\hat{h}^{\tau'}$, with τ possibly different from τ',

$$n(h^\tau) = n(\hat{h}^{\tau'}) \Rightarrow \psi_e(\mu, h^\tau) = \psi_e(\mu, \hat{h}^{\tau'}).$$

We proceed by conditioning on G (the unconditional expectation is then the average of the conditioning on G and the symmetric term from B). Moreover, for large t, conditional on G, the probability that $n(h^t)$ is negative goes to zero sufficiently fast, that it is enough to show that

$$\Pr\{n(h^\tau) \geq 0 \text{ for } \tau = 0, \ldots, t-1\} \times$$
$$E\left\{\left.\sum_{\tau=0}^{t-1}[\psi_e(\pi, h^\tau) - \psi_e(\mu, h^\tau)]\right| G, a^\tau = e, n(h^\tau) \geq 0\right\} \quad \text{(A.2)}$$

is bounded. Moreover, we can also assume $\mu > 1/2$, since conditional on G, the probability that $n(h^\tau)$ is small becomes arbitrarily small as t becomes large.

From Lemma A.1 (using the value of σ from (A.1)), we have that for $\sigma := 4r(1-r) \in (0,1)$, if $\pi, \mu > \frac{1}{2}$, then

$$\psi_e(\pi, n(h^\tau)) - \psi_e(\mu, n(h^\tau)) < \sigma^{n(h^\tau)}(\pi - \mu).$$

Then the expression in (A.2) is bounded above by

$$\sum_{\tau=0}^{t-1}\sum_{n=0}^{\tau}\Pr(n(h^\tau) = n)\sigma^n(\pi - \mu)$$
$$= (\pi - \mu)\sum_{n=0}^{t-1}\sigma^n\sum_{\tau=n}^{t-1}\Pr(n(h^\tau) = n)$$
$$\leq (\pi - \mu)\sum_{n=0}^{\infty}\sigma^n\sum_{\tau=n}^{\infty}\Pr(n(h^\tau) = n). \quad \text{(A.3)}$$

We first bound

$$\Pr(n(h^\tau) = n) = b((\tau + n)/2; \tau, p) = \binom{\tau}{(\tau + n)/2}r^{(\tau+n)/2}(1-r)^{(\tau-n)/2}.$$

Using Stirling's formula[8]

$$\sqrt{2\pi}\, m^{m+1/2}e^{-m} \leq m! \leq e\, m^{m+1/2}e^{-m} \text{ for all positive integers } m,$$

we bound the binomial coefficients as follows

$$\binom{\tau}{(\tau + n)/2} = \frac{\tau!}{\frac{(\tau+n)}{2}!\frac{(\tau-n)}{2}!}$$

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[8]See, for example, Abramowitz and Stegun (1972, 6.1.38).

$$\leq \frac{e\ \tau^{\tau+\frac{1}{2}}e^{-\tau}}{2\pi\left[\frac{(\tau+n)}{2}\right]^{\frac{(\tau+n)}{2}+\frac{1}{2}}e^{-\frac{(\tau+n)}{2}}\left[\frac{(\tau-n)}{2}\right]^{\frac{(\tau-n)}{2}+\frac{1}{2}}e^{-\frac{(\tau-n)}{2}}}$$

$$\leq \frac{\tau^{\tau+\frac{1}{2}}}{\sqrt{2}\left[\frac{(\tau+n)}{2}\right]^{\frac{(\tau+n)}{2}+\frac{1}{2}}\left[\frac{(\tau-n)}{2}\right]^{\frac{(\tau-n)}{2}+\frac{1}{2}}}$$

$$= \frac{(2\tau)^{\tau+\frac{1}{2}}}{(\tau^2-n^2)^{\frac{\tau}{2}+\frac{1}{2}}}\times\left(\frac{\tau-n}{\tau+n}\right)^{n/2}$$

$$\leq \frac{(2\tau)^{\tau+\frac{1}{2}}}{(\tau^2-n^2)^{\frac{\tau}{2}+\frac{1}{2}}}$$

$$\leq 2^{\tau+\frac{1}{2}}\left(\frac{\tau^2}{\tau^2-n^2}\right)^{\frac{\tau}{2}+\frac{1}{4}}.$$

We also need the following calculation. Setting $k := \sqrt{(1+\sigma)/(1-\sigma)}$, gives for all $\tau > kn$,

$$\frac{\tau^2}{\tau^2-n^2}\sigma < \frac{k^2n^2}{k^2n^2-n^2}\sigma$$

$$= \frac{k^2}{k^2-1}\sigma$$

$$= \frac{1+\sigma}{2\sigma}\sigma = \frac{1+\sigma}{2} =: y < 1,$$

where the final inequality holds because $\sigma < 1$.

We are now in a position to bound (A.3), since

$$\sum_{n=0}^{\infty}\sigma^n\sum_{\tau=n}^{\infty}\Pr(n(h^\tau)=n) = \sum_{n=0}^{\infty}\sigma^n\sum_{\tau=n}^{kn}\Pr(n(h^\tau)=n)$$

$$+ \sum_{n=0}^{\infty}\left(\sigma^2\frac{r}{1-r}\right)^{n/2}\sum_{\tau=kn+1}^{\infty}\binom{\tau}{(\tau+n)/2}[r(1-r)]^{\tau/2}$$

$$\leq \sum_{n=0}^{\infty}\sigma^n(k-1)n$$

$$+ \sqrt{2}\sum_{n=0}^{\infty}\left(\sigma^2\frac{r}{1-r}\right)^{n/2}\sum_{\tau=kn+1}^{\infty}\left(\frac{\tau^2}{\tau^2-n^2}\right)^{\frac{\tau}{2}+\frac{1}{4}}[4r(1-r)]^{\tau/2}$$

$$\leq \sum_{n=0}^{\infty}\sigma^n(k-1)n$$

$$+ \sqrt{2} \sum_{n=0}^{\infty} \left(\sigma^2 \frac{r}{1-r} \right)^{n/2} \sum_{\tau=kn+1}^{\infty} \left(\frac{\tau^2}{\tau^2 - n^2} \right)^{\frac{1}{4}} y^{\tau/2}.$$

Since $\sigma < 1$ and $y < 1$, this expression is bounded if

$$1 > \sigma^2 \frac{r}{1-r} = 16r^2(1-r)^2 \frac{r}{1-r} = 16r^3(1-r).$$

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