

A Tractable Monetary Model under General Preferences

TSZ-NGA WONG*

The Bank of Canada

First version received November 2013; final version accepted June 2015 (Eds.)

Abstract

This paper studies an economy with both centralised and decentralised monetary exchanges under search frictions. A degenerate asset distribution is featured under a broad class of preferences including, for example, constant return to scale, constant elasticity of substitution, CARA and others from a range of macroeconomic literatures. Some novel applications impossible under quasi-linear preferences, for example endogenous growth, are illustrated under this class of preferences. This paper finds that the welfare cost and growth loss of inflation can be much higher in these applications than previous estimates.

Keywords: Money, Search and Matching, Bargaining and General Trading Protocols, Endogenous Growth, Heterogeneous-Agent Model.

* *Acknowledgement.* For their helpful comments, I thank Gabriele Camera, Jonathan Chiu, Tai-Wai Hu, Ricardo Lagos, Miguel Molico, Daniel Sanches, Enchuan Shao, Eric Swanson, Steve Williamson, and especially Guillaume Rocheteau and Randy Wright, as well as the participants of the Money, Banking, Payment and Finance Workshop 2013 at the Federal Reserve Bank of Chicago, the Midwest Theory Meeting 2013 at Michigan State University as well as the seminars at the Bank of Canada and UC Irvine for comments. I also thank three anonymous referees for very helpful suggestions. I am deeply grateful for comments from the editor Phillip Kircher and one referee, which substantially improved this paper. The views expressed in the paper are mine and nothing should be attributed to the Bank of Canada.

1 Introduction

In their seminal work, Lagos and Wright (2005, henceforth LW) devise a monetary model with uninsurable risks of decentralised trades that features a degenerate distribution of money holding in the equilibrium.¹ The degeneracy makes keeping track of distribution manageable in a dynamic economy; otherwise the analysis can be challenging as in the incomplete markets literature. Two key features in LW are to allow agents to trade in centralised markets between decentralised trades, and to assume quasi-linear preferences in those centralised markets.

This paper generalises LW's results to a broad class of preferences without quasi-linearity. This class of preferences includes the Stone-Geary utility, Greenwood–Hercowitz–Huffman utility, and utility functions with features like constant return to scale, constant elasticity of substitution (and hence Cobb-Douglas), CARA, quasi-linearity certainly, and many others. With this class of preferences, the history of money holdings does not affect the optimal money holding; nevertheless it is affected by preferences, technology and policy. Such a feature allows endogenous, degenerate distribution of money holding in the simple setting without ex-ante heterogeneity; and even with large ex-ante heterogeneity it allows a non-degenerate but still tractable distribution.

But, beyond the technical contribution, why generalise the preferences? Of course, it never hurts to work with a broader class of preferences. Yet a more important argument is that the quasi-linearity assumption undermines some of LW's intended advantages. On one hand the introduction of centralised markets taps into the potential of connecting a search model of money to a significant body of macroeconomic literature; however, on the other hand, the quasi-linearity assumption can be too restrictive for relevant macroeconomic applications. One important example of this conflict can be found when applying LW to study inflation and endogenous growth. In Section 5 I show that incorporating the standard endogenous growth model of Lucas (1988) in LW fails to generate endogenous growth, essentially because of the absence of curvature under the quasi-linear preferences. Instead, formulating endogenous growth with this class of preferences can easily capture a growth mechanism of inflation, which is also consistent with data. This example makes the point that the generalisation of preferences offers the capacity to model applications impossible under quasi-linear preferences. It also illustrates that these new applications of LW can have important policy

¹See Lagos, Rocheteau and Wright (2014) for a recent survey. Also see Shi (1997) for an alternative "big household" approach, where the trading risks of members are pooled within the household. See Lagos and Wright (2005) for a comparison of the two approaches.

implications. For instance, when growth is endogenous, inflation has a larger impact on welfare because not only is current output affected, but also the entire path of future output. In sum, generalising preferences helps open up a wide new possibility of applications leading to a very different understanding of policies, without compromising tractability.

The generalisation does not only widen the scope of the search models of money, but also other macroeconomic literatures. Going back to the previous example, it introduces to the endogenous growth literature new features such as search, bargaining and various trading protocols; novel mechanisms such as the strategic complementarity between buyers and sellers; and additional sources of inefficiency such as the two-sided hold-up problem. While these are impossible under the standard paradigm of Walrasian markets, their synthesis is now possible with the search models since this class of preferences has been used in many areas of macroeconomics; see Section 4 for a survey.

While in this paper I only focus on the monetary models, in a broader sense, in any intertemporal model with heterogeneity inserting a subperiod with this class of preferences will lead to a tractable wealth distribution - an analytical environment not commonly found in the literature.² Section 6 elaborates this feature. Alternatively, while the numerical approach is also useful and complementary, it has less to say about properties such as existence, uniqueness or multiplicity, and dynamics. A tractable model also provides a useful benchmark to validate numerical methods. Furthermore, a tractable model makes heterogeneous-agent models easily extendable to other applications, for example, even to the non-stationary environment with endogenous growth shown in Section 5. Finally, a tractable model helps explain the economy, especially when several mechanisms come into effect, which are difficult to single out from numerical examples.

2 Basic Model

Overview. The environment is the same as LW except this paper use more general preferences. Time is discrete and infinite, indexed by $t = 0, 1, \dots$. Alternating in each period are two markets: a frictional decentralised market (DM) where agents match bilaterally and bargain,

²See Bewley (1980), Aiyagari (1994), and Buera and Shin (2013) for the challenge of distribution under incomplete markets. One can still proceed with the distribution directly but restricting the terms of trade as well as the domain and divisibility of assets, as in Kiyotaki and Wright (1989) and Trejos and Wright (1995). Or one can apply numerical methods "approximating" the equilibrium distribution like Krusell and Smith (1996), Molico (2006) and Chiu and Molico (2010). Also see Menzies, Shi and Sun (2013) for a directed search approach for a relatively tractable computational model that produces non-degenerate distribution. See Rocheteau, Weill and Wong (2015) for a distribution model formulated in the continuous time.

and a frictionless centralised market (CM) where agents trade with each other at Walrasian prices. In the DM agents can only observe the actions and outcomes of their trades, and are anonymous. There is no technology for recordkeeping, commitment or coordinating global punishment. As a result, debt contract is infeasible and a medium of exchange - money - is essential for trades in the DM.³ Money supply grows exogenously, $M_{t+1} = (1 + \tau) M_t$, by lump-sum transfers to agents in the CM. Let ϕ be the price of money in terms of the CM goods (numeraire). There can be inflation in this economy; denote the inflation rate as $\pi_t \equiv \phi_t / \phi_{t+1} - 1$. As seen later, the real balances $z \equiv \phi m$ is the relevant state variable for agent's decision.

Technology and preference. A unit measure of agents live forever with a discount factor $\beta \in (0, 1)$. Utility in a period involves actions in the CM and DM. In the CM, agents consume the numeraire goods X , where the production function is AH using labour H ; in the DM, agents consume a different goods x , produced one-to-one with the DM labour q . The following form of utility function is maintained throughout this paper:

$$\tilde{U}(X, H, x, q) = u(x) - c(q) + U(X, \bar{H} - H),$$

where \bar{H} is the maximal H an agent can work. The agent's lifetime preferences are given by $\mathbb{E} \sum \beta^t \tilde{U}(X_t, H_t, x_t, q_t)$. The following standard assumptions are thus made:⁴

Assumption 1 (Smoothness) $U \in \mathcal{C}^2$ and $u \in \mathcal{C}^2$, where $U_{XX} < 0$, $U_{HH} \leq 0$, $U_{XH} \leq 0$, $u_x > 0$, $u_{xx} < 0$, $c_q > 0$, $c_{qq} \geq 0$ and u satisfies the Inada condition. Also, $u(0) = c(0) = 0$.

Assumption 2 (Large labour endowment) For any finite ω , there exists a unique solution $X > 0$ and $H \in (0, \bar{H})$ to $\max_{X, H} U(X, \bar{H} - H)$ s.t. $X = AH + \omega$, if \bar{H} is sufficiently large.

The agent's problem is as follows. Let $W_t(s)$ and $V_t(z)$ denote the value functions in the CM and DM with real balances s and z respectively. The CM problem is

$$W_t(s) = \max_{X, H, z} \{U(X, \bar{H} - H) + \beta V_{t+1}(z)\}, \quad (1)$$

³See Kocherlakota (1998) for the role of recordkeeping, and Wallace (2010) for details on the essentiality of money. The DM is a useful modelling device to capture all these frictions.

⁴Assumption 1 guarantees that the economy is "smooth", and there exist some trades in the DM with a positive social surplus (otherwise no trade in the DM is preferable). Assumption 2 guarantees an interior solution in the equilibrium. Both assumptions are satisfied in Lagos and Wright (2005). For simplicity I do not consider any upper bound for q .

$$\text{s.t. } X = AH + s - (1 + \pi_{t+1})z + T_t, \quad (2)$$

$$X \geq 0, H \in [0, \overline{H}], z \geq 0, \quad (3)$$

where (2) is the agent's budget constraint, (3) are boundary constraints, T_t is the lump sum injection (drain if negative) of money from the government, and the inflation factor $1 + \pi_{t+1} = \phi_t / \phi_{t+1}$ captures changes in the real prices across periods. Let $X_t(s)$, $H_t(s)$ and $z_t(s)$ denote the solutions to (1) given s units of real balances and the DM value function $V_{t+1}(z)$. The Bellman equation for $V_{t+1}(z)$ is given below. The solution $\{X_t(s), H_t(s), z_t(s)\}$ to (1) is interior if the boundary constraints (3) are slack.

Decentralised trades. Now I turn to the heart of the double-coincidence problem in the DM that gives rise to monetary exchange. Some agents (sellers) can produce but do not want to consume, while others (buyers) want to consume but cannot produce, and hence trades emerge. In the DM, with probability $\alpha \leq 1/2$, the agent matches and buys x from a seller with payoff $u(x)$. By symmetry, there is also probability α that the agent matches and sells q to a buyer with cost $c(q)$.

Suppose the real balances held by the buyer and seller are z and z' respectively. In a bilateral match in the DM, the seller's sales q must equal the buyer's purchase x . Denote their common value as $q(z, z', W_t)$, and the real balances that change hands to the seller by $d(z, z', W_t)$, which in general depends on the real balances and the continuation value. Denote the buyer's surplus and seller's surplus as

$$\begin{aligned} S^b(z, z', W_t) &\equiv u[q(z, z', W_t)] + W_t[z - d(z, z', W_t)] - W_t(z), \\ S^s(z, z', W_t) &\equiv -c[q(z, z', W_t)] + W_t[z' + d(z, z', W_t)] - W_t(z'). \end{aligned}$$

At the beginning of DM, given real balances z , the agent's value function $V_t(z)$ is given by the following Bellman equation

$$V_t(z) = \alpha \int S^b(z, z', W_t) dF_t(z') + \alpha \int S^s(z', z, W_t) dF_t(z') + W_t(z), \quad (4)$$

where $F_t(z)$ is the distribution of z in the beginning of the DM. There is probability density $\alpha dF_t(z')$ that an agent meets a seller (buyer) with real balances z' in the DM and results in the buyer surplus $S^b(z, z', W_t)$ [seller surplus $S^s(z', z, W_t)$]. Thus, $S^b(z, z', W_t)$ and $S^s(z, z', W_t)$ summarize the essential information about the trading protocols $\{q(z, z', W_t), d(z, z', W_t)\}$ for agents to choose their money holdings.

Trading protocols. Without loss of generality, I take the surplus functions $S^b(z, z', W_t)$ and $S^s(z, z', W_t)$ as primitives, since the mappings only depend on the exogenous trading protocol $\{q(\cdot), d(\cdot)\}$ rather than any endogenous variable. This formulation effectively covers any trading protocol without going into details. For example, consider the trading protocol $\{q(z, z', W_t), d(z, z', W_t)\}$ that solves the Nash bargaining problem:

$$\begin{aligned} & \max_{q,d} \left(S^b \right)^\theta (S^s)^{1-\theta} \text{ s.t.} & (5) \\ S^b &= u(q) + W_t(z - d) - W_t(z), \\ S^s &= -c(q) + W_t(z' + d) - W_t(z'), \\ d &\in [-z', z]. \end{aligned}$$

The buyer's bargaining power is given by $\theta \in (0, 1]$. The Nash bargaining problem can be sufficiently represented by the corresponding $S^b(z, z', W_t)$ and $S^s(z, z', W_t)$ to (5). In general, I maintain the following assumption on $S^b(z, z', W_t)$ and $S^s(z, z', W_t)$.⁵

Assumption 3 (Trading protocol) For any $z, z' > 0$ and $j = b, s$,

- (i) $S^j(z, z', W)$ is upper semi-continuous in z and z' , $S^j(z, z', W) \geq 0$ and $S_1^b(0_+, z', W) + S_2^s(z', 0_+, W) > \frac{1+\tau-\beta}{\alpha\beta} W_1(0_+)$;
- (ii) $S^j(z, z', W) = S^j(z, z', W + w_0)$ for any finite w_0 .
- (iii) There exists $\bar{z} < \infty$ such that $S^b(z, z', W) \leq S^b(\bar{z}, z', W)$ and $S^s(z', z, W) \leq S^s(z', \bar{z}, W)$ for any $z \geq \bar{z}$ and $z' \geq 0$.

Assumption 3 is weak and natural; it is satisfied under protocols like Nash bargaining, proportional bargaining, random ultimatum (illustrated in the later sections), competitive pricing, pairwise core and many others.⁶

Equilibrium. The law of motions of the individual real balances s_t and z_t depends on

⁵Gu, Mattesini and Wright (2014) also propose a general trading protocol based on a tighter structure of axioms. Assumption 3 is tailored to establish the proof of degeneracy in this economy. Assumption 3i guarantees the solution to the DM trades exists, which involves a positive payment; Assumption 3ii states that a "parallel shift" to the continuation value does not change the trading protocol; Assumption 3iii guarantees infinite money holding is never optimal.

⁶Notice that with the generalized Nash bargaining $S^b(z, z', W)$ may not increase in z ; with the proportional bargaining $S_1^b(0, z', W)$ is bounded even though $u_x(0) = \infty$; with random ultimatum $q(z, z', W)$ and $d(z, z', W)$ can be stochastic; and with pairwise core $S^b(z, z', W)$ may not be continuous. These are allowed in my formulation of the general trading protocol.

the trading protocol, which is given by

$$s_t = \begin{cases} z - d(z, z', W_t) & \text{with density } \alpha dF_t(z'), \\ z + d(z', z, W_t) & \text{with density } \alpha dF_t(z'), \\ z & \text{with probability } 1 - 2\alpha, \end{cases} \quad (6)$$

$$z_{t+1} = z_t(s), \text{ where } s \text{ follows } G_t(s), \quad (7)$$

where $G_t(s)$ is the distribution of s according to (6) and $F_{t+1}(z)$ is the distribution of z according to (7). Suppose F_0 is degenerate at $z = \phi_0 M_0$. Define a (monetary) equilibrium:

Definition 1 *An equilibrium consists of positive real prices $\{\phi_t\}_{t=0}^\infty$, allocation $\{X_t, H_t, z_t, q_t, d_t\}_{t=0}^\infty$ and distributions $\{F_t, G_t\}_{t=0}^\infty$ such that*

- (i. agent optimization) *For any $s \in \text{dom}(G_t)$, $\{X_t(s), H_t(s), z_t(s)\}$ solves the agent's optimization problem (1), and $\{W_t(s), V_t(z)\}$ solves the Bellman equations (1) and (4);*
- (ii. bargaining) *$q_t = q(z, z', W_t)$ and $d_t = d(z, z', W_t)$ for any $z, z' \in \text{dom}(F_t)$;*
- (iii. goods markets clear) *$\int X_t(s) dG_t(s) = A \int H_t(s) dG_t(s)$;*
- (iv. money markets clear) *$\int z_t(s) dG_t(s) = \phi_{t+1} M_{t+1}$;*
- (v. government budget) *$T_t = \tau \phi_t M_t$;*
- (vi. law of motion) *$G_t(s)$ and $F_t(z)$ are given by (6) and (7).*

A degenerate (and stationary monetary) equilibrium is an equilibrium that also satisfies $z_t(s) = z^*$ for all $s \in \text{dom}(G_t)$, i.e., all agents always exit the CM with the same real balances, regardless of the ones they enter the CM. In general, there can be degenerate equilibria that are, for example, non-monetary, non-stationary, cyclic, chaotic or driven by sunspots; they are not analysed in this paper.⁷

3 Main Results

I now specify a general environment that admits a degenerate equilibrium. To ease the presentation, I first digress to introduce an assumption on preferences and then verify the degenerate equilibrium.

Assumption 4 (General preferences) $U \in \mathcal{C}^2$ satisfies

$$U_{XX}U_{HH} - (U_{XH})^2 = 0. \quad (8)$$

⁷These examples of equilibria can be found in Lagos and Wright (2003).

I refer the set $\mathcal{U} \subseteq \mathcal{C}^2$ as the collection of utility functions U satisfying (8). Obviously, the quasi-linear preferences in LW satisfy Assumption 3 with $U_{HH} = U_{XH} = 0$. An important property of U is given by the following lemma:

Lemma 1 *If $U_{XX} \neq 0$, then U satisfies (8) if and only if there exists a function $\Lambda \in \mathcal{C}^1$ such that $U_H = \Lambda \circ U_X$.*

Lemma 1 is helpful to give some sense about how degeneracy is possible. Suppose solutions are interior and V_{t+1} is differentiable. From the first order condition with respect to z in (1), I have

$$(1 + \pi)U_{X,t} = \beta V_{z,t+1}.$$

Also, from the first order condition with respect to X in (1), I have the intertemporal Euler's equation:

$$AU_{X,t} = -U_{H,t}.$$

Combining these two equations with $U_{H,t} = \Lambda(U_{X,t})$ from Lemma 1, I have

$$\frac{\beta A}{1 + \pi} = \frac{\Lambda \left[\frac{\beta}{1 + \pi} V_{z,t+1}(z) \right]}{V_{z,t+1}(z)}.$$

Since the right side is a function of z only, not s , it implies $z_t(s)$ is a constant function on the equilibrium path. Thus, the history of money holding does not affect the optimal money holding. For example, it is straightforward to verify the non quasi-linear case $U = X^{1-\sigma}(\bar{H} - H)^\sigma$. In general V_{t+1} might not be differentiable (for example, under the pairwise-core trading protocol used in Hu, Kennan and Wallace 2009) and the solutions might not be interior, so a more sophisticated argument is needed to show the degenerate equilibrium, summarized by the following proposition.

Proposition 1 *If $\tau > \beta - 1$ and \bar{H} is sufficiently large, then there exists a degenerate equilibrium, which is unique for generic values of the other parameters.*

Sketch of the proof. The strategy of proof is sketched as follows. I first establish an auxiliary problem that agents choose X and H to maximize U in the CM, given s_t and z_{t+1} . It becomes a static problem and yields a value function in s_t and z_{t+1} . One complication is to check interior solutions under the general preferences. A key part of the proof is that I transform the value function of the static problem as if it were some

quasi-linear utilities in the CM. I formulate a new problem in an auxiliary economy with this quasi-linear utility in the CM, and keep everything in the DM unchanged. I show that the solutions in the auxiliary economy are interior, and the value functions are the fixed point of a contraction mapping from a space of linear functions into itself. This step is done without relying on the differentiability of V_t in the original economy. Finally, I show the existence and uniqueness of the degenerate equilibrium under the auxiliary economy. I translate the equilibrium allocation of the auxiliary economy back to an allocation in the original economy, which is also degenerate and satisfies all the equilibrium conditions of the original economy under the same supporting prices and laws of motion. By construction, the translated allocation also maximizes the agent's preferences under the original economy. Hence, the translated allocation constitutes a degenerate equilibrium.

Proposition 1 states that agents always exit the CM with the same real balances regardless of the real balances entering the CM. The intuition is as follows. Consider the agent has purchases in the DM, so some of the real balances s brought to the CM are depleted. Then, due to the wealth effect, the agent wants to reduce his consumption X and increase his labour supply H . If the labour endowment \bar{H} is sufficiently large, then X is also large enough to buffer any reduction in X from the zero lower bound, and the increase in labour supply is not restricted by \bar{H} . As a result, the decrease in X and increase in H completely "rebalance" the falling short of real balances s after the DM purchases. With quasi-linear preferences, agents only rebalance the depleted real balances by increasing his labour supply H . With the general preferences, it can also be rebalanced by reducing his consumption X as well.

The conditions of $\tau > \beta - 1$ and a sufficiently large \bar{H} are common in the literature: $\tau > \beta - 1$ guarantees the return of money is not too high otherwise money is hoarded rather than circulating; a sufficiently large \bar{H} guarantees an interior allocation. The uniqueness holds for generic parameter space since there may be multiple degenerate equilibria under some special combination of parameters.⁸

4 Exact Solutions

To make use of the utility function U , it is useful to obtain exact solutions rather than in the PDE form $U_{XX}U_{HH} - (U_{XH})^2 = 0$. The following lemma solves for the exact solutions:

Lemma 2 *(i) U satisfies (8) if it has one of the following two forms:*

⁸See Wright (2010) for a related discussion.

$$a. U = (C_1X + C_2H + C_3) \varphi \left(\frac{C_4X + C_5H + C_6}{C_1X + C_2H + C_3} \right) + C_7X + C_8H + C_9;$$

$$b. U = (C_1X + C_2H) \varphi \left(\frac{X}{H} \right) + C_3X + C_4H + C_5,$$

where $C_i, i = 1 \dots 9$, are arbitrary constants and $\varphi \in \mathcal{C}^2$.

(ii) U satisfies (8) if and only if it has the following parametric form:

$$U = pX + \Lambda(p)H + \chi(p), \text{ where } p \text{ solves } -\chi'(p) = X + \Lambda'(p)H, \Lambda \in \mathcal{C}^1 \text{ and } \chi \in \mathcal{C}^1.$$

Lemma 2 includes some utilities, such as the class of constant return to scale (CRS) preferences, that have been extensively studied in many macroeconomic literatures. A popular example of CRS is the utility with constant elasticity of substitution (CES), which has the form $U = \left[(1 - \sigma)X^\psi + \sigma(\bar{H} - H)^\psi \right]^{1/\psi}$, for all $\sigma \in (0, 1)$ and $\psi \in (-\infty, 1)$. A Cobb-Douglas utility $U = X^{1-\sigma}(\bar{H} - N - H)^\sigma$ is widely used to study endogenous growth with leisure, which will be illustrated in the next section. Heckman (1976) uses the CRS utility $U[X, A(\bar{H} - N - H)]$ to capture the effective leisure $A(\bar{H} - N - H)$ on endogenous growth. Barro and Becker (1989) use the CRS utility $U = HU_0\left(\frac{X}{H}\right)$, where U_0 is some increasing and strictly concave function, to study fertility by interpreting H as the size of a family.

The utility class \mathcal{U} also includes non-CRS functions. A prime example is the quasi-linear utility $U = \tilde{U}(X) - H$ used in LW. Another form of the quasi-linear utility $U = X - g(H)$, where g is an increasing convex function, is a special case of Greenwood–Hercowitz–Huffman (GHH) utility, commonly used in international economics and business cycles when it is desirable not to have any income effect in the labour supply curve $w = g_H(H)$. Another non-CRS preference is the Stone-Geary utility $U = (X - x_0)^{1-\sigma}(\bar{H} - H)^\sigma$, used extensively in studies of international trade and structural change. Here, x_0 is the level of subsistence consumption. Other examples include the constant-absolute-risk-aversion (CARA) utility $U = -\exp[-\alpha_X X - \alpha_H(\bar{H} - H)] + \alpha_0 X$, where $\alpha_0 > 0$ can be arbitrarily small just to rule out a bang-bang solution. CARA is widely used in, for example, the literatures of both asset pricing and incomplete markets, since the associated consumption function is tractable even in a stochastic environment. Some applications using these utility functions are illustrated in the following sections.

5 Application to Endogenous Growth

In this section I illustrate how to make use of the general preferences to incorporate the standard endogenous growth model à la Lucas (1988) in a decentralised economy of monetary

exchange. This application also highlights the aspects where the generalisation of preferences matters, qualitatively and quantitatively. Last but not least, it demonstrates how to incorporate particular bargaining protocols even when deviating from a stationary environment.

Engine of long-run growth. The basic environment is extended as follows. Agents trade in the CM and DM as before, but now agents can also invest some time N in the CM to improve their human capital A , which evolves according to the following law of motion:

$$A_{t+1} = \delta(N_t) A_t, \quad (9)$$

where $\delta(N)$ is positive, increasing and strictly concave, and satisfies the Inada condition. In the growing economy, the relevant individual states are human capital and real balances, which are denoted as $\mathbf{a}^D = (A, z)$ in the beginning of the DM and $\mathbf{a}^C = (A, s)$ in the beginning of the CM. Compared with the literature, Lucas (1988) is the special case without decentralised markets, and Lagos and Wright (2005) is the special case with $\delta(N) = 1$.

Bargaining protocol. To illustrate as well as to simplify the environment, I assume a random ultimatum game in the DM: there is probability θ (probability $1 - \theta$) that the buyer (seller) can make a take-it-or-leave-it offer.⁹ Similar to Nash and proportional bargaining alike, $\theta \in (0, 1]$ captures the buyer's bargaining power. When a buyer of state $\mathbf{a} = (A, z)$ can make an offer to a seller of state $\mathbf{a}' = (A', z')$, he will propose the offer $[q_b(\mathbf{a}, \mathbf{a}'), d_b(\mathbf{a}, \mathbf{a}')] that maximizes his continuation value subject to the seller's participation constraint:$

$$\max_{q_b, d_b \in [-z', z]} \{u(q_b) + W(A, z - d_b)\} \text{ s.t. } W(A', z') = -c(q_b) + W(A', z' + d_b). \quad (10)$$

On the other hand, when a seller of state \mathbf{a}' can make an offer, his proposal $[q_s(\mathbf{a}, \mathbf{a}'), d_s(\mathbf{a}, \mathbf{a}')] is given by$

$$\max_{q_s, d_s \in [-z', z]} \{-c(q_s) + W(A', z' + d_s)\} \text{ s.t. } W(A, z) = u(q_s) + W(A, z - d_s). \quad (11)$$

I first hypothesize, and then verify, that the equilibrium is degenerate, i.e. the individual states for the rest of the agents is \mathbf{a}' . By summarizing all the cases, the (ex-ante) DM value

⁹Under other bargaining protocols, agents may also want to accumulate human capital in order to improve their bargaining solution when they turn out to be *buyers*. However, assuming a random ultimatum simplifies the economy by shutting down this interesting channel.

function $V(\mathbf{a})$ is given by the following Bellman equation:

$$\begin{aligned} V(\mathbf{a}) &= \alpha\theta \{u[q_b(\mathbf{a}, \mathbf{a}')] + W[\mathbf{a}_1, \mathbf{a}_2 + d_b(\mathbf{a}, \mathbf{a}')]\} \\ &\quad + \alpha(1-\theta) \{-c[q_s(\mathbf{a}', \mathbf{a})] + W[\mathbf{a}_1, \mathbf{a}_2 + d_s(\mathbf{a}', \mathbf{a})]\} + (1-\alpha)W(\mathbf{a}). \end{aligned} \quad (12)$$

The first (second) term is the ex-post DM value if the agent can match and make a offer to a seller (buyer), which happens with probability $\alpha\theta$ [probability $\alpha(1-\theta)$]. When the agent cannot match anyone in the DM, or can match someone but cannot propose, which happens with probability $1-\alpha$, the ex-post DM value is simply $W(\mathbf{a})$, given the result that the participation constraints in (10) and (11) are always binding. Moving to the next subperiod, the CM value function $W(A, z)$ is given by the following Bellman equation:

$$\begin{aligned} W(A, s) &= \max_{X, H, N, z} \{U(X, \bar{H} - H - N) + \beta V[\delta(N)A, z]\}, \text{ s.t.} \\ X &= AH + s - (1 + \pi)z + T, \\ X &\geq 0, N \in [0, \bar{H}], H \in [0, \bar{H} - N], z \geq 0. \end{aligned} \quad (13)$$

Denote $\{X(\mathbf{a}_t^C), H(\mathbf{a}_t^C), N(\mathbf{a}_t^C), z(\mathbf{a}_t^C)\}$ the solution to (13), which can be growing unbounded following \mathbf{a}_t^C .

Equilibrium.¹⁰ Define a degenerate balanced growth path as follows: a degenerate equilibrium such that $z(\mathbf{a}_t^C)/A_t = z^*$ and $N(\mathbf{a}_t^C) = N^*$ for any \mathbf{a}_t^C on the equilibrium path; as well as the CM and DM outputs, $\int A_t H(\mathbf{a}_t^C) dG_t(\mathbf{a}_t^C)$, $q_{b,t}$ and $q_{s,t}$, all grow at the same factor $\delta(N^*)$. Notice that, unlike the static economy, on the degenerate balanced growth path inflation is given by $1 + \pi = (1 + \tau)\delta(N^*)^{-1}$; inflation and money growth are no longer the same. Without loss of generality, I assume that the government sets τ as passive to directly target π in the equilibrium; hence π is treated as an exogenous policy variable.

General preferences. So far I have abstracted from any specification of preferences. Following Lucas (1988), I assume that the CM utility is given by the following member of \mathcal{U} :

$$U(X, \bar{H} - H - N) = B \left(\frac{X}{1 - \sigma} \right)^{1 - \sigma} \left(\frac{\bar{H} - H - N}{\sigma} \right)^\sigma, \quad \sigma \in (0, 1), \quad (14)$$

where $B > 0$ is the weight of the CM utility. I also use general CRRA functions for the

¹⁰Here I assume that Bellman equations are valid under unbounded growth. See Alvarez and Stokey (1998). However, Bellman equations may no longer be valid when utility is not bounded, for example, for a log utility function. As seen later, the utilities given by (14), (15) and (16) are homogenous of degree $1 - \sigma$. Nevertheless, the sufficient conditions given in Alvarez and Stokey (1996) are indeed satisfied under the general preferences.

preferences in the DM, which are given by

$$u(q) = D \frac{q^{1-\gamma}}{1-\gamma}, \quad \gamma \in [0, 1], \quad (15)$$

$$c(q) = \frac{q^{1+\eta} A^{-\varphi}}{\varphi}, \quad \eta, \varphi \geq 0, \quad (16)$$

where $D > 0$ is the weight of the DM consumption. The case $\gamma = 1$ is interpreted as log utility as usual. All parameters are to be estimated in the later quantitative exercise. Before that, however, I need the following proposition to show the conditions for the existence of a degenerate balanced growth path with the utility function (14).

Proposition 2 (*Existence under General Preferences*) *Given the utility function of form (14), a degenerate balanced growth path exists if $\pi \geq \beta [\max_N \delta(N)]^{-\sigma} - 1$, \bar{H} is sufficiently large, and the parameters satisfy*

$$\gamma = \sigma, \quad (17)$$

$$\varphi = \sigma + \eta. \quad (18)$$

Condition (17) [(18)] guarantees a degenerate balanced growth path in the DM when the seller (buyer) proposes. The condition of a large labour endowment \bar{H} guarantees an interior solution, as before. The condition $\pi \geq \beta [\max_N \delta(N)]^{-\sigma} - 1$ is the growth version of the standard condition $\pi > \beta - 1$: the return of money $-\pi$ cannot be too high to be used as a medium of exchange in the DM.

What happens if (14) is replaced with the quasi-linear preferences $U(X) - H - N$ in a growing economy? As Waller (2011) finds but with other bargaining protocols, the existence of a balanced growth path under the quasi-linear preference requires extra conditions.¹¹

Proposition 3 (*Waller 2011*) *If quasi-linear preferences are assumed, then the necessary conditions for the existence of a degenerate balanced growth path include: (a) $U(X) = \log X$, (b) $\theta = \gamma = 1$ and (c) $\varphi = 1 + \eta$.*

Compared with the general preferences, under the quasi-linear preferences conditions (a) and (b) in Proposition 3 are the extra conditions for the existence of a degenerate balanced growth path. The intuition is as follows. The intratemporal Euler's equation under the

¹¹For readers familiar with the literature, condition (a) is necessary for a balanced growth path in the CM, and condition (b) is necessary for a balanced growth path in the DM, which is not considered in Waller (2011). Also, he restricts $\eta = 0$.

quasi-linear preferences requires $U_X(X) = A^{-1}$. To have both sides of equality grow at the same rate, log utility is necessary. Instead, under the general preferences, both labour and consumption can be used to satisfy $U_X = -A^{-1}U_H$, so there is a larger degree of freedom for parameters. Second, log utility also means that the shadow value of real balances, $U_X z = z/A$, is constant on the balanced growth path, so the DM goods traded in the decentralised markets are also constant when sellers can make offers. To have consumption growth in the decentralised markets, any chance that sellers can propose must be shut down, by restricting the economy such that buyers have all the bargaining power, $\theta = 1$. In sum, a quasi-linear economy needs extra conditions to maintain a knife-edge growth path.

The following proposition gives an implication of these restrictions: in a quasi-linear economy inflation does not affect the output growth rate.

Proposition 4 (*No Endogenous Growth under QL*) *On any degenerate balanced growth path under quasi-linear preferences, $\delta(N^*)$ solves $\delta(N^*) = \beta\delta'(N^*)$.*

Under the quasi-linear preferences, the output growth does not depend on inflation. To see this, recall that in the Lucas model the growth rate is endogenously chosen to balance the opportunity cost and benefit of accumulating human capital. With quasi-linear preferences, the cost is independent to inflation as the marginal disutility of labour supply is constant. Also, agents no longer have any benefit from investing human capital for the DM production since sellers have no bargaining power (necessary for the balanced growth under quasi-linear preferences). Thus, a quasi-linear economy essentially shuts down any potential mechanism that inflation can affect growth. Of course, the absence of the growth mechanism is no longer the case under the general preferences. The following proposition summarizes the key result in this section:

Proposition 5 (*Growth Mechanism under General Preferences*) *Given the utility function of the form (14), on the degenerate balanced growth path, the comparative statics are given by $dz^*/d\pi < 0$ and $d\delta(N^*)/d\pi \leq 0$. The inequality is strict if the equilibrium features $z^* < \frac{D}{B(1-\gamma)} \left(\frac{\varphi D}{1+\eta} \right)^{\frac{1-\gamma}{\eta+\gamma}}$.*

A lesson from comparing Proposition 4 with Proposition 5 is that formulating an LW model with the general preferences can feature this growth mechanism but is impossible under the quasi-linear preferences. The growth mechanism under the general preferences relates to the *strategic complementarity* between human capital and real balances. On one

hand, a lower level of a buyer's real balances affords less purchases, and hence the seller wants to invest less human capital to justify the lower level of production. On the other hand, a lower level of a seller's human capital worsens the buyer's terms of trade, and hence the buyer wants to hold less real balances. Since a higher inflation raises the shadow cost of holding real balances, it reduces the investment in human capital through the strategic complementarity, and eventually triggers a further cut in real balances holding, and so on.¹² The strategic complementarity amplifies the negative effect of inflation.

It is also useful to notice a new source of inefficiency under the general preferences: this economy features a *two-sided hold-up* problem. On average, only portion $1 - \theta$ (portion θ) of the time can the sellers (buyers) get the entire ex-post trade surplus, but the sellers (buyers) must bear the entire cost of investing in human capital (real balances) - a hold-up problem for both sides of trade. Under the quasi-linear preference the economy is in the corner case $\theta = 1$: it reduces to a one-sided hold-up problem without strategic complementarity. The welfare cost of inflation can be very different under the general preferences.

Preliminary quantitative finding. To give some sense of the inflation cost under the general preferences, I perform a quantitative exercise closely following LW. This section serves an illustrative purpose, due to the obvious difficulty in assessing the relevance of inflation for future output growth.¹³ The annual rate of time preference is $r = \beta^{-1} - 1 = 0.04$, the matching rate is $\alpha = 0.5$, and the buyer's bargaining power is $\theta = 0.5$. I normalize $\bar{H} = 1$ since existence is no longer a concern in the quantitative exercise. The remaining parameters are estimated based on the same U.S. data from 1900 to 2000 used by Lucas (2000), LW and many others. Figure 1 displays the plots of annual observations (the circles in the figures) on the nominal interest rate i , on the money-output ratio $L \equiv M/(PY)$, and on the growth-interest rate differential $\rho \equiv \frac{P_{+1}Y_{+1}}{PY} (1+i)^{-1} - 1$. The data on L and ρ capture the money demand and effective growth rate respectively. In the model the nominal

¹²Sellers can propose DM trades at the constrained efficient level (condition on A) when the buyers hold higher real balances than the threshold in Proposition 5. In this case a higher inflation does not affect seller's surplus or the incentive to invest human capital.

¹³Some features of the model may result in overstating the welfare cost. First, as in the Lucas (1988) model, agents can invest in human capital with their labour in every period. In practice, while human capital can be accumulated by a continuous learning process (e.g. through experience and skill), a significant share of human capital is formed at the early stage of the life cycle and at much lower frequency than in the model. Second, I estimate the model with the data of output growth rate rather than some direct observation on human capital such as years and quality of schooling. While it avoids taking a stance on what human capital really is and how to measure properly, still an ongoing issue in the growth literature, my empirical exercise does not directly verify the mechanism with the micro data. In particular, the negative effect of inflation on human capital accumulation suggested by the aggregate data might be weaker in the micro data. Last but not least, money is the only medium of exchange. Further research will be required.

interest rate is given by Fisher's equation (the growth version) $i = \beta^{-1} (1 + \pi) \delta (N^*)^\sigma - 1$. Assume that $\delta(N) = (\delta_0 N + N_0)^\kappa$, where $\delta_0, N_0 \geq 0$ and $\kappa \in [0, 1]$. This configuration nests the exogenous-growth models when $\delta_0 = 0$, and the static models when $\kappa = 0$. See Online Appendix C for details and robustness checks.

The data present a negative correlation between the effective growth rate and the nominal interest rate. The model is capable of capturing that with the fitted curve (solid line) in the right panel, as well as the fitted money demand (solid line) in the left panel. Going from 0% to 10% inflation for 10 years reduces the output growth rate of 1.3%. The welfare cost of 10% inflation is worth 7.3% of consumption, larger than LW ($\leq 4.6\%$) under the quasi-linear preferences without endogenous growth. Inflation can have such a large impact because of the growth mechanism: affecting not only the current output but also the entire path of future output. It is much larger than Lucas' ($\leq 1\%$) with exogenous growth, because the latter measures the welfare cost by the area under the reduced-form money demand curve. It ignores the social benefit of monetary exchange, which can be seriously affected by inflation under the growth mechanism.

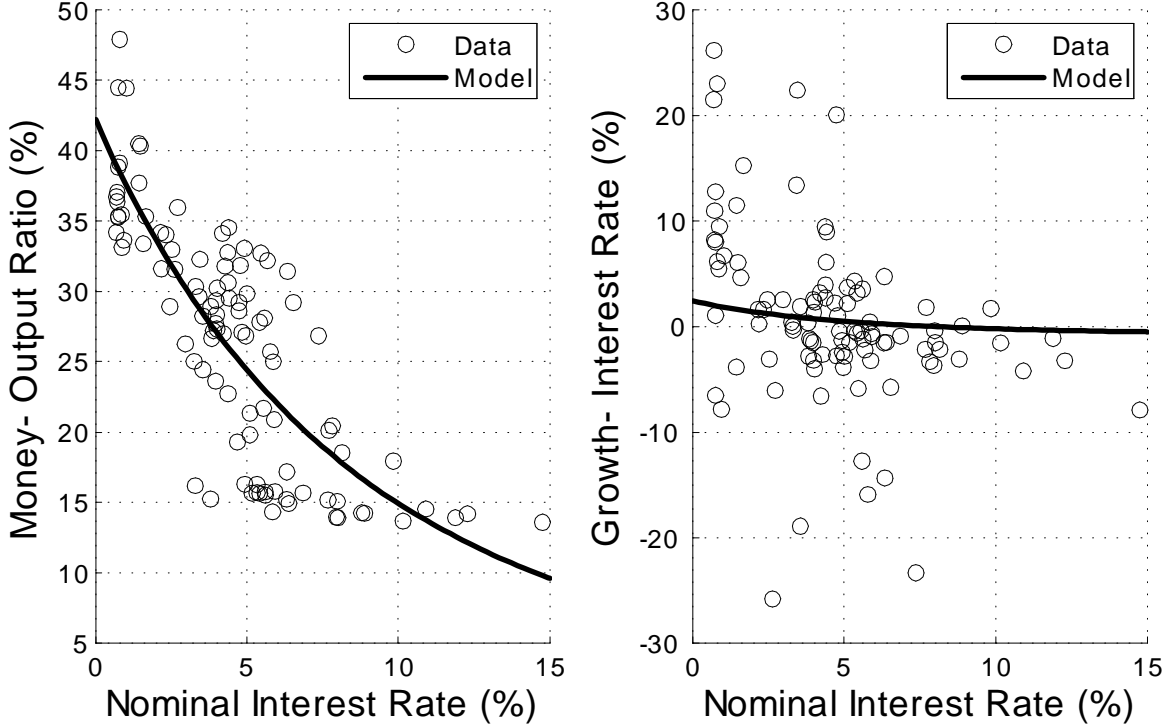


Figure 1: U.S. money demand and output growth, 1900-2000. Each circle represents a sample of annual data. Sources: see Online Appendix C.

6 Application to a Heterogeneous-Agent Economy

The main insight of this paper is that under the general preferences the optimal money holdings for the future periods are not a function of how much money they obtained in the past. Proposition 1 concludes that ex-ante homogeneous agents hold identical amounts of money. But the basic insight is broader. Even when agents are ex-ante heterogeneous, the amount of money they want to take into the future does not depend on money earned in previous trades; nevertheless it is affected by ex-ante heterogeneity in preferences and productivities. In such environments, allowing general utility functions rather than restricting attention only to quasi-linear utilities generates some advantages that this section will outline.

I introduce ex-ante heterogeneity as in standard incomplete-market models but keep the changes to the benchmark model minimal. In this case, money holding is similar to capital in the incomplete-market models. Under a sufficiently large ex-ante heterogeneity, the distribution may cease to be tractable under the quasi-linear preferences. This section thus provides another example to illustrate some advantages of modeling with this class of general preferences: an economy can still feature a tractable distribution under a sufficiently large ex-ante heterogeneity, which helps to obtain some useful results analytically. Again, it is for illustrational purpose; there are ongoing researches with more thorough examination.¹⁴

Compared to the benchmark case in Section 2, in this economy agents are subject to uninsurable i.i.d. shocks to A over time, which are realized at the beginning of the CM.¹⁵ The CM value function is given by

$$\begin{aligned} W(A, s) &= \max_{X, H, z} \{U(X, \bar{H} - H) + \beta V(z)\}, \text{ s.t.} \\ X &= AH + s - (1 + \pi)z + T, X \geq 0, H \in [0, \bar{H}], z \geq 0. \end{aligned} \tag{19}$$

Denote $W(s) = \mathbb{E}_A W(A, s)$ the expected CM value function before the realization of A . For the sake of brevity, I also assume $c(q) = q$ and $\theta = 1$ (ie, buyers can make TIOLI offers), so $S^s(z, z'; W) = 0$ and $q = W(z' + d) - W(z')$. None of these assumptions is essential. The

¹⁴For example, see Wen (2015) for a similar environment but with uninsurable shocks to wealth instead.

¹⁵The main results still hold when A is persistent, or has a component shared among agents which is shocked by some aggregate uncertainties as in Krusell and Smith (1998). Alternatively, having endowment shocks will not have significant differences from the basic model: the money distribution is still degenerate.

DM value function is then given by

$$V(z) = \alpha \int S^b(z, z'; W) dF(z') + W(z), \quad (20)$$

where now $S^b(z, z'; W) = \max_{d \leq z} \{u[W(z' + d) - W(z')] + W(z - d) - W(z)\}$.

To sharpen the comparison, the domain of A is given by $[A_0, \infty)$, $A_0 > 0$. The unbounded domain captures a large heterogeneity among agents, which could make the distribution intractable under the quasi-linear preferences. To see this, suppose there exists an equilibrium featuring an interior solution and a linear $W(s)$. This equilibrium implies $q(z, z'; W) = \min(\lambda z, q^*)$, where $\lambda = W_1(0) > 0$ and $u_x(q^*) = 1$. For a typical agent with an individual state $\{s, A\}$, the first order condition of z implies

$$-\frac{1 + \pi}{A} U_H[X(s, A), \bar{H} - H(s, A)] = \beta \lambda [\alpha \max\{u_x[\lambda z(s, A)] - 1, 0\} + 1]. \quad (21)$$

Consider the quasi-linear preferences, then $-U_H = 1$, $\lambda = \mathbb{E}_A A^{-1}$ and (21) becomes

$$\frac{1 + \pi}{A} = \beta \lambda [\alpha \max\{u_x[\lambda z(s, A)] - 1, 0\} + 1], \quad (22)$$

which does not have a solution for a sufficiently large A , always possible with an unbounded domain. The intuition is that under the quasi-linear preferences, the marginal disutility of labour is constant, and thus the labour supply can have bang-bang solutions. For agents with sufficiently large A , their money holding cannot be rebalanced to the level unrestricted by their labour endowment \bar{H} . Thus, the optimal money holdings out of the CM will depend on the money holding brought to the CM, and the tractability is lost. With homogeneous agents, this problem can be avoided by assuming a sufficiently large \bar{H} . When the heterogeneity is sufficiently large (for example with a unbound domain of A), however, there does not exist any finite \bar{H} such that the labour endowment is not binding for *all* agents.

With heterogeneous agents, the problem of binding labour endowment can be avoided if the utility function has sufficient curvature in leisure, allowable by modeling with the general preferences. For example, consider the following CES member of the general preferences:

$$U(X, \bar{H} - H) = \left[(1 - \sigma) \left(\frac{X}{1 - \sigma} \right)^\psi + \sigma \left(\frac{\bar{H} - H}{\sigma} \right)^\psi \right]^{1/\psi}, \quad \sigma \in (0, 1), \psi < 1. \quad (23)$$

Then the intratemporal Euler's equation $AU_X = -U_H$ implies $X = (\sigma^{-1} - 1) A^{1/(1-\psi)} (\bar{H} - H)$

and $-U_H = \left[(1 - \sigma) A^{\frac{\psi}{1-\psi}} + \sigma \right]^{\frac{1-\psi}{\psi}}$; the indirect marginal disutility of labour now increases in A . Presume that there exists an equilibrium with $z(s, A) = z(A)$, thus (21) becomes

$$(1 + \pi) \left(1 - \sigma + \sigma A^{\frac{-\psi}{1-\psi}} \right)^{\frac{1-\psi}{\psi}} = \beta \lambda [\alpha \max \{u_x[\lambda z(A)] - 1, 0\} + 1], \quad (24)$$

where $\lambda = \mathbb{E}_A \left(1 - \sigma + \sigma A^{\frac{-\psi}{1-\psi}} \right)^{\frac{1-\psi}{\psi}}$. Thus, there exists a solution $z(A)$ to (24) for all $A \in [A_0, \infty)$ only if $\psi \in (0, 1)$, and

$$\tau = \pi > \beta \mathbb{E}_A \left[1 + \left(\frac{\sigma}{1 - \sigma} \right) A^{\frac{-\psi}{1-\psi}} \right]^{\frac{1-\psi}{\psi}} - 1. \quad (25)$$

The conditions become "if and only if" when the economy also features a sufficiently large \bar{H} (to guarantee an interior solution as before). Under the general preferences, the optimal money demand, $z(A)$, can be tractably solved by rearranging terms in (24). Its distribution only depends on the current distribution of A but not the history of the previous money holding distribution - the distribution of money holding is thus degenerate conditional on A .

Besides the tractability of distribution, some useful results are also readily available. In the incomplete-market literature, Bewley (1980) considers the effect of paying interest for holding money (financed by lump-sum taxation), which is the same as the deflation by lump-sum taxation in this economy. He finds that a stationary monetary equilibrium exists only if the interest rate on money, $(1 + \tau)^{-1} - 1$, is sufficiently lower than the rate of time preferences, $\beta^{-1} - 1$ (the interest rate of Friedman's rule in his environment). Otherwise there does not exist a stationary monetary equilibrium since the supporting lump-sum tax is too high to be feasible for poor agents in his endowment economy. Condition (25) is similar to Bewley's finding, but for a different reason: if the money growth rate is too low that (25) fails, then the equilibrium may no longer be stationary: money holding could be increasingly concentrated in the hands of agents with a long history of high productivities. Thus, unlike the benchmark of ex-ante homogeneity, Friedman's rule ($\tau = \beta - 1$) is no longer implementable when there is a large heterogeneity.

In the literature of search models, Galenianos and Kircher (2008) study an economy that also features a non-degenerate distribution but with the quasi-linear preferences, by allowing auctions over indivisible DM goods. As in their economy, one can show that in this one the per-period utility declines with inflation for all agents. Unlike their economy, however, in general the declines can be larger or smaller for the high- A agents, depending on the sign

of $U_{X,H}$, i.e., whether or not consumption and leisure are strategic complements.¹⁶ Thus, inflation can be a progressive or regressive form of taxation under the general preferences.

7 Conclusion

This paper generalises the seminal work by Lagos and Wright (2005) to a broad class of preferences. Examples are designed to highlight when, why and how the generalisation is useful in modelling important issues that could be impossible under the quasi-linear preferences. On top of these examples, other studies also show that the generalisation matters. For instance, Swanson (2012) illustrates that the traditional measure of risk aversion can be misleading in asset pricing; Gu, Mattesini and Wright (2014) show that under the general preferences credit can no longer be neutral in a monetary equilibrium with capital.

I demonstrate that degeneracy is always preserved under general trading protocols. This model can go further and incorporate other matching and exchange mechanisms, for example, different search mechanisms and competing media of exchange.¹⁷ It is also useful in maintaining tractability for models with heterogeneous agents and other uninsurable shocks. The key is to insert a subperiod with this class of preferences that allows agents to reset the individual state by consuming and working. While in general these applications will have non-degenerate money holdings, the technique still eliminates the further heterogeneity in money holdings arising from previous trades.

There are also economies where *some equilibria* looks like LW, so the quasi-linear preferences could be good approximation. Rocheteau, Rupert, Shell and Wright (2008) consider an economy with indivisible goods where agents behave as if they have quasi-linear preferences under the sunspot equilibria. Similar properties are also shown by Faig (2008), where a lottery of agents' money balances is available. A common feature of these works is the exploitation of the non-concavity of the value functions: if there is a randomization device and agents can coordinate on it, then there is a region such that the value function is linear, as if driven by some quasi-linear preferences. This argument extends to this class of general preferences as well.

¹⁶The CES preferences (23) always feature an progressive inflation. Furthermore, inflation can be shown as regressive only if $U_{X,H} > 0$, for example $U(X, \bar{H} - H) = -\exp(-X - \bar{H} + H) + X$.

¹⁷See Lagos and Rocheteau (2005) for a comitative search model with endogenous search intensity; and Rocheteau and Wright (2005) for the one with free entry. There also have been extensive studies in this literature on the co-existence of money and interest-bearing assets. For recent studies highlighting the competition among media of exchange, see Lagos and Rocheteau (2008) on money v.s. assets; Gu, Mattesini and Wright (2012) on money v.s. credit; Venkateswaran and Wright (2013) on money v.s. collateral.

8 Appendix A: Proofs of Main Results

8.1 Proof of Lemma 1

If. Since $U \in \mathcal{C}^2$ and $\Lambda \in \mathcal{C}^1$, differentiating both sides of $U_H = \Lambda(U_X)$ with respect to X and H and eliminating $\Lambda'(U_X)$, I have $U_{HH}U_{XX} = (U_{XH})^2$.

Only if. Since $U_{XX} \neq 0$, U_{HX}/U_{XX} is well-defined. Fix continuous functions $x(\rho)$ and $h(\rho)$ such that $\rho = U_X[x(\rho), \bar{H} - h(\rho)]$ for all ρ in the range of U_X (the existence follows Michael selection theorem). For example, consider $U(X, \bar{H} - H) = X^{1-\sigma}(\bar{H} - H)^\sigma$, then fix any differentiable function $h(\rho) \in [0, \bar{H}]$ and I have $x(\rho) = [\bar{H} - h(\rho)] [(1-\sigma)/\rho]^{1/\sigma}$. For any p_0 and p in the range of U_X , construct $\Lambda(p)$ as the following path integral from p_0 to p :

$$\Lambda(p) \equiv \int_{p_0}^p \frac{U_{HX}[x(\rho), \bar{H} - h(\rho)]}{U_{XX}[x(\rho), \bar{H} - h(\rho)]} d\rho + U_H[x(p_0), \bar{H} - h(p_0)]. \quad (26)$$

Using the construction $\rho = U_X[x(\rho), \bar{H} - h(\rho)]$, I have

$$\begin{aligned} \Lambda(p) &= \int_{p_0}^p \frac{U_{HX}[x(\rho), \bar{H} - h(\rho)]}{U_{XX}[x(\rho), \bar{H} - h(\rho)]} dU_X[x(\rho), \bar{H} - h(\rho)] + U_H[x(p_0), \bar{H} - h(p_0)], \\ &= \int_{p_0}^p U_{HX}[x(\rho), \bar{H} - h(\rho)] dx(\rho) + \int_{p_0}^p \frac{U_{HX}[x(\rho), \bar{H} - h(\rho)]^2}{U_{XX}[x(\rho), \bar{H} - h(\rho)]} dh(\rho) + U_H[x(p_0), \bar{H} - h(p_0)], \\ &= \int_{p_0}^p U_{HX}[x(\rho), \bar{H} - h(\rho)] dx(\rho) + \int_{p_0}^p U_{HH}[x(\rho), \bar{H} - h(\rho)] dh(\rho) + U_H[x(p_0), \bar{H} - h(p_0)], \\ &= U_H[x(p), \bar{H} - h(p)], \end{aligned}$$

where the third line utilizes the fact that $U_{XX} \neq 0$ and $U_{HH} = (U_{XH})^2/U_{XX}$ for any $U \in \mathcal{U}$. Notice that by construction I have $\Lambda(p) \in \mathcal{C}^1$. Define $p = U_X(X, \bar{H} - H)$ where $X = x(p)$ and $H = h(p)$, then I establish the result $U_H(X, \bar{H} - H) = \Lambda \circ U_X(X, \bar{H} - H)$.

8.2 Proof of Proposition 1

Step 1. First I need the following lemma to show a useful property of \mathcal{U} .

Lemma 3 *Suppose $U \in \mathcal{U}$. For any $A > 0$, for any (X_1, H_1) and (X_2, H_2) which satisfy*

$$AU_X(X_i, \bar{H} - H_i) = -U_H(X_i, \bar{H} - H_i), \quad i = 1, 2, \quad (27)$$

I have $U_X(X_1, \bar{H} - H_1) = U_X(X_2, \bar{H} - H_2)$.

Proof. Denote $p_1 = U_X (X_1, \bar{H} - H_1)$ and $p_2 = U_X (X_2, \bar{H} - H_2)$. Suppose Lemma 3 is not true and there exist (X_1, H_1) and (X_2, H_2) which satisfy (27) but $p_1 \neq p_2$. Using Lemma 1, I have

$$Ap_i = -\Lambda(p_i), i = 1, 2.$$

Since $\Lambda \in \mathcal{C}^1$, the premise that there are at least two distinct roots to $Ap = -\Lambda(p)$ implies there also exists two distinct roots p_i and $p_{i'}$ to $Ap = -\Lambda(p)$ such that $\Lambda'(p_i) > -A^{-1}$ and $\Lambda'(p_{i'}) < -A^{-1}$. Also, from the proof of Lemma 1 I have $\Lambda' = U_{XH}/U_{XX} \geq 0$, which contradicts to the fact that $\Lambda'(p_{i'}) < -A^{-1} < 0$. ■

Step 2. Fix some $\bar{s} > 0$. Define $U^0(s)$ for all $s \in [-\bar{s}, \bar{s}]$ as

$$\begin{aligned} U^0(s) &\equiv \max_{X, H} U(X, \bar{H} - H) \\ \text{s.t. } X &\leq AH + s, X \geq 0, \text{ and } H \in [0, \bar{H}] \end{aligned} \quad (28)$$

By Assumption 2, for a sufficiently large \bar{H} , the problem (28) has a unique interior solution $X(s) > 0$ and $H(s) \in (0, \bar{H})$. Then the first order conditions of (28) with respect to X and H imply

$$AU_X [X(s), \bar{H} - H(s)] = -U_H [X(s), \bar{H} - H(s)]. \quad (29)$$

By Lemma 3, there exists a constant λ such that $\lambda = U_X [X(s), \bar{H} - H(s)]$ for any $X(s)$ and $H(s)$ satisfies (29). Applying an envelope theorem to $U^0(s)$, I have $U_s^0(s) = U_X [X(s), \bar{H} - H(s)] = \lambda$. So for all $s \in [-\bar{s}, \bar{s}]$, we can write

$$U^0(s) = \lambda_0 + \lambda s, \quad (30)$$

where λ_0 is some constant solving $\lambda_0 = U^0(0)$. Substituting (30) into (1), consider the value function $W_t(s)$ given by

$$W_t(s) = \max_z \{U^0[s - (1 + \pi)z + T_t] + \beta V_{t+1}(z)\}, s \in [0, 2\bar{s}], \quad (31)$$

where in the degenerate equilibrium (to be verified) I have $\pi_{t+1} = \tau$, $\tau z = T_t$ and $z = \bar{s}$.

Step 3. Notice that for a sufficiently large \overline{H} , there exist some increasing, strictly concave function $U^1(X)$ and constant $X^* > 0$ such that

$$\begin{aligned} X^* &= \arg \max_X \{U^1(X) - \lambda X\}, \\ \lambda_0 &= U^1(X^*) - \lambda X^*, \\ 2\overline{s} + 2|\tau|\overline{z} &< X^* < A\overline{H} - (1 + 2|\tau|)\overline{z}, \end{aligned}$$

where \overline{z} is given by Assumption 3iii with λ given by (30).

Now define an auxiliary economy with the quasi-linear preferences $U^1(X) - A\lambda H$ in the CM and everything the same in the DM. Guess an equilibrium with $\pi = \tau$. Given any distribution $F(z)$ and T , the CM value function of the auxiliary economy is the fixed point of the following functional equation $C(W)$

$$C(W) \equiv \max_{X, H, z} \left\{ U^1(X) - A\lambda H + \beta\alpha \int S^b(z, z', W) dF(z') + \beta\alpha \int S^s(z', z, W) dF(z') + \beta W(z) \right\}, \text{ s.t.} \quad (32)$$

$$\begin{aligned} X &= AH + s - (1 + \tau)z + T, \\ X &\geq 0, H \in [0, \overline{H}] \text{ and } z \geq 0. \end{aligned}$$

Denote \mathcal{S}_λ the space of linear function W in the form $W(s) = w_0 + \lambda s$, where $s \in [0, 2\overline{s}]$. With the sup norm $\|\cdot\|$, \mathcal{S}_λ is a complete metric space.

I want to show that given a sufficiently large \overline{H} , $C(W)$ is a contraction mapping from \mathcal{S}_λ to \mathcal{S}_λ . First I want to show $C(W) \in \mathcal{S}_\lambda$ for any $W(s) \in \mathcal{S}_\lambda$. Consider any $W(s) \in \mathcal{S}_\lambda$. Using the premise $\tau > \beta - 1$ from Proposition 1 and Assumption 3iii, I have $z \leq \overline{z}$. Given the construction of X^* , the solution X to (32) is $X = X^*$ and the solution $H = H^*(s)$ is given by

$$H^* = \frac{X^* - s + (1 + \tau)z - T}{A} \in \left[\frac{X^* - 2\overline{s} - 2|\tau|\overline{z}}{A}, \frac{X^* + (1 + 2|\tau|)\overline{z}}{A} \right] \subseteq (0, \overline{H}),$$

where from the government budget constraint I have $T \in [-|\tau|\overline{z}, |\tau|\overline{z}]$. Thus, the solution H^* to (32) is interior for all $W(s) \in \mathcal{S}_\lambda$. Then $C(W)$ is equivalent to the following linear function

$$C(W) = v_0(W) + \lambda s, \text{ where}$$

$$v_0(W) \equiv \lambda_0 + \lambda T + \beta w_0 + \alpha \max_z \left\{ -\frac{\lambda}{\alpha} (1 + \tau - \beta) z + \beta \int S^b(z, z'; W) dF(z') + \beta \int S^s(z', z; W) dF(z') \right\}.$$

Thus, given a sufficiently large \bar{H} , I have established $C(W) \in \mathcal{S}_\lambda$ for any $W(s) \in \mathcal{S}_\lambda$. Finally, to show $C(W)$ is a contraction mapping, I want to show $\|C(W) - C(W')\| \leq \beta \|W - W'\|$ for any $W(s) \in \mathcal{S}_\lambda$ and $W'(s) \in \mathcal{S}_\lambda$. Consider $W(s) = w + \lambda s$ and $W'(s) = w' + \lambda s$. By Assumption 3ii I have $S^b(z, z', W) = S^b(z, z', W')$ and $S^s(z', z, W) = S^s(z', z, W')$, so I have $v_0(W) - v_0(W') = \beta(w - w')$ and thus $\|C(W) - C(W')\| = \beta |w - w'| = \beta \|W - W'\|$. Finally, given $\tau > \beta - 1$ and Assumption 3i, there is generically unique solution $z^* \in (0, \bar{z}]$ to $\max_z \left\{ -\frac{\lambda}{\alpha} (1 + \tau - \beta) z + \beta \int S^b(z, z', W) dF(z') + \beta \int S^s(z', z, W) dF(z') \right\}$. It implies that F is degenerate in the equilibrium of the auxiliary economy. I verify that $\{X^*, H^*(s), z^*\}$ constitute an degenerate equilibrium in auxiliary economy with $\pi_{t+1} = \tau$ and $\tau z^* = T_t$.

Step 4. I recover the equilibrium allocation of the original economy. Given $\pi_t = \tau$ and the unique solution z^* in the fixed-point function $W^* = C(W^*)$ from Step 3, set $z_t(s_t) = z^*$ and $\bar{s} = z^*$. The money market is cleared by setting $\phi_t M_t = z^*$; the government budget is satisfied by setting and $T_t = T = \tau z^*$. Again for a sufficiently large \bar{H} , the CM problem (28) in Step 2 has an interior solution $X'(s)$ and $H'(s)$, which is unique. Set $X_t(s) = X'[s - (1 + \pi) z^* + T]$ and $H_t(s) = H'[s - (1 + \pi) z^* + T]$, which are interior and also satisfy the goods market condition. Verify that $W_t(s) = W^*(s)$ is also the CM value function given by (31). Hence, $\{X_t(s), H_t(s), z_t(s)\}$ attains the maximal lifetime utility. The allocation constitutes a unique degenerate equilibrium in the original economy.

References

- [1] Aiyagari, S. R. (1994). Uninsured idiosyncratic risk and aggregate saving. *The Quarterly Journal of Economics*, 659-684.
- [2] Alvarez, F., & N. L. Stokey (1998). Dynamic programming with homogeneous functions. *Journal of Economic Theory*, 82(1), 167-189.
- [3] Barro, R. J., & G. S. Becker (1989). Fertility choice in a model of economic growth. *Econometrica: journal of the Econometric Society*, 481-501.
- [4] Bewley, T. (1980) *The Optimum Quantity of Money*, in *Models of Monetary Economics*, ed. by John Kareken and Neil Wallace. Minneapolis, Minnesota: Federal Reserve Bank.

- [5] Buera, F., & Y. Shin (2013). Financial Frictions and the Persistence of History: A Quantitative Exploration. *Journal of Political Economy*, 121(2), 221-272.
- [6] Chiu, J., & M. Molico (2010). Liquidity, redistribution, and the welfare cost of inflation. *Journal of Monetary Economics*, 57(4), 428-438.
- [7] Faig, M. (2008). Endogenous buyer–seller choice and divisible money in search equilibrium. *Journal of Economic Theory*, 141(1), 184-199.
- [8] Galenianos, M., & P. Kircher (2008). A model of money with multilateral matching. *Journal of Monetary Economics*, 55(6), 1054-1066.
- [9] Gu, C., F. Mattesini, & R. Wright (2014). Money and Credit Redux, Mimeo, University of Wisconsin, Madison.
- [10] Heckman, J., (1976). A life-cycle model of earnings, learning, and consumption. *Journal of political economy*, 84(4), 11-44.
- [11] Hu, T-W., J. Kennan, N. Wallace (2009). Coalition-Proof Trade and the Friedman Rule in the Lagos-Wright Model. *Journal of Political Economy*, 117(1), 116-137.
- [12] Kiyotaki, N., and R. Wright (1989). On money as a medium of exchange. *The Journal of Political Economy*, 927-954.
- [13] Kocherlakota, N. (1998). Money is memory. *journal of economic theory*, 81(2), 232-251.
- [14] Krusell, P., & A. A. Jr. Smith (1998). Income and wealth heterogeneity in the macroeconomy. *Journal of Political Economy*, 106(5), 867-896.
- [15] Lagos, R., & G. Rocheteau (2005). Inflation, output and welfare. *International Economic Review*, 46(2), 495-522.
- [16] Lagos, R., & G. Rocheteau (2008). Money and capital as competing media of exchange. *Journal of Economic Theory*, 142(1), 247-258.
- [17] Lagos, R., G. Rocheteau, G., & R. Wright (2014). The Art of Monetary Theory: A New Monetarist Perspective. Mimeo, University of Wisconsin, Madison.
- [18] Lagos, R. and R. Wright (2003). Dynamics, cycles, and sunspot equilibria in ‘genuinely dynamic, fundamentally disaggregative’ models of money. *Journal of Economic Theory*, 109(2), 156-171.

- [19] Lagos, R. and R. Wright (2005). A Unified Framework for Monetary Theory and Policy Analysis. *Journal of Political Economy*, 113(3), 463-484.
- [20] Lucas Jr, R. E. (1988). On the mechanics of economic development. *Journal of monetary economics*, 22(1), 3-42.
- [21] Lucas Jr, R. E. (2000). Inflation and welfare. *Econometrica*, 68(2), 247-274.
- [22] Menzio, G., S. Shi, and H. Sun (2013). A monetary theory with non-degenerate distributions. *Journal of Economic Theory*, 148(6), 2266-2312.
- [23] Molico, M. (2006). The Distribution Of Money And Prices In Search Equilibrium," *International Economic Review*, 47(3), 701-722.
- [24] Rocheteau, G., P. Rupert, K. Shell, and R. Wright, 2008, "General equilibrium with nonconvexities, sunspots, and money," *Journal of Economic Theory*, 142, 294-317
- [25] Rocheteau, G., P.-O. Weill, and T.-N. Wong (2015). A tractable model of monetary exchange with ex-post heterogeneity. *NBER Working Paper* 21179.
- [26] Rocheteau, G., and R. Wright (2005). Money in Search Equilibrium, in *Competitive Equilibrium, and in Competitive Search Equilibrium*. *Econometrica: Journal of the Econometric Society*. 73(1), 175-202.
- [27] Shi, S. (1997). A divisible search model of fiat money. *Econometrica: Journal of the Econometric Society*, 75-102.
- [28] Swanson, E. T. (2012). Risk aversion and the labor margin in dynamic equilibrium models. *The American Economic Review*, 102(4), 1663-1691.
- [29] Venkateswaran, V., & R. Wright (2013). Pledgability and Liquidity: A New Monetarist Model of Financial and Macroeconomic Activity. *NBER Macroeconomics Annual*, 28(1), 227-270.
- [30] Wallace, N. (2010). The Mechanism-Design Approach to Monetary Theory. *Handbook of Monetary Economics*, 3, 3-23.
- [31] Waller, C. J. (2011). Random matching and money in the neoclassical growth model: Some analytical results. *Macroeconomic Dynamics*, 15(S2), 293-312.

- [32] Wen, Y. (2015). Money, liquidity and welfare. *European Economic Review*, 76, 1-24.
- [33] Wright, R. (2010). A uniqueness proof for monetary steady state. *Journal of Economic Theory*, 145(1), 382-391.