Scoring auctions with non-quasilinear scoring rules*

Krishnendu Ghosh Dastidar
Centre for Economic Studies and Planning,
School of Social Sciences,
Jawaharlal Nehru University,
New Delhi-110067, INDIA.
email: krishnendu.dastidar@gmail.com, kgd12@yahoo.com

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Abstract

In this paper we analyse scoring auctions with general non-quasilinear scoring rules. We assume that cost function of each firm is additively separable in quality and type. In sharp contrast to the recent results in the literature we show the following. (i) Equilibria in scoring auctions can be computed without any endogeneity problems and we get explicit solutions. (ii) We provide a complete characterisation of such equilibria and compare quality, price and expected scores across first-score and second-score auctions. (iii) We show that expected score will be higher in second-score auctions for most scoring rules, provided a restriction on the distribution function of types is satisfied.

JEL Classification: D44, H57, L13

1 Introduction

Most papers on scoring auctions, except a very few recent ones, have used quasilinear scoring rules. In this paper we allow for general non-quasilinear scoring rules and take a step forward. The reasons behind this exercise are twofold: (i) equilibrium properties of scoring auctions with general non-quasilinear scoring rules have not been fully worked out, and (ii) non-quasilinear scoring rules are often used in real life.

In the modern world, auctions are used to conduct a huge volume of economic transactions. Governments use them to sell treasury bills, foreign exchange, mineral rights

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including oil fields, and other assets such as firms to be privatized. Government contracts are typically awarded by procurement auctions, which are also often used by firms subcontracting work or buying services and raw materials. Government procurement expenditure, on an average, constitute about thirteen percent of the GDP (OECD, 2013). Clearly, public procurements constitute a significant part of the economic activities in many countries.[1]

The theory of auctions provides the necessary analytical framework to study such procurements. In the canonical model there is one indivisible object up for sale and there are some potential bidders. In any standard auction the object is sold to the highest bidder. In a procurement auction, where the auctioneer is the buyer, the object is sold to the lowest bidder. The payment by each bidder depends on the type of auction used by the seller. There is a huge literature around this model.[2]

It may be noted that the benchmark model of auctions is really a *price-only* auctions. For example, in the traditional theory of standard procurement auctions, the auctioneer cares only about the price of the object, but not the other attributes. However, in many procurement situations, *the buyer cares about attributes other than price* when evaluating the offers submitted by suppliers. Non-monetary attributes that buyers care about include quality, time to completion etc. For example, in the contract for the construction of a new aircraft, the specification of its characteristics is probably as important as its price. Under these circumstances, auctions are usually multidimensional. The essential element of such multi-dimensional auctions is a *scoring rule*. In a *scoring auction*, the bidders are asked to submit multidimensional bids that include price and some non-price attributes, such as quality. The bids are then transformed into a score by an ex ante publicly announced scoring rule, and the bidder whose score is the highest is awarded the contract.

1.1 Scoring auction: the baseline model

A more formal description of the above scenario is as follows. First, the buyer announces how offers will be evaluated. The scoring rule is given by $S(p,q)$ where p is the price and q is the quality. The score, $S(p,q)$, is increasing in quality and decreasing in price. The suppliers submit (p,q) pairs. The buyer awards the contract to a firm whose offer achieves the highest score. The cost to the supplier is $C(q,\theta)$ where θ is the type. Types are assumed to be identically and independently distributed. It is typically assumed that $C_q > 0$ and $C_\theta > 0$. The winner's payoff is $p - C(q,\theta)$. Losers earn zero. In a **first-score** auction the winning firm's offer is finalized as the contract. This auction rule is a two-dimensional analogue of the first price auction. In a **second-score** auction the winning firm is required to match the highest rejected score. In meeting this score, the firm is free to choose any quality-price combination.[3]

A scoring rule, $S(p,q)$ is quasilinear if it can be expressed as $\phi(q) - p$. For **non-quasilinear** rules we must have at least one of the following: $S_{pp} \neq 0$ or $S_{pq} \neq 0$.

[1] See Koning and van de Meerendonk (2014).

[2] See Krishna (2010) for all the standard results around the benchmark model.

[3] We provide the following example to illustrate the above two auctions. Let the scoring rule be $S(p,q) = 2q - p$. Suppose two firms A and B offer $(5,7)$ and $(3,5)$ as their (p,q) pairs. We have $S(5,7) = 9$ and $S(3,5) = 7$. Under both auction formats (first-score and second-score) firm A is declared the winner. The final contract awarded to firm A is $(5,7)$ under the first-score auction and any (p,q) satisfying $S(p,q) = 7$ under the second-score auction.

1.2 Examples of scoring auctions

We now provide some examples of both quasilinear and non-quasilinear scoring rules that are used in real life.

Quasilinear scoring rule: The Department of Defence in USA often relies on competitive source selection to procure weapon systems. Each individual component of a bid of the weapon system is evaluated and assigned a score, these scores are summed to yield a total score, and the firm achieving the highest score wins the contract.[4]

Non-quasilinear scoring rule: For highway construction projects, states like Alaska, Colorado, Florida, Michigan, North Carolina, and South Dakota use quality-over-price ratio rules, in which the score is computed based on the quality divided by price (i.e. $S(p, q) = \frac{q}{p}$). This scoring rule is also extensively used in Japan. Ministry of Land, Infrastructure and Transportation in Japan allocates most of the public construction project contracts through scoring auctions based on quality-over-price ratio rules.[5]

We now proceed to provide a brief literature review.

1.3 Relevant Literature

Che (1993) is a pioneer in analysing such scoring auctions. In his model both the quality and the bidder's types are single-dimensional, and the scoring rule is quasilinear. Che (1993) computes equilibria in first-score and second-score auctions and also analyses optimal mechanisms when types are identically and independently distributed. Branco (1997) analyses the properties of optimal mechanisms when types are single-dimensional but correlated.

The paper by Asker and Cantillon (2008) deals with multidimensional types in a scoring auction. This paper defines a 'pseudotype' and shows that if the scoring rule is quasilinear and types are independently distributed then every equilibrium in the scoring auction is typewise outcome equivalent to an equilibrium in the scoring auction where suppliers are constrained to bid only on the basis of their pseudotypes.

Asker and Cantillon (2010) analyses optimal mechanisms with one-dimensional quality and two-dimensional discrete types. Nishimura (2015) computes optimal mechanisms with multidimensional quality and single-dimensional types that are identically and independently distributed.

In may be noted that in **all** the above papers the scoring rule in **quasilinear**.

Very few papers in the literature have dealt with non-quasilinear scoring rules. This is surprising given the fact that such rules are often used by public authorities in many countries. Hanazono, Nakabayashi and Tsuruoka (2015) is the only paper till date that analyses general non-quasilinear scoring rules. This paper considers a broad class of scoring rules and computes equilibria for first-score and second-score auctions and compares

[4] Examples of other scoring auctions include "A+B bidding" for highway construction work in the United States, where the highway procurement authorities evaluate offers on the basis of their costs as well as time to completion, weighted by a road user cost (see Asker and Cantillon, 2008 and Che, 1993 for other examples).

[5] This scoring rule (quality over price ratio) is also used in Australia. In addition, some governments in EU countries use the scoring auction in which the score is the sum of the price and quality measurements but the score is nonlinear in the price bid (see Nakabayashi and Hirose, 2014 for other details).

expected scores. Hanazono (2010) provides an example with a specific non-quasilinear scoring rule and a specific cost function.[6] Wang and Liu (2014) analyse equilibrium in first-score auctions with another specific non-quasilinear scoring rule.[7]

However, it may be noted that in the above mentioned papers that analyse non-quasilinear scoring rules, explicit solutions for the equilibrium strategies are not always obtained. For example, in Hanazono, Nakabayashi and Tsuruoka (2015) the choice of 'quality' in equilibrium is endogenous in the 'score' under the general scoring function. Moreover, the comparison of expected scores is based on properties of induced utility whose arguments are implicitly defined.[8]

1.4 Contributions of this paper

In this paper we ask the following questions.

(i) Can we get explicit solutions for equilibrium strategies with general non-quasilinear scoring rules? (ii) Can we provide a complete characterisation of such equilibria? (iii) Also, can we get a clear ranking of the expected scores in first-score and second-score auctions?

We show that all of the above can be achieved if the cost function of each firm is additively separable in quality and type (i.e. $C(q,\theta) = c(q) + \theta$). Our approach helps in dealing with most non-quasilinear scoring rules. Our main results are as follows.

1. We first provide explicit solutions for equilibrium bidding strategies in first-score and second-score auctions (propositions 1 and 2). Our computations provide a much simpler way to derive equilibria in scoring auctions without any endogeneity problems. We also provide a couple of examples to illustrate our point.

2. Next, we provide a complete characterisation of such equilibria.

   (a) We first show that the score quoted by any type in equilibrium is strictly higher in the second-score auction as compared to the score quoted in first score-auction (proposition 3). This is analogous to the standard benchmark model where for any particular type, the bid in the second-price auction is always higher than the bid in the first-price auction.[9]

   (b) We also demonstrate that the equilibrium scores are decreasing in type, θ. This means the winner in any auction is the firm with the lowest type (least cost). That is, the symmetric equilibria are always efficient.

   (c) Thereafter, we provide sufficient conditions under which the equilibrium quality/price quoted by any type in first-score auction is higher (or lower) than the quality/price quoted in a second-score auction (propositions 4 and 5).

[6] Hanazono (2010) uses the quality-to-price scoring rule. That is, here $S(p,q) = \frac{q}{p}$. It may be noted that this short note is written in Japanese. I am grateful to Masaki Aoyagi for helping me understand the results of this paper.

[7] In Wang and Liu (2014) the scoring rule is as follows: $S(p,q) = \omega_1 \frac{\bar{p}}{p} + \omega_2 \frac{q}{\underline{q}}$, where weights ω_1, ω_2 satisfy $\omega_1 + \omega_2 = 1$, $\bar{p}$ is the highest acceptable bidding price and $\underline{q}$ is the lowest acceptable quality.

[8] This paper avoids specific functional forms but instead imposes some restrictions on the induced utility.

[9] In a second-price auction of the canonical model bidders bid their valuations. In a first-price auction bids are strictly less than valuations.

3. Next, we discuss the impact of increase in the number of bidders on equilibrium configurations in both auctions.

    (a) We demonstrate that the quality/price/score quoted by any type in equilibrium of a second-score auction is invariant with respect to the number of bidders (proposition 6). This is similar to the second-price auction in the benchmark model, where, regardless of the number of bidders, all bidders bid their valuations.

    (b) However, the quality and price quoted by any type in equilibrium of a first-score auction depend on the number of bidders. Consequently, the equilibrium score quoted in a first-score auction depends on the number of bidders. We first identify sufficient conditions under which quality/price quoted in a first-score auction increase (or decrease) with an increase in the number of bidders (proposition 7).

    (c) Thereafter, in proposition 8 we show that the score quoted by any type in a first-score auction always increases as the competition intensifies (the number of bidders increases). This is because any increase in competition induces a bidder to quote a higher score. This is similar to the first-price auction in the canonical model where bids increase with the number of bidders.

4. Lastly, we compare expected scores in first-score and second-score auction with non-quasilinear scoring rules. Let Σ^I be the expected score in a first-score auction and Σ^{II} be the expected score in a second-score auction. It is well known that when the scoring rule is quasilinear and types are identically and independently distributed then $\Sigma^I = \Sigma^{II}$. This result on expected score equivalence is the analogue of revenue equivalence theorem of the canonical model.

    (a) We first identify sufficient conditions under which we get $\Sigma^I = \Sigma^{II}$ even with non-quasilinear scoring rules (proposition 10).

    (b) Finally, we provide our main result on expected scores (proposition 11). *With mild restrictions on the scoring rules and distribution function of types we show that* $\Sigma^I < \Sigma^{II}$. This has interesting policy implications as well. In real life second-score auctions are never used. Our result suggests that in many cases an auctioneer will be better off using second-score auctions than using first-score auctions. Proposition 11 is also interesting as it emphasises the need to put restrictions on the distribution function of types to get a ranking of expected scores. This stands in sharp contrast to the other papers in the literature. We also illustrate propositions 10 and 11 with numerical examples.

Plan of the paper In section 2 we provide the model of our exercise. In section 3 we compute the equilibria for first-score and second-score auctions. Section 4 provides the equilibrium characterisations. In the first part of section 4 we discuss the properties of score/price/quality quoted in equilibrium of the two auction formats. In the second part of this section we discuss the effects of an increase in the number of bidders. In section 5 we give the main results on the comparison of expected scores. Lastly, we provide some concluding remarks and possible scope for future research in this area. All proofs are provided in the appendix.

We now proceed to provide the model of our exercise.

2 The Model

A buyer solicits bids from n firms. Each bid, (p, q), specifies an offer of promised quality, q and price, p, at which a fixed quantity of products with the offered level of quality q is delivered. The quantity is normalized to one. For simplicity quality is modelled as a one-dimensional attribute.

A scoring rule is a function $S : \mathbb{R}^2_{++} \longrightarrow \mathbb{R} : (p, q) \longrightarrow S(p, q)$ that associates a score to any potential contract and represents a continuous preference relation over contract characteristics (p, q).

Assumption 1 $S(.)$ is strictly decreasing in p and strictly increasing in q. That is, $S_p < 0$ and $S_q > 0$. We assume that the partial derivatives S_p, S_q, S_{pp}, S_{pq}, S_{qq} exist and they are continuous in all $(p, q) \in \mathbb{R}^2_{++}$.

As noted before, a scoring rule is **quasilinear** if it can be expressed as $\phi(q) - p$. For quasilinear rules we must have $S_{pp} = 0$ and $S_{pq} = 0$. For **non-quasi-linear** rules we must have at least one of the following: $S_{pp} \neq 0$ or $S_{pq} \neq 0$.

The cost to the supplier is $C(q, x)$ where x is the type.

Assumption 2 We assume $C_q > 0$, $C_{qq} \geq 0$ and $C_x > 0$.

Prior to bidding each firm i learns its cost parameter x_i as private information. The buyer and *other* firms (i.e. other than firm i) do not observe x_i but only knows the distribution function of the cost parameter. It is assumed that x_is are identically and independently distributed over $[\underline{x}, \bar{x}]$ where $0 \leq \underline{x} < \bar{x}$.

If supplier i wins the contract, its payoff is $p - C(q, x_i)$. Losers earn zero.

We now provide our most important assumption which separates our paper from the rest of the papers of this genre.

Assumption 3 *Cost is additively separable in quality and type.*

That is, $C(q, x) = c(q) + \alpha(x)$ where $c'(.) > 0, c''(.) \geq 0, \alpha(\underline{x}) \geq 0$ and $\alpha'(.) > 0$.

Define $\theta_i = \alpha(x_i)$. Let $\underline{\theta} = \alpha(\underline{x})$ and let $\bar{\theta} = \alpha(\bar{x})$. Clearly, $0 \leq \underline{\theta} < \bar{\theta}$. Since x_is are identically and independently distributed over $[\underline{x}, \bar{x}]$, so are the θ_is over $\left[\underline{\theta}, \bar{\theta}\right]$. Let the distribution function of θ_i be $F(.)$ and the density function be $f(.)$. Note that $f(\theta) \geq 0$ $\forall \theta \in \left[\underline{\theta}, \bar{\theta}\right]$.

We can now write the cost for supplier i as

$$C(q, \theta_i) = c(q) + \theta_i,$$

where θ_i is the type of supplier i.

We also assume the following.

Assumption 4

$$-\frac{(S_q)^2}{S_p}S_{pp} + 2S_qS_{pq} - S_pS_{qq} - (S_p)^2 c'' < 0 \text{ for all } (p,q) \in \mathbb{R}^2_{++}$$

This assumption ensures that the second order condition for payoff maximisation is satisfied during equilibrium computations. It may also be noted that when $c''(.) > 0$ then both for the quasilinear rule $(S(p,q) = \phi(q) - p)$ and the quality-to-price ratio $\left(S(p,q) = \frac{q}{p}\right)$ (which is a non-quasilinear rule) the above is always satisfied.

The following may be noted.

1. The assumption (cost is additively separable in quality and type) is consistent with the set of assumptions in Hanazono et al (2015) and Asker and Cantillon (2008).

2. Additive separability implies $C_{q\theta}(.) = 0$. This is different from Che (1993), Branco (1997) and Nishimura (2015).[10]

3. Our cost, $C(q,\theta_i) = c(q) + \theta_i$, can be interpreted in the following way. $c(q)$ is the variable cost and θ_i is the fixed cost of firm i. This means, the variable costs are same across firms but the fixed costs are private information. θ_i can be interpreted to be the inverse of managerial efficiency which is private information to the firm. Higher is θ_i, lower is the managerial efficiency, and consequently, higher will be the cost.

3 Equilibrium in first-score and second-score auctions

We now provide the equilibrium for first-score and second-score auctions. The proofs are given in the appendix.

Proposition 1 In a *first-score* auction there is a symmetric equilibrium where a supplier with type θ chooses $\left(p^I(\theta), q^I(\theta)\right)$. Such $p^I(.)$ and $q^I(.)$ are obtained by solving the following equations:

$$\begin{aligned} -\frac{S_q(.)}{S_p(.)} &= c'(.) \\ p - c(q) &= \theta + \gamma(\theta) \end{aligned}$$

where

$$\gamma(\theta) = \frac{1}{(1-F(\theta))^{n-1}} \int_{\theta}^{\bar{\theta}} (1-F(t))^{n-1}\, dt$$

[10] In Che (1993) we have $C_{q\theta}(.) > 0$ and in Branco (1997) we have $C_{q\theta} < 0$. In Nishimura (2015) C_θ has strictly increasing differences in (q,θ).

Proposition 2 In a *second-score* auction there is a weakly dominant strategy equilibrium where a supplier with type θ chooses $\left(p^{II}(\theta), q^{II}(\theta)\right)$. Such $p^{II}(.)$ and $q^{II}(.)$ are obtained by solving the following equations:

$$\begin{aligned} -\frac{S_q(.)}{S_p(.)} &= c'(.) \\ p - c(q) &= \theta \end{aligned}$$

Comment In the appendix we provide a proof of the above two propositions. Here we provide a brief sketch of the argument.

First, consider proposition 1. For any quality, q, let $\Psi(s,q)$ be the price required to generate a score of s. That is, $S(\Psi(s,q), q) = s$. Clearly, $\Psi(.)$ is well defined and it is strictly decreasing in s and strictly increasing in q.

Consider any symmetric equilibrium of first-score auction where a bidder with type θ bids (p,q). Let the score generated by such a bid be $S(p,q) = s$. Since $\Psi(.)$ is well defined and is strictly decreasing in s, we can think of the equilibrium as where a bidder bids a score s and quality q. The payoff (conditional on winning) with a score s to a bidder with type θ is

$$\Psi(s,q) - c(q) - \theta.$$

In any equilibrium, for any type θ, the quality choice, q, must be such so as to maximise $\Psi(s,q) - c(q) - \theta$. The FOC and SOC for such a maximisation are as follows:

$$\begin{aligned} \Psi_q(.) - c'(.) &= 0 - - - - (1a) \\ \Psi_{qq}(.) - c''(.) &< 0 - - - - (1b) \end{aligned}$$

Note that

$$\Psi_{qq}(.) - c''(.) < 0 \Longleftrightarrow -\frac{(S_q)^2}{S_p} S_{pp} + 2 S_q S_{pq} - S_p S_{qq} - (S_p)^2 c'' < 0$$

Given our assumption 4, the SOC (which is (1b)) will always be satisfied.

Note that we have the following[11]:

$$\Psi_q(.) = -\frac{S_q(.)}{S_p(.)} \text{ and } \Psi_s(.) = \frac{1}{S_p(.)} - - - - (2)$$

Hence we can rewrite (1a) as follows:

$$-\frac{S_q(.)}{S_p(.)} = c'(.) - - - - (3)$$

Consequently, in any equilibrium (3) will be satisfied. Now let's suppose that all firms other than firm i choose (p,q) according to the equations in proposition 1 (i.e. $-\frac{S_q(.)}{S_p(.)} = c'(.)$ and $p - c(q) = \theta + \gamma(\theta)$). Thereafter, using standard auction theoretic techniques we can show that it is optimal for firm i to choose (p,q) by following the same equations.

[11] From $S(p,q) - s = 0$ we can implicitly solve for p and then use the implicit function theorem.

Now consider the case of second-score auction (proposition 2). What matters to any firm i is the maximum of scores quoted by *other* firms[12]. Let the maximum of the scores quoted by firms other than i be δ. Now let firm i choose $\left(p^{II}, q^{II}\right)$ by following the two equations in proposition 2 (i.e. $-\frac{S_q(.)}{S_p(.)} = c'(.)$ and $p - c(q) = \theta$) and thereby pick up a score $s = S\left(p^{II}, q^{II}\right)$. Using standard techniques it can be shown that regardless of δ, it is always better for firm i to choose $\left(p^{II}, q^{II}\right)$ by following these two equations.

A couple of observations can be made.

1. In our model, the cost function is additively separable in quality and type and we get explicit solutions for equilibrium strategies for both kinds of scoring rules: quasilinear and non-quasilinear. Additive separability of the cost function makes equilibrium computations very simple. This stands in sharp contrast to the recent papers that deal with non-quasilinear scoring rules.

2. When the scoring rule is quasilinear, $S_p(.)$ is a constant and S_q is independent of p (since $S_{pp} = S_{qp} = 0$). Note that in any auction the equation $-\frac{S_q(.)}{S_p(.)} = c'(.)$ is satisfied. This means the quality, q, is constant and same for the two auctions.

We illustrate the above two propositions in two examples given below.

Example 1 (non-quasilinear scoring rule) Let $S(p,q) = \frac{q}{p}$ and $C(q,\theta) = \frac{1}{2}q^2 + \theta$. Let θ be uniformly distributed over $[1,2]$ and $n = 2$.

In a first-score auction the symmetric equilibrium is as follows.

$$p^I(\theta) = 2 + \theta, \quad q^I(\theta) = \sqrt{2+\theta} \; \forall \theta \in [1,2].$$

In a second-score auction the symmetric equilibrium is as follows.

$$p^{II}(\theta) = 2\theta, \; q^{II}(\theta) = \sqrt{2\theta} \; \forall \theta \in [1,2].$$

Example 2 (quasilinear scoring rule) Let $S(p,q) = q - p$ and $C(q,\theta) = \frac{1}{2}q^2 + \theta$. Let θ be uniformly distributed over $[1,2]$ and $n = 2$.

In a first-score auction the symmetric equilibrium is as follows.

$$p^I(\theta) = \frac{3}{2} + \frac{1}{2}\theta, \quad q^I(\theta) = 1 \; \forall \theta \in [1,2].$$

In a second-score auction the symmetric equilibrium is as follows.

$$p^{II}(\theta) = \frac{1}{2} + \theta, \; q^{II}(\theta) = 1 \; \forall \theta \in [1,2].$$

[12] Note that in a second-score auction the winning firm is required to match the highest rejected score. In meeting this score, the firm is free to choose any quality-price combination.

4 Equilibrium Characterisation

We now provide some properties of the symmetric equilibria that were derived in the previous section. All proofs are given in the appendix. First, we define the following:

$$
\begin{aligned}
A(p,q) &= -\frac{S_q(p,q)}{S_p(p,q)} S_{pp}(p,q) + S_{qp}(p,q) \\
B(p,q) &= -\frac{S_q(p,q)}{S_p(p,q)} S_{pq}(p,q) + S_p(p,q)\, c''(q) + S_{qq}(p,q)
\end{aligned}
$$

4.1 Equilibrium score, quality and price

Lemma 1 $p^I(\bar{\theta}) = p^{II}(\bar{\theta})$ and $q^I(\bar{\theta}) = q^{II}(\bar{\theta})$.

Comment A firm with the highest type $(\bar{\theta})$ quotes the same price and quality across first-score and second-score auctions (lemma 1). This is true regardless of the fact whether the scoring rule is quasilinear or not.

We now proceed to consider scoring rules that are non-quasilinear. Note that for such rules we must have at least one of the following: $S_{pp} \neq 0$, $S_{pq} \neq 0$.

The next proposition compares the equilibrium scores quoted first-score and second-score auctions. Let $S^I(\theta) = S\left(p^I(\theta), q^I(\theta)\right)$ and $S^{II}(\theta) = S^{II}\left(p^{II}(\theta), q^{II}(\theta)\right)$. In the first-score and second-score auctions the equilibrium scores quoted by a firm with type θ is $S^I(\theta)$ and $S^{II}(\theta)$ respectively.

Proposition 3 If $A(p,q) \neq 0 \ \ \forall (p,q) \in \mathbb{R}^2_{++}$ then $S^I(\theta) < S^{II}(\theta) \ \forall \theta \in \left[\underline{\theta}, \bar{\theta}\right)$. Also, $\frac{d}{d\theta} S^I(\theta), \frac{d}{d\theta} S^{II}(\theta) < 0 \ \forall \theta \in \left(\underline{\theta}, \bar{\theta}\right)$.

Comment The equilibrium score quoted by any type $\theta \in \left[\underline{\theta}, \bar{\theta}\right)$ is strictly higher in the second-score auction as compared to the equilibrium score in first score-auction. This is analogous to the standard benchmark model where for any particular type, the bid in a second-price auction is always higher than the bid in a first-price auction. Proposition 3 also shows that equilibrium scores are decreasing in type, θ. This means the winner in any auction is the firm with the lowest type (least cost). That is, the symmetric equilibria are always efficient.

Proposition 3 requires $A(p,q) \neq 0$. For quasilinear scoring rules we have $S_{pp} = S_{qp} = 0$. This means that for quasilinear scoring rules $A(p,q) = 0$. It may be noted that the result, $S^I(\theta) < S^{II}(\theta)$, also holds for quasilinear scoring rules. The reason is as follows. In a first-score auction (or second-score auction) the price and quality in equilibrium will satisfy the equation, $p - c(q) = \theta + \gamma(\theta)$ (or $p - c(q) = \theta$). We have earlier noted that if the scoring rule is quasilinear then quality quoted in equilibrium is constant and same for the two auctions. This means price quoted in a first-score auction will be higher than the price quoted in a second-score auction. Since $S_p < 0$, the score quoted in a first-score auction will be lower than the score quoted in a second-score auction when the scoring rule in quasilinear.[13]

[13] It is possible to have $A(p,q) = 0 \ \ \forall (p,q) \in \mathbb{R}^2_{++}$ even with non-quasilinear rules (for example, take $S(p,q) = e^{q-p}$). Proposition 4 shows that when $A(p,q) = 0$ then $q^I(\theta) = q^{II}(\theta)$. In this case also it can be shown that $S^I(\theta) < S^{II}(\theta) \ \forall \theta \in \left[\underline{\theta}, \bar{\theta}\right)$.

In proposition 4 below we show that whether the quality quoted by any type $\theta \in \left[\underline{\theta}, \bar{\theta}\right)$ in first-score auction is higher (or lower) than the quality quoted in second-score auction depends crucially on the sign of the term $A\left(p, q\right)$. In fact, this term plays a crucial role in determining whether the equilibrium quality quoted in any auction is increasing in θ or not. Proposition 5 shows that comparison of price quoted by any type $\theta \in \left[\underline{\theta}, \bar{\theta}\right)$ in first-score auction with the one quoted in second-score auction depends crucially on the sign of the term $B\left(p, q\right)$. This term also determines whether the equilibrium price quoted in any auction is increasing in θ or not.

Proposition 4 (i) If $A\left(p, q\right) > 0 \;\; \forall \left(p, q\right) \in \mathbb{R}^2_{++}$ then $q^I\left(\theta\right) > q^{II}\left(\theta\right) \forall \theta \in \left[\underline{\theta}, \bar{\theta}\right)$. Also, $\frac{dq^I(\theta)}{d\theta}, \frac{dq^{II}(\theta)}{d\theta} > 0 \;\; \forall \theta \in \left(\underline{\theta}, \bar{\theta}\right)$.

(ii) If $A\left(p, q\right) < 0 \; \forall \left(p, q\right) \in \mathbb{R}^2_{++}$ then $q^I\left(\theta\right) < q^{II}\left(\theta\right) \; \forall \theta \in \left[\underline{\theta}, \bar{\theta}\right)$. Also, $\frac{dq^I(\theta)}{d\theta}, \frac{dq^{II}(\theta)}{d\theta} < 0 \;\; \forall \theta \in \left(\underline{\theta}, \bar{\theta}\right)$.

(iii) If $A\left(p, q\right) = 0 \;\; \forall \left(p, q\right) \in \mathbb{R}^2_{++}$ then $q^I\left(\theta\right) = q^{II}\left(\theta\right) \;\; \forall \theta \in \left[\underline{\theta}, \bar{\theta}\right)$. Also, $\frac{dq^I(\theta)}{d\theta}, \frac{dq^{II}(\theta)}{d\theta} = 0 \;\; \forall \theta \in \left(\underline{\theta}, \bar{\theta}\right)$.

Proposition 5 Suppose $A\left(p, q\right) \neq 0 \; \forall \left(p, q\right) \in \mathbb{R}^2_{++}$.

(i) If $B\left(p, q\right) < 0 \, \forall \left(p, q\right) \in \mathbb{R}^2_{++}$ then $p^I\left(\theta\right) > p^{II}\left(\theta\right) \; \forall \theta \in \left[\underline{\theta}, \bar{\theta}\right)$. Also, $\frac{dp^I(\theta)}{d\theta}, \; \frac{dp^{II}(\theta)}{d\theta} > 0 \; \forall \theta \in \left(\underline{\theta}, \bar{\theta}\right)$.

(ii) If $B\left(p, q\right) > 0 \, \forall \left(p, q\right) \in \mathbb{R}^2_{++}$ then $p^I\left(\theta\right) < p^{II}\left(\theta\right) \; \forall \theta \in \left[\underline{\theta}, \bar{\theta}\right)$. Also, $\frac{dp^I(\theta)}{d\theta}, \; \frac{dp^{II}(\theta)}{d\theta} < 0 \; \forall \theta \in \left(\underline{\theta}, \bar{\theta}\right)$.

(iii) If $B\left(p, q\right) = 0 \, \forall \left(p, q\right) \in \mathbb{R}^2_{++}$ then $p^I\left(\theta\right) = p^{II}\left(\theta\right) \; \forall \theta \in \left[\underline{\theta}, \bar{\theta}\right]$. Also, $\frac{dp^I(\theta)}{d\theta}, \; \frac{dp^{II}(\theta)}{d\theta} = 0 \; \forall \theta \in \left(\underline{\theta}, \bar{\theta}\right)$.

Discussions We now try to provide a discussion of the above results. As mentioned before, for any quality, q, $\Psi\left(s, q\right)$ is the price required to generate a score of s. That is, $S\left(\Psi\left(s, q\right), q\right) = s$. Routine computation show that $\Psi_{qs} = -\frac{A(.)}{(S_p)^2}$. This implies that Ψ_{qs} has the opposite sign of $A\left(.\right)$. We know that in equilibrium for both auctions $\Psi_q\left(s, q\right) = c'\left(q\right)$. From this equation we can derive that $\frac{dq}{ds} = -\frac{\Psi_{qs}}{\Psi_{qq} - c''}$. Since $\Psi_{qq} - c'' < 0$, $\frac{dq}{ds}$ has the same sign as Ψ_{qs}. This means that $\frac{dq}{ds}$ has the opposite sign of $A\left(.\right)$ in both auctions. When $A\left(.\right) > 0$ then $\frac{dq}{ds} < 0$. Since $S^I\left(.\right) < S^{II}\left(.\right)$ (see proposition 3) we must have $q^I\left(.\right) > q^{II}\left(.\right)$. Again, when $A\left(.\right) < 0$ then $\frac{dq}{ds} > 0$. Using a similar logic we must have $q^I\left(.\right) < q^{II}\left(.\right)$.

We now try to provide an alternative interpretation of our results. Note that $\epsilon^p_p = \frac{\partial S_p}{\partial p} \frac{p}{S_p} = \frac{S_{pp} p}{S_p}$ is the price elasticity of S_p. Similarly, $\epsilon^q_p = \frac{\partial S_q}{\partial p} \frac{p}{S_q} = \frac{S_{pq} p}{S_q}$ is the price elasticity of S_q. Now

$$A = -\frac{S_q}{S_p} S_{pp} + S_{qp} = \frac{S_q}{p} \left[-\frac{S_{pp}}{S_p} p + \frac{S_{qp}}{S_q} p \right] = \frac{S_q}{p} \left[\epsilon^q_p - \epsilon^p_p \right].$$

The above means that $A > 0 \iff \epsilon^q_p > \epsilon^p_p$.

First, take the case of $B < 0$. From proposition 5 we get that $B < 0 \Longrightarrow p^I\left(\theta\right) > p^{II}\left(\theta\right)$. In the equilibrium of both first-score and second-score auctions we have $-\frac{S_q}{S_p} = c'$ (see propositions 1 and 2). If $\epsilon^q_p > \epsilon^p_p$, then intuitively it means that any increase in price would lead to increase in $-\frac{S_q}{S_p}$ (as the proportionate change in S_q is higher than the proportionate change in S_p). This means in equilibrium c' must be higher for first-score auction. Since $c'' \geq 0$ then it is only possible with higher levels of quality. So if price quoted by any type is higher in first-score auction then the quality quoted in first-score auction will also be higher if $\epsilon_p^q > \epsilon_p^p$. By a similar logic, we can intuitively argue that if price quoted by any type is higher in first-score auction then the quality quoted in first-score auction will be lower (or same) if $\epsilon_p^q < \epsilon_p^p$ $\left(\text{or } \epsilon_p^q = \epsilon_p^p\right)$.

Now take the case of $B \geq 0$. From proposition 5 we know that in this case $p^I(\theta) \leq p^{II}(\theta)$. From proposition 3 we know that $S^I(\theta) < S^{II}(\theta)$. Since $S_p < 0$ and $S_q > 0$ then we must have $q^I(\theta) < q^{II}(\theta)$ in this case.

Since S_P can be interpreted as 'price sensitivity' of the scoring rule and S_q as the 'quality sensitivity' of the scoring rule, we can say that ϵ_p^q is the elasticity of 'quality sensitivity' and ϵ_p^p is the *elasticity of 'price sensitivity'. Note that $A > 0 \iff \epsilon_p^q > \epsilon_p^p$. Hence, proposition 4 shows that the quality quoted by any type in a first-score auction will be higher than the quality quoted in a second-score auction iff the elasticity of 'quality sensitivity' is higher than the elasticity of 'price sensitivity'.*

We now claim that the signs of $B(.)$ and $A(.)$ are related. We state this in terms of a lemma. The proof is provided in the appendix.

Lemma 2 *Suppose* $A(p,q) \neq 0 \; \forall (p,q) \in \mathbb{R}^2_{++}$. $B(p,q) \geq 0 \Longrightarrow A(p,q) < 0$.

Comment From lemma 2 we know that $A(p,q) > 0 \Longrightarrow B(p,q) < 0$. Proposition 4 demonstrates that $A(p,q) > 0 \quad \Longrightarrow q^I(\theta) > q^{II}(\theta)$. From proposition 5 we get $B(p,q) < 0 \Longrightarrow p^I(\theta) > p^{II}(\theta)$. This clearly means $A(p,q) > 0 \Longrightarrow q^I(\theta) > q^{II}(\theta)$ and $p^I(\theta) > p^{II}(\theta)$. From proposition 5 we also get that $B(p,q) \geq 0 \Longrightarrow p^I(\theta) \leq p^{II}(\theta)$. Lemma 2 shows that $B(p,q) \geq 0 \Longrightarrow A(p,q) < 0$. Combining this with proposition 4 we get that $B(p,q) \geq 0 \Longrightarrow p^I(\theta) \leq p^{II}(\theta)$ and $q^I(\theta) < q^{II}(\theta)$.

We now provide a few examples to illustrate propositions 4 and 5. The point is to show that scoring rules and cost functions exist that satisfy all our assumptions and the conditions of propositions 4 and 5.

1. We first consider conditions mentioned in proposition 4.
   (a) $S(p,q) = \frac{q}{p}$ and $C(q,\theta) = \frac{1}{2}q^2 + \theta$. In this example $A(.) > 0 \; \forall (p,q) \in \mathbb{R}^2_{++}$.
   (b) $S(p,q) = 10q - p^2$ and $C(q,\theta) = q + \theta$. In this example $A(.) < 0 \; \forall (p,q) \in \mathbb{R}^2_{++}$.
   (c) $S(p,q) = e^{q-p}$ and $C(q,\theta) = \frac{1}{2}q^2 + \theta$. In this example $A(.) = 0 \; \forall (p,q) \in \mathbb{R}^2_{++}$.
2. We now consider conditions mentioned in proposition 5.
   (a) $S(p,q) = \frac{q}{p}$ and $C(q,\theta) = \frac{1}{2}q^2 + \theta$. In this example $B(.) < 0 \; \forall (p,q) \in \mathbb{R}^2_{++}$.
   (b) $S(p,q) = e^{q-p} - p$ and $C(q,\theta) = \frac{1}{2}q + \theta$. In this example $B(.) > 0 \; \forall (p,q) \in \mathbb{R}^2_{++}$.
   (c) $S(p,q) = 10q - p^2$ and $C(q,\theta) = q + \theta$. In this example $B(.) = 0 \; \forall (p,q) \in \mathbb{R}^2_{++}$.

4.2 Impact of an increase in n (the number of bidders)

We now proceed to discuss the impact of increase in n (the number of bidders) on equilibrium quality and price in both auctions. For any given θ, let $q^{I}(n;\theta)$ and $q^{II}(n;\theta)$ be the quality quoted in first-score and second score auctions respectively when the number of bidders is n. Similarly, for any given θ, let $p^{I}(n;\theta)$ and $p^{II}(n;\theta)$ be the price quoted in first-score and second-score auctions respectively when the number of bidders is n.

Proposition 6 For all $n > m$
(i) $q^{II}(n;\theta) = q^{II}(m;\theta)$.
(ii) If $A(p,q) > 0 \;\; \forall (p,q) \in \mathbb{R}^2_{++}$ then $q^{I}(n;\theta) < q^{I}(m;\theta)$.
(iii) If $A(p,q) < 0 \;\; \forall (p,q) \in \mathbb{R}^2_{++}$ then $q^{I}(n;\theta) > q^{I}(m;\theta)$.

Proposition 7 Suppose $A(p,q) \neq 0 \; \forall (p,q) \in \mathbb{R}^2_{++}$. Then for all $n > m$
(i) $p^{II}(n;\theta) = p^{II}(m;\theta)$.
(ii) If $B(p,q) = 0 \; \forall (p,q) \in \mathbb{R}^2_{++}$ then $p^{I}(n;\theta) = p^{I}(m;\theta)$.
(iii) If $B(p,q) > 0 \; \forall (p,q) \in \mathbb{R}^2_{++}$ then $p^{I}(n;\theta) > p^{I}(m;\theta)$.
(iv) If $B(p,q) < 0 \; \forall (p,q) \in \mathbb{R}^2_{++}$ then $p^{I}(n;\theta) < p^{I}(m;\theta)$.

For any given type θ let $S^{I}(n;\theta)$ and $S^{II}(n;\theta)$ be the scores quoted in equilibrium in first-score and second-score auction respectively when the number of bidders is n. That is, $S^{I}(n;\theta) = S\left(p^{I}(n;\theta), q^{I}(n;\theta)\right)$ and $S^{II}(n;\theta) = S\left(p^{II}(n;\theta), q^{II}(n;\theta)\right)$. The next proposition explores how the equilibrium score quoted changes with an increase in the number of bidders. The proof is in the appendix.

Proposition 8 (i) For all $n > m$, $S^{II}(n;\theta) = S^{II}(m;\theta)$.
(ii) For all $n > m$, $S^{I}(n;\theta) > S^{I}(m;\theta)$.

Comment In the second-score auction the quality and price quoted in equilibrium are independent of the number of bidders (see proposition 2). Consequently, the score quoted in equilibrium is invariant with respect to the number of bidders. This is similar to the second-price auction in the benchmark model, where, regardless of the number of bidders, all bidders bid their valuations.

As $\gamma(\theta)$ depends on n, the quality and price quoted in equilibrium of a first-score auction depend on number of bidders (see proposition 1). Consequently, the equilibrium score quoted in a first-score auction depends on n. Proposition 8 shows that in the first-score auction the score quoted by any type increases as the competition intensifies (n increases). This is because any increase in competition induces a bidder with type θ to quote a higher score. This is also similar to the first-price auction in the benchmark model where bids increase with the number of bidders.

5 Expected Scores

The previous section provided equilibrium characterisation for first-score and second-score auctions. We now proceed to give our results on expected scores. Before giving our main results we need to discuss some preliminaries on order statistics.

5.1 Order Statistics : some notations and preliminaries

Let $y_1, y_2..y_n$ denote a random sample of size n drawn from $F(.)$. Then $x_1 \leq x_2... \leq x_n$ where $x_i s$ are $y_i s$ arranged in increasing magnitudes, are defined to be the order statistics corresponding to the random sample $y_1, y_2....y_n$.

We would be interested in x_1 (lowest order statistic) and x_2 (second lowest order statistic). The corresponding distribution functions and density functions are $F_1(.)$, $F_2(.)$ and $f_1(.)$, $f_2(.)$. Note that

$$\begin{aligned} F_1(x) &= 1-(1-F(x))^n \text{ and } F_2(x) = 1-(1-F(x))^n - nF(x)(1-F(x))^{n-1} \\ f_1(x) &= n(1-F(x))^{n-1} f(x) \text{ and } f_2(x) = n(n-1)F(x)(1-F(x))^{n-2} f(x) \end{aligned}$$

$$\text{Note that } F_2(x) = F_1(x) - nF(x)(1-F(x))^{n-1}$$

5.2 Some preliminary results

In proposition 3 it is shown that $S^I(\theta) < S^{II}(\theta)$ for all $\theta \in [\underline{\theta}, \bar{\theta})$ and both $S^I(\theta)$ and $S^{II}(\theta)$ are strictly decreasing in θ. As noted before, the winner in any auction is the firm with the lowest type.

The following two lemmas will help us in comparing the expected scores across auctions. The proofs appear in the appendix.

Lemma 3 (i) In a first-score auction the expected score is as follows:

$$\begin{aligned} \Sigma^I &= \int_{\underline{\theta}}^{\bar{\theta}} S^I(\theta) f_1(\theta)\, d\theta \\ &= S\left(p^I(\bar{\theta}), q^I(\bar{\theta})\right) - \int_{\underline{\theta}}^{\bar{\theta}} F_1(\theta)(1+\gamma'(\theta))\, S_p\left(p^I(\theta), q^I(\theta)\right) d\theta \end{aligned}$$

(ii) In a second-score auction the expected score is as follows:

$$\begin{aligned} \Sigma^{II} &= \int_{\underline{\theta}}^{\bar{\theta}} S^{II}(\theta) f_2(\theta)\, d\theta \\ &= S\left(p^{II}(\bar{\theta}), q^{II}(\bar{\theta})\right) - \int_{\underline{\theta}}^{\bar{\theta}} F_2(\theta)\, S_p\left(p^{II}(\theta), q^{II}(\theta)\right) d\theta \end{aligned}$$

Lemma 4

$$\int_{\underline{\theta}}^{\bar{\theta}} F_1(\theta)(1+\gamma'(\theta))\, d\theta = \int_{\underline{\theta}}^{\bar{\theta}} F_2(\theta)\, d\theta$$

where

$$\gamma(\theta) = \frac{1}{(1-F(\theta))^{n-1}} \int_{\theta}^{\bar{\theta}} (1-F(t))^{n-1}\, dt$$

5.3 Expected scores: second-score vs first-score

We will now compare the expected score in a second-score auction $\left(\Sigma^{II}\right)$ with that in a first-score auction $\left(\Sigma^{I}\right)$.

From lemma 1 we know $p^{I}\left(\bar{\theta}\right) = p^{II}\left(\bar{\theta}\right)$ and $q^{I}\left(\bar{\theta}\right) = q^{II}\left(\bar{\theta}\right)$. This means

$$S\left(p^{I}\left(\bar{\theta}\right), q^{I}\left(\bar{\theta}\right)\right) = S\left(p^{II}\left(\bar{\theta}\right), q^{II}\left(\bar{\theta}\right)\right).$$

Using this and lemma 3 one clearly gets that to compare Σ^{I} and Σ^{II} we need to compare the following terms:

$$\left[\int_{\underline{\theta}}^{\bar{\theta}} F_1(\theta)\left(1+\gamma'(\theta)\right) S_p\left(p^{I}(\theta), q^{I}(\theta)\right) d\theta\right] \text{ and } \left[\int_{\underline{\theta}}^{\bar{\theta}} F_2(\theta) S_p\left(p^{II}(\theta), q^{II}(\theta)\right) d\theta\right].$$

Note that if the scoring rule is **quasilinear** (i.e. $S(p,q) = \phi(q) - p$) then $S_p = -1$. Hence, from lemma 4 the next result follows.

Proposition 9 If the scoring rule is quasilinear then $\Sigma^{I} = \Sigma^{II}$.

Comment The above result is well known (See Che, 1993 and Asker and Cantillon, 2008). For scoring auctions this is the analogue of revenue equivalence theorem of the canonical model.

We now proceed to provide our main results on expected scores when the scoring rules are **non-quasilinear**. Note that for non-quasilinear scoring rules we must have at least one of the following: $S_{pp} \neq 0$, $S_{pq} \neq 0$. *We first demonstrate the possibility of expected score equivalence even with non-quasilinear scoring rules.*

Proposition 10 If $\forall (p,q) \in \mathbb{R}^2_{++}$, $A(.) \neq 0$ and $S_{pp}\frac{B(.)}{A(.)} - S_{pq} = 0$ then $\Sigma^{I} = \Sigma^{II}$.

Comment We illustrate proposition 10 with a couple of examples. In one example $S_{pq} = 0$ and in the other example $S_{pq} \neq 0$.

Example 3: Let $S(p,q) = 10q - p^2$, $C(q,\theta) = q + \theta$ and θ is uniformly distributed over $[1,2]$. The scoring rule is non-quasilinear and satisfies all our assumptions. Here it can be easily shown that $\Sigma^{I} = \Sigma^{II} = \frac{25}{3}$.

Example 4: Let $S(p,q) = e^{q-p} - p$, $C(q,\theta) = \frac{1}{2}q + \theta$ and θ is uniformly distributed over $\left[\frac{1}{4}, \frac{1}{2}\right]$. The scoring rule is non-quasilinear and satisfies all our assumptions. Here we have $\Sigma^{I} = \Sigma^{II} = \frac{1}{6}$.

From proposition 10 we get that

$$\Sigma^{I} \neq \Sigma^{II} \Longrightarrow S_{pp}\frac{B(.)}{A(.)} - S_{pq} \neq 0 \text{ for some } (p,q) \in \mathbb{R}^2_{++}.$$

Now suppose the scoring rule is such that $S_{pp}\frac{B(.)}{A(.)} - S_{pq} \neq 0$ for some $(p,q) \in \mathbb{R}^2_{++}$. We now show that a restriction on the distribution function of types ensures $\Sigma^{I} < \Sigma^{II}$. We now provide this ranking result in proposition 11 below.

Proposition 11 Suppose the scoring rule, $S(.)$, is non-quasilinear and $S_{pp}\frac{B(.)}{A(.)} - S_{pq} \neq 0$ for some $(p,q) \in \mathbb{R}^2_{++}$. If $f'(\theta) \leq 0$ for $\theta \in [\underline{\theta}, \bar{\theta}]$ and $f(\bar{\theta})$ is large enough then $\Sigma^I < \Sigma^{II}$.

Comment *Proposition 11 is interesting as it demonstrates the need to put restrictions on the distribution function of types to get a ranking of expected scores. This stands in sharp contrast to the other papers in the literature.*

It may be noted that most non-quasilinear scoring rules, including the quality over price ratio, satisfy the restriction $S_{pp}\frac{B(.)}{A(.)} - S_{pq} \neq 0$. Also, the restriction, $f'(\theta) \leq 0$, is satisfied by many distribution functions (including the uniform distribution). As such, the expected scores will be strictly higher with second-score auctions for most scoring rules and many distribution functions.

This has interesting policy implications as well. In real life second-score auctions are never used. Our result suggests that in a large number of cases an auctioneer will be better off using second-score auctions than using first-score auctions.

We now illustrate this result with two examples. We take the 'quality over price' scoring rule and the same quadratic cost function in both examples. Note that the restriction $S_{pp}\frac{B(.)}{A(.)} - S_{pq} \neq 0$ is satisfied for this scoring rule and cost function. The distribution function of types are different in the two examples.

Proposition 11 demonstrates that when $S_{pp}\frac{B(.)}{A(.)} - S_{pq} \neq 0$ for some $(p,q) \in \mathbb{R}^2_{++}$ then $\Sigma^I \geq \Sigma^{II}$ implies that at least one of the following is true: (i) $f'(\theta) > 0$ or (ii) $f(\bar{\theta})$ is not large enough. In example 5 we take a uniform distribution, where $f'(\theta) = 0$ and show that $\Sigma^I < \Sigma^{II}$. In example 6, we take a different distribution function where $f'(.) > 0$ and we get $\Sigma^I > \Sigma^{II}$. Clearly, examples 5 and 6 illustrate proposition 11.[14]

Example 5 Let $S(p,q) = \frac{q}{p}$ and $C(q,\theta) = \frac{1}{2}q^2 + \theta$. Suppose θ be uniformly distributed over $[1,2]$ and $n = 2$. For this distribution we have

$$f_1(\theta) = 2(2-\theta) \quad \text{and } f_2(\theta) = 2(\theta - 1)$$

The equilibria are as follows:

First-score auction:

$$\begin{aligned}
\text{price:} \quad & p^I(\theta) = 2 + \theta \\
\text{quality} \quad : \quad & q^I(\theta) = \sqrt{2+\theta} \\
\text{score:} \quad & s^I(\theta) = \frac{q^I(\theta)}{p^I(\theta)} = \frac{1}{\sqrt{2+\theta}} \\
\text{Expected score:} \quad & \Sigma^I = \int_1^2 s^I(\theta) f_1(\theta)\, d\theta = 0.548\,72
\end{aligned}$$

[14] I must thank Kasunori Yamada and Diganta Mukherjee for helping me with the computations using MATLAB.

Second-score auction:

$$\begin{aligned}\text{price:} \quad & p^{II}(\theta) = 2\theta \\ \text{quality:} \quad & q^{II}(\theta) = \sqrt{2\theta} \\ \text{score:} \quad & s^{II}(\theta) = \frac{q^{II}(\theta)}{p^{II}(\theta)} = \frac{1}{\sqrt{2\theta}} \\ \text{Expected score:} \quad & = \int_1^2 s^{II}(\theta) f_2(\theta)\, d\theta = 0.552\,28\end{aligned}$$

Note that in this example $\Sigma^{II} > \Sigma^I$.

Example 6 Let $S(p,q) = \frac{q}{p}$ and $C(q,\theta) = \frac{1}{2}q^2 + \theta$. Now suppose $n = 2$ and θ is distributed over $[1.2, 1.203731]$ with density $f(x) = 500x^3 - 600$ and distribution function $F(x) = 125x^4 - 600x + \frac{2304}{5}$. For this distribution we have

$$\begin{aligned} f_1 &= 2\left(-125x^4 + 600x - \frac{2299}{5}\right)\left(500x^3 - 600\right) \text{ and} \\ f_2 &= 2\left(125x^4 - 600x + \frac{2304}{5}\right)\left(500x^3 - 600\right)\end{aligned}$$

Now the equilibria are as follows:

First-score auction:

$$\text{price:} \quad p^I(\theta) = 2\left(\theta + \frac{25\theta^5 - 300\theta^2 + \frac{45598\theta}{10} - \frac{18197}{100}}{-125\theta^4 + 600\theta - \frac{2299}{5}}\right)$$

$$\text{quality} \ : \quad q^I(\theta) = \sqrt{2\left(\theta + \frac{25\theta^5 - 300\theta^2 + \frac{45598\theta}{10} - \frac{18197}{100}}{-125\theta^4 + 600\theta - \frac{2299}{5}}\right)}$$

$$\text{score:} \quad s^I(\theta) = \frac{q^I(\theta)}{p^I(\theta)} = \frac{1}{\sqrt{2\left(\theta + \frac{25\theta^5 - 300\theta^2 + \frac{45598\theta}{10} - \frac{18197}{100}}{-125\theta^4 + 600\theta - \frac{2299}{5}}\right)}}$$

$$\text{Expected score:} \quad \Sigma^I = \int_{1.2}^{1.203731} s^I(\theta) f_1(\theta)\, d\theta = 0.6469$$

Second-score auction:

$$\begin{aligned}\text{price:} \quad & p^{II}(\theta) = 2\theta \\ \text{quality:} \quad & q^{II}(\theta) = \sqrt{2\theta} \\ \text{score:} \quad & s^{II}(\theta) = \frac{q^{II}(\theta)}{p^{II}(\theta)} = \frac{1}{\sqrt{2\theta}} \\ \text{Expected score:} \quad & \Sigma^{II} = \int_{1.2}^{1.203731} s^{II}(\theta) f_2(\theta)\, d\theta = 0.6449\end{aligned}$$

Note that in this example $\Sigma^{II} < \Sigma^I$.

6 Conclusion

In this paper we analysed scoring auctions with general non-quasilinear scoring rules. We demonstrated that additive separability of cost functions vastly simplifies the equilibrium computations. Unlike recent papers, we get explicit solutions for the Bayesian-Nash equilibrium without any endogeneity problems. Moreover, we analyse the properties of such equilibria and the ranking of expected scores across first-score and second-score auctions and demonstrate that they depend only on the curvature properties of the scoring rule and distribution function of types. Our approach helps in dealing with most non-quasilinear scoring rules. The following may be noted.

1. In this paper we concentrated mainly on single dimensional quality. Characterisation of equilibrium and ranking of expected scores when quality is multidimensional is an open question and is left for future research.

2. Optimal mechanisms (that maximise expected scores) have been derived in the literature for quasi-linear scoring rules (See Che, 1993, Asker Cantillon, 2010 and Nishimura, 2015). However, such optimal mechanisms for general non-quasilinear scoring rules have not been adequately analysed. This is an open question and is left for future research.

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Appendix

Proof of Proposition 1 In the main body of the paper (see the comment after proposition 2) we have introduced $\Psi(q,s)$ and have given a sketch of the proof. For the sake of convenience we reproduce the discussion on $\Psi(q,s)$.

For any quality, q, let $\Psi(s,q)$ be the price required to generate a score of s. That is, $S(\Psi(s,q),q)=s$. Clearly, $\Psi(.)$ is well defined and it is strictly decreasing in s and strictly increasing in q.

Consider any symmetric equilibrium of first-score auction where a bidder with type θ bids (p,q). Let the score generated by such a bid be $S(p,q)=s$. Since $\Psi(.)$ is well defined and is strictly decreasing in s we can think of the equilibrium as where a bidder bids a score s and quality q. The payoff (conditional on winning) with a score s to a bidder with type θ is

$$\Psi(s,q)-c(q)-\theta.$$

In any equilibrium, for any type θ, the quality choice, q, must be such so as to maximise $\Psi(s,q)-c(q)-\theta$. The FOC and SOC for such a maximisation are as follows:

$$\begin{aligned}\Psi_q(.)-c'(.) &= 0----(1a)\\ \Psi_{qq}(.)-c''(.) &< 0----(1b)\end{aligned}$$

Note that

$$\Psi_{qq}(.)-c''(.)<0 \Longleftrightarrow -\frac{(S_q)^2}{S_p}S_{pp}+2S_qS_{pq}-S_pS_{qq}-(S_p)^2c''<0$$

Given our assumption 4, the SOC (which is (1b)) will always be satisfied.

Note that we have the following:

$$\Psi_q(.)=-\frac{S_q(.)}{S_p(.)} \text{ and } \Psi_s(.)=\frac{1}{S_p(.)}----(2)$$

Hence we can rewrite (1a) as follows:

$$-\frac{S_q(.)}{S_p(.)}=c'(.)----(3)$$

Consequently, in any equilibrium (3) will be satisfied.

We now show that there is a symmetric equilibrium where a bidder with type θ chooses $p^I(\theta)$ and $q^I(\theta)$. Such $p^I(.)$ and $q^I(.)$ are obtained by solving the following equations:

$$
\begin{aligned}
-\frac{S_q(.)}{S_p(.)} &= c'(.) ----(4a) \\
p - c(q) &= \theta + \gamma(\theta) ----(4b) \\
\text{where } \gamma(\theta) &= \frac{1}{(1-F(\theta))^{n-1}} \int_{\theta}^{\bar{\theta}} (1-F(t))^{n-1} dt ----(4c)
\end{aligned}
$$

First note that (4a) is same as (3) and it is true at any equilibrium. Now we show why (4b) is needed. To do this let's suppose that all firms $j = 2, 3..n$ choose $p^I(\theta_j)$ and $q^I(\theta_j)$ according to (4a) and (4b). Then we show that it is optimal for firm 1 to choose the same strategy. Note that from (4b) we have

$$
\forall \theta \in [\underline{\theta}, \bar{\theta}],\ p^I(\theta) - c\left(q^I(\theta)\right) = \theta + \gamma(\theta) ----(5)
$$

Differentiating both sides of (5) w.r.t. θ we have

$$
\begin{aligned}
\forall \theta &\in \left(\underline{\theta}, \bar{\theta}\right),\ \frac{dp^I(\theta)}{d\theta} - c'(q(\theta)) \frac{dq^I(\theta)}{d\theta} = 1 + \gamma'(\theta) \\
&= \frac{(n-1) f(\theta)}{(1-F(\theta))^n} \int_{\theta}^{\bar{\theta}} (1-F(t))^{n-1} dt ----(6)
\end{aligned}
$$

From (6) we clearly have

$$
\frac{dp^I(\theta)}{d\theta} - c'(q(\theta)) \frac{dq^I(\theta)}{d\theta} > 0 ----(7)
$$

For any firm $j \in \{2, 3..n\}$ the choice of $p^I(\theta_j), q^I(\theta_j)$ leads to score $S\left(p^I(\theta_j), q^I(\theta_j)\right)$. Then we can say that any firm $j \in \{2, 3..n\}$ with type θ_j chooses score $S\left(p^I(\theta_j), q^I(\theta_j)\right)$ and quality $q^I(\theta_j)$.

Let

$$
\hat{S}(\theta) = S\left(p^I(\theta), q^I(\theta)\right)
$$

Then, we have that any firm $j \in \{2, 3..n\}$ with type θ chooses score $\hat{S}(\theta)$ and qualities $q(\theta)$. Now note the following:

$$
\begin{aligned}
\frac{d}{d\theta}\hat{S}(\theta) &= S_p(.) \frac{dp^I(\theta)}{d\theta} + S_q(.) \frac{dq^I(\theta)}{d\theta} \\
&= S_p(.) \frac{dp^I(\theta)}{d\theta} - C_q(.) S_p(.) \frac{dq^I(\theta)}{d\theta} \text{ (using (4a))} \\
&= S_p(.) \left[\frac{dp^I(\theta)}{d\theta} - C_q(.) \frac{dq^I(\theta)}{d\theta}\right] ----(8)
\end{aligned}
$$

From (8) we have

$$\frac{d}{d\theta}\hat{S}(\theta) < 0 \text{ (since } S_p(.) < 0 \text{ and } \frac{dp^I(\theta)}{d\theta} - C_q(.)\frac{dq^I(\theta)}{d\theta} > 0 \text{ (from 7)}$$

The above means that for any firm $j \in \{2, 3..n\}$ the score quoted is strictly decreasing in θ. Hence, the scores of firms $2, 3..n$ lie in the interval $\left[\hat{S}(\bar{\theta}), \hat{S}(\underline{\theta})\right]$.

Now take the case of firm 1. It has to choose a score, s_1 and a quality, q, given the choice of firms $2, 3..n$. Clearly $s_1 \in \left[\hat{S}(\bar{\theta}), \hat{S}(\underline{\theta})\right]$. Note that choosing s_1 is equivalent to choosing z s.t. $s_1 = \hat{S}(z)$. Hence, the probability of winning for firm 1 is as follows:

$$\begin{aligned} & \text{Prob. } \left\{\hat{S}(z) > \max_{j\neq 1}\left(\hat{S}(\theta_j)\right)\right\} \\ = & \text{ Prob. } \left\{\hat{S}(z) > \left(\hat{S}\left(\min_{j\neq 1}(\theta_j)\right)\right)\right\} \text{ (since } \hat{S}'(.) < 0) \\ = & \text{ Prob. } \left\{z < \min_{j\neq 1}(\theta_j)\right\} - - - - (9). \end{aligned}$$

We know that θ is distributed over $[\underline{\theta}, \bar{\theta}]$ with distribution function $F(.)$ and density function $f(.)$. From the basic theory of order statistics (see section 5.1) we also know that the lowest order statistic from among $(n-1)$ i.i.d random variables has a distribution function $G(.) = 1 - (1 - F(.))^{n-1}$. That is, for the random variables $\theta_2, \theta_3..\theta_n$

$$\text{Prob}\left\{\min_{j\neq 1}(\theta_j) < \Sigma\right\} = G(\Sigma) = 1 - (1 - F(\Sigma))^{n-1}.$$

Using (9) we can write

$$\begin{aligned} & \text{Prob. } \left\{\hat{S}(z) > \max_{j\neq 1}\left(\hat{S}(\theta_j)\right)\right\} \\ = & \text{ Prob. } \left\{z < \min_{j\neq 1}(\theta_j)\right\} \\ = & \ 1 - G(z) = (1 - F(z))^{n-1}. \end{aligned}$$

That is, if firm 1 chooses a score of $s_1 = \hat{S}(z)$ it wins with probability $(1 - F(z))^{n-1}$. Let it choose quality x and let it's type be θ_1. Then, firm 1's cost is $c(x) + \theta_1$. Therefore, firm 1's expected payoff by choosing a score $s_1 = \hat{S}(z)$ and quality x is

$$\pi_1 = (1 - F(z))^{n-1}\left[\Psi\left(x, \hat{S}(z)\right) - c(x) - \theta_1\right] - - - - (10)$$

Firm 1's choice variables are x and z. Note that from the 1OCs for an optimum we have

$$\frac{\partial \pi_1}{\partial x} = 0 \Longrightarrow \Psi_q\left(x, \hat{S}(z)\right) - c'(x) = 0 - - - - (11)$$

From earlier discussions we know that (11) is equivalent to

$$-\frac{S_q(.)}{S_p(.)} = c'(.) - - - - (12)$$

The above is same as (4a).

We now proceed to deal with the optimal choice of z. It may be noted that

$$\frac{\partial\Psi\left(x,\hat{S}(z)\right)}{\partial z}=\Psi_s\left(x,\hat{S}(z)\right)\hat{S}'(z)$$

By using (2) and (8)

$$\begin{aligned}\Psi_s\left(x,\hat{S}(z)\right)\hat{S}'(z) &= \frac{1}{S_p(p(z),q(z))}S_p\left(p^I(z),q^I(z)\right)\left[\frac{dp^I(z)}{dz}-c'(q(z))\frac{dq^I(z)}{dz}\right]\\ &= \frac{dp^I(z)}{dz}-c'(q(z))\frac{dq^I(z)}{dz}\\ &= 1+\gamma'(z) \;=\frac{(n-1)f(z)}{(1-F(z))^n}\int_z^{\bar{\theta}}(1-F(t))^{n-1}dt \text{ (from 6)}\end{aligned}$$

That is,

$$\frac{\partial\Psi\left(x,\hat{S}(z)\right)}{\partial z}=1+\gamma'(z)\;=\frac{(n-1)f(z)}{(1-F(z))^n}\int_z^{\bar{\theta}}(1-F(t))^{n-1}dt----(13)$$

Now note that from (10) and (13) we have

$$\begin{aligned}\frac{\partial}{\partial z}\pi_1 &= -(n-1)(1-F(z))^{n-2}f(z)\left[\Psi\left(x,\hat{S}(z)\right)-c(x)-\theta_1\right]+(1-F(z))^{n-1}\frac{\partial\Psi\left(x,\hat{S}(z)\right)}{\partial z}\\ &= (1-F(z))^{n-2}\left[\begin{array}{c}-(n-1)f(z)\left\{\Psi\left(x,\hat{S}(z)\right)-c(x)-\theta_1\right\}\\ +(1-F(z))\frac{(n-1)f(z)}{(1-F(z))^n}\int_z^{\bar{\theta}}(1-F(t))^{n-1}dt\end{array}\right]\\ &= (n-1)(1-F(z))^{n-2}f(z)\left[\begin{array}{c}\frac{1}{(1-F(z))^{n-1}}\int_z^{\bar{\theta}}(1-F(t))^{n-1}dt\\ -\left\{\Psi\left(x,\hat{S}(z)\right)-c(x)-\theta_1\right\}\end{array}\right]\end{aligned}$$

From above and using definition of $\gamma(z)$ (see 4c) we get that

$$\frac{\partial}{\partial z}\pi_1=(n-1)(1-F(z))^{n-2}f(z)\left[\gamma(z)-\left\{\Psi\left(x,\hat{S}(z)\right)-c(x)-\theta_1\right\}\right]\quad----(14)$$

Note that $(n-1)(1-F(z))^{n-2}f(z)>0$ for all $z\in\left(\underline{\theta},\bar{\theta}\right)$.

Also note that by using (13) we get that

$$\begin{aligned}&\frac{\partial}{\partial z}\left[\gamma(z)-\left\{\Psi\left(x,\hat{S}(z)\right)-c(x)-\theta_1\right\}\right]\\ &= \gamma'(z)-\frac{\partial\Psi\left(x,\hat{S}(z)\right)}{\partial z}\\ &= \gamma'(z)-\left[1+\gamma'(z)\right]\text{ using (13)}\\ &= -1<0\;------(15)\end{aligned}$$

From (4b) we know that

$$p^I(\theta) - c\left(q^I(\theta)\right) = \theta + \gamma(\theta)$$

$$\Longrightarrow \Psi\left(q^I(\theta), \hat{S}(\theta)\right) - c\left(q^I(\theta)\right) = \theta + \gamma(\theta) - - - - (16)$$

We know that firm 1's choice of x is such that (12) (which is same as 4a) is satisfied. Using this fact and (16) we get that

$$\text{if } z = \theta_1 \text{ then } \gamma(z) - \left\{\Psi\left(x, \hat{S}(z)\right) - c(x) - \theta_1\right\} = 0 - - - - (17)$$

This means (see 14 and 17)

$$\frac{\partial}{\partial z}\pi_1 = 0 \text{ at } z = \theta_1 - - - - (18)$$

Moreover, from (14), (15) and (18) we clearly get that

$$\begin{aligned} z &< \theta_1 \Longrightarrow \frac{\partial}{\partial z}\pi_1 > 0 \text{ and} \\ z &> \theta_1 \Longrightarrow \frac{\partial}{\partial z}\pi_1 < 0. \quad - - - - (19) \end{aligned}$$

(18) and (19) implies that $z = \theta_1$ is the optimal choice for firm 1. Therefore firm 1's choice of quality, x and score, $\hat{S}(z)$ must satisfy (12) and (17). This is same as 4a and 4b.

That is, we have proved that in a first-score auction there is a symmetric equilibrium where a bidder with pseudo-type θ chooses $p^I(\theta)$ and $q^I(\theta)$. Such $p^I(.)$ and $q^I(.)$ are obtained by solving the following equations:

$$\begin{aligned} -\frac{S_{q_1}(.)}{S_p(.)} &= c'(.) \\ p - c(q) &= \theta + \frac{1}{(1-F(\theta))^{n-1}} \int_\theta^{\bar{\theta}} (1-F(t))^{n-1}\, dt. \end{aligned}$$

This completes our proof for proposition 1.■

Proof of Proposition 2 We will now show that in a second-score auction the weakly dominant strategy for each firm with type θ is to choose $p(\theta)$ and $q(\theta)$ that are obtained by solving the following equations:

$$\begin{aligned} -\frac{S_q(.)}{S_p(.)} &= c'(.) - - - - (20a) \\ p &= c(q) + \theta - - - - (20b) \end{aligned}$$

Let the score quoted by firm i by following 20a and 20b be s. That is, $s = S(p(\theta), q(\theta))$. It may be recalled from our earlier discussions that (20a) which is same as (3) and it is equivalent to (1a) reproduced below.

$$\Psi_q(.) - c'(.) = 0 - - - - (1a)$$

From (1a) we get q as a function of s. From earlier discussion we know that for any s, the quality choice, q, (as in 1a above) is such so as to maximise $\Psi(q, s) - c(q) - \theta$. Then, using the envelope theorem we get

$$\frac{d}{ds}[\Psi(q(s), s) - c(q) - \theta] = \Psi_s = \frac{1}{S_p(.)} < 0 \text{ (see (2))} \quad - - - - (21).$$

Now clearly (by using the equivalence of (1a) and (20a),

$$\Psi(q(s), s) = p^{II}(\theta)$$

The above implies from (20b)

$$\Psi(q(s), s) - c(q(s)) - \theta = 0 - - - - (22)$$

Now let firm i follow (20a) and (20b) and thereby pick up a score s. Let the maximum of the scores quoted by firms other than i be δ. Now if $s > \delta$ then by following (20a) and (20b) firm i wins the contract. As per the rules of the second score auction, the winner is required to match the highest rejected score which is δ. In meeting this score, the firm is free to choose any quality-price combination. Clearly, firm i will choose qualities so as to maximise $\Psi(q, \delta) - c(q) - \theta$. Those choice of qualities must satisfy the following equation:

$$\Psi_q(q, \delta) - c'(q) = 0 - - - - (23)$$

The firm's profit by meeting a score δ is therefore $\Psi(q(\delta), \delta) - c(q(\delta)) - \theta$. Since $\delta < s$ and using (21) and (22)

$$\Psi(q(\delta), \delta) - c(q(\delta)) - \theta > \Psi(q(s), s) - c(q(s)) - \theta = 0 - - (24)$$

If firm i decides to pick up any score, $\phi \neq s$ (by choosing (p, q) *other than* as in 20a and 20b),then it would not matter as long as $\phi > \delta$. If $\phi < \delta$, then firm would not win the contract and earn zero payoff. Hence if $s > \delta$ then the firm's best strategy is to quote a score s. Similarly, it can be shown that is $s < \delta$ then also the firm's best strategy is to quote a score s. In other words, choice of s is a weakly dominant strategy.■

Proof of Lemma 1 Note that by using the L'Hospital's rule we get

$$\begin{aligned}
\lim_{\theta \longrightarrow \bar{\theta}} \gamma(\theta) &= \lim_{\theta \longrightarrow \bar{\theta}} \frac{1}{(1 - F(\theta))^{n-1}} \int_\theta^{\bar{\theta}} (1 - F(t))^{n-1} dt \\
&= \lim_{\theta \longrightarrow \bar{\theta}} \frac{\frac{d}{d\theta}\left(\int_\theta^{\bar{\theta}} (1 - F(t))^{n-1} dt\right)}{\frac{d}{d\theta}(1 - F(\theta))^{n-1}} \\
&= \lim_{\theta \longrightarrow \bar{\theta}} \frac{1 - F(\theta)}{(n-1) f(\theta)} = 0.
\end{aligned}$$

Hence, using propositions 1 and 2, for the type $\bar{\theta}$, in both first-score and second-score auctions, $p\left(\bar{\theta}\right)$, $q\left(\bar{\theta}\right)$ is obtained by solving the following equations.

$$\begin{aligned} -\frac{S_q(.)}{S_p(.)} &= c'(.) \\ p &= c(q)+\theta \end{aligned}$$

This shows that $p^I\left(\bar{\theta}\right) = p^{II}\left(\bar{\theta}\right)$ and $q^I\left(\bar{\theta}\right) = q^{II}\left(\bar{\theta}\right)$. ■

Proof of Proposition 3 In equilibrium, in both first-score and second-score auctions the following is true:

$$-\frac{S_q(p,q)}{S_p(p,q)} = c'(q) - - - - (25)$$

From (25) we get p implicitly a function of q. That is, $p = \sigma(q)$ and we have

$$\begin{aligned} -\frac{S_q(\sigma(q),q)}{S_p(\sigma(q),q)} - c'(q) &= 0 \\ &\Longleftrightarrow \\ S_p(\sigma(q),q)\,c'(q) + S_q(\sigma(q),q) &= 0 - - - - (26) \end{aligned}$$

Using the implicit function theorem we get that

$$\sigma'(q) = -\left[\frac{c'S_{pq} + S_p c'' + S_{qq}}{c'S_{pp} + S_{qp}}\right] - - - - (27).$$

Using (25) we have

$$\sigma'(q) = -\left[\frac{-\frac{S_q}{S_p}S_{pq} + S_p c'' + S_{qq}}{-\frac{S_q}{S_p}S_{pp} + S_{qp}}\right] = -\frac{B(.)}{A(.)} - - - - (28)$$

Note that $\sigma'(q)$ is well defined since $\forall (p,q) \in \mathbb{R}^2_{++}$, $-\frac{S_q}{S_p}S_{pp} + S_{qp} \neq 0$.

Since by assumption S_{pp} and S_{qp} are continuous $\forall (p,q) \in \mathbb{R}^2_{++}$, then $A(p,q) = -\frac{S_q}{S_p}S_{pp} + S_{qp} \neq 0 \ \forall (p,q) \in \mathbb{R}^2_{++}$ implies either (a) $\forall (p,q) \in \mathbb{R}^2_{++} \ \ A(p,q) > 0$ **or** (b) $\forall (p,q) \in \mathbb{R}^2_{++} \ \ A(p,q) < 0$.

Note that for **both** auctions (from (25) using the fact that $c'(.) = -\frac{S_q}{S_p}$)

$$\begin{aligned} \sigma'(q) - c'(q) &= \sigma'(q) + \frac{S_q}{S_p} \\ &= \frac{\left[-\frac{(S_q)^2}{S_p}S_{pp} + 2S_qS_{qp} - S_pS_{qq} - (S_p)^2 c''\right]}{S_p\left[-\frac{S_q}{S_p}S_{pp} + S_{qp}\right]} - - - - (29) \end{aligned}$$

Note that by assumption the numerator of (31) is strictly negative. Since $S_p < 0$ we have that

$$\begin{aligned} \text{if } \forall (p,q) &\in \mathbb{R}^2_{++},\ A(p,q) = -\frac{S_q}{S_p}S_{pp} + S_{qp} > 0 \text{ then } \sigma'(q) - c'(q) > 0 \text{ and} \\ \text{if } \forall (p,q) &\in \mathbb{R}^2_{++},\ A(p,q) = -\frac{S_q}{S_p}S_{pp} + S_{qp} < 0 \text{ then } \sigma'(q) - c'(q) < 0 \ - - - - (30) \end{aligned}$$

Now note the following.

$$
\begin{aligned}
\frac{d}{dq}S(\sigma(q),q) &= S_p\sigma'(q)+S_q \\
&= S_p\sigma'(q)-S_pc'(q) \;\; \text{(from 25)} \\
&= S_p[\sigma'(q)-c'(q)] ----(31)
\end{aligned}
$$

From (30) we know that $\sigma'(q)-c'(q)$ has the same sign as $A(p,q)=\left(-\frac{S_q}{S_p}S_{pp}+S_{qp}\right)$. Since $S_p<0$ from (31) we get that $\frac{d}{dq}S(\sigma(q),q)$ has the opposite sign of $A(p,q)$.

Now suppose $A(p,q)>0$. This means $\frac{d}{dq}S(\sigma(q),q)<0$. Since $q^I(\theta)>q^{II}(\theta)$ when $A(p,q)>0$ we must have $S\left(\sigma\left(q^I(\theta)\right),q^I(\theta)\right)<S\left(\sigma\left(q^{II}(\theta)\right),q^{II}(\theta)\right)$. Now since $p^I(\theta)=\sigma\left(q^I(\theta)\right)$ and $p^{II}(\theta)=\sigma\left(q^{II}(\theta)\right)$ for $\theta\in\left[\underline{\theta},\bar{\theta}\right]$ this implies $S\left(p^I(\theta),q^I(\theta)\right)<S\left(p^{II}(\theta),q^{II}(\theta)\right)$. This means $S^I(\theta)<S^{II}(\theta)$.

Now suppose $A(p,q)=-\frac{S_q}{S_p}S_{pp}+S_{qp}<0$. This means $\frac{d}{dq}S(\sigma(q),q)>0$. Since $q^I(\theta)<q^{II}(\theta)$ when $A(p,q)<0$ we must have $S\left(\sigma\left(q^I(\theta)\right),q^I(\theta)\right)<S\left(\sigma\left(q^{II}(\theta)\right),q^{II}(\theta)\right)$. Now since $p^I(\theta)=\sigma\left(q^I(\theta)\right)$ and $p^{II}(\theta)=\sigma\left(q^{II}(\theta)\right)$ for all $\theta\in\left[\underline{\theta},\bar{\theta}\right]$ this implies $S\left(p^I(\theta),q^I(\theta)\right)<S\left(p^{II}(\theta),q^{II}(\theta)\right)$.

Using (7) and (8) we know that $\frac{d}{d\theta}S\left(p^I(\theta),q^I(\theta)\right)<0$. Using a exactly similar method we can show that $\frac{d}{d\theta}S\left(p^{II}(\theta),q^{II}(\theta)\right)<0$.

This completes proof of proposition 3.■

Proof of Proposition 4 (i) and (ii) Note that in equilibrium, in both first-score and second-score auctions $-\frac{S_q(p,q)}{S_p(p,q)}=c'(q)$. As in (25), we get p implicitly a function of q. That is, $p=\sigma(q)$. In a first-score auction we have (see proposition 1)

$$
\begin{aligned}
p^I-c\left(q^I\right) &= \theta+\gamma(\theta) \\
&\Longrightarrow \sigma\left(q^I\right)-c\left(q^I\right)=\theta+\gamma(\theta) ----(32)
\end{aligned}
$$

In second-score auction we have (see proposition 2)

$$
\begin{aligned}
p^{II}-c\left(q^{II}\right) &= \theta \\
&\Longrightarrow \sigma\left(q^{II}\right)-c\left(q^{II}\right)=\theta ----(33)
\end{aligned}
$$

Now using (32) and (33), for any $\theta\in\left[\underline{\theta},\bar{\theta}\right]$ we get

$$
\begin{aligned}
\sigma\left(q^I(\theta)\right)-c\left(q^I(\theta)\right) &= \theta+\gamma(\theta) \;\text{ and} \\
\sigma\left(q^{II}(\theta)\right)-c\left(q^{II}(\theta)\right) &= \theta ----(34)
\end{aligned}
$$

From (34) it is clear that for any $\theta\in\left[\underline{\theta},\bar{\theta}\right)$

$$
\sigma\left(q^I(\theta)\right)-c\left(q^I(\theta)\right)>\sigma\left(q^{II}(\theta)\right)-c\left(q^{II}(\theta)\right) ----(34a)
$$

Now let $\forall(p,q)\in\mathbb{R}^2_{++}$, $A(p,q)=-\frac{S_q}{S_p}S_{pp}+S_{qp}>0$. For any $\theta\in\left[\underline{\theta},\bar{\theta}\right)$ if possible let's suppose $q^I(\theta)\leq q^{II}(\theta)$. Since from (30) we have $\sigma'(q)-c'(q)>0$ when $A(p,q)>0$, we must have $\sigma\left(q^I(\theta)\right)-c\left(q^I(\theta)\right)\leq\sigma\left(q^{II}(\theta)\right)-c\left(q^{II}(\theta)\right)$. But this contradicts (34a). Hence, when $A(p,q)>0$ we must have $q^I(\theta)>q^{II}(\theta)$.

Now let $\forall (p,q) \in \mathbb{R}^2_{++}$, $A(p,q) = -\frac{S_q}{S_p} S_{pp} + S_{qp} < 0$. From (30) we have $\sigma'(q) - c'(q) < 0$. Now using a logic similar to the one used in the previous paragraph we get $q^I(\theta) < q^{II}(\theta)$.

From (32) we get that in a first-score auction the following is true for all $\theta \in [\underline{\theta}, \bar{\theta}]$

$$\sigma\left(q^I(\theta)\right) - c\left(q^I(\theta)\right) = \theta + \gamma(\theta) - - - - (35)$$

From (35) we get that for all $\theta \in (\underline{\theta}, \bar{\theta})$ we have

$$\left[\sigma'\left(q^I(\theta)\right) - c'\left(q^I(\theta)\right)\right] \frac{dq^I(\theta)}{d\theta} = 1 + \gamma'(\theta) = \frac{(n-1) f(\theta)}{(1-F(\theta))^n} \int_\theta^{\bar{\theta}} (1-F(t))^{n-1} dt - - - (36).$$

Since $\frac{(n-1)f(\theta)}{(1-F(\theta))^n} \int_\theta^{\bar{\theta}} (1-F(t))^{n-1} dt > 0$ from (36) we get that $\frac{dq^I(\theta)}{d\theta}$ has the same sign as $\sigma'\left(q^I(\theta)\right) - c'\left(q^I(\theta)\right)$. From (32) we know that

$$\begin{aligned} &\text{if } \forall (p,q) \in \mathbb{R}^2_{++}, A(p,q) > 0 \text{ then } \sigma'(.) - c'(.) > 0 \text{ and} \\ &\text{if } \forall (p,q) \in \mathbb{R}^2_{++}, A(p,q) < 0 \text{ then } \sigma'(.) - c'(.) < 0. \end{aligned}$$

This shows that

$$\begin{aligned} &\text{if } \forall (p,q) \in \mathbb{R}^2_{++}, A(p,q) > 0 \text{ then } \frac{dq^I(\theta)}{d\theta} > 0 \ \forall \theta \in (\underline{\theta}, \bar{\theta}) \text{ and} \\ &\text{if } \forall (p,q) \in \mathbb{R}^2_{++}, A(p,q) < 0 \text{ then } \frac{dq^I(\theta)}{d\theta} < 0 \ \forall \theta \in (\underline{\theta}, \bar{\theta}). \end{aligned}$$

Using a similar logic we can show that in a second-score auction,

$$\begin{aligned} &\text{if } \forall (p,q) \in \mathbb{R}^2_{++}, A(p,q) > 0 \text{ then } \frac{dq^{II}(\theta)}{d\theta} > 0 \ \forall \theta \in (\underline{\theta}, \bar{\theta}) \text{ and} \\ &\text{if } \forall (p,q) \in \mathbb{R}^2_{++}, A(p,q) < 0 \text{ then } \frac{dq^{II}(\theta)}{d\theta} < 0 \ \forall \theta \in (\underline{\theta}, \bar{\theta}) \end{aligned}$$

(iii) Now suppose $A(p,q) = -\frac{S_q}{S_p} S_{pp} + S_{qp} = 0$ for all $(p,q) \in \mathbb{R}^2_{++}$. Note that from propositions 1 and 2 we get that for both first-score and second-score auctions $S_q + S_p c' = 0$. Differentiating this equation w.r.t θ we get that for both auctions

$$S_{qp} p'(\theta) + S_{qq} q'(\theta) + c'\left[S_{pp} p'(\theta) + S_{pq} q'(\theta)\right] + S_p c'' q'(\theta) = 0 - - - - (37)$$

Since $-\frac{S_q}{S_p} = c'$, by substituting for c' and rearranging terms in (37) we get

$$p'(\theta)\left[-\frac{S_q}{S_p} S_{pp} + S_{qp}\right] + \frac{q'(\theta)}{S_p}\left[S_p S_{qq} - S_q S_{pq} + (S_p)^2 c''\right] = 0$$

Since $-\frac{S_q}{S_p} S_{pp} + S_{qp} = 0$ the above implies that

$$\frac{q'(\theta)}{S_p}\left[S_p S_{qq} - S_q S_{pq} + (S_p)^2 c''\right] = 0 - - - - (37a)$$

From assumption 4 we know $S_p S_{qq} - S_q S_{pq} + (S_p)^2 c'' > -\frac{(S_q)^2}{S_p} S_{pp} + S_q S_{qp} = S_q \left[-\frac{S_q}{S_p} S_{pp} + S_{qp}\right] = 0$ since $-\frac{S_q}{S_p} S_{pp} + S_{qp} = 0$. This means $S_p S_{qq} - S_q S_{pq} + (S_p)^2 c'' > 0$. Since $S_p < 0$ from (37a) we get that for both auctions $q'(\theta) = 0$ for all θ. That is, $\frac{dq^I(\theta)}{d\theta} = \frac{dq^{II}(\theta)}{d\theta} = 0$. This means that for all θ, $q^I(\theta) = q^I(\bar{\theta})$ and $q^{II}(\theta) = q^{II}(\bar{\theta})$. From lemma 1 we know that $q^I(\bar{\theta}) = q^{II}(\bar{\theta})$ and this implies that for all θ, $q^I(\theta) = q^{II}(\theta)$. This completes our proof of proposition 3.■

Proof of Proposition 5 (i) and (ii) Since by assumption S_{pp} and S_{qp} are continuous $\forall (p,q) \in \mathbb{R}^2_{++}$, then $A(p,q) = -\frac{S_q}{S_p}S_{pp} + S_{qp} \neq 0 \; \forall (p,q) \in \mathbb{R}^2_{++}$ implies either (a) $\forall (p,q) \in \mathbb{R}^2_{++}$, $A(p,q) = -\frac{S_q}{S_p}S_{pp} + S_{qp} > 0$ **or** (b) $\forall (p,q) \in \mathbb{R}^2_{++}$, $A(p,q) = -\frac{S_q}{S_p}S_{pp} + S_{qp} < 0$.

Now suppose $B(p,q) = -\frac{S_q}{S_p}S_{pq} + S_p c'' + S_{qq} < 0$. Note that $p^I(\theta) = \sigma\left(q^I(\theta)\right)$ and $p^{II}(\theta) = \sigma\left(q^{II}(\theta)\right)$. Also note from (28) when $-\frac{S_q}{S_p}S_{pp} + S_{qp} > 0$ we have that $\sigma'(.) > 0$ and $q^I(\theta) > q^{II}(\theta)$ (shown in proposition 3). Since $q^I(\theta) > q^{II}(\theta)$ and $\sigma'(.) > 0$ we get $\sigma\left(q^I(\theta)\right) > \sigma\left(q^{II}(\theta)\right) \Longrightarrow p^I(\theta) > p^{II}(\theta)$. Again, when $-\frac{S_q}{S_p}S_{pp} + S_{qp} < 0$ we have that $\sigma'(.) < 0$ and $q^I(\theta) < q^{II}(\theta)$. Since $q^I(\theta) < q^{II}(\theta)$ and $\sigma'(.) < 0$ we get $\sigma\left(q^I(\theta)\right) > \sigma\left(q^{II}(\theta)\right) \Longrightarrow p^I(\theta) > p^{II}(\theta)$.

Now suppose $B(p,q) = -\frac{S_q}{S_p}S_{pq} + S_p c'' + S_{qq} > 0$. This implies $A(p,q) = -\frac{S_q}{S_p}S_{pp} + S_{qp} < 0$ (see lemma 2). From (28) and proposition 3 we know that when $-\frac{S_q}{S_p}S_{pp} + S_{qp} < 0$ we have $\sigma'(.) \geq 0$ and $q^I(\theta) < q^{II}(\theta)$. Since $q^I(\theta) < q^{II}(\theta)$ and $\sigma'(.) \geq 0$ we get $\sigma\left(q^I(\theta)\right) < \sigma\left(q^{II}(\theta)\right) \Longrightarrow p^I(\theta) < p^{II}(\theta)$.

Now since $p^I(\theta) = \sigma\left(q^I(\theta)\right)$ and $p^{II}(\theta) = \sigma\left(q^{II}(\theta)\right)$ for $\theta \in \left[\underline{\theta}, \bar{\theta}\right]$, we get that for all $\theta \in \left(\underline{\theta}, \bar{\theta}\right)$,

$$\frac{dp^I(\theta)}{d\theta} = \sigma'\left(q^I(\theta)\right)\frac{dq^I(\theta)}{d\theta} = -\left[\frac{-\frac{S_q}{S_p}S_{pq} + S_p c'' + S_{qq}}{-\frac{S_q}{S_p}S_{pp} + S_{qp}}\right]\frac{dq^I(\theta)}{d\theta} \quad ---(38)$$

$$\text{and } \frac{dp^{II}(\theta)}{d\theta} = \sigma'\left(q^{II}(\theta)\right)\frac{dq^{II}(\theta)}{d\theta} = -\left[\frac{-\frac{S_q}{S_p}S_{pq} + S_p c'' + S_{qq}}{-\frac{S_q}{S_p}S_{pp} + S_{qp}}\right]\frac{dq^{II}(\theta)}{d\theta} \quad ---(38a).$$

Note that from proposition 3 we get that $\frac{dq^I(\theta)}{d\theta}$, $\frac{dq^{II}(\theta)}{d\theta}$ have the same sign as $A(p,q) = -\frac{S_q}{S_p}S_{pp} + S_{qp}$. This means $\frac{dp^I(\theta)}{d\theta}$, $\frac{dp^{II}(\theta)}{d\theta}$ has the same sign as $-\left(-\frac{S_q}{S_p}S_{pq} + S_p c'' + S_{qq}\right)$.

This means $B(p,q) = -\frac{S_q}{S_p}S_{pq} + S_p c'' + S_{qq} < 0$ implies $\frac{dp^I(\theta)}{d\theta}$, $\frac{dp^{II}(\theta)}{d\theta} > 0$. Similarly $B(p,q) = -\frac{S_q}{S_p}S_{pq} + S_p c'' + S_{qq} > 0$ implies $\frac{dp^I(\theta)}{d\theta}$, $\frac{dp^{II}(\theta)}{d\theta} < 0$.

(iii) Now suppose $B(p,q) = -\frac{S_q}{S_p}S_{pq} + S_p c'' + S_{qq} = 0$. Using (38) and (38a) we get that $\frac{dp^I(\theta)}{d\theta} = \frac{dp^{II}(\theta)}{d\theta} = 0$. This means that for all θ, $p^I(\theta) = p^I\left(\bar{\theta}\right)$ and $p^{II}(\theta) = p^{II}\left(\bar{\theta}\right)$. From lemma 1 we know that $p^I\left(\bar{\theta}\right) = p^{II}\left(\bar{\theta}\right)$ and this implies that for all θ, $p^I(\theta) = p^{II}(\theta)$. This completes our proof of proposition 3.■

Proof of Lemma 2 Since by assumption S_{pp} and S_{qp} are continuous $\forall (p,q) \in \mathbb{R}^2_{++}$, then $A(p,q) = -\frac{S_q}{S_p}S_{pp} + S_{qp} \neq 0 \; \forall (p,q) \in \mathbb{R}^2_{++}$ implies either (a) $\forall (p,q) \in \mathbb{R}^2_{++}$ $A(p,q) > 0$ **or** (b) $\forall (p,q) \in \mathbb{R}^2_{++}$ $A(p,q) < 0$.

It may be noted that

$$\begin{aligned} B(p,q) &= -\frac{S_q}{S_p}S_{pq} + S_p c'' + S_{qq} = -\frac{1}{S_p}\left[S_q S_{pq} - (S_p)^2 c'' - S_p S_{qq}\right] \\ &< -\frac{1}{S_p}\left[\frac{(S_q)^2}{S_p}S_{pp} - S_q S_{pq}\right] \text{ (assumption 4 of our model)} \\ &= \frac{S_q}{S_p}\left[-\frac{S_q(.)}{S_p(.)}S_{pp}(.) + S_{qp}(.)\right]. \end{aligned}$$

Since $S_p < 0$ and $S_q > 0$ the above means that

$$\begin{aligned} -\frac{S_q(.)}{S_p(.)}S_{pp}(.) + S_{qp}(.) &> 0 \Longrightarrow -\frac{S_q}{S_p}S_{pq} + S_p c'' + S_{qq} < 0. \\ &\Longleftrightarrow \\ -\frac{S_q}{S_p}S_{pq} + S_p c'' + S_{qq} &\geq 0 \Longrightarrow -\frac{S_q}{S_p}S_{pp} + S_{qp} < 0. \\ &\Longleftrightarrow \\ B(p,q) &\geq 0 \Longrightarrow A(p,q) < 0 \end{aligned}$$

Proof of Proposition 6 (i) Note that from proposition 2 it is clear that $q^{II}(\theta)$ does not depend on n. This means $q^{II}(n;\theta) = q^{II}(m;\theta)$.

(ii) Using (29) and the definition of $\gamma(\theta)$ we know that

$$\sigma\left(q^I(n;\theta)\right) - c\left(q^I(n;\theta)\right) = \theta + \gamma(\theta) = \theta + \int_\theta^{\bar{\theta}}\left(\frac{1-F(t)}{1-F(\theta)}\right)^{n-1} dt - - - - (39)$$

Since $\left(\frac{1-F(t)}{1-F(\theta)}\right) < 1$ for all $t \in \left(\theta, \bar{\theta}\right)$, $\left(\frac{1-F(t)}{1-F(\theta)}\right)^{n-1}$ strictly decreases with an increase in n. That is, $\theta + \gamma(\theta)$ strictly decreases with an increase in n.

Now suppose $\forall (p,q) \in \mathbb{R}^2_{++}$, $A(p,q) = -\frac{S_q}{S_p}S_{pp} + S_{qp} > 0$. This implies $\sigma'(q) - c'(q) > 0$ (from (30). If possible let $q^I(n;\theta) \geq q^I(m;\theta)$. But this means $\sigma\left(q^I(n;\theta)\right) - c\left(q^I(n;\theta)\right) \geq \sigma\left(q^I(m;\theta)\right) - c\left(q^I(m;\theta)\right)$. But $\theta + \int_\theta^{\bar{\theta}}\left(\frac{1-F(t)}{1-F(\theta)}\right)^{n-1} dt < \theta + \int_\theta^{\bar{\theta}}\left(\frac{1-F(t)}{1-F(\theta)}\right)^{m-1} dt$. But this is a contradiction as we must have $\sigma\left(q^I(n;\theta)\right) - c\left(q^I(n;\theta)\right) = \theta + \int_\theta^{\bar{\theta}}\left(\frac{1-F(t)}{1-F(\theta)}\right)^{n-1} dt$ and $\sigma\left(q^I(m;\theta)\right) - c\left(q^I(m;\theta)\right) = \theta + \int_\theta^{\bar{\theta}}\left(\frac{1-F(t)}{1-F(\theta)}\right)^{m-1} dt$ (from (39). This means if $n > m$ then $q^I(n;\theta) < q^I(m;\theta)$.

(iii) Now suppose $\forall (p,q) \in \mathbb{R}^2_{++}$, $A(p,q) = -\frac{S_q}{S_p}S_{pp} + S_{qp} < 0$. Using an exactly similar logic as above we can show that if $n > m$ then $q^I(n;\theta) > q^I(m;\theta)$.■

Proof of Proposition 7 (i) Note that from proposition 2 it is clear that $p^{II}(\theta)$ does not depend on n. This means $p^{II}(n;\theta) = p^{II}(m;\theta)$.

(ii) Note that $p^{I}(n;\theta) = \sigma\left(q^{I}(n;\theta)\right)$.

Suppose $B(p,q) = -\frac{S_q}{S_p} S_{pq} + S_p c'' + S_{qq} = 0$, $\forall (p,q) \in \mathbb{R}^2_{++}$. This means $\sigma'(.) = 0$ (using 28). This in turn implies $\sigma\left(q^{I}(n;\theta)\right) = \sigma\left(q^{I}(m;\theta)\right)$. But this means $p^{I}(n;\theta) = p^{I}(m;\theta)$.

(iii) Now suppose $B(p,q) = -\frac{S_q}{S_p} S_{pq} + S_p c'' + S_{qq} > 0$, $\forall (p,q) \in \mathbb{R}^2_{++}$. Using lemma 2 this implies $A(p,q) = -\frac{S_q}{S_p} S_{pp} + S_{qp} < 0$. This means $\sigma'(.) > 0$ (using 28). Since $A(p,q) = -\frac{S_q}{S_p} S_{pp} + S_{qp} < 0$ we get that if $n > m$ we have $q^{I}(n;\theta) > q^{I}(m;\theta)$ (proposition 5). This in turn implies $\sigma\left(q^{I}(n;\theta)\right) > \sigma\left(q^{I}(m;\theta)\right)$. This means $p^{I}(n;\theta) > p^{I}(m;\theta)$.

(iv) First, suppose $\forall (p,q) \in \mathbb{R}^2_{++}$, $B(p,q) = -\frac{S_q}{S_p} S_{pq} + S_p c'' + S_{qq} < 0$ and $A(p,q) = -\frac{S_q}{S_p} S_{pp} + S_{qp} > 0$. This means $\sigma'(.) > 0$ (using 28). Since $-\frac{S_q}{S_p} S_{pp} + S_{qp} > 0$ we get that if $n > m$ we have $q^{I}(n;\theta) < q^{I}(m;\theta)$ (proposition 5). This in turn implies $\sigma\left(q^{I}(n;\theta)\right) < \sigma\left(q^{I}(m;\theta)\right)$. This means $p^{I}(n;\theta) < p^{I}(m;\theta)$.

Now suppose $\forall (p,q) \in \mathbb{R}^2_{++}$, $B(p,q) = -\frac{S_q}{S_p} S_{pq} + S_p c'' + S_{qq} < 0$ and $A(p,q) = -\frac{S_q}{S_p} S_{pp} + S_{qp} < 0$. This means $\sigma'(.) < 0$ (using 28). Since $-\frac{S_q}{S_p} S_{pp} + S_{qp} < 0$ we get that if $n > m$ we have $q^{I}(n;\theta) > q^{I}(m;\theta)$ (proposition 5). This in turn implies $\sigma\left(q^{I}(n;\theta)\right) < \sigma\left(q^{I}(m;\theta)\right)$. This means $p^{I}(n;\theta) < p^{I}(m;\theta)$.■

Proof of Proposition 8 (i) From propositions 5 and 6 we get that or all $n > m$ $q^{II}(n;\theta) = q^{II}(m;\theta)$ and $p^{II}(n;\theta) = p^{II}(m;\theta)$. This implies

$$S^{II}(n;\theta) = S\left(p^{II}(n;\theta), q^{II}(n;\theta)\right) = S\left(p^{II}(m;\theta), q^{II}(m;\theta)\right) = S^{II}(m;\theta).$$

(ii) Given any θ, using (4b) and (4c) we have

$$p^{I}(n;\theta) = c\left(q^{I}(n;\theta)\right) + \theta + \frac{\int_{\theta}^{\bar{\theta}} (1 - F(t))^{n-1} dt}{(1 - F(\theta))^{n-1}}$$

Differentiating the above w.r.t n we get

$$\frac{\partial}{\partial n} p^{I}(n;\theta) = c'\left(q^{I}(n;\theta)\right) \frac{\partial}{\partial n} q^{I}(n;\theta) + \frac{\partial}{\partial n}\left(\frac{\int_{\theta}^{\bar{\theta}} (1 - F(t))^{n-1} dt}{(1 - F(\theta))^{n-1}}\right) - - - - (40)$$

Note that

$$\frac{\partial}{\partial n} S^{I}(n;\theta) = \frac{\partial}{\partial n} S\left(p^{I}(n;\theta), q^{I}(n;\theta)\right) = S_p(.) \frac{\partial}{\partial n} p^{I}(n;\theta) + S_q(.) \frac{\partial}{\partial n} q^{I}(n;\theta)$$

Using (40) the above can be written as

$$\begin{aligned} \frac{\partial}{\partial n} S^{I}(n;\theta) &= S_p(.) \left[c'\left(q^{I}(n;\theta)\right) \frac{\partial}{\partial n} q^{I}(n;\theta) + \frac{\partial}{\partial n}\left(\int_{\theta}^{\bar{\theta}} \left(\frac{1 - F(t)}{1 - F(\theta)}\right)^{n-1} dt\right)\right] \\ &\quad + S_q(.) \frac{\partial}{\partial n} q^{I}(n;\theta) - - - - (40a) \end{aligned}$$

Note that in equilibrium $c'\left(q^I(n;\theta)\right) = -\frac{S_q\left(p^I(n;\theta),q^I(n;\theta)\right)}{S_p\left(p^I(n;\theta),q^I(n;\theta)\right)}$ (see 25). Using this in (40a) together with the fact that $S_p < 0$ and $\frac{\partial}{\partial n}\left(\int_\theta^{\bar{\theta}}\left(\frac{1-F(t)}{1-F(\theta)}\right)^{n-1} dt\right) < 0$ we get

$$\frac{\partial}{\partial n} S^I(n;\theta) = S_p(.) \frac{\partial}{\partial n}\left(\int_\theta^{\bar{\theta}}\left(\frac{1-F(t)}{1-F(\theta)}\right)^{n-1} dt\right) > 0.$$

This completes proof of proposition 7.■

Proof of lemma 3 In a first-score auction the expected score is as follows:

$$\begin{aligned}\Sigma^I &= \int_{\underline{\theta}}^{\bar{\theta}} S\left(p^I(\theta), q^I(\theta)\right) f_1(\theta)\, d\theta = \int_{\underline{\theta}}^{\bar{\theta}} S\left(p^I(\theta), q^I(\theta)\right) dF_1(\theta) \\ &= \left[S\left(p^I(\theta), q^I(\theta)\right) F_1(\theta)\right]_{\underline{\theta}}^{\bar{\theta}} - \int_{\underline{\theta}}^{\bar{\theta}} F_1(\theta)\, dS\left(p^I(\theta), q^I(\theta)\right) - - - (43)\end{aligned}$$

Note that from (25) we have

$$-\frac{S_q\left(p^I(\theta), q^I(\theta)\right)}{S_p\left(p^I(\theta), q^I(\theta)\right)} = c'\left(q^I(\theta)\right). - - - - (44)$$

Also, from (6) we have

$$\forall\theta \in \left(\underline{\theta}, \bar{\theta}\right),\ \frac{dp^I(\theta)}{d\theta} - c'\left(q^I(\theta)\right)\frac{dq^I(\theta)}{d\theta} = 1 + \gamma'(\theta) - - - - (45)$$

Now we have

$$\begin{aligned}dS\left(p^I(\theta), q^I(\theta)\right) &= S_p\left(p^I(\theta), q^I(\theta)\right)\frac{dp^I(\theta)}{d\theta} + S_q\left(\left(p^I(\theta), q^I(\theta)\right)\right)\frac{dq^I(\theta)}{d\theta} \\ &= S_p\left(p^I(\theta), q^I(\theta)\right)\left[\frac{dp^I(\theta)}{d\theta} - c'\left(q^I(\theta)\right)\frac{dq^I(\theta)}{d\theta}\right] \text{ (using 44)} \\ &= S_p\left(p^I(\theta), q^I(\theta)\right)\left[1 + \gamma'(\theta)\right] \text{ (using 45)}\end{aligned}$$

Using the above in (43) we get

$$\Sigma^I = S\left(p^I\left(\bar{\theta}\right), q^I\left(\bar{\theta}\right)\right) - \int_{\underline{\theta}}^{\bar{\theta}} F_1(\theta)\left(1 + \gamma'(\theta)\right) S_p\left(p^I(\theta), q^I(\theta)\right) d\theta.$$

By a similar logic we can show that

$$\Sigma^{II} = S\left(p^{II}\left(\bar{\theta}\right), q^{II}\left(\bar{\theta}\right)\right) - \int_{\underline{\theta}}^{\bar{\theta}} F_2(\theta)\, S_p\left(p^{II}(\theta), q^{II}(\theta)\right) d\theta.$$

This completes our proof for lemma 3.■

Proof of lemma 4 In the proof of lemma 1 we have shown that

$$\lim_{\theta \longrightarrow \bar{\theta}} \gamma(\theta) = 0 - - - - (46)$$

Now

$$\int_{\underline{\theta}}^{\bar{\theta}} F_1(\theta)(1+\gamma'(\theta))\,d\theta = \int_{\underline{\theta}}^{\bar{\theta}} F_1(\theta)\,d\theta + \int_{\underline{\theta}}^{\bar{\theta}} F_1(\theta)\,d\gamma(\theta) - - - - (47)$$

Note that

$$\int_{\underline{\theta}}^{\bar{\theta}} F_1(\theta)\,d\gamma(\theta) = [F_1(\theta)\gamma(\theta)]_{\underline{\theta}}^{\bar{\theta}} - \int_{\underline{\theta}}^{\bar{\theta}} \gamma(\theta)\,dF_1(\theta) - - - - (48)$$

Using (46) we know that $[F_1(\theta)\gamma(\theta)]_{\underline{\theta}}^{\bar{\theta}} = 0$. Since

$$dF_1(\theta) = f_1(\theta)\,d\theta = n(1-F(\theta))^{n-1} f(\theta)\,d\theta \text{ and } \gamma(\theta) = \frac{1}{(1-F(\theta))^{n-1}} \int_{\theta}^{\bar{\theta}} (1-F(t))^{n-1}\,dt$$

from (48) we get

$$\begin{aligned}\int_{\underline{\theta}}^{\bar{\theta}} F_1(\theta)\,d\gamma(\theta) &= -\int_{\underline{\theta}}^{\bar{\theta}} \frac{1}{(1-F(\theta))^{n-1}} \left(\int_{\theta}^{\bar{\theta}} (1-F(t))^{n-1}\,dt\right) n(1-F(\theta))^{n-1} f(\theta)\,d\theta \\ &= -\int_{\underline{\theta}}^{\bar{\theta}} \left[\int_{\theta}^{\bar{\theta}} (1-F(t))^{n-1}\,dt\right] nf(\theta)\,d\theta - - - (49)\end{aligned}$$

Changing the order of integration in (49) we have

$$\begin{aligned}\int_{\underline{\theta}}^{\bar{\theta}} F_1(\theta)\,d\gamma(\theta) &= -n\int_{\underline{\theta}}^{\bar{\theta}} \left[\int_{\underline{\theta}}^{t} f(\theta)\,d\theta\right] (1-F(t))^{n-1}\,dt \\ &= -n\int_{\underline{\theta}}^{\bar{\theta}} F(t)(1-F(t))^{n-1}\,dt \\ &= -n\int_{\underline{\theta}}^{\bar{\theta}} F(\theta)(1-F(\theta))^{n-1}\,d\theta - - - - (50)\end{aligned}$$

Hence using (50) in (47) we have

$$\int_{\underline{\theta}}^{\bar{\theta}} F_1(\theta)(1+\gamma'(\theta))\,d\theta = \int_{\underline{\theta}}^{\bar{\theta}} F_1(\theta)\,d\theta - n\int_{\underline{\theta}}^{\bar{\theta}} F(\theta)(1-F(\theta))^{n-1}\,d\theta - - - - (51)$$

Now note that

$$F_2(\theta) = F_1(\theta) - nF(\theta)(1-F(\theta))^{n-1} - - - -(52)$$

Therefore, from (51) and (52) we get

$$\int_{\underline{\theta}}^{\bar{\theta}} F_1(\theta)(1+\gamma'(\theta))\,d\theta = \int_{\underline{\theta}}^{\bar{\theta}} F_2(\theta)\,d\theta$$

This completes our proof for lemma 4.■

Proof of Proposition 10 In the proof of proposition 3 we defined $\sigma(q)$. Note that using (25) and (26) we get

$$\begin{aligned} -S_p\left(p^I, q^I\right) &= -S_p\left(\sigma\left(q^I\right), q^I\right) \text{ and} \\ -S_p\left(p^{II}, q^{II}\right) &= -S_p\left(\sigma\left(q^{II}\right), q^{II}\right). \end{aligned}$$

From (28) we get that

$$\sigma'(q) = -\frac{B}{A} ----(53).$$

Now note that using (53) we have

$$\begin{aligned} \frac{d}{dq}\left[-S_p(\sigma(q), q)\right] &= -S_{pp}\sigma'(q) - S_{pq} \\ &= S_{pp}\frac{B}{A} - S_{pq} ----(54) \end{aligned}$$

By the hypothesis $\forall (p,q) \in \mathbb{R}^2_{++}$, $A(.) \neq 0$. This means we have either $A(.) > 0$ or $A(.) < 0$ $\forall (p,q) \in \mathbb{R}^2_{++}$. From proposition 4 we get that for any given θ, $A(.) > 0$ implies $q^I > q^{II}$ and $A(.) < 0$ implies $q^I < q^{II}$. If $S_{pp}\frac{B}{A} - S_{pq} = 0$ $\forall (p,q) \in \mathbb{R}^2_{++}$, then $\frac{d}{dq}\left[-S_p(\sigma(q), q)\right] = 0$ for all q. This means for all $\theta \in \left[\underline{\theta}, \bar{\theta}\right]$

$$\begin{aligned} -S_p\left(\sigma\left(q^I(\theta)\right), q^I(\theta)\right) &= -S_p\left(\sigma\left(q^I\left(\bar{\theta}\right)\right), q^I\left(\bar{\theta}\right)\right) \text{ and} \\ -S_p\left(\sigma\left(q^{II}(\theta)\right), q^{II}(\theta)\right) &= -S_p\left(\sigma\left(q^{II}\left(\bar{\theta}\right)\right), q^{II}\left(\bar{\theta}\right)\right) ----(55) \end{aligned}$$

Note that $q^I\left(\bar{\theta}\right) = q^{II}\left(\bar{\theta}\right)$ (lemma 1). This means (by using (55)) that for all $\theta \in \left[\underline{\theta}, \bar{\theta}\right]$

$$-S_p\left(\sigma\left(q^I(\theta)\right), q^I(\theta)\right) = -S_p\left(\sigma\left(q^{II}(\theta)\right), q^{II}(\theta)\right) ----(56)$$

Note that $S\left(p^I\left(\bar{\theta}\right), q^I\left(\bar{\theta}\right)\right) = S\left(p^{II}\left(\bar{\theta}\right), q^{II}\left(\bar{\theta}\right)\right)$ (see lemma 1). Using this,lemma 3. (55) and (56) above we get

$$\begin{aligned} &\Sigma^I - \Sigma^{II} \\ = &\int_{\underline{\theta}}^{\bar{\theta}} \left[F_2(\theta) S_p\left(p^{II}(\theta), q^{II}(\theta)\right) - F_1(\theta)\left(1+\gamma'(\theta)\right) S_p\left(p^I(\theta), q^I(\theta)\right)\right] d\theta \\ = &\int_{\underline{\theta}}^{\bar{\theta}} \left[F_2(\theta) S_p\left(\sigma\left(q^{II}(\theta)\right), q^{II}(\theta)\right) - F_1(\theta)\left(1+\gamma'(\theta)\right) S_p\left(\sigma\left(q^I(\theta)\right), q^I(\theta)\right)\right] d\theta \\ = &\int_{\underline{\theta}}^{\bar{\theta}} S_p\left(\sigma\left(q^I\left(\bar{\theta}\right)\right), q^I\left(\bar{\theta}\right)\right)\left[F_2(\theta) - F_1(\theta)\left(1+\gamma'(\theta)\right)\right] d\theta \\ = &S_p\left(\sigma\left(q^I\left(\bar{\theta}\right)\right), q^I\left(\bar{\theta}\right)\right) \int_{\underline{\theta}}^{\bar{\theta}} \left[F_2(\theta) - F_1(\theta)\left(1+\gamma'(\theta)\right)\right] d\theta \\ = &0 \text{ (using lemma 4).} ----(57) \end{aligned}$$

(57) above proves proposition 10.■

Proof of Proposition 11 Note that

$$\begin{aligned}
&\Sigma^{II}-\Sigma^{I}\\
=&\int_{\underline{\theta}}^{\bar{\theta}}\left[S^{II}(\theta)f_2(\theta)-S^{I}(\theta)f_1(\theta)\right]d\theta\\
=&\int_{\underline{\theta}}^{\bar{\theta}}n\left(1-F(\theta)\right)^{n-2}f(\theta)\left[(n-1)F(\theta)S^{II}(\theta)-S^{I}(\theta)\left(1-F(\theta)\right)\right]d\theta\\
=&\int_{\underline{\theta}}^{\bar{\theta}}\left[(n-1)F(\theta)S^{II}(\theta)-S^{I}(\theta)\left(1-F(\theta)\right)\right]d\left(-\frac{n}{n-1}\left(1-F(\theta)\right)^{n-1}\right)\\
=&-\frac{n}{n-1}S^{I}(\underline{\theta})+\frac{n}{n-1}I----(58)\\
\text{where } I =&\int_{\underline{\theta}}^{\bar{\theta}}\left(1-F(\theta)\right)^{n-1}\left[\begin{array}{c}(n-1)F(\theta)\frac{d}{d\theta}S^{II}(\theta)+(n-1)S^{II}(\theta)f(\theta)\\-\left(1-F(\theta)\right)\frac{d}{d\theta}S^{I}(\theta)+S^{I}(\theta)f(\theta)\end{array}\right]d\theta
\end{aligned}$$

Now

$$\begin{aligned}
I =&\int_{\underline{\theta}}^{\bar{\theta}}\left(1-F(\theta)\right)^{n-1}f(\theta)\left[(n-1)S^{II}(\theta)+S^{I}(\theta)\right]d\theta\\
&+\int_{\underline{\theta}}^{\bar{\theta}}\left(1-F(\theta)\right)^{n-1}\left[\begin{array}{c}(n-1)F(\theta)\frac{d}{d\theta}S^{II}(\theta)\\-\left(1-F(\theta)\right)\frac{d}{d\theta}S^{I}(\theta)\end{array}\right]d\theta---(59)
\end{aligned}$$

Note that

$$\begin{aligned}
&\int_{\underline{\theta}}^{\bar{\theta}}\left(1-F(\theta)\right)^{n-1}f(\theta)\left[(n-1)S^{II}(\theta)+S^{I}(\theta)\right]\\
=&\int_{\underline{\theta}}^{\bar{\theta}}\left[(n-1)S^{II}(\theta)+S^{I}(\theta)\right]d\left(-\frac{\left(1-F(\theta)\right)^{n}}{n}\right)\\
=&\frac{n-1}{n}S^{II}(\underline{\theta})+\frac{1}{n}S^{I}(\underline{\theta})\\
&+\int_{\underline{\theta}}^{\bar{\theta}}\frac{\left(1-F(\theta)\right)^{n}}{n}\left[(n-1)\frac{d}{d\theta}S^{II}(\theta)+\frac{d}{d\theta}S^{I}(\theta)\right]d\theta---(60)
\end{aligned}$$

Therefore, using (59) and (60) we get

$$\begin{aligned}
I =&\frac{n-1}{n}S^{II}(\underline{\theta})+\frac{1}{n}S^{I}(\underline{\theta})\\
&+\int_{\underline{\theta}}^{\bar{\theta}}\frac{\left(1-F(\theta)\right)^{n}}{n}\left[(n-1)\frac{d}{d\theta}S^{II}(\theta)+\frac{d}{d\theta}S^{I}(\theta)\right]d\theta\\
&+\int_{\underline{\theta}}^{\bar{\theta}}\left(1-F(\theta)\right)^{n-1}\left[\begin{array}{c}(n-1)F(\theta)\frac{d}{d\theta}S^{II}(\theta)\\-\left(1-F(\theta)\right)\frac{d}{d\theta}S^{I}(\theta)\end{array}\right]d\theta---(61)
\end{aligned}$$

Now note that

$$\begin{aligned}
&\int_{\underline{\theta}}^{\bar{\theta}} \frac{(1-F(\theta))^n}{n}\left[(n-1)\frac{d}{d\theta}S^{II}(\theta)+\frac{d}{d\theta}S^{I}(\theta)\right]d\theta \\
&+\int_{\underline{\theta}}^{\bar{\theta}} (1-F(\theta))^{n-1}\left[\begin{array}{c}(n-1)F(\theta)\frac{d}{d\theta}S^{II}(\theta)\\ -(1-F(\theta))\frac{d}{d\theta}S^{I}(\theta)\end{array}\right]d\theta \\
=&\int_{\underline{\theta}}^{\bar{\theta}} (1-F(\theta))^{n-1}\left[\begin{array}{c}(n-1)F(\theta)\frac{d}{d\theta}S^{II}(\theta)-(1-F(\theta))\frac{d}{d\theta}S^{I}(\theta)\\ +\frac{(n-1)}{n}(1-F(\theta))\frac{d}{d\theta}S^{II}(\theta)\\ +\frac{1}{n}(1-F(\theta))\frac{d}{d\theta}S^{I}(\theta)\end{array}\right]d\theta \\
=&\int_{\underline{\theta}}^{\bar{\theta}} (1-F(\theta))^{n-1}\left[\begin{array}{c}\frac{(n-1)}{n}((n-1)F(\theta)+1)\frac{d}{d\theta}S^{II}(\theta)\\ -\frac{(n-1)}{n}(1-F(\theta))\frac{d}{d\theta}S^{I}(\theta)\end{array}\right]d\theta \\
>&\int_{\underline{\theta}}^{\bar{\theta}} (1-F(\theta))^{n-1}\left[\frac{(n-1)}{n}((n-1)F(\theta)+1)\frac{d}{d\theta}S^{II}(\theta)\right]d\theta---(62)
\end{aligned}$$

since $\frac{d}{d\theta}S^{I}(\theta)<0$ for all $\theta\in(\underline{\theta},\bar{\theta})$ (see proposition 3) and $F(\theta)<1$ for $\theta<\bar{\theta}$.

Therefore using (61) and (62)

$$\begin{aligned}
I >& \frac{n-1}{n}S^{II}(\underline{\theta})+\frac{1}{n}S^{I}(\underline{\theta}) \\
&+\int_{\underline{\theta}}^{\bar{\theta}} (1-F(\theta))^{n-1}\left[\frac{(n-1)}{n}((n-1)F(\theta)+1)\frac{d}{d\theta}S^{II}(\theta)\right]d\theta--(63)
\end{aligned}$$

Using (58) and (63) we get

$$\begin{aligned}
&\Sigma^{II}-\Sigma^{I} \\
>&\frac{n-1}{n}\left[\begin{array}{c}S^{II}(\underline{\theta})-S^{I}(\underline{\theta})\\ +\int_{\underline{\theta}}^{\bar{\theta}}(1-F(\theta))^{n-1}\left[((n-1)F(\theta)+1)\frac{d}{d\theta}S^{II}(\theta)\right]d\theta\end{array}\right]--(64)
\end{aligned}$$

Now since $f'(\theta)\leq 0$ for all $\theta\in[\underline{\theta},\bar{\theta}]$ we get that $\frac{F(\theta)}{f(\theta)}$ and $\frac{1}{f(\theta)}$ is increasing in θ. This means

$$\begin{aligned}
\frac{F(\theta)}{f(\theta)} &\leq \frac{F(\bar{\theta})}{f(\bar{\theta})}=\frac{1}{f(\bar{\theta})} \text{ and} \\
\frac{1}{f(\theta)} &\leq \frac{1}{f(\bar{\theta})}----(65)
\end{aligned}$$

We also know that $\frac{d}{d\theta}S^{II}(\theta)<0$ for all $\theta\in[\underline{\theta},\bar{\theta}]$ (see proposition 3). Therefore,

$$\begin{aligned}
&\int_{\underline{\theta}}^{\bar{\theta}} (1-F(\theta))^{n-1}\left[((n-1)F(\theta)+1)\frac{d}{d\theta}S^{II}(\theta)\right]d\theta \\
=&\int_{\underline{\theta}}^{\bar{\theta}} (1-F(\theta))^{n-1}f(\theta)\left[\left((n-1)\frac{F(\theta)}{f(\theta)}+\frac{1}{f(\theta)}\right)\frac{d}{d\theta}S^{II}(\theta)\right]d\theta \\
\geq&\int_{\underline{\theta}}^{\bar{\theta}} (1-F(\theta))^{n-1}f(\theta)\left[\left((n-1)\frac{1}{f(\bar{\theta})}+\frac{1}{f(\bar{\theta})}\right)\frac{d}{d\theta}S^{II}(\theta)\right]d\theta \\
=&\frac{1}{f(\bar{\theta})}\int_{\underline{\theta}}^{\bar{\theta}}\left[\frac{d}{d\theta}S^{II}(\theta)\right]\left[n(1-F(\theta))^{n-1}f(\theta)\right]d\theta----(66)
\end{aligned}$$

$$\text{Let } \frac{d}{d\theta}S^{II}(\theta) = k(\theta) ----(67)$$

Hence, from (66) we get that

$$\int_{\underline{\theta}}^{\bar{\theta}} (1-F(\theta))^{n-1} \left[((n-1)F(\theta)+1) \frac{d}{d\theta}S^{II}(\theta) \right] d\theta$$
$$\geq \frac{1}{f(\bar{\theta})} \int_{\underline{\theta}}^{\bar{\theta}} [k(\theta)] \left[n(1-F(\theta))^{n-1} f(\theta) \right] d\theta ----(68)$$

Since $n(1-F(\theta))^{n-1} f(\theta) \geq 0$ for all $\theta \in \left[\underline{\theta}, \bar{\theta}\right]$, by using the mean value theorem for integrals we know that there exists $\varepsilon \in \left[\underline{\theta}, \bar{\theta}\right]$ such that

$$\frac{1}{f(\bar{\theta})} \int_{\underline{\theta}}^{\bar{\theta}} [k(\theta)] \left[n(1-F(\theta))^{n-1} f(\theta) \right] d\theta$$
$$= \frac{1}{f(\bar{\theta})} k(\varepsilon) \int_{\underline{\theta}}^{\bar{\theta}} \left[n(1-F(\theta))^{n-1} f(\theta) \right] d\theta ---(69)$$

Now note that

$$\int_{\underline{\theta}}^{\bar{\theta}} \left[n(1-F(\theta))^{n-1} f(\theta) \right] d\theta$$
$$= \int_{\underline{\theta}}^{\bar{\theta}} d\left(-(1-F(\theta))^{n}\right) = 1 ----(70)$$

Hence from (69) and (70) we get that

$$\frac{1}{f(\bar{\theta})} \int_{\underline{\theta}}^{\bar{\theta}} \left[\frac{d}{d\theta}S^{II}(\theta) \right] \left[n(1-F(\theta))^{n-1} f(\theta) \right] d\theta = \frac{1}{f(\bar{\theta})} k(\varepsilon) ---(71)$$

Hence, from (68) and (71) we get that

$$\int_{\underline{\theta}}^{\bar{\theta}} (1-F(\theta))^{n-1} \left[((n-1)F(\theta)+1) \frac{d}{d\theta}S^{II}(\theta) \right] d\theta \geq \frac{1}{f(\bar{\theta})} k(\varepsilon) ----(72)$$

Now from (64) and (72)

$$\Sigma^{II} - \Sigma^{I} > \frac{n-1}{n} \left[S^{II}(\underline{\theta}) - S^{I}(\underline{\theta}) + \frac{1}{f(\bar{\theta})} k(\varepsilon) \right] ----(73)$$

From proposition 2 we know that $k(\varepsilon)$ is independent of $f(.)$. From proposition 3 we know that $S^{II}(\underline{\theta}) - S^{I}(\underline{\theta}) > 0$ and $k(\varepsilon) < 0$. Hence, if $f(\bar{\theta})$ is large enough then $S^{II}(\underline{\theta}) - S^{I}(\underline{\theta}) + \frac{1}{f(\bar{\theta})} k(\varepsilon) > 0$. Using (73) this implies $\Sigma^{II} - \Sigma^{I} > 0$.■