Oracle Estimation of a Change Point in High Dimensional Quantile Regression*

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Abstract

In this paper, we consider a high dimensional quantile regression model where the sparsity structure may differ between the two sub-populations. We develop ℓ_1-penalized estimators of both regression coefficients and the threshold parameter. Our penalized estimators not only select covariates but also discriminate between a model with homogeneous sparsity and a model with a change point. As a result, it is not necessary to know or pretest whether the change point is present, or where it occurs. Our estimator of the change point achieves an oracle property in the sense that its asymptotic distribution is the same as if the unknown active sets of regression coefficients were known. Importantly, we establish this oracle property without a perfect covariate selection, thereby avoiding the need for the minimum level condition on the signals of active covariates. Dealing with high dimensional quantile regression with an unknown change point calls for a new proof technique since the quantile loss function is non-smooth and furthermore the corresponding objective function is non-convex with respect to the change point. The technique developed in this paper is applicable to a general M-estimation framework with a change point, which may be of independent interest. The proposed methods are then illustrated via Monte Carlo experiments and an application to tipping in the dynamics of racial segregation.

Keywords: change-point, variable selection, quantile regression, high-dimensional M-estimation, sparsity, LASSO, SCAD

AMS 2000 subject classifications: Primary 62H12, 62J05; secondary 62J07

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1 Introduction

In this paper, we consider a high dimensional quantile regression model where the sparsity structure (e.g., identities and effects of important or contributing regressors) may differ between the two sub-populations, thereby allowing for a possible change point in the model. Let $Y \in \mathbb{R}$ be a response variable, $Q \in \mathbb{R}$ be a scalar random variable that determines a possible change point, and $X \in \mathbb{R}^p$ be a p-dimensional vector of covariates. Here, Q can be a component of X, and p is potentially much larger than the sample size n. Specifically, high dimensional quantile regression with a change point is modelled as follows:

$$Y = X^T \beta_0 + X^T \delta_0 1\{Q > \tau_0\} + U, \tag{1.1}$$

where $(\beta_0, \delta_0, \tau_0)$ is a vector of unknown parameters and the regression error U satisfies $\mathbb{P}(U \leq 0|X, Q) = \gamma$ for some known $\gamma \in (0, 1)$. Unlike the mean regression, quantile regression analyzes the effects of active regressors on different parts of the conditional distribution of a response variable. Therefore, it allows the sparsity patterns to differ at different quantiles and also handles heterogeneity due to either heteroscedastic variance or other forms of non-location-scale covariate effects. By taking into account a possible change point in the model, we provide a more realistic picture of the sparsity patterns. For instance, when analyzing high-dimensional gene expression data, the identities of contributing genes may depend on the environmental or demographical variables (e.g., exposed temperature, age or weights).

Our paper is closely related to the literature on models with unknown change points (e.g., Tong (1990), Chan (1993), Hansen (2000), Pons (2003), Kosorok and Song (2007), Seijo and Sen (2011a,b) and Li and Ling (2012) among many others). Recent papers on change points under high-dimensional setups include Enikeeva and Harchaoui (2013); Chan et al. (2014), Frick et al. (2014), Cho and Fryzlewicz (2015), Chan et al. (2016), Callot et al. (2016), and Lee et al. (2016) among others; however, none of these papers consider a change point in high dimensional quantile regression. The literature on high dimensional

quantile regression includes Belloni and Chernozhukov (2011), Bradic et al. (2011), Wang et al. (2012), Wang (2013), and Fan et al. (2014) among others. All the aforementioned papers on quantile regression are under the homogeneous sparsity framework (equivalently, assuming that $\delta_0 = 0$ in the quantile regression model). Ciuperca (2013) considers penalized estimation of a quantile regression model with breaks, but the corresponding analysis is restricted to the case when p is small.

In this paper, we consider estimating regression coefficients $\alpha_0 \equiv (\beta_0^T, \delta_0^T)^T$ as well as the threshold parameter τ_0 and selecting the contributing regressors based on ℓ_1-penalized estimators. One of the strengths of our proposed procedure is that it does not require to know or pretest whether $\delta_0 = 0$ or not, that is, whether the population's sparsity structure and covariate effects are invariant or not. In other words, we do not need to know whether the threshold τ_0 is present in the model.

For a sparse vector $v \in \mathbb{R}^p$, we denote the active set of v as $J(v) \equiv \{j : v_j \neq 0\}$. One of the main contributions of this paper is that our proposed estimator of τ_0 achieves an *oracle property* in the sense that its asymptotic distribution is the same as if the unknown active sets $J(\beta_0)$ and $J(\delta_0)$ were known. Importantly, we establish this oracle property without assuming a perfect covariate selection, thereby avoiding the need for the minimum level condition on the signals of active covariates.

Dealing with high dimensional quantile regression with an unknown change point calls for a new proof technique since the quantile loss function is non-smooth and furthermore the corresponding objective function is non-convex with respect to the threshold parameter τ_0. The technique developed in this paper is applicable to a general M-estimation framework with a change point, which may be of independent interest.

The proposed estimation method in this paper consists of three main steps: in the first step, we obtain initial estimators of α_0 and τ_0, whose rates of convergence may be suboptimal; in the second step, we re-estimate τ_0 to obtain an improved estimator of τ_0 that converges at the rate of n and achieves the oracle property mentioned above; in the third step, using

the second step estimator of τ_0, we update the estimator of α_0. In particular, we propose alternative estimators of α_0, depending on the purpose of estimation (prediction vs. variable selection).

One particular application of (1.1) comes from tipping in the racial segregation in social sciences (see, e.g. Card et al., 2008). The empirical question addressed in Card et al. (2008) is whether and the extent to which the neighborhood's white population decreases substantially when the minority share in the area exceeds a tipping point (or change point). In Section 8, we use the US Census tract dataset constructed by Card et al. (2008) and find that the tipping exists in the neighborhoods of Chicago and Pittsburgh.

The remainder of the paper is organized as follows. Section 2 provides an informal description of our estimation methodology. In Section 3, we derive the consistency of the estimators in terms of the excess risk. Section 4 presents regularity assumptions we need to establish further asymptotic properties of the proposed estimators, which are given in Sections 5 and 6. In Section 7, we provide discussions how to choose tuning parameters and present the results of some Monte Carlo experiments. Section 8 illustrates the usefulness of our method by applying it to tipping in the racial segregation. Section 9 concludes and Appendix A describes in detail regarding how to construct the confidence interval for τ_0. Apendices B and C contain high-level regularity conditions on the loss function and the proofs of all the theoretical results, respectively.

Notation. Throughout the paper, we use $|v|_q$ for the ℓ_q norm for a vector v with $q = 0, 1, 2$. We use $|v|_\infty$ to denote the sup norm. For two sequences of positive real numbers a_n and b_n, we write $a_n \ll b_n$ and equivalently $b_n \gg a_n$ if $a_n = o(b_n)$. If there exists a positive finite constant c such that $a_n = c \cdot b_n$, then we write $a_n \propto b_n$. Let $\lambda_{\min}(A)$ denote the minimum eigenvalue of a matrix A. We use w.p.a.1 to mean "with probability approaching one." We write $\theta_0 \equiv \beta_0 + \delta_0$. For a $2p$ dimensional vector α, let α_J and α_{J^c} denote its subvectors formed by indices in $J(\alpha_0)$ and $\{1, ..., 2p\}/J(\alpha_0)$, respectively. Likewise, let $X_J(\tau)$ denote the subvector of $X(\tau) \equiv (X^T, X^T 1\{Q > \tau\})^T$ whose indices are in $J(\alpha_0)$. The

true parameter vectors β_0, δ_0 and θ_0 (except τ_0) are implicitly indexed by the sample size n, and we allow that the dimensions of $J(\beta_0)$, $J(\delta_0)$ and $J(\theta_0)$ can go to infinity as $n \to \infty$. For simplicity, we omit their dependence on n in our notation.

2   Estimators

In this section, we describe our estimation method. We take the check function approach of Koenker and Bassett (1978). Let $\rho(t_1, t_2) \equiv (t_1 - t_2)(\gamma - 1\{t_1 - t_2 \leq 0\})$ denote the loss function for quantile regression. Let $\mathcal{A}$ and $\mathcal{T}$ denote the parameter spaces for $\alpha_0 \equiv (\beta_0, \delta_0)$ and τ_0, respectively. For each $\alpha \equiv (\beta, \delta) \in \mathcal{A}$ and $\tau \in \mathcal{T}$, we write $X^T \beta + X^T \delta 1\{Q > \tau\} = X(\tau)^T \alpha$ with the shorthand notation that $X(\tau) \equiv (X^T, X^T 1\{Q > \tau\})^T$. We suppose that the vector of true parameters is defined as the minimizer of the expected loss:

$$(\alpha_0, \tau_0) = \underset{\alpha \in \mathcal{A}, \tau \in \mathcal{T}}{\operatorname{argmin}} \mathbb{E}\left[\rho(Y, X(\tau)^T \alpha)\right]. \tag{2.1}$$

By construction, τ_0 is not unique when $\delta_0 = 0$.

Suppose we observe independent and identically distributed samples $\{Y_i, X_i, Q_i\}_{i \leq n}$. Let $X_i(\tau)$ and $X_{ij}(\tau)$ denote the i-th realization of $X(\tau)$ and j-th element of $X_i(\tau)$, respectively, $i = 1, \ldots, n$ and $j = 1, \ldots, 2p$, so that $X_{ij}(\tau) \equiv X_{ij}$ if $j \leq p$ and $X_{ij}(\tau) \equiv X_{i,j-p} 1\{Q_i > \tau\}$ otherwise. Define

$$R_n(\alpha, \tau) \equiv \frac{1}{n} \sum_{i=1}^{n} \rho(Y_i, X_i(\tau)^T \alpha) = \frac{1}{n} \sum_{i=1}^{n} \rho(Y_i, X_i^T \beta + X_i^T \delta 1\{Q_i > \tau\}).$$

In addition, let $D_j(\tau) \equiv \{n^{-1} \sum_{i=1}^{n} X_{ij}(\tau)^2\}^{1/2}$, $j = 1, \ldots, 2p$.

We describe the main steps of our ℓ_1-penalized estimation method. For some tuning parameter κ_n, define:

$$\textbf{Step 1: } (\breve{\alpha}, \breve{\tau}) = \operatorname{argmin}_{\alpha \in \mathcal{A}, \tau \in \mathcal{T}} R_n(\alpha, \tau) + \kappa_n \sum_{j=1}^{2p} D_j(\tau)|\alpha_j|. \tag{2.2}$$

This step produces an initial estimator $(\breve{\alpha}, \breve{\tau})$. The tuning parameter κ_n is required to satisfy

$$\kappa_n \propto (\log p)(\log n)\sqrt{\frac{\log p}{n}}. \tag{2.3}$$

Note that we take κ_n that converges to zero at a rate slower than the standard $(\log p/n)^{1/2}$ rate in the literature. This modified rate of κ_n is useful in our context to deal with an unknown τ_0. A data-dependent method of choosing κ_n is discussed in Section 7.1.

Remark 2.1. Define $d_j \equiv (\frac{1}{n}\sum_{i=1}^n X_{ij}^2)^{1/2}$ and $d_j(\tau) \equiv (\frac{1}{n}\sum_{i=1}^n X_{ij}^2 1\{Q_i > \tau\})^{1/2}$. Note that $\sum_{j=1}^{2p} D_j(\tau)|\alpha_j| = \sum_{j=1}^p d_j|\beta_j| + \sum_{j=1}^p d_j(\tau)|\delta_j|$, so that the weight $D_j(\tau)$ adequately balances the regressors; the weight d_j regarding $|\beta_j|$ does not depend on τ, while the weight $d_j(\tau)$ with respect to $|\delta_j|$ does, which takes into account the effect of the threshold τ on the parameter change δ.

The main purpose of the first step is to obtain an initial estimator of α_0. The achieved convergence rates of this step might be suboptimal due to the uniform control of the score functions over the space $\mathcal{T}$ of the unknown τ_0.

In the second step, we introduce our improved estimator of the change point τ_0. It does not use a penalty term, while using the first step estimator of α_0. Define:

$$\textbf{Step 2: } \widehat{\tau} = \underset{\tau \in \mathcal{T}}{\operatorname{argmin}} R_n(\breve{\alpha}, \tau), \tag{2.4}$$

where $\breve{\alpha}$ is the first step estimator of α_0 in (2.2). In Section 5, we show that when τ_0 is identifiable, $\widehat{\tau}$ is consistent for τ_0 at a rate of n^{-1}. Furthermore, we obtain the limiting distribution of $n(\widehat{\tau} - \tau_0)$, and establish conditions under which its asymptotic distribution is the same as if the true α_0 were known, without a perfect model selection on α_0, nor assuming the minimum signal condition on the nonzero elements of α_0.

In the third step, we update the Lasso estimator of α_0 using a different value of the penalization tuning parameter and the second step estimator of τ_0. In particular, we recom-

mend two different estimators of α_0: one for the prediction and the other for the variable selection, serving for different purposes of practitioners.

For two different tuning parameters ω_n and μ_n whose rates will be specified later by (3.2) and (5.1), define:

Step 3a (for prediction):

$$\widehat{\alpha} = \mathrm{argmin}_{\alpha \in \mathcal{A}} R_n(\alpha, \widehat{\tau}) + \omega_n \sum_{j=1}^{2p} D_j(\widehat{\tau})|\alpha_j|, \tag{2.5}$$

Step 3b (for variable selection):

$$\widetilde{\alpha} = \mathrm{argmin}_{\alpha \in \mathcal{A}} R_n(\alpha, \widehat{\tau}) + \mu_n \sum_{j=1}^{2p} w_j D_j(\widehat{\tau})|\alpha_j|, \tag{2.6}$$

where $\widehat{\tau}$ is the second step estimator of τ_0 in (2.4), and the "signal-adaptive" weight w_j in (2.6), motivated by the local linear approximation of the SCAD penalties (Fan and Li, 2001; Zou and Li, 2008), is calculated based on the Step 3a estimator $\widehat{\alpha}$ from (2.5):

$$w_j \equiv \begin{cases} 1, & |\widehat{\alpha}_j| < \mu_n \\ 0, & |\widehat{\alpha}_j| > a\mu_n \\ \frac{a\mu_n - |\widehat{\alpha}_j|}{\mu_n(a-1)} & \mu_n \leq |\widehat{\alpha}_j| \leq a\mu_n. \end{cases}$$

Here $a > 1$ is some prescribed constant, and $a = 3.7$ is often used in the literature. We take this as our choice of a.

Remark 2.2. For $\widehat{\alpha}$ in (2.5), we set ω_n to converge to zero at a rate of $(\log(p \vee n)/n)^{1/2}$ (a more standard rate compared to κ_n in (2.3)). Therefore, the estimator $\widehat{\alpha}$ converges in probability to α_0 faster than $\breve{\alpha}$. In addition, μ_n in (2.6) is chosen to be slightly larger than ω_n for the purpose of the variable selection. A data-dependent method of choosing ω_n as well as μ_n is discussed in Section 7.1. In Sections 5 and 6, we establish conditions under which $\widehat{\alpha}$ achieves the (minimax) optimal rate of convergence in probability for α_0 regardless

of the identifiability of τ_0.

Remark 2.3. It is well known in linear models without the presence of an unknown τ_0 (see, e.g. Bühlmann and van de Geer (2011)) that the Lasso estimator may not perform well for the purpose of the variable selection. The estimator $\widetilde{\alpha}$ defined in Step 3b uses an entry-adaptive weight w_j that corrects the shrinkage bias, and possesses similar merits of the asymptotic unbiasedness of the SCAD penalty. Therefore, we recommend $\widehat{\alpha}$ for the prediction; while suggesting $\widetilde{\alpha}$ for the variable selection.

Remark 2.4. Note that the objective function is non-convex with respect to τ in the first and second steps. However, the proposed estimators can be calculated efficiently using existing algorithms, and we describe the computation algorithms in Section 7.1.

Remark 2.5. Note that Step 2 can be repeated using the updated estimator of α_0 in Step 3. Analogously, Step 3 can be iterated after that. This would give asymptotically equivalent estimators but might improve the finite-sample performance especially when p is very large. Repeating Step 2 might be useful especially when $\breve{\delta} = 0$ in the first step. In this case, there is no unique $\widehat{\tau}$ in Step 2. So, we skip the second step by setting $\widehat{\tau} = \breve{\tau}$ and move to the third step directly. If a preferred estimator of δ_0 in the third step (either $\widehat{\delta}$ or $\widetilde{\delta}$), depending on the estimation purpose, is different from zero, we could go back to Step 2 and re-estimate τ_0. If the third step estimator of δ_0 is also zero, then we conclude that there is no change point and disregard the first-step estimator $\breve{\tau}$ since τ_0 is not identifiable in this case.

3 Risk Consistency

Given the loss function $\rho(t_1, t_2) \equiv (t_1 - t_2)(\gamma - 1\{t_1 - t_2 \leq 0\})$ for the quantile regression model, define the *excess risk* to be

$$R(\alpha, \tau) \equiv \mathbb{E}\rho(Y, X(\tau)^T \alpha) - \mathbb{E}\rho(Y, X(\tau_0)^T \alpha_0). \tag{3.1}$$

By the definition of (α_0, τ_0) in (2.1), we have that $R(\alpha, \tau) \geq 0$ for any $\alpha \in \mathcal{A}$ and $\tau \in \mathcal{T}$. What we mean by the "risk consistency" here is that the excess risk converges in probability to zero for the proposed estimators. The other asymptotic properties of the proposed estimators will be presented in Sections 5 and 6.

In this section, we begin by stating regularity conditions that are needed to develop our first theoretical result. Recall that X_{ij} denotes the jth element of X_i.

Assumption 3.1 (Setting). *(i) The data $\{(Y_i, X_i, Q_i)\}_{i=1}^n$ are independent and identically distributed with $\mathbb{E}\,|X_{ij}|^m \leq \frac{m!}{2} K_1^{m-2}$ for all j and some constant $K_1 < \infty$.*

(ii) $\mathbb{P}(\tau_1 < Q \leq \tau_2) \leq K_2(\tau_2 - \tau_1)$ for any $\tau_1 < \tau_2$ and some constant $K_2 < \infty$.

(iii) $\alpha_0 \in \mathcal{A} \equiv \{\alpha : |\alpha|_\infty \leq M_1\}$ for some constant $M_1 < \infty$, and $\tau_0 \in \mathcal{T} \equiv [\underline{\tau}, \overline{\tau}]$. Furthermore, the probability of $\{Q < \underline{\tau}\}$ and that of $\{Q > \overline{\tau}\}$ are strictly positive, and

$$\sup_{j \leq p} \sup_{\tau \in \mathcal{T}} \mathbb{E}[X_{ij}^2 | Q = \tau] < \infty.$$

(iv) There exist universal constants $\underline{D} > 0$ and $\overline{D} > 0$ such that w.p.a.1,

$$0 < \underline{D} \leq \min_{j \leq 2p} \inf_{\tau \in \mathcal{T}} D_j(\tau) \leq \max_{j \leq 2p} \sup_{\tau \in \mathcal{T}} D_j(\tau) \leq \overline{D} < \infty.$$

(v) $\mathbb{E}\left[\left(X^T \delta_0\right)^2 | Q = \tau\right] \leq M_2 |\delta_0|_2^2$ for all $\tau \in \mathcal{T}$ and for some constant M_2 satisfying $0 < M_2 < \infty$.

In addition to the random sampling assumption, condition (i) imposes mild moment restrictions on X. Condition (ii) imposes a weak restriction that the probability that $Q \in (\tau_1, \tau_2]$ is bounded by a constant times $(\tau_2 - \tau_1)$. Condition (iii) assumes that the parameter space is compact and that the support of Q is strictly larger than $\mathcal{T}$. These conditions are standard in the literature on change-point and threshold models (e.g., Seijo and Sen (2011a,b)). Condition (iii) also assumes that the conditional expectation of $\mathbb{E}[X_{ij}^2 | Q = \cdot]$

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is bounded on $\mathcal{T}$ uniformly in j. Condition (iv) requires that each regressor be of the same magnitude uniformly over the threshold τ. As the data-dependent weights $D_j(\tau)$ are the sample second moments of the regressors, it is not stringent to assume them to be bounded away from both zero and infinity. Condition (v) puts some weak upper bound on $\mathbb{E}[(X^T\delta_0)^2 | Q = \tau]$ for all $\tau \in \mathcal{T}$ when $\delta_0 \neq 0$. A simple sufficient condition for condition (v) is that the eigenvalues of $\mathbb{E}[X_{J(\delta_0)} X_{J(\delta_0)}^T | Q = \tau]$ are bounded uniformly in τ, where $X_{J(\delta_0)}$ denotes the subvector of X corresponding to the nonzero components of δ_0.

Throughout the paper, we let $s \equiv |J(\alpha_0)|_0$, namely the cardinality of $J(\alpha_0)$. We allow that $s \to \infty$ as $n \to \infty$ and will give precise regularity conditions regarding its growth rates. The following theorem is concerned about the convergence of $R(\breve{\alpha}, \breve{\tau})$ with the first step estimator.

Theorem 3.1 (Risk Consistency). *Let Assumption 3.1 hold. Suppose that the tuning parameter κ_n satisfies* (2.3). *Then, $R(\breve{\alpha}, \breve{\tau}) = O_P(\kappa_n s)$.*

Note that Theorem 3.1 holds regardless of the identifiability of τ_0 (that is, whether $\delta_0 = 0$ or not). Theorem 3.1 implies the risk consistency immediately if $\kappa_n s \to 0$ as $n \to \infty$. The restriction on s is slightly stronger than that of the standard result $s = o(\sqrt{n/\log p})$ in the literature for the M-estimation (see, e.g. van de Geer (2008) and Chapter 6.6 of Bühlmann and van de Geer (2011)) since the objective function $\rho(Y, X(\tau)^T\alpha)$ is non-convex in τ, due to the unknown change-point.

Remark 3.1. The extra logarithmic factor $(\log p)(\log n)$ in the definition of κ_n (see (2.3)) is due to the existence of the unknown and possibly non-identifiable threshold parameter τ_0. In fact, an inspection of the proof of Theorem 3.1 reveals that it suffices to assume that κ_n satisfies $\kappa_n \gg \log_2(p/s)[\log(np)/n]^{1/2}$. The term $\log_2(p/s)$ and the additional $(\log n)^{1/2}$ term inside the brackets are needed to establish the stochastic equicontinuity of the empirical process

$$\nu_n(\alpha, \tau) \equiv \frac{1}{n} \sum_{i=1}^{n} \left[\rho\left(Y_i, X_i(\tau)^T \alpha\right) - \mathbb{E}\rho\left(Y, X(\tau)^T \alpha\right) \right]$$

uniformly over $(\alpha, \tau) \in \mathcal{A} \times \mathcal{T}$.

The following theorem shows that an improved rate of convergence is possible for the excess risk by taking the second and third steps of estimation.

Theorem 3.2 (Improved Risk Consistency). *Let Assumption 3.1 hold. In addition, assume that $|\widehat{\tau} - \tau_0| = O_P(n^{-1})$ when $\delta_0 \neq 0$. Let*

$$\omega_n \propto \sqrt{\frac{\log(p \vee n)}{n}} \ . \tag{3.2}$$

Then, whether $\delta_0 = 0$ or not,

$$R\left(\widehat{\alpha}, \widehat{\tau}\right) = O_p\left(\omega_n s\right).$$

For the sake of not introducing additional assumptions at this stage, we have assumed in Theorem 3.2 that $|\widehat{\tau} - \tau_0| = O_P(n^{-1})$ when τ_0 is identifiable. Its formal statement is delegated to Theorem 5.3 in Section 5.

Remark 3.2. As in Theorem 3.1, the risk consistency part of Theorem 3.2 holds whether or not $\delta_0 = 0$. We obtain the improved rate of convergence in probability for the excess risk by combining the fact that our objective function is convex with respect to α given each τ with the second-step estimation results that (i) if $\delta \neq 0$, then $\widehat{\tau}$ is within a shrinking local neighborhood of τ_0, and (ii) when $\delta_0 = 0$, $\widehat{\tau}$ does not affect the excess risk in the sense that $R\left(\alpha_0, \tau\right) = 0$ for all $\tau \in \mathcal{T}$.

4   Assumptions for Oracle Properties

In this section, we list a set of assumptions that will be useful to derive asymptotic properties of the proposed estimators in Sections 5 and 6. In the following, we divide our discussions into two important cases: (i) $\delta_0 \neq 0$ and τ_0 is identified, and (ii) $\delta_0 = 0$ and thus τ_0 is not identified. The asymptotic properties are derived under both cases. Note that such

a distinction is only needed for presenting our theoretical results. In practice, we do not need to know whether $\delta_0 = 0$ or not.

Assumption 4.1 (Underlying Distribution).   *(i) The conditional distribution $Y|X, Q$ has a continuously differentiable density function $f_{Y|X,Q}(y|x,q)$ with respect to y, whose derivative is denoted by $\tilde{f}_{Y|X,Q}(y|x,q)$.*

(ii) There are constants $C_1, C_2 > 0$ such that for all (y, x, q) in the support of (Y, X, Q),

$$|\tilde{f}_{Y|X,Q}(y|x,q)| \le C_1, \quad f_{Y|X,Q}(x(\tau_0)^T \alpha_0 | x, q) \ge C_2.$$

(iii) When $\delta_0 \ne 0$, $\Gamma(\tau, \alpha_0)$ is positive definite uniformly in a neighborhood of τ_0, where

$$\Gamma(\tau, \alpha_0) \equiv \frac{\partial^2 \mathbb{E}[\rho(Y, X_J(\tau)^T \alpha_{0J})]}{\partial \alpha_J \partial \alpha_J^T} = \mathbb{E}[X_J(\tau) X_J(\tau)^T f_{Y|X,Q}(X(\tau)^T \alpha_0 | X, Q)].$$

When $\delta_0 = 0$, the matrix $\mathbb{E}[X_{J(\beta_0)} X_{J(\beta_0)}^T f_{Y|X,Q}(X_{J(\beta_0)}^T \beta_{0J(\beta_0)} | X, Q)]$ is positive definite.

Conditions (i) and (ii) are standard assumptions for quantile regression models. To follow the notation in condition (iii), recall that α_J denotes the subvector of α whose indices are in $J(\alpha_0)$. Expressions $X_J(\tau)$, $X_{J(\beta_0)}$, α_{0J} and $\beta_{0J(\beta_0)}$ can be understood similarly. Condition (iii) is a weak condition that imposes non-singularity of the Hessian matrix of the population objective function uniformly in a neighborhood of τ_0 in case of $\delta_0 \ne 0$. This condition reduces to the usual non-singularity condition when $\delta_0 = 0$.

4.1   Compatibility Conditions

We now make an assumption that is an extension of the well-known *compatibility condition* (see Bühlmann and van de Geer (2011), Chapter 6). In particular, the following condition is a uniform-in-τ version of the compatibility condition. Recall that for a $2p$ dimensional vector α, we use α_J and α_{J^c} to denote its subvectors formed by indices in $J(\alpha_0)$ and $\{1, ..., 2p\}/J(\alpha_0)$, respectively.

Assumption 4.2 (Compatibility Condition). *(i) When $\delta_0 \neq 0$, there is a neighborhood $\mathcal{T}_0 \subset \mathcal{T}$ of τ_0, and a constant $\phi > 0$ such that for all $\tau \in \mathcal{T}_0$ and all $\alpha \in \mathbb{R}^{2p}$ satisfying $|\alpha_{J^c}|_1 \leq 5|\alpha_J|_1$,*

$$\phi|\alpha_J|_1^2 \leq s\alpha^T \mathbb{E}[X(\tau)X(\tau)^T]\alpha. \tag{4.1}$$

(ii) When $\delta_0 = 0$, there is a constant $\phi > 0$ such that for all $\tau \in \mathcal{T}$ and all $\alpha \in \mathbb{R}^{2p}$ satisfying $|\alpha_{J^c}|_1 \leq 4|\alpha_J|_1$,

$$\phi|\alpha_J|_1^2 \leq s\alpha^T \mathbb{E}[X(\tau)X(\tau)^T]\alpha. \tag{4.2}$$

Assumption 4.2 requires that the compatibility condition hold uniformly in τ over a neighbourhood of τ_0 when $\delta_0 \neq 0$ and over the entire parameter space $\mathcal{T}$ when $\delta_0 = 0$. Note that this assumption is imposed on the population covariance matrix $\mathbb{E}[X(\tau)X(\tau)^T]$; thus, a simple sufficient condition of Assumption 4.2 is that the smallest eigenvalue of $\mathbb{E}[X(\tau)X(\tau)^T]$ is bounded away from zero uniformly in τ. Even if $p > n$, the population covariance can still be strictly positive definite while the sample covariance is not.

4.2 Restricted Nonlinearity Conditions

In this subsection, we make an assumption called a *restricted nonlinear condition* to deal with the quantile loss function. We extend condition D.4 in Belloni and Chernozhukov (2011) to accommodate the possible existence of the unknown threshold in our model (specifically, a uniform-in-τ version of the restricted nonlinear condition as in the compatibility condition).

Note that when $Q \leq \tau_0$, $X(\tau_0)^T\alpha_0 = X^T\beta_0$, while when $Q > \tau_0$, $X(\tau_0)^T\alpha_0 = X^T\theta_0$, where $\theta_0 \equiv \beta_0 + \delta_0$. Hence we define the "prediction balls" with radius r and corresponding centers as follows:

$$\begin{aligned}
\mathcal{B}(\beta_0, r) &= \{\beta \in \mathcal{B} \subset \mathbb{R}^p : \mathbb{E}[(X^T(\beta - \beta_0))^2 1\{Q \leq \tau_0\}] \leq r^2\}, \\
\mathcal{G}(\theta_0, r) &= \{\theta \in \mathcal{G} \subset \mathbb{R}^p : \mathbb{E}[(X^T(\theta - \theta_0))^2 1\{Q > \tau_0\}] \leq r^2\},
\end{aligned} \tag{4.3}$$

where $\mathcal{B}$ and $\mathcal{G}$ are parameter spaces for β_0 and θ_0, respectively. To deal with the case that $\delta_0 = 0$, we also define

$$\tilde{\mathcal{B}}(\beta_0, r, \tau) = \{\beta \in \mathcal{B} \subset \mathbb{R}^p : \mathbb{E}[(X^T(\beta - \beta_0))^2 1\{Q \leq \tau\}] \leq r^2\},$$
$$\tilde{\mathcal{G}}(\beta_0, r, \tau) = \{\theta \in \mathcal{G} \subset \mathbb{R}^p : \mathbb{E}[(X^T(\theta - \beta_0))^2 1\{Q > \tau\}] \leq r^2\}. \tag{4.4}$$

Assumption 4.3 (Restricted Nonlinearity)**.** *The following holds for the constants C_1 and C_2 defined in Assumption 4.1 (ii).*

(i) *When $\delta_0 \neq 0$, there exists a constant $r_{QR}^* > 0$ such that*

$$\inf_{\beta \in \mathcal{B}(\beta_0, r_{QR}^*), \beta \neq \beta_0} \frac{\mathbb{E}[|X^T(\beta - \beta_0)|^2 1\{Q \leq \tau_0\}]^{3/2}}{\mathbb{E}[|X^T(\beta - \beta_0)|^3 1\{Q \leq \tau_0\}]} \geq r_{QR}^* \frac{2C_1}{3C_2} > 0 \tag{4.5}$$

and that

$$\inf_{\theta \in \mathcal{G}(\theta_0, r_{QR}^*), \theta \neq \theta_0} \frac{\mathbb{E}[|X^T(\theta - \theta_0)|^2 1\{Q > \tau_0\}]^{3/2}}{\mathbb{E}[|X^T(\theta - \theta_0)|^3 1\{Q > \tau_0\}]} \geq r_{QR}^* \frac{2C_1}{3C_2} > 0. \tag{4.6}$$

(ii) *When $\delta_0 = 0$, there exists a constant $r_{QR}^* > 0$ such that*

$$\inf_{\tau \in \mathcal{T}} \inf_{\beta \in \tilde{\mathcal{B}}(\beta_0, r_{QR}^*, \tau), \beta \neq \beta_0} \frac{\mathbb{E}[|X^T(\beta - \beta_0)|^2 1\{Q \leq \tau\}]^{3/2}}{\mathbb{E}[|X^T(\beta - \beta_0)|^3 1\{Q \leq \tau\}]} \geq r_{QR}^* \frac{2C_1}{3C_2} > 0 \tag{4.7}$$

and that

$$\inf_{\tau \in \mathcal{T}} \inf_{\theta \in \tilde{\mathcal{G}}(\beta_0, r_{QR}^*, \tau), \beta \neq \beta_0} \frac{\mathbb{E}[|X^T(\theta - \theta_0)|^2 1\{Q > \tau\}]^{3/2}}{\mathbb{E}[|X^T(\theta - \theta_0)|^3 1\{Q > \tau\}]} \geq r_{QR}^* \frac{2C_1}{3C_2} > 0. \tag{4.8}$$

Remark 4.1. As pointed out by Belloni and Chernozhukov (2011), If $X^T c$ follows a log-concave distribution conditional on Q for any nonzero c (e.g. if the distribution of X is multivariate normal), then Theorem 5.22 of Lovász and Vempala (2007) and the Hölder

inequality imply that for all $\alpha \in \mathcal{A}$,

$$\mathbb{E}[|X(\tau_0)^T(\alpha - \alpha_0)|^3|Q] \leq 6\left\{\mathbb{E}[\{X(\tau_0)^T(\alpha - \alpha_0)\}^2|Q]\right\}^{3/2},$$

which provides a sufficient condition for Assumption 4.3. On the other hand, this assumption can hold more generally since equations (4.5)-(4.8) in Assumption 4.3 need to hold only locally around true parameters α_0.

4.3 Additional Assumptions When $\delta_0 \neq 0$

We first describe the additional conditions on the distribution of (X, Q).

Assumption 4.4 (Additional Conditions on the Distribution of (X, Q)). *Assume $\delta_0 \neq 0$. In addition, there exists a neighborhood $\mathcal{T}_0 \subset \mathcal{T}$ of τ_0 that satisfies the following.*

(i) Q has a density function $f_Q(\cdot)$ that is continuous and bounded away from zero on $\mathcal{T}_0$.

(ii) Let $\tilde{X}$ denote all the components of X excluding Q in case that Q is an element of X. The conditional distribution of Q given $\tilde{X}$ has a density function $f_{Q|\tilde{X}}(q|\tilde{x})$ that is bounded uniformly in both $q \in \mathcal{T}_0$ and $\tilde{x}$.

(iii) There exists $M_3 > 0$ such that $M_3^{-1} \leq \mathbb{E}[(X^T\delta_0)^2|Q = \tau] \leq M_3$ for all $\tau \in \mathcal{T}_0$.

Condition (i) implies that $\mathbb{P}\{|Q - \tau_0| < \varepsilon\} > 0$ for any $\varepsilon > 0$, and condition (ii) requires that the conditional density of Q given $\tilde{X}$ be uniformly bounded. When τ_0 is identified, we require δ_0 to be considerably different from zero. This requirement is given in condition (iii). Note that this condition is concerned with $\mathbb{E}[(X^T\delta_0)^2|Q = \tau]$, which is an important quantity to develop asymptotic results when $\delta_0 \neq 0$. Note that condition (iii) is a local condition with respect to τ in the sense that it has to hold only locally in a neighborhood of τ_0.

The following additional moment conditions are useful to derive our theoretical results.

Assumption 4.5 (Moment Bounds). *(i) There exist constants $0 < \widetilde{C}_1 \leq \widetilde{C}_2 < 1$ such that for all $\beta \in \mathbb{R}^p$ satisfying $\mathbb{E}|X^T\beta| \neq 0$,*

$$\widetilde{C}_1 \leq \frac{\mathbb{E}[|X^T\beta|1\{Q > \tau_0\}]}{\mathbb{E}|X^T\beta|} \leq \widetilde{C}_2.$$

(ii) There exist positive constants M, r and the neighborhood $\mathcal{T}_0$ of τ_0 such that

$$\mathbb{E}\left[(X^T[(\theta - \beta) - (\theta_0 - \beta_0)])^2 \big| Q = \tau\right] \leq M,$$

$$\mathbb{E}[|X^T(\beta - \beta_0)| \big| Q = \tau] \leq M,$$

$$\mathbb{E}[|X^T(\theta - \theta_0)| \big| Q = \tau] \leq M,$$

$$\sup_{\tau \in \mathcal{T}_0 : \tau > \tau_0} \mathbb{E}\left[|X^T(\beta - \beta_0)|\frac{1\{\tau_0 < Q \leq \tau\}}{(\tau - \tau_0)}\right] \leq M\mathbb{E}[|X^T(\beta - \beta_0)|1\{Q \leq \tau_0\}],$$

$$\sup_{\tau \in \mathcal{T}_0 : \tau < \tau_0} \mathbb{E}\left[|X^T(\theta - \theta_0)|\frac{1\{\tau < Q \leq \tau_0\}}{(\tau_0 - \tau)}\right] \leq M\mathbb{E}[|X^T(\theta - \theta_0)|1\{Q > \tau_0\}],$$

uniformly in $\beta \in \mathcal{B}(\beta_0, r)$, $\theta \in \mathcal{G}(\theta_0, r)$ and $\tau \in \mathcal{T}_0$.

Remark 4.2. Condition (i) requires that Q have non-negligible support on both sides of τ_0. Note that it is equivalent to

$$\left(\frac{1}{\widetilde{C}_2} - 1\right)\mathbb{E}[|X^T\beta|1\{Q > \tau_0\}] \leq \mathbb{E}\left[|X^T\beta|1\{Q \leq \tau_0\}\right]$$

$$\leq \left(\frac{1}{\widetilde{C}_1} - 1\right)\mathbb{E}|X^T\beta|1\{Q > \tau_0\}. \tag{4.9}$$

Hence this assumption prevents the conditional expectation of $X^T\beta$ given Q from changing too dramatically across regimes. Condition (ii) requires the boundedness and certain smoothness of the conditional expectation functions $\mathbb{E}[(X^T[(\theta - \beta) - (\theta_0 - \beta_0)])^2 | Q = \tau]$, $\mathbb{E}[|X^T(\beta - \beta_0)| | Q = \tau]$, and $\mathbb{E}[|X^T(\theta - \theta_0)| | Q = \tau]$, and prohibits degeneracy in one regime. The last two inequalities in condition (ii) are satisfied if

$$\frac{\mathbb{E}\left[|X^T\beta| | Q = \tau\right]}{\mathbb{E}\left[|X^T\beta|\right]} \leq M$$

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for all $\tau \in \mathcal{T}_0$ and for all β satisfying $0 < \mathbb{E}\left|X^T\beta\right| \le c$ for some small $c > 0$. In this view, we may regard condition (ii) as a local version of condition (i).

5   Asymptotic Properties: Case I. $\delta_0 \neq 0$

We first establish the consistency of $\breve{\tau}$ for τ_0.

Theorem 5.1 (Consistency of $\breve{\tau}$). *Let Assumptions 3.1, 4.1, 4.4, and 4.5 hold. Furthermore, assume that $\kappa_n s = o(1)$. Then, $\breve{\tau} \xrightarrow{p} \tau_0$.*

The following theorem presents the rates of convergence for the first step estimators of α_0 and τ_0. Recall that κ_n is the first-step penalization tuning parameter that satisfies (2.3).

Theorem 5.2 (Rates of Convergence When $\delta_0 \neq 0$). *Suppose that $\kappa_n s^2 \log p = o(1)$. Then under Assumptions 3.1-4.5, we have:*

$$|\breve{\alpha} - \alpha_0|_1 = O_P(\kappa_n s), \ R(\breve{\alpha}, \breve{\tau}) = O_P(\kappa_n^2 s), \quad and \quad |\breve{\tau} - \tau_0| = O_P(\kappa_n^2 s).$$

It is worth noting that $\breve{\tau}$ converges to τ_0 faster than the standard parametric rate of $n^{-1/2}$, as long as $s^2(\log p)^6(\log n)^4 = o(n)$. The main reason for such *super-consistency* is that the objective function behaves locally linearly around τ_0 with a kink at τ_0, unlike in the regular estimation problem where the objective function behaves locally quadratically around the true parameter value. Moreover, the achieved convergence rate for $\breve{\alpha}$ is nearly minimax optimal, with an additional factor $(\log p)(\log n)$ compared to the rate of regular Lasso estimation (e.g., Bickel et al. (2009); Raskutti et al. (2011)). This factor arises due to the unknown change-point τ_0. We will improve the rates of convergence for both τ_0 and α_0 further by taking the second and third steps of estimation.

Recall that the second-step estimator of τ_0 is defined as

$$\widehat{\tau} = \underset{\tau \in \mathcal{T}}{\operatorname{argmin}} R_n(\breve{\alpha}, \tau),$$

where $\breve{\alpha}$ is the first step estimator of α_0 in (2.2). Consider an oracle case for which α in $R_n(\alpha, \tau)$ is fixed at α_0. Let $R_n^*(\tau) = R_n(\alpha_0, \tau)$ and

$$\widetilde{\tau} = \operatorname*{argmin}_{\tau \in \mathcal{T}} R_n^*(\tau).$$

Define $\dot{\rho}(t) \equiv t(\gamma - 1\{t \leq 0\})$, so that $\rho(t, s) = \dot{\rho}(t - s)$. For each $i = 1, \ldots, n$, let $U_i \equiv Y_i - X_i^T \beta_0 - X_i^T \delta_0 1\{Q_i > \tau_0\}$, $\dot{\rho}_{1i} \equiv \dot{\rho}(U_i - X_i^T \delta_0) - \dot{\rho}(U_i)$ and $\dot{\rho}_{2i} \equiv \dot{\rho}(U_i + X_i^T \delta_0) - \dot{\rho}(U_i)$.

We now give one of the main results of this paper.

Theorem 5.3 (Oracle Estimation of τ_0). *Let Assumptions 3.1-4.5 hold. Furthermore, suppose that $\kappa_n s^2 \log p = o(1)$. Then, we have that*

$$\widehat{\tau} - \widetilde{\tau} = o_p\left(n^{-1}\right)$$

and $n(\widehat{\tau} - \tau_0)$ converges in distribution to the smallest minimizer of a compound Poisson process, which is given by

$$M(h) \equiv \sum_{i=1}^{N_1(-h)} \rho_{1i} 1\{h < 0\} + \sum_{i=1}^{N_2(h)} \rho_{2i} 1\{h \geq 0\},$$

where N_1 and N_2 are Poisson processes with the same jump rate $f_Q(\tau_0)$ and $\{\rho_{1i}\}$ and $\{\rho_{2i}\}$ are two sequences of independent and identically distributed random variables. The two common distributions are identical to the conditional distributions of $\dot{\rho}_{1i}$ and $\dot{\rho}_{2i}$ given $Q_i = \tau_0$, respectively. Here, N_1, N_2, $\{\rho_{1i}\}$ and $\{\rho_{2i}\}$ are mutually independent.

The first conclusion of Theorem 5.3 establishes that the second step estimator of τ_0 is an oracle estimator in the sense that it is asymptotically equivalent to the infeasible, oracle estimator $\widetilde{\tau}$. As emphasized in the introduction, we obtain the oracle property without relying on the perfect model selection in the first step nor on the existence of the minimum signal condition on active covariates. The second conclusion of Theorem 5.3 follows from combining well-known weak convergence results in the literature (see e.g. Pons (2003);

Kosorok and Song (2007); Lee and Seo (2008)) with the argmax continuous mapping theorem by Seijo and Sen (2011b).

Remark 5.1. Li and Ling (2012) propose a numerical approach for constructing a confidence interval by simulating a compound Poisson process in the context of least squares estimation. We adopt their approach to simulate the compound Poisson process for quantile regression. See Section 8 for a detailed description of how to construct a confidence interval for τ_0.

We now consider the Step 3a estimator of α_0 defined in (2.5). Recall that ω_n is the Step 3a penalization tuning parameter that satisfies (3.2).

Theorem 5.4 (Improved Rates of Convergence When $\delta_0 \neq 0$). *Suppose that $\kappa_n s^2 \log p = o(1)$. Then under Assumptions 3.1-4.5,*

$$|\widehat{\alpha} - \alpha_0|_1 = O_P(\omega_n s) \quad and \quad R(\widehat{\alpha}, \widehat{\tau}) = O_P(\omega_n^2 s).$$

Theorem 5.4 shows that the estimator $\widehat{\alpha}$ defined in Step 3a achieves the optimal rate of convergence in terms of prediction and estimation. In other words, when ω_n is proportional to $\{\log(p \vee n)/n\}^{1/2}$ in equation (3.2) and p is larger than n, it obtains the minimax rates as in e.g., Raskutti et al. (2011).

As we mentioned in Section 2, the Step 3b estimator of α_0 has the purpose of the variable selection. The nonzero components of $\widetilde{\alpha}$ are expected to identify the important regressors. Partition $\widetilde{\alpha} = (\widetilde{\alpha}_J, \widetilde{\alpha}_{J^c})$ such that $\widetilde{\alpha}_J = (\widetilde{\alpha}_j : j \in J(\alpha_0))$ and $\widetilde{\alpha}_{J^c} = (\widetilde{\alpha}_j : j \notin J(\alpha_0))$. Note that $\widetilde{\alpha}_J$ consists of the estimators of β_{0J} and δ_{0J}, whereas $\widetilde{\alpha}_{J^c}$ consists of the estimators of all the zero components of β_0 and δ_0. Let $\alpha_{0J}^{(j)}$ denote the j-th element of α_{0J}.

We now establish conditions under which the estimator $\widetilde{\alpha}$ defined in Step 3b has the *change-point-oracle properties*, meaning that it achieves the variable selection consistency and has the limiting distributions as though the identities of the important regressors and the location of the change point were known.

Theorem 5.5 (Variable Selection When $\delta_0 \neq 0$)**.** *Suppose that $\kappa_n s^2 \log p = o(1)$, $s^4 \log s = o(n)$, and*

$$\omega_n + s\sqrt{\frac{\log s}{n}} \ll \mu_n \ll \min_{j \in J(\alpha_0)} |\alpha_{0J}^{(j)}|. \tag{5.1}$$

Then under Assumptions 3.1-4.5, we have:

$$|\widetilde{\alpha}_J - \alpha_{0J}|_2 = O_P\left(\sqrt{\frac{s \log s}{n}}\right), \quad |\widetilde{\alpha}_J - \alpha_{0J}|_1 = O_P\left(s\sqrt{\frac{\log s}{n}}\right)$$

and

$$P(\widetilde{\alpha}_{J^c} = 0) \to 1.$$

We see that (5.1) provides a condition on the strength of the signal via $\min_{j \in J(\alpha_0)} |\alpha_{0J}^{(j)}|$, and the tuning parameter in Step 3b should satisfy $\omega_n \ll \mu_n$ and $s^2 \log s/n \ll \mu_n^2$. Hence the variable selection consistency demands a larger tuning parameter than in Step 3a.

To conduct statistical inference, we now discuss the asymptotic distribution of $\widetilde{\alpha}_J$. Define $\widehat{\alpha}_J^* \equiv \mathrm{argmin}_{\alpha_J} R_n^*(\alpha_J, \tau_0)$. Note that the asymptotic distribution for $\widehat{\alpha}_J^*$ corresponds to an oracle case that we know τ_0 as well as the true active set $J(\alpha_0)$ *a priori*. The limiting distribution of $\widetilde{\alpha}_J$ is the same as that of $\widehat{\alpha}_J^*$. Hence, we call this result the *change-point-oracle property* of the Step 3b estimator and the following theorem establishes this property.

Theorem 5.6 (Change-Point-Oracle Properties)**.** *Suppose that all the conditions imposed in Theorem 5.5 are satisfied. Furthermore, assume that $\frac{\partial}{\partial \alpha} E\left[\rho\left(Y, X^T \alpha\right) | Q = t\right]$ exists for all t in a neighborhood of τ_0 and all its elements are continuous and bounded, and that $s^3(\log s)(\log n) = o(n)$. Then, we have that $\widetilde{\alpha}_J = \widehat{\alpha}_J^* + o_p(n^{-1/2})$.*

Since the sparsity index (s) grows at a rate slower than the sample size (n), it is straightforward to establish the the asymptotic normality of a linear transformation of $\widetilde{\alpha}_J$, i.e., $\mathbf{L}\widetilde{\alpha}_J$, where $\mathbf{L} : \mathbb{R}^s \to \mathbb{R}$ with $|\mathbf{L}|_2 = 1$, by combing the existing results on quantile regression with parameters of increasing dimension (see, e.g. He and Shao (2000)) with Theorem 5.6.

Remark 5.2. Without the condition on the strength of minimal signals, it may not be possi-

ble to achieve the variable selection consistency or establish change-point-oracle properties. However, the following theorem shows that the SCAD-weighted penalized estimation still can achieve a satisfactory rate of convergence in estimation of α_0 without the condition that $\mu_n \ll \min_{j \in J(\alpha_0)} |\alpha_{0J}^{(j)}|$.

Theorem 5.7 (Satisfactory Rates Without Minimum Signal Condition). *Assume that Assumptions 3.1-4.5 hold. Suppose that $\kappa_n s^2 \log p = o(1)$ and $\omega_n \ll \mu_n$. Then, without the lower bound requirement on $\min_{j \in J(\alpha_0)} |\alpha_{0J}^{(j)}|$, we have that $|\widetilde{\alpha} - \alpha_0|_1 = O_P(\mu_n s)$.*

6 Asymptotic Properties: Case II. $\delta_0 = 0$

In this section, we show that our estimators have desirable results even if there is no change point in the true model. The case of $\delta_0 = 0$ corresponds to the high dimensional linear quantile regression model. Since $X^T \beta_0 + X^T \delta_0 1\{Q > \tau_0\} = X^T \beta_0$, τ_0 is non-identifiable, and there is no structural change on the coefficient. But a new analysis different from that of the standard high-dimensional model is still required because in practice we do not know whether $\delta_0 = 0$ or not. Thus, the proposed estimation method still estimates τ_0 to account for possible structural changes. The following results show that in this case, the first step estimator of α_0 will asymptotically behave as if $\delta_0 = 0$ were *a priori* known.

Theorem 6.1 (Rates of Convergence When $\delta_0 = 0$). *Suppose that $\kappa_n s = o(1)$. Then under Assumptions 3.1-4.3, we have that*

$$|\breve{\alpha} - \alpha_0|_1 = O_P(\kappa_n s) \quad and \quad R(\breve{\alpha}, \breve{\tau}) = O_P(\kappa_n^2 s).$$

The results obtained in Theorem 6.1 combined with those obtained in Theorem 5.2 imply that the first step estimatior performs equally well in terms of rates of convergence for both the ℓ_1 loss for $\breve{\alpha}$ and the excess risk regardless of the existence of the threshold effect. It is straightforward to obtain an improved rate result for the Step 3a estimator, equivalent to

Theorem 5.4 under Assumptions 3.1-4.3. We omit the details for brevity.

We now give a result that is equivalent to Theorem 5.5.

Theorem 6.2 (Variable Selection When $\delta_0 = 0$). *Suppose that $\kappa_n s = o(1)$, $s^4 \log s = o(n)$, and*

$$\omega_n + s \sqrt{\frac{\log s}{n}} \ll \mu_n \ll \min_{j \in J(\alpha_0)} |\alpha_{0J}^{(j)}|.$$

Then under Assumptions 3.1-4.3, we have:

$$\left| \widetilde{\beta}_J - \beta_{0J} \right|_2 = O_P\left(\sqrt{\frac{s \log s}{n}} \right), \quad \left| \widetilde{\beta}_J - \beta_{0J} \right|_1 = O_P\left(s \sqrt{\frac{\log s}{n}} \right),$$

and

$$P(\widetilde{\beta}_{J^c} = 0) \to 1, \quad P(\widetilde{\delta} = 0) \to 1.$$

Theorem 6.2 demonstrates that when there is in fact no change point, our estimator for δ_0 is exactly zero with a high probability. Therefore, the estimator can also be used as a diagnostic tool to check whether there exists any structural change. Results similar to Theorems 5.6 and 5.7 can be established straightforwardly as well; however, their details are omitted for brevity.

7   Simulation Results

7.1   Tuning parameter selection

Recall that our estimators are obtained by three steps, which involve three tuning parameters in the penalization: (1) κ_n in Step 1 ought to dominate the score function uniformly over the range of τ, and hence should be slightly larger than the others; (2) ω_n is used in Step 3a for the prediction, and (3) μ_n in Step 3b for the variable selection should be larger than ω_n. Note that the tuning parameters in both Steps 3a and 3b are similar to those of the existing literature since the change point $\widehat{\tau}$ has been estimated.

In the following Monte Carlo experiments, we build on the data-dependent selection method in Belloni and Chernozhukov (2011). Define

$$\Lambda(\tau) := \max_{1 \leq j \leq 2p} \left| \frac{1}{n} \sum_{i=1}^{n} \frac{X_{ij}(\tau) \left(\gamma - 1\{U_i \leq \gamma\} \right)}{D_j(\tau)} \right|, \qquad (7.1)$$

where U_i is simulated from the *i.i.d.* uniform distribution on the interval $[0, 1]$; γ is the fixed quantile level (for median regression $\gamma = 0.5$). Note that $\Lambda(\tau)$ is a stochastic process indexed by τ. Let $\overline{\Lambda}_{1-q}$ be the $(1-q)$-quantile of $\sup_{\tau \in \mathcal{T}} \Lambda(\tau)$. Then, we select the tuning parameter in Step 1 by $\kappa_n = c_1 \cdot \overline{\Lambda}_{1-q}$. Similarly, let $\Lambda_{1-q}(\widehat{\tau})$ be the $(1-q)$-quantile of $\Lambda(\widehat{\tau})$, where $\widehat{\tau}$ is chosen in Step 2. We select ω_n and μ_n in Step 3 by $\omega_n = c_1 \cdot \Lambda_{1-q}(\widehat{\tau})$ and $\mu_n = c_2 \cdot \omega_n$.

Based on the suggestions of Belloni and Chernozhukov (2011) and some preliminary simulations, we decide to set $c_1 = 1.1$, $c_2 = \log \log n$, and $q = 0.1$. In addition, recall that we set $a = 3.7$ when calculating the SCAD weights w_j in Step 3b following the convention in the literature (e.g. Fan and Li (2001) and Loh and Wainwright (2013)). In Step 1, we first solve the lasso problem for α given each grid point of $\tau \in \mathcal{T}$. Then, we choose $\breve{\tau}$ and the corresponding $\breve{\alpha}(\breve{\tau})$ that minimize the objective function. Step 2 can be solved simply by the grid search. Step 3 is a standard lasso quantile estimation given $\widehat{\tau}$, whose numerical implementation is well established.

7.2   Monte Carlo Experiments

In this section we provide the results of Monte Carlo simulation studies. The baseline model is the following median regression: for $i = 1, \ldots, n$,

$$Y_i = X_i^T \beta_0 + X_i^T \delta_0 1\{Q_i > \tau_0\} + U_i, \qquad (7.2)$$

where U_i follows the standard normal distribution, and Q_i follows the uniform distribution on the interval $[0, 1]$. The p-dimensional covariate X_i is composed of a constant and Z_i,

i.e. $X := (1, Z_i^T)^T$, where Z_i follows the multivariate normal distribution $N(0, \Sigma)$ with a covariance matrix $\Sigma_{ij} = (1/2)^{|i-j|}$. Here, the variables U_i, Q_i and Z_i are independent of each other.

The p-dimensional parameters β_0 and δ_0 are set to $\beta_0 = (1, 1, 1/2, 0, \ldots, 0)$ and $\delta_0 = (2, 2, 1, 0, \ldots, 0)$ in Design 1 and $\beta_0 = (1, 1, 1/2, 1/3, 1/4, 1/5, 0, \ldots, 0)$ and $\delta_0 = (2, 2, 1, 1/2, 1/3, 1/4, 0, \ldots, 0)$ in Design 2. Note that the parameters in Design 2 are decaying, indicating that the minimal signal of active regressors is weaker in Design 2 than in Design 1. In both designs, we set the change point parameter $\tau_0 = 0.5$. The sample sizes are set to $n = 200$ and 400. The dimension of X_i is set to $p = 250$. Note that we have 500 regressors in total. The range of τ is set to $\mathcal{T} = [0.15, 0.85]$. We conduct 1,000 replications of each design.

We compare estimation results of each step. To assess the performance of our estimators, we also compare the results with two "oracle estimators". Specifically, Oracle 1 knows the true active set $J(\alpha_0)$ and the change point parameter τ_0, and Oracle 2 knows only $J(\alpha_0)$. The threshold parameter τ_0 is re-estimated in Steps 3a and 3b using updated estimates of α_0.

Tables 1–2 summarize the simulation results. We abuse notation slightly and denote all estimators by $(\widehat{\alpha}, \widehat{\tau})$. They would be understood as $(\breve{\alpha}, \breve{\tau})$ in Step 1, $\widehat{\tau}$ in Step 2, and so on. We report Excess Risk, the average number of parameters selected, $\mathbb{E}[J(\widehat{\alpha})]$, and the ℓ_2-norm of the bias, $|\mathbb{E}[\widehat{\alpha}] - \alpha_0|_2$. For each sample, the excess risk is calculated by the simulation, $S^{-1} \sum_{s=1}^{S} \left[\rho(Y_s, X_s^T(\widehat{\tau})\widehat{\alpha}) - \rho(Y_s, X_s^T(\tau_0)\alpha_0) \right]$, where $S = 10{,}000$ is the number of simulations; then we report the average value of 1,000 replications. Similarly, we also calculate prediction errors by the simulation, $\left(S^{-1} \sum_{s=1}^{S} \left(X_s^T(\widehat{\tau})\widehat{\alpha} - X_s^T(\tau_0)\alpha_0 \right)^2 \right)^{1/2}$, and report the average value.

We also report the root mean square error (RMSE) and the coverage probability of the 95% confidence interval (Cov. Prob. of CI) of $\widehat{\tau}$. The confidence intervals for τ_0 are calculated by simulating the two-sided compound Poisson process in Theorem 5.3 by adopting the approach proposed by Li and Ling (2012). The details are provided in Section A. Note that

the root mean square error of $\widehat{\tau}$ and the coverage probability of the confidence interval at the rows of Step 3a and Step 3b in the tables are estimation results of updated $\widehat{\tau}$: we re-estimate τ as in Step 2 using $(\widehat{U}_i, \widehat{\alpha})$ and $(\widetilde{U}_i, \widetilde{\alpha})$ from Step 3a and Step 3b instead of $(\breve{U}_i, \breve{\alpha})$.

Table 1: Simulation Results of Design 1

	Excess Risk	$\mathbb{E}[J(\hat{\alpha})]$	$\vert\mathbb{E}[\widehat{\alpha}] - \alpha_0\vert_2$	Prediction Error	RMSE of $\widehat{\tau}$	Cov. Prob. of CI
$n = 200$						
Oracle 1	0.014	NA	0.008	0.263	NA	NA
Oracle 2	0.016	NA	0.007	0.319	0.009	0.930
Step 1	0.052	20.010	0.330	0.584	0.051	0.820
Step 2	0.046	NA	NA	0.541	0.045	0.900
Step 3a	0.043	21.010	0.314	0.535	0.042	0.920
Step 3b	0.033	6.170	0.259	0.456	0.020	0.920
$n = 400$						
Oracle 1	0.002	NA	0.007	0.118	NA	NA
Oracle 2	0.006	NA	0.007	0.194	0.003	0.980
Step 1	0.018	20.730	0.186	0.314	0.005	0.920
Step 2	0.017	NA	NA	0.307	0.003	0.950
Step 3a	0.017	21.670	0.183	0.303	0.003	0.950
Step 3b	0.008	6.090	0.042	0.215	0.003	0.960

Note: Oracle 1 knows both $J(\alpha_0)$ and τ_0 and Oracle 2 knows only $J(\alpha_0)$. Expectations ($\mathbb{E}$) is calculated by the average of 1,000 iterations in each design. Note that $J(\alpha_0) = 6$. 'NA' denotes 'Not Available' as the parameter is not estimated in the step. The estimation results for τ at the rows of Step 3a and Step 3b are based on the re-estimation of τ given estimates from Step 3a ($\widehat{\alpha}$) and Step 3b ($\widetilde{\alpha}$).

Overall, the simulation results confirm the asymptotic theory developed in the previous sections. First, when we compare prediction errors of Oracle 1 and those of the proposed estimators in Step 3, they are mostly inside the bound of $\sqrt{\log p}$. Only the case of $n = 400$ in Design 2 shows that the empirical risk is slightly bigger than the bound. With a weak signal in this design, this is likely the case. Second, the root mean square error of $\widehat{\tau}$ decreases quickly and confirms the super-consistency result of $\widehat{\tau}$. As theoretically ensured, the super-consistency holds regardless of the signal strength. As a result, $\widehat{\tau}$ performs relatively satisfactorily even in Design 2. Third, the model selection in Step 3b is quite satisfactory in Design 1. It slightly under-select the relevant regressors in Design 2 but it is a natural result considering that some signals are quite weak in the decaying design. Fourth, the coverage

Table 2: Simulation Results of Design 2

	Excess Risk	$\mathbb{E}[J(\hat{\alpha})]$	$|\mathbb{E}[\hat{\alpha}] - \alpha_0|_2$	Prediction Error	RMSE of $\hat{\tau}$	Cov. Prob. of CI
$n = 200$						
Oracle 1	0.029	NA	0.020	0.392	NA	NA
Oracle 2	0.031	NA	0.020	0.429	0.008	0.960
Step 1	0.094	23.700	0.507	0.842	0.101	0.830
Step 2	0.090	NA	NA	0.802	0.085	0.910
Step 3a	0.084	24.880	0.451	0.754	0.076	0.910
Step 3b	0.095	9.150	0.278	0.789	0.034	0.950
$n = 400$						
Oracle 1	0.005	NA	0.014	0.151	NA	NA
Oracle 2	0.014	NA	0.015	0.279	0.003	0.990
Step 1	0.027	25.270	0.211	0.393	0.005	0.930
Step 2	0.029	NA	NA	0.382	0.004	0.970
Step 3a	0.027	26.300	0.206	0.379	0.004	0.970
Step 3b	0.036	9.990	0.136	0.442	0.004	0.990

Note: Oracle 1 knows both $J(\alpha_0)$ and τ_0 and Oracle 2 knows only $J(\alpha_0)$. Expectations ($\mathbb{E}$) is calculated by the average of 1,000 iterations in each design. Note that $J(\alpha_0) = 12$. 'NA' denotes 'Not Available' as the parameter is not estimated in the step. The estimation results for τ at the rows of Step 3a and Step 3b are based on the re-estimation of τ given estimates from Step 3a ($\hat{\alpha}$) and Step 3b ($\tilde{\alpha}$).

Figure 1: Histogram of the True Covariate Selection in Design 1, $J(\alpha_0) = 6$
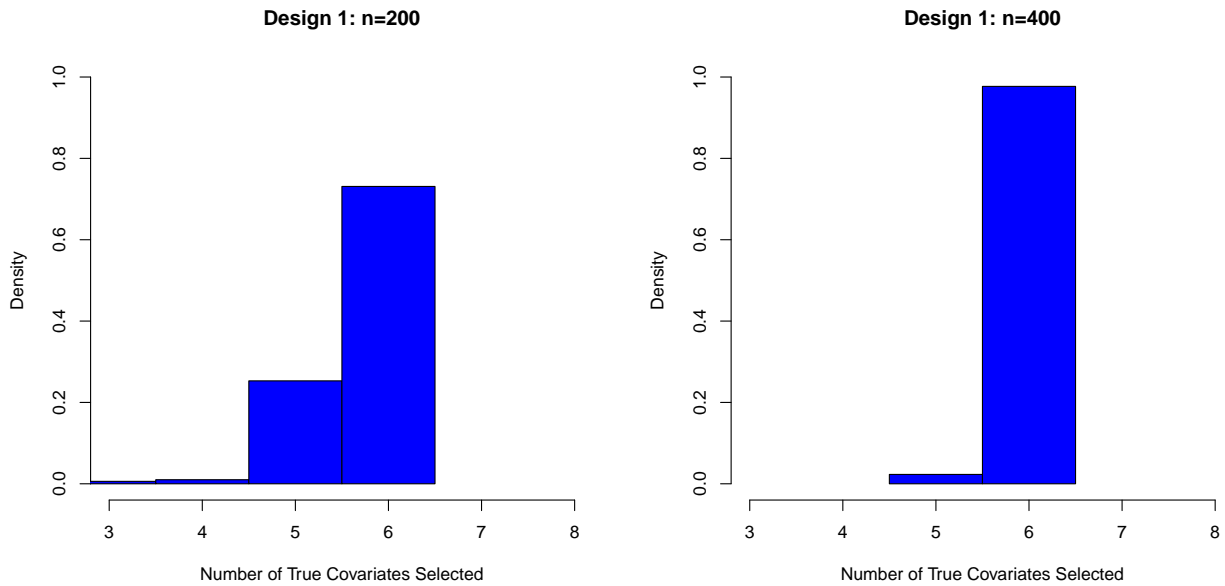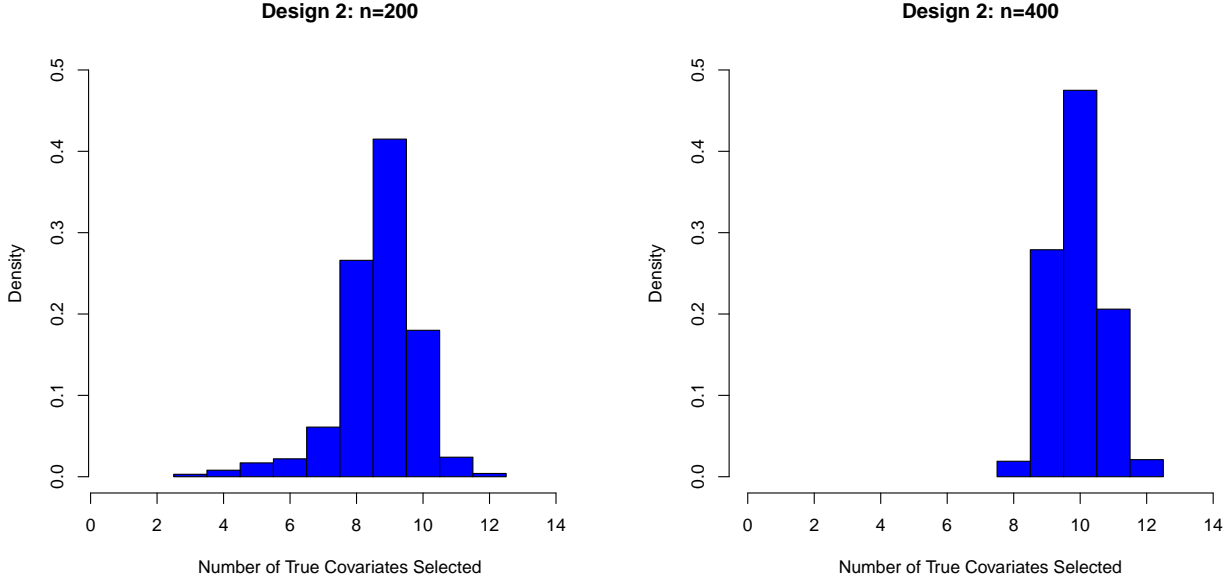

Design 1: n=200

Design 1: n=400

Figure 2: Histogram of the True Covariate Selection in Design 2, $J(\alpha_0) = 12$

probabilities of the confidence interval are close to 95% except Step 1 of each design with $n = 200$. Thus, we recommend practitioners to use $\widehat{\tau}$ in Step 2 or the re-estimated version of it based on the estimates from Step 3a or Step 3b. Finally, Figures 1–2 show the frequency of the true covariates selected from Step 3b in each design. In Design 1, the true covariates are almost perfectly selected when $n = 400$, while we miss some small-sized coefficients when the parameters are decaying in Design 2.

In summary, the proposed estimation procedure works well in finite samples and confirms the theoretical results developed earlier. We emphasize that the recommended estimator for α_0 is either obtained in Step 3a (for the prediction) or Step 3b (for the variable selection). We see from Tables 1–2 that the measures of both the prediction and the variable selection are improved in Step 3a and Step 3b, respectively, compared to the preliminary estimator in Step 1.

8 Estimating a Change Point in Racial Segregation

As an empirical illustration, we investigate the existence of tipping in the dynamics of racial segregation using the dataset constructed by Card et al. (2008). They show that the neighborhood's white population decreases substantially when the minority share in the area exceeds a tipping point (or threshold point), using U.S. Census tract-level data. Lee et al. (2011) develop a test for the existence of threshold effects and apply their test to this dataset. Different from these existing studies, we consider a high-dimensional setup by allowing both possibly highly nonlinear effects of the main covariate (minority share in the neighborhood) and possibly higher-order interactions between additional covariates.

We build on the specifications used in Card et al. (2008) and Lee et al. (2011) to choose the following median regression with a constant shift due to the tipping effect:

$$Y_i = g_0(Q_i) + \delta_0 1\{Q_i > \tau_0\} + X_i'\beta_0 + U_i, \tag{8.1}$$

where for census tract i, the dependent variable Y_i is the ten-year change in the neighborhood's white population, Q_i is the base-year minority share in the neighborhood, and X_i is a vector of six tract-level control variables and their various interactions depending on the model specification. The basic six variables in X_i include the unemployment rate, the log of mean family income, the fractions of single-unit, vacant, and renter-occupied housing units, and the fraction of workers who use public transport to travel to work. The function $g(\cdot)$ is approximated by the cubic b-splines with 20 knots over equi-quantile locations, so the degree of freedom is 24 including the intercept term.

In the first set of models, we consider possible interactions among the six tract-level control variables up to six-way interactions. Specifically, the vector X in the six-way interactions will be composed of the following 63 regressors,

$$\{X^{(1)}, \ldots, X^{(6)}, X^{(1)}X^{(2)}, \ldots, X^{(5)}X^{(6)}, \ldots, X^{(1)}X^{(2)}X^{(3)}X^{(4)}X^{(5)}X^{(6)}\},$$

Table 3: Median Regression with a Tipping Effect (Chicago)

	No. of Reg.	No. of Selected Reg. in Step 3b	$\widehat{\tau}$	CI for τ_0
6 control variables				
No Interaction	31	18	48.74	[46.50, 52.20]
Two-way Interaction	46	23	48.74	[43.28, 59.13]
Three-way Interaction	66	25	48.74	[46.87, 51.40]
Four-way Interaction	81	26	48.74	[47.06, 51.72]
Five-way Interaction	87	25	48.74	[46.76, 51.11]
Six-way Interaction	88	25	48.74	[46.86, 51.24]
12 control variables				
No Interaction	37	21	48.74	[44.21, 55.65]
Two-way Interaction	103	23	48.74	[46.28, 51.31]
Three-way Interaction	323	28	48.74	[46.69, 51.05]
Four-way Interaction	818	28	48.74	[47.42, 50.24]
Five-way Interaction	1610	28	48.74	[46.76, 51.10]
Six-way Interaction	2534	27	48.74	[46.19, 51.59]

Note: The sample size is $n = 1,813$. The parameter τ is estimated by the grid search over $\{Q_i\} \in [10, 60]$. Both $\widehat{\tau}$ and the 95% confidence interval are the results of re-estimation after Step 3b: τ is estimated again using $(\widetilde{U}_i, \widetilde{\alpha})$ from Step 3b.

Table 4: Median Regression with a Tipping Effect (Pittsburgh)

|	No. of Reg.	No. of Selected Reg. in Step 3b	$\widehat{\tau}$	CI for τ_0
6 control variables				
No Interaction	31	8	53.45	[45.59, 60.00]
Two-way Interaction	46	6	53.45	[44.81, 60.00]
Three-way Interaction	66	6	53.45	[45.20, 60.00]
Four-way Interaction	81	6	53.45	[45.12, 60.00]
Five-way Interaction	87	6	53.45	[45.73, 60.00]
Six-way Interaction	88	6	53.45	[46.02, 60.00]
12 control variables				
No Interaction	37	5	53.45	[45.89, 60.00]
Two-way Interaction	103	6	53.45	[44.63, 60.00]
Three-way Interaction	323	10	53.45	[46.03, 60.00]
Four-way Interaction	818	10	53.45	[44.65, 60.00]
Five-way Interaction	1610	10	53.45	[45.74, 60.00]
Six-way Interaction	2534	9	53.45	[46.25, 60.00]

Note: The sample size is $n = 663$. The parameter τ is estimated by the grid search over $\{Q_i\} \in [10, 60]$. Both $\widehat{\tau}$ and the 95% confidence interval are the results of re-estimation after Step 3b: τ is estimated again using $(\widetilde{U}_i, \widetilde{\alpha})$ from Step 3b.

where $X^{(j)}$ is the j-th element among those tract-level control variables. Note that the lower order interaction vector (e.g. two-way or three-way) is nested by the higher order interaction vector (e.g. three-way or four-way). The total number of regressors varies from 31 when there is no interaction to 88 when there are full six-way interactions. In the next set of models, we add the square of each tract-level control variable and generate similar interactions up to six. In this case the total number of regressors varies from 37 to 2,534. For example, the number of regressors in the largest model consists of $\#$(b-sline basis) $+$ $\#$(indicator function) $+$ $\#$(interactions up to six-way out of 12) $= 24 + 1 + \sum_{k=1}^{6} \binom{12}{k} =$ $2,534$. We use the census-tract-level samples of Chicago and Pittsburgh whose base year is 1980. The sample size of Chicago is 1,813 and that of Pittsburgh is 663. Note that the number of regressors is much larger than the sample size in some model specifications.

Tables 3–4 summarize the estimation results. We report the total number of regressors in each model and the number of selected regressors in Step 3b. The change point τ is estimated by the grid search over $\{Q_i\} \in [10, 60]$. In this empirical example, we report the estimates of τ_0 and the confidence intervals updated after Step 3b (that is, τ is re-estimated using the estimates of α_0 in Step 3b). If this estimate is different from the previous one in Step 2, then we repeat Step 3 and Step 2 until it converges. The reported confidence intervals for τ_0 is the intersection of the simulated confidence intervals and the parameter space $[10, 60]$.

The estimation results suggest several interesting points. First, the proposed method selects sparse representations in all model specifications even when the number of regressors is relatively large. Furthermore, the sizes of selected regressors are stable and do not vary much with the number of regressors. Second, the estimated change points are quite robust to the model specification. This result is reconfirmed in Figure 3, where we plot the predicted values over Q_i at the sample median of X_i with observations in the data. They are from the model of six-way interactions with 12 control variables and the vertical line indicates the location of a tipping point. In these estimation results, white population at the threshold

Figure 3: Estimation Results: Six-way Interactions with 12 Control Variables
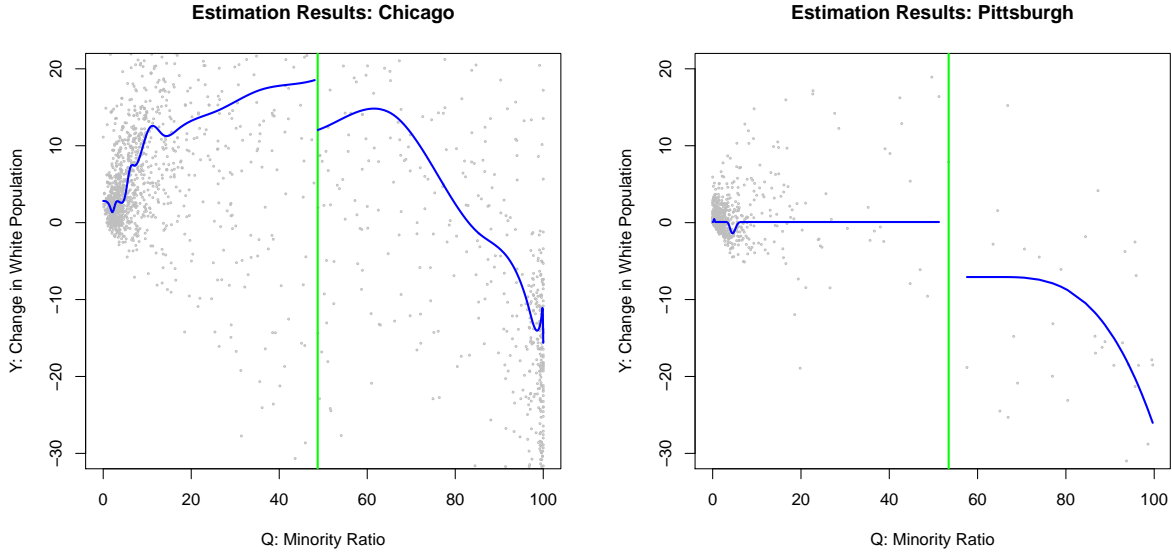
Note: Each dot denotes the tract-level observation of Minority Share and Change in White Population from 1980 to 1990. The vertical line stands for the tipping point $(\widehat{\tau})$, which is re-estimated after Step 3b. The graph (blue line) represents the predicted value of Y_i given Q_i and $\text{med}(X_i)$. Specifically, $\widehat{Y}_i = \sum_{j=1}^{24} B(Q_i)\widetilde{\beta}_{spline}^{(j)} + 1(Q_i > \widehat{\tau})\widetilde{\delta} + \text{med}(X_i)^T\widetilde{\beta}$, where $B(\cdot)$ is the cubic b-spline basis with 20 knots. Parameters $\widetilde{\beta}_{spline}$, $\widetilde{\delta}$ and $\widetilde{\beta}$ are estimation results from Step 3b. Notice that $\widetilde{g}(Q_i) = \sum_{j=1}^{24} B(Q_i)\widetilde{\beta}_{spline}^{(j)}$.

point dropped 6.61 percentage points in Chicago and 7.15 percentage points in Pittsburgh, respectively. Finally, the confidence intervals are quite tight in all cases and they provide convincing evidence of the tipping effect.

In summary, this empirical example shows that the proposed method works well in the real empirical setup. The estimation results also confirm that there exists a tipping point in the racial segregation when we consider high-dimensional median regression.

9   Conclusions

In this paper, we have developed ℓ_1-penalized estimators of a high dimensional quantile regression model with an unknown change point due to a covariate threshold. We have shown among other things that our estimator of the change point achieves an oracle property without relying on a perfect covariate selection, thereby avoiding the need for the minimum

level condition on the signals of active covariates. We have illustrated the usefulness of our estimation methods via Monte Carlo experiments and an application to tipping in the racial segregation.

In a recent working paper, Leonardi and Bühlmann (2016) consider a high dimensional *mean* regression model with multiple change points whose number may grow as the sample size increases. They have proposed a binary search algorithm to choose the number of change points. It is an important future research topic to develop a computationally efficient algorithm to detect multiple changes for high dimensional quantile regression models.

Appendices

We first provide the algorithm of constructing the confidence interval for τ_0 in Appendix A. To provide theoretical results, we consider a general M-estimation framework that includes quantile regression as a special case. We provide high-level regularity conditions on the loss function in Appendix B. Under these conditions, we derive asymptotic properties and then we verify all the high level assumptions for the quantile regression model in Appendix C. Hence, our general results are of independent interest and can be applicable to other models, for example logistic regression models.

A   The Algorithm of Constructing the Confidence Interval for τ_0

The detailed algorithm for constructing the confidence interval based on the Step 2 estimator is as follows:

1. Simulate two independent Poisson processes $N_1(-h)$ for $h < 0$ and $N_2(h)$ for $f > 0$ with the same jump rate $\widehat{f}_Q(\widehat{\tau})$ over $h \in [-\overline{H}n, \overline{H}n]$, where $f_Q(\cdot)$ is the pdf of Q, n is the sample size, and $\overline{H} > 0$ is a large constant. For estimating $f_Q(\cdot)$, we use the kernel density estimator with a normal density kernel and the rule-of-thumb bandwidth, $1.06 \cdot \min\{s, (Q_{0.75} - Q_{0.25})/1.34\} \cdot n^{-1/5}$, where s is the standard deviation of Q and $Q_{0.75} - Q_{0.25}$ is the interquartile range of Q. A Poisson process $N(h)$ is generated by the following algorithm:

   (a) Set $h = 0$ and $k = 0$.
   (b) Generate ϵ from the uniform distribution on $[0, 1]$.
   (c) $h = h + [-(1/\widehat{f}_Q(\widehat{\tau})) \log(\epsilon)]$.
   (d) If $h > n\overline{H}$, then stop and goto Step (f). Otherwise, set $k = k + 1$ and $h_k = h$.
   (e) Repeat Steps (b)–(d).

(f) The algorithm generates $\{h_k\}$ for $k = 1, \ldots, \overline{K}$. Transform it into the Poisson process $N(h) \equiv \sum_{k=1}^{\overline{K}} 1\{h_k \leq h\}$ for $h \in [0, n\overline{H}]$.

2. Using the residuals $\{\breve{U}_i\}$ and the estimate $\breve{\delta}$ from Step 1, construct the empirical distributions of $\breve{\dot{\rho}}_{1i} \equiv \dot{\rho}(\breve{U}_i - X_i^T \breve{\delta}) - \dot{\rho}(\breve{U}_i)$ and $\breve{\dot{\rho}}_{2i} \equiv \dot{\rho}(\breve{U}_i + X_i^T \breve{\delta}) - \dot{\rho}(\breve{U}_i)$ for $i = 1, \ldots, n$, where $\dot{\rho}(t) \equiv t(\gamma - 1\{t \leq 0\})$ is the check function as defined in Section 5.

3. Simulate $\dot{\rho}_{1j}$ for $j = 1, \ldots, N_1(-h)$ and $\dot{\rho}_{2j}$ for $j = 1, \ldots, N_2(h)$.

4. Recall that
$$M(h) \equiv \sum_{i=1}^{N_1(-h)} \rho_{1i} 1\{h < 0\} + \sum_{i=1}^{N_2(h)} \rho_{2i} 1\{h \geq 0\}$$
from Section 5. Construct the function $M(\cdot)$ for $h \in [-\overline{H}n, \overline{H}n]$ using values from Steps 1–3 above. Find the smallest minimizer h of $M(\cdot)$.

5. Repeat Steps 1–4 above and generate $\{h_1, \ldots, h_B\}$.

6. Construct the 95% confidence interval of $\hat{\tau}$ from the empirical distribution of $\{h_b\}$ by $[\hat{\tau} + h_{0.025}/n, \hat{\tau} + h_{0.975}/n]$, where $h_{0.025}$ and $h_{0.975}$ are 2.5 and 97.5 percentiles of $\{h_b\}$, respectively.

It is straightforward to modify the algorithm above for the confidence intervals with Step 3a and Step 3b estimators. We set $\overline{H} = 0.5$, and $B = 1,000$ in this simulation studies.

B Regularity conditions on the general loss function

Let Y be a scalar variable of outcome and X be a vector of p-dimensional observed characteristics. Suppose there is an observable scalar variable Q such that the conditional distribution of Y or some feature of that (given X) depends on:

$$X^T \beta_0 1\{Q \leq \tau_0\} + X^T \theta_0 1\{Q > \tau_0\} = X^T \beta_0 + X^T \delta_0 1\{Q > \tau_0\},$$

where $\delta_0 = \theta_0 - \beta_0$. Let $\rho : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ be a loss function under consideration, whose analytical form is clear in specific models. Suppose the true parameters are defined as the minimizer of the expected loss:

$$(\beta_0, \delta_0, \tau_0) \equiv \underset{(\beta,\delta) \in \mathcal{A}, \tau \in \mathcal{T}}{\mathrm{argmin}} \, \mathbb{E}\left[\rho(Y, X^T\beta + X^T\delta 1\{Q > \tau\})\right], \tag{B.1}$$

where $\mathcal{A}$ and $\mathcal{T}$ denote the parameter spaces for (β_0, δ_0) and τ_0. Here β represents the components of "baseline parameters", while δ represents the structural changes; τ is the change point value where the structural changes occur, if any. By construction, τ_0 is not unique when $\delta_0 = 0$. For each $(\beta, \delta) \in \mathcal{A}$ and $\tau \in \mathcal{T}$, define $2p \times 1$ vectors:

$$\alpha \equiv (\beta^T, \delta^T)^T, \quad X(\tau) \equiv (X^T, X^T 1\{Q > \tau\})^T.$$

Then $X^T\beta + X^T\delta 1\{Q > \tau\} = X(\tau)^T \alpha$, and by letting $\alpha_0 \equiv (\beta_0^T, \delta_0^T)^T$, we can write (B.1) more compactly as:

$$(\alpha_0, \tau_0) = \underset{\alpha \in \mathcal{A}, \tau \in \mathcal{T}}{\mathrm{argmin}} \, \mathbb{E}\left[\rho(Y, X(\tau)^T \alpha)\right]. \tag{B.2}$$

In quantile regression models, for a given quantile $\gamma \in (0, 1)$, recall that

$$\rho(t_1, t_2) = (t_1 - t_2)(\gamma - 1\{t_1 - t_2 \le 0\}).$$

B.1 When $\delta_0 \neq 0$ and τ_0 is identified

For a constant $\eta > 0$, define

$$r_1(\eta) \equiv \sup_r \left\{ r : \mathbb{E}\left(\left[\rho\left(Y, X^T\beta\right) - \rho\left(Y, X^T\beta_0\right)\right] 1\left\{Q \le \tau_0\right\}\right) \right.$$
$$\left. \ge \eta \mathbb{E}[(X^T(\beta - \beta_0))^2 1\{Q \le \tau_0\}] \text{ for all } \beta \in \mathcal{B}(\beta_0, r) \right\}$$

and

$$r_2(\eta) \equiv \sup_r \left\{ r : \mathbb{E}\left(\left[\rho\left(Y, X^T\theta \right) - \rho\left(Y, X^T\theta_0 \right) \right] 1\left\{ Q > \tau_0 \right\} \right) \right.$$
$$\left. \geq \eta \mathbb{E}[(X^T(\theta - \theta_0))^2 1\{Q > \tau_0\}] \text{ for all } \theta \in \mathcal{G}(\theta_0, r) \right\},$$

where $\mathcal{B}(\beta_0, r)$ and $\mathcal{G}(\theta_0, r)$ are defined in (4.3). Note that $r_1(\eta)$ and $r_2(\eta)$ are the maximal radii over which the excess risk can be bounded below by the quadratic loss on $\{Q \leq \tau_0\}$ and $\{Q > \tau_0\}$, respectively.

Assumption B.1.   *(i) Let $\mathcal{Y}$ denote the support of Y. There is a Liptschitz constant $L > 0$ such that for all $y \in \mathcal{Y}$, $\rho(y, \cdot)$ is convex, and*

$$|\rho(y, t_1) - \rho(y, t_2)| \leq L|t_1 - t_2|, \forall t_1, t_2 \in \mathbb{R}.$$

(ii) For all $\alpha \in \mathcal{A}$, almost surely,

$$\mathbb{E}\left[\rho(Y, X(\tau_0)^T\alpha) - \rho(Y, X(\tau_0)^T\alpha_0)|Q \right] \geq 0.$$

(iii) There exist constants $\eta^* > 0$ and $r^* > 0$ such that $r_1(\eta^*) \geq r^*$ and $r_2(\eta^*) \geq r^*$.

(iv) There is a constant $c_0 > 0$ such that for all $\tau \in \mathcal{T}_0$,

$$\mathbb{E}\left[\left(\rho\left(Y, X^T\theta_0 \right) - \rho\left(Y, X^T\beta_0 \right) \right) 1\left\{ \tau < Q \leq \tau_0 \right\} \right] \geq c_0 \mathbb{E}\left[\left(X^T\left(\beta_0 - \theta_0 \right) \right)^2 1\left\{ \tau < Q \leq \tau_0 \right\} \right],$$
$$\mathbb{E}\left[\left(\rho\left(Y, X^T\beta_0 \right) - \rho\left(Y, X^T\theta_0 \right) \right) 1\left\{ \tau_0 < Q \leq \tau \right\} \right] \geq c_0 \mathbb{E}\left[\left(X^T\left(\beta_0 - \theta_0 \right) \right)^2 1\left\{ \tau_0 < Q \leq \tau \right\} \right].$$

In this paper, we focus on a convex Lipchitz loss function, which is assumed in condition (i). It might be possible to weaken the convexity to a "restricted strong convexity condition" as in Loh and Wainwright (2013). For simplicity, we focus on the case of a convex loss, which is satisfied for quantile regression. However, unlike the framework of M-estimation

in Negahban et al. (2012) and Loh and Wainwright (2013), we do allow $\rho(t_1, t_2)$ to be non-differentiable, which admits the quantile regression model as a special case.

Condition (iii) requires that the excess risk can be bounded below by a quadratic function locally when τ is fixed at τ_0, while condition (iv) is an analogous condition when α is fixed at α_0. conditions (iii) and (iv), combined with the convexity of $\rho(Y, \cdot)$, helps us derive the rates of convergence (in the ℓ_1 norm) of the Lasso estimators of (α_0, τ_0). Furthermore, these two conditions separate the conditions for α and τ, making them easier to interpret and verify.

Remark B.1. Condition (iii) of Assumption B.1 is similar to *the restricted nonlinear impact (RNI) condition* of Belloni and Chernozhukov (2011). One may consider an alternative formulation as in van de Geer (2008) and Bühlmann and van de Geer (2011) (Chapter 6), which is known as the *margin condition*. But the margin condition needs to be adjusted to account for structural changes as in condition (iv). It would be an interesting future research topic to develop a general theory of high-dimensional M-estimation with an unknown sparsity-structural-change with general margin conditions.

Remark B.2. Assumptions B.1 (iv) and 4.4 (iii) together imply that for all $\tau \in \mathcal{T}_0$, there exists a constant $c_0 > 0$ such that

$$
\begin{aligned}
\Delta_1(\tau) &\equiv \mathbb{E}\left[\left(\rho\left(Y, X^T\theta_0\right) - \rho\left(Y, X^T\beta_0\right)\right) \mathbf{1}\left\{\tau < Q \leq \tau_0\right\}\right] \geq c_0^2 \mathbb{P}\left[\tau < Q \leq \tau_0\right], \\
\Delta_2(\tau) &\equiv \mathbb{E}\left[\left(\rho\left(Y, X^T\beta_0\right) - \rho\left(Y, X^T\theta_0\right)\right) \mathbf{1}\left\{\tau_0 < Q \leq \tau\right\}\right] \geq c_0^2 \mathbb{P}\left[\tau_0 < Q \leq \tau\right].
\end{aligned}
\tag{B.3}
$$

Note that Assumption B.1 (ii) implies that $\Delta_1(\tau)$ is monotonely non-increasing when $\tau < \tau_0$, and $\Delta_2(\tau)$ is monotonely non-decreasing when $\tau > \tau_0$, respectively. Therefore, Assumptions B.1 (ii), B.1 (iv) and 4.4 (iii) all together imply that (B.3) holds for all τ in the $\mathcal{T}$, not just in the $\mathcal{T}_0$ since $\mathcal{T}$ is compact. Equation (B.3) plays an important role in achieving a super-efficient convergence rate for τ_0, since it states the presence of a kink in the expected loss and that of a jump in the loss function at τ_0.

We now move to the set of assumptions that are useful to deal with the Step 3b estimator.

Define
$$m_j(\tau, \alpha) \equiv \frac{\partial \mathbb{E}[\rho(Y, X(\tau)^T \alpha)]}{\partial \alpha_j}, \quad m(\tau, \alpha) \equiv (m_1(\tau, \alpha), ..., m_{2p}(\tau, \alpha))^T.$$

Also, let $m_J(\tau, \alpha) \equiv (m_j(\tau, \alpha) : j \in J(\alpha_0))$.

Assumption B.2. $\mathbb{E}[\rho(Y, X(\tau)^T \alpha)]$ *is three times continuously differentiable with respect to* α, *and there are constants* $c_1, c_2, L > 0$ *and a neighborhood* $\mathcal{T}_0$ *of* τ_0 *such that the following conditions hold: for all large* n *and all* $\tau \in \mathcal{T}_0$,

(i) *there is* $M_n > 0$, *which may depend on the sample size* n, *such that*

$$\max_{j \leq 2p} |m_j(\tau, \alpha_0) - m_j(\tau_0, \alpha_0)| < M_n |\tau - \tau_0|;$$

(ii) *there is* $r > 0$ *such that for all* $\beta \in \mathcal{B}(\beta_0, r)$, $\theta \in \mathcal{G}(\theta_0, r)$, $\alpha = (\beta^T, \theta^T - \beta^T)^T$ *satisfies:*

$$\max_{j \leq 2p} \sup_{\tau \in \mathcal{T}_0} |m_j(\tau, \alpha) - m_j(\tau, \alpha_0)| < L |\alpha - \alpha_0|_1;$$

(iii) α_0 *is in the interior of the parameter space* $\mathcal{A}$, *and*

$$\inf_{\tau \in \mathcal{T}_0} \lambda_{\min} \left(\frac{\partial^2 \mathbb{E}[\rho(Y, X_J(\tau)^T \alpha_{0J})]}{\partial \alpha_J \partial \alpha_J^T} \right) > c_1,$$

$$\sup_{|\alpha_J - \alpha_{0J}|_1 < c_2,} \sup_{\tau \in \mathcal{T}_0} \max_{i,j,k \in J} \left| \frac{\partial^3 \mathbb{E}[\rho(Y, X_J(\tau)^T \alpha_J)]}{\partial \alpha_i \partial \alpha_j \partial \alpha_k} \right| < L.$$

The score-condition in the population level is expressed by $m(\tau_0, \alpha_0) = 0$ since α_0 is in the interior of $\mathcal{A}$ by condition (iii). Conditions (i) and (ii) regulate the continuity of the score $m(\tau, \alpha)$, and condition (iii) assumes the higher-order differentiability of the expectation of the loss function. Condition (i) requires the Lipschitz continuity of the score function with respect to the threshold. The Lipschitz constant may grow with n, since it is assumed uniformly over $j \leq 2p$. In many examples, M_n in fact grows slowly; as a result, it does not affect the asymptotic behavior of $\widetilde{\alpha}$. For quantile regression models, we will show that

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$M_n = Cs^{1/2}$ for some constant $C > 0$. Condition (ii) requires the local equicontinuity at α_0 in the ℓ_1 norm of the class

$$\{m_j(\tau, \alpha) : \tau \in \mathcal{T}_0, j \leq 2p\}.$$

We now establish that Assumptions B.1 and B.2 are satisfied for quantile regression models.

Lemma B.1. *Suppose that Assumptions 3.1 and 4.1 hold. Then Assumptions B.1 and B.2 are satisfied by the loss function for the quantile regression model, with $M_n = Cs^{1/2}$ for some constant $C > 0$.*

B.1.1 Proof of Lemma B.1

Verification of Assumption B.1 (i). It is straightforward to show that the loss function for quantile regression is convex and satisfies the Liptschitz condition. ∎

Verification of Assumption B.1 (ii). Note that $\rho(Y, t) = h_\gamma(Y - t)$, where $h_\gamma(t) = t(\gamma - 1\{t \leq 0\})$. By (B.3) of Belloni and Chernozhukov (2011),

$$h_\gamma(w - v) - h_\gamma(w) = -v(\gamma - 1\{w \leq 0\}) + \int_0^v (1\{w \leq z\} - 1\{w \leq 0\})dz \qquad \text{(B.4)}$$

where $w = Y - X(\tau_0)^T \alpha_0$ and $v = X(\tau_0)^T(\alpha - \alpha_0)$. Note that

$$\mathbb{E}[v(\gamma - 1\{w \leq 0\})|Q] = -\mathbb{E}[X(\tau_0)^T(\alpha - \alpha_0)(\gamma - 1\{U \leq 0\})|Q] = 0,$$

since $\mathbb{P}(U \leq 0|X, Q) = \gamma$. Let $F_{Y|X,Q}$ denote the CDF of the conditional distribution $Y|X, Q$.

Then

$$\mathbb{E}\left[\rho(Y, X(\tau_0)^T\alpha) - \rho(Y, X(\tau_0)^T\alpha_0)|Q\right]$$

$$= \mathbb{E}\left[\int_0^{X(\tau_0)^T(\alpha-\alpha_0)} (1\{U \leq z) - 1\{U \leq 0\})dz \middle| Q\right]$$

$$= \mathbb{E}\left[\int_0^{X(\tau_0)^T(\alpha-\alpha_0)} [F_{Y|X,Q}(X(\tau_0)^T\alpha_0 + z|X, Q) - F_{Y|X,Q}(X(\tau_0)^T\alpha_0|X, Q)]dz \middle| Q\right]$$

$$\geq 0,$$

where the last inequality follows immediately from the fact that $F_{Y|X,Q}(\cdot|X, Q)$ is the CDF. Hence, we have verified Assumption B.1 (ii). ∎

Verification of Assumption B.1 (iii). Following the arguments analogous those used in (B.4) of Belloni and Chernozhukov (2011), the mean value expansion implies:

$$\mathbb{E}\left[\rho(Y, X(\tau_0)^T\alpha) - \rho(Y, X(\tau_0)^T\alpha_0)|Q\right]$$

$$= \mathbb{E}\left\{\int_0^{X(\tau_0)^T(\alpha-\alpha_0)} \left[zf_{Y|X,Q}(X(\tau_0)^T\alpha_0|X, Q) + \frac{z^2}{2}\tilde{f}_{Y|X,Q}(X(\tau_0)^T\alpha_0 + t|X, Q)\right] dz \middle| Q\right\}$$

$$= \frac{1}{2}(\alpha - \alpha_0)^T\mathbb{E}\left[X(\tau_0)X(\tau_0)^T f_{Y|X,Q}(X(\tau_0)^T\alpha_0|X, Q)|Q\right](\alpha - \alpha_0)$$

$$+ \mathbb{E}\left\{\int_0^{X(\tau_0)^T(\alpha-\alpha_0)} \frac{z^2}{2}\tilde{f}_{Y|X,Q}(X(\tau_0)^T\alpha_0 + t|X, Q)dz \middle| Q\right\}$$

for some intermediate value t between 0 and z. By condition (ii) of Assumption 4.1,

$$|\tilde{f}_{Y|X,Q}(X(\tau_0)^T\alpha_0 + t|X, Q)| \leq C_1 \quad \text{and} \quad f_{Y|X,Q}(X(\tau_0)^T\alpha_0|X, Q) \geq C_2.$$

Hence, taking the expectation on $\{Q \leq \tau_0\}$ gives

$$
\begin{aligned}
\mathbb{E} &\left[\rho(Y, X^T\beta) - \rho(Y, X^T\beta_0)1\{Q \leq \tau_0\} \right] \\
&\geq \frac{C_2}{2}\mathbb{E}[(X^T(\beta - \beta_0))^2 1\{Q \leq \tau_0\}] - \frac{C_1}{6}\mathbb{E}[|X^T(\beta - \beta_0)|^3 1\{Q \leq \tau_0\}] \\
&\geq \frac{C_2}{4}\mathbb{E}[|X^T(\beta - \beta_0)|^2 1\{Q \leq \tau_0\}],
\end{aligned}
$$

where the last inequality follows from

$$
\frac{C_2}{4}\mathbb{E}[|X^T(\beta - \beta_0)|^2 1\{Q \leq \tau_0\}] \geq \frac{C_1}{6}\mathbb{E}[|X^T(\beta - \beta_0)|^3 1\{Q \leq \tau_0\}]. \tag{B.5}
$$

To see why (B.5) holds, note that by (4.5), for any nonzero $\beta \in \mathcal{B}(\beta_0, r_{QR}^*)$,

$$
\frac{\mathbb{E}[|X^T(\beta - \beta_0)|^2 1\{Q \leq \tau_0\}]^{3/2}}{\mathbb{E}[|X^T(\beta - \beta_0)|^3 1\{Q \leq \tau_0\}]} \geq r_{QR}^* \frac{2C_1}{3C_2} \geq \frac{2C_1}{3C_2}\mathbb{E}[|X^T(\beta - \beta_0)|^2 1\{Q \leq \tau_0\}]^{1/2},
$$

which proves (B.5) immediately. Thus, we have shown that Assumption B.1 (iii) holds for $r_1(\eta)$ with $\eta^* = C_2/4$ and $r^* = r_{QR}^*$ defined in (4.5) in Assumption 4.1. The case for $r_2(\eta)$ is similar and hence is omitted. ∎

Verification of Assumption B.1 (iv). We again start from (B.4) but with different choices of (w, v) such that $w = Y - X(\tau_0)^T\alpha_0$ and $v = X^T\delta_0[1\{Q \leq \tau_0\} - 1\{Q > \tau_0\}]$. Then arguments similar to those used in verifying Assumptions B.1 (ii)-(iii) yield that for $\tau < \tau_0$,

$$
\mathbb{E}\left[\rho\left(Y, X^T\theta_0\right) - \rho\left(Y, X^T\beta_0\right)|Q = \tau \right] \tag{B.6}
$$

$$
= \mathbb{E}\left\{ \int_0^{X^T\delta_0} z f_{Y|X,Q}(X^T\beta_0 + t|X, Q)dz \Big| Q = \tau \right\} \tag{B.7}
$$

$$
\geq \mathbb{E}\left\{ \int_0^{\tilde{\varepsilon}(X^T\delta_0)} z f_{Y|X,Q}(X^T\beta_0 + t|X, Q)dz \Big| Q = \tau \right\} \tag{B.8}
$$

$$
\geq \frac{\tilde{\varepsilon}^2 C_3}{2}\mathbb{E}\left[(X^T\delta_0)^2 | Q = \tau \right], \tag{B.9}
$$

where t is an intermediate value t between 0 and z. Thus, we have that

$$\mathbb{E}\left[\left(\rho\left(Y, X^T\theta_0\right) - \rho\left(Y, X^T\beta_0\right)\right) 1\left\{\tau < Q \leq \tau_0\right\}\right] \geq \frac{\bar{\varepsilon}^2 C_3}{2}\mathbb{E}\left[\left(X^T\left(\beta_0 - \theta_0\right)\right)^2 1\left\{\tau < Q \leq \tau_0\right\}\right].$$

The case that $\tau > \tau_0$ is similar. ∎

Verification of Assumption B.2. Note that that

$$m_j(\tau, \alpha) = \mathbb{E}[X_j(\tau)(1\{Y - X(\tau)^T\alpha \leq 0\} - \gamma)].$$

Hence, $m_j(\tau_0, \alpha_0) = 0$, for all $j \leq 2p$. For condition (i) of Assumption B.2, for all $j \leq 2p$,

$$|m_j(\tau, \alpha_0) - m_j(\tau_0, \alpha_0)|$$

$$= |\mathbb{E}X_j(\tau)[1\{Y \leq X(\tau)^T\alpha_0\} - 1\{Y \leq X(\tau_0)^T\alpha_0\}]|$$

$$= |\mathbb{E}X_j(\tau)[\mathbb{P}(Y \leq X(\tau)^T\alpha_0|X, Q) - \mathbb{P}(Y \leq X(\tau_0)^T\alpha_0|X, Q)]|$$

$$\leq C\mathbb{E}|X_j(\tau)||(X(\tau) - X(\tau_0))^T\alpha_0|$$

$$= C\mathbb{E}|X_j(\tau)||X^T\delta_0(1\{Q > \tau\} - 1\{Q > \tau_0\})|$$

$$\leq C\mathbb{E}|X_j(\tau)||X^T\delta_0|(1\{\tau < Q < \tau_0\} + 1\{\tau_0 < Q < \tau\})$$

$$\leq C(\mathbb{P}(\tau_0 < Q < \tau) + \mathbb{P}(\tau < Q < \tau_0)) \sup_{\tau, \tau' \in \mathcal{T}_0} \mathbb{E}(|X_j(\tau)X^T\delta_0||Q = \tau')$$

$$\leq C(\mathbb{P}(\tau_0 < Q < \tau) + \mathbb{P}(\tau < Q < \tau_0)) \sup_{\tau, \tau' \in \mathcal{T}_0} [\mathbb{E}(|X_j(\tau)|^2||Q = \tau')]^{1/2}[\mathbb{E}(|X^T\delta_0|^2|Q = \tau')]^{1/2}$$

$$\leq CM_2 K_2|\delta_0|_2|\tau_0 - \tau|$$

for some constant C, where the last inequality follows from conditions (ii), (iii) and (v) of Assumption 3.1. Therefore, we have verified condition (i) of Assumption B.2 with $M_n = CM_2 K_2|\delta_0|_2$.

We now verify condition (ii) of Assumption B.2. For all j and τ in a neighborhood of τ_0,

$$
\begin{aligned}
|m_j(\tau, \alpha) - m_j(\tau, \alpha_0)| &= |\mathbb{E}X_j(\tau)(1\{Y \le X(\tau)^T\alpha\} - 1\{Y \le X(\tau)^T\alpha_0\})| \\
&= |\mathbb{E}X_j(\tau)(\mathbb{P}(Y \le X(\tau)^T\alpha|X,Q) - \mathbb{P}(Y \le X(\tau)^T\alpha_0|X,Q))| \\
&\le C\mathbb{E}|X_j(\tau)||X(\tau)^T(\alpha - \alpha_0)| \le C|\alpha - \alpha_0|_1 \max_{j \le 2p, i \le 2p} \mathbb{E}|X_j(\tau)X_i(\tau)|,
\end{aligned}
$$

which implies the result immediately in view of Assumption 3.1. Finally, it is straightforward to verify condition (iii) using Assumption 4.1 (iii). ∎

B.2   When $\delta_0 = 0$

We now consider the case when $\delta_0 = 0$. In this case, τ_0 is not identifiable, and there is actually no structural change in the sparsity. If α_0 is in the interior of $\mathcal{A}$, then $m(\tau, \alpha_0) = 0$ for all $\tau \in \mathcal{T}$.

For a constant $\eta > 0$, define

$$
\begin{aligned}
\tilde{r}_1(\eta) \equiv \sup_r \Big\{ r : \mathbb{E}\left(\left[\rho\left(Y, X^T\beta\right) - \rho\left(Y, X^T\beta_0\right)\right]1\{Q \le \tau\}\right) \\
\ge \eta\mathbb{E}[(X^T(\beta - \beta_0))^2 1\{Q \le \tau\}] \text{ for all } \beta \in \tilde{\mathcal{B}}(\beta_0, r, \tau) \text{ and for all } \tau \in \mathcal{T} \Big\}
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{r}_2(\eta) \equiv \sup_r \Big\{ r : \mathbb{E}\left(\left[\rho\left(Y, X^T\theta\right) - \rho\left(Y, X^T\beta_0\right)\right]1\{Q > \tau\}\right) \\
\ge \eta\mathbb{E}[(X^T(\theta - \beta_0))^2 1\{Q > \tau\}] \text{ for all } \theta \in \tilde{\mathcal{G}}(\beta_0, r, \tau) \text{ and for all } \tau \in \mathcal{T} \Big\},
\end{aligned}
$$

where $\tilde{\mathcal{B}}(\beta_0, r, \tau)$ and $\tilde{\mathcal{G}}(\beta_0, r, \tau)$ are defined in (4.4).

Assumption B.3.   *(i) Let $\mathcal{Y}$ denote the support of Y. There is a Liptschitz constant*

$L > 0$ such that for all $y \in \mathcal{Y}$, $\rho(y, \cdot)$ is convex, and

$$|\rho(y, t_1) - \rho(y, t_2)| \le L|t_1 - t_2|, \forall t_1, t_2 \in \mathbb{R}.$$

(ii) For all $\alpha \in \mathcal{A}$ and for all $\tau \in \mathcal{T}$, almost surely,

$$\mathbb{E}[\rho(Y, X(\tau)^T \alpha) - \rho(Y, X^T \beta_0)|Q] \ge 0,$$

(iii) There exist constants $\eta^* > 0$ and $r^* > 0$ such that $\tilde{r}_1(\eta^*) \ge r^*$ and $\tilde{r}_2(\eta^*) \ge r^*$.

(iv) $\mathbb{E}[\rho(Y, X(\tau)^T \alpha)]$ is three times differentiable with respect to α, and there are universal constants $r > 0$ and $L > 0$ such that for all $\beta \in \tilde{\mathcal{B}}(\beta_0, r, \tau)$, $\theta \in \tilde{\mathcal{G}}(\beta_0, r, \tau)$, $\alpha = (\beta^T, \theta^T - \beta^T)^T$ satisfies:

$$\max_{j \le 2p} |m_j(\tau, \alpha) - m_j(\tau, \alpha_0)| < L |\alpha - \alpha_0|_1.$$

for all large n and for all $\tau \in \mathcal{T}$.

(v) α_0 is in the interior of the parameter space $\mathcal{A}$, and there are constants c_1 an $c_2 > 0$ such that

$$\lambda_{\min} \left(\frac{\partial^2 \mathbb{E}[\rho(Y, X_{J(\beta_0)}^T \beta_{0J})]}{\partial \beta_J \partial \beta_J^T} \right) > c_1,$$

$$\sup_{|\alpha_J - \alpha_{0J}|_1 < c_2,} \max_{i,j,k \in J(\beta_0)} \left| \frac{\partial^3 \mathbb{E}[\rho(Y, X_{J(\beta_0)}^T \beta_J)]}{\partial \beta_i \partial \beta_j \partial \beta_k} \right| < L.$$

As in Lemma B.1, we now establish that Assumption B.3 is satisfied for quantile regression models when $\delta_0 = 0$.

Lemma B.2. *Suppose that Assumptions 3.1 and 4.1 hold. Then Assumption B.3 is satisfied.*

B.2.1 Proof of Lemma B.2

Verification of Assumption B.3 (i). This is the same as the verification of Assumption B.1 (i). ∎

Verification of Assumption B.3 (ii). This can be verified exactly as in verification of Assumption B.1 (ii) with $\alpha_0 = \beta_0$ now. ∎

Verification of Assumption B.3 (iv). By the arguments identical to those used to verify Assumption B.1 (iii), we have that

$$
\begin{aligned}
&\mathbb{E}\left[\rho(Y, X^T\beta) - \rho(Y, X^T\beta_0)1\{Q \leq \tau\}\right] \\
&\geq \frac{C_2}{2}\mathbb{E}[(X^T(\beta - \beta_0))^2 1\{Q \leq \tau\}] - \frac{C_1}{6}\mathbb{E}[|X^T(\beta - \beta_0)|^3 1\{Q \leq \tau\}] \\
&\geq \frac{C_2}{4}\mathbb{E}[|X^T(\beta - \beta_0)|^2 1\{Q \leq \tau\}],
\end{aligned}
$$

where the last inequality follows from (4.7). This proves the case for $\tilde{r}_1(\eta)$. The case for $\tilde{r}_2(\eta)$ is similar and hence is omitted. ∎

Verification of Assumptions B.3 (iv) and (v). They can be verified similarly as in verification of Assumption B.2 in the proof of Lemma Lemma B.1. For all j and $\tau \in \mathcal{T}$,

$$
\begin{aligned}
|m_j(\tau, \alpha) - m_j(\tau, \alpha_0)| &= |\mathbb{E}X_j(\tau)(1\{Y \leq X(\tau)^T\alpha\} - 1\{Y \leq X(\tau)^T\alpha_0\})| \\
&= |\mathbb{E}X_j(\tau)(\mathbb{P}(Y \leq X(\tau)^T\alpha|X, Q) - \mathbb{P}(Y \leq X(\tau)^T\alpha_0|X, Q))| \\
&\leq C\mathbb{E}|X_j(\tau)||X(\tau)^T(\alpha - \alpha_0)| \leq C|\alpha - \alpha_0|_1 \max_{j \leq 2p, i \leq 2p}\mathbb{E}|X_j(\tau)X_i(\tau)|,
\end{aligned}
$$

which implies condition B.3 (iv) in view of Assumption 3.1. It also is straightforward to verify condition B.3 (v) using Assumption 4.1 (iii). ∎

C   Proofs of Theorems

Throughout the proofs, we define

$$\nu_n\left(\alpha,\tau\right) \equiv \frac{1}{n}\sum_{i=1}^{n}\left[\rho\left(Y_i, X_i\left(\tau\right)^T\alpha\right) - \mathbb{E}\rho\left(Y, X\left(\tau\right)^T\alpha\right)\right].$$

Without loss of generality let $\nu_n\left(\alpha_J,\tau\right) = n^{-1}\sum_{i=1}^{n}\left[\rho\left(Y_i, X_{iJ}\left(\tau\right)^T\alpha_J\right) - \mathbb{E}\rho\left(Y, X_J\left(\tau\right)^T\alpha_J\right)\right]$.

In this section, we suppose that Assumptions B.1 and B.2 hold when $\delta_0 \neq 0$ and that Assumption B.3 holds when $\delta_0 = 0$, respectively.

C.1   Useful Lemmas

For the positive constant K_1 in Assumption 3.1 (i), define

$$c_{np} \equiv \sqrt{\frac{2\log\left(4np\right)}{n}} + \frac{K_1\log\left(4np\right)}{n}.$$

Let $\lceil x \rceil$ denote the smallest integer greater than or equal to a real number x. The following lemma bounds $\nu_n\left(\alpha,\tau\right)$.

Lemma C.1. *For any positive sequences m_{1n} and m_{2n}, and any $\widetilde{\delta} \in (0,1)$, there are constants L_1, L_2 and $L_3 > 0$ such that for $a_n = L_1 c_{np}\widetilde{\delta}^{-1}$, $b_n = L_2 c_{np}\lceil\log_2\left(m_{2n}/m_{1n}\right)\rceil\widetilde{\delta}^{-1}$, and $c_n = L_3 n^{-1/2}\widetilde{\delta}^{-1}$,*

$$\mathbb{P}\left\{\sup_{\tau\in\mathcal{T}}\sup_{|\alpha-\alpha_0|_1\leq m_{1n}}\left|\nu_n\left(\alpha,\tau\right) - \nu_n\left(\alpha_0,\tau\right)\right| \geq a_n m_{1n}\right\} \leq \widetilde{\delta}, \tag{C.1}$$

$$\mathbb{P}\left\{\sup_{\tau\in\mathcal{T}}\sup_{m_{1n}\leq|\alpha-\alpha_0|_1\leq m_{2n}}\frac{\left|\nu_n\left(\alpha,\tau\right) - \nu_n\left(\alpha_0,\tau\right)\right|}{|\alpha-\alpha_0|_1} \geq b_n\right\} \leq \widetilde{\delta}, \tag{C.2}$$

and for any $\eta > 0$ and $\mathcal{T}_\eta = \{\tau\in\mathcal{T}: |\tau-\tau_0|\leq\eta\}$,

$$\mathbb{P}\left\{\sup_{\tau\in\mathcal{T}_\eta}\left|\nu_n\left(\alpha_0,\tau\right) - \nu_n\left(\alpha_0,\tau_0\right)\right| \geq c_n|\delta_0|_2\sqrt{\eta}\right\} \leq \widetilde{\delta}. \tag{C.3}$$

Proof of (C.1): Let $\epsilon_1, ..., \epsilon_n$ denote a Rademacher sequence, independent of $\{Y_i, X_i, Q_i\}_{i \leq n}$. By the symmetrization theorem (see, for example, Theorem 14.3 of Bühlmann and van de Geer (2011)) and then by the contraction theorem (see, for example, Theorem 14.4 of Bühlmann and van de Geer (2011)),

$$
\mathbb{E} \left(\sup_{\tau \in \mathcal{T}} \sup_{|\alpha - \alpha_0|_1 \leq m_{1n}} |\nu_n(\alpha, \tau) - \nu_n(\alpha_0, \tau)| \right)
$$
$$
\leq 2\mathbb{E} \left(\sup_{\tau \in \mathcal{T}} \sup_{|\alpha - \alpha_0|_1 \leq m_{1n}} \left| \frac{1}{n} \sum_{i=1}^{n} \epsilon_i \left[\rho\left(Y_i, X_i(\tau)^T \alpha\right) - \rho\left(Y_i, X_i(\tau)^T \alpha_0\right)\right] \right| \right)
$$
$$
\leq 4L\mathbb{E} \left(\sup_{\tau \in \mathcal{T}} \sup_{|\alpha - \alpha_0|_1 \leq m_{1n}} \left| \frac{1}{n} \sum_{i=1}^{n} \epsilon_i X_i(\tau)^T (\alpha - \alpha_0) \right| \right).
$$

Note that

$$
\sup_{\tau \in \mathcal{T}} \sup_{|\alpha - \alpha_0|_1 \leq m_{1n}} \left| \frac{1}{n} \sum_{i=1}^{n} \epsilon_i X_i(\tau)^T (\alpha - \alpha_0) \right|
$$
$$
= \sup_{\tau \in \mathcal{T}} \sup_{|\alpha - \alpha_0|_1 \leq m_{1n}} \left| \sum_{j=1}^{2p} (\alpha_j - \alpha_{0j}) \frac{1}{n} \sum_{i=1}^{n} \epsilon_i X_{ij}(\tau) \right|
$$
$$
\leq \sup_{|\alpha - \alpha_0|_1 \leq m_{1n}} \sum_{j=1}^{2p} |\alpha_j - \alpha_{0j}| \sup_{\tau \in \mathcal{T}} \max_{j \leq 2p} \left| \frac{1}{n} \sum_{i=1}^{n} \epsilon_i X_{ij}(\tau) \right| \tag{C.4}
$$
$$
\leq m_{1n} \sup_{\tau \in \mathcal{T}} \max_{j \leq 2p} \left| \frac{1}{n} \sum_{i=1}^{n} \epsilon_i X_{ij}(\tau) \right|.
$$

For all $\tilde{L} > K_1$,

$$
\mathbb{E} \left(\sup_{\tau \in \mathcal{T}} \max_{j \leq 2p} \left| \sum_{i=1}^{n} \epsilon_i X_{ij}(\tau) \right| \right) \leq_{(1)} \tilde{L} \log \mathbb{E} \left[\exp \left(\tilde{L}^{-1} \sup_{\tau \in \mathcal{T}} \max_{j \leq 2p} \left| \sum_{i=1}^{n} \epsilon_i X_{ij}(\tau) \right| \right) \right]
$$
$$
\leq_{(2)} \tilde{L} \log \mathbb{E} \left[\exp \left(\tilde{L}^{-1} \max_{\tau \in \{Q_1, ..., Q_n\}} \max_{j \leq 2p} \left| \sum_{i=1}^{n} \epsilon_i X_{ij}(\tau) \right| \right) \right]
$$
$$
\leq_{(3)} \tilde{L} \log \left[4np \exp \left(\frac{n}{2(\tilde{L}^2 - \tilde{L}K_1)} \right) \right],
$$

where inequality (1) follows from Jensen's inequality, inequality (2) comes from the fact that

$X_{ij}(\tau)$ is a step function with jump points on $\mathcal{T} \cap \{Q_1, \ldots, Q_n\}$, and inequality (3) is by Bernstein's inequality for the exponential moment of an average (see, for example, Lemma 14.8 of Bühlmann and van de Geer (2011)), combined with the simple inequalities that $\exp(|x|) \leq \exp(x) + \exp(-x)$ and that $\exp(\max_{1 \leq j \leq J} x_j) \leq \sum_{j=1}^{J} \exp(x_j)$. Then it follows that

$$\mathbb{E}\left(\sup_{\tau \in \mathcal{T}} \max_{j \leq 2p} \left| \frac{1}{n} \sum_{i=1}^{n} \epsilon_i X_{ij}(\tau) \right| \right) \leq \frac{\tilde{L} \log(4np)}{n} + \frac{1}{2(\tilde{L} - K_1)} = c_{np}, \qquad \text{(C.5)}$$

where the last equality follows by taking $\tilde{L} = K_1 + \sqrt{n/[2\log(4np)]}$. Thus, by Markov's inequality,

$$\mathbb{P}\left\{ \sup_{\tau \in \mathcal{T}} \sup_{|\alpha - \alpha_0|_1 \leq m_{1n}} |\nu_n(\alpha, \tau) - \nu_n(\alpha_0, \tau)| > a_n m_{1n} \right\} \leq (a_n m_{1n})^{-1} 4L m_{1n} c_{np} = \tilde{\delta},$$

where the last equality follows by setting $L_1 = 4L$.

Proof of (C.2): Recall that $\epsilon_1, \ldots, \epsilon_n$ is a Rademacher sequence, independent of $\{Y_i, X_i, Q_i\}_{i \leq n}$. Note that

$$\mathbb{E}\left(\sup_{\tau \in \mathcal{T}} \sup_{m_{1n} \leq |\alpha - \alpha_0|_1 \leq m_{2n}} \frac{|\nu_n(\alpha, \tau) - \nu_n(\alpha_0, \tau)|}{|\alpha - \alpha_0|_1}\right)$$

$$\leq_{(1)} 2\mathbb{E}\left(\sup_{\tau \in \mathcal{T}} \sup_{m_{1n} \leq |\alpha - \alpha_0|_1 \leq m_{2n}} \left| \frac{1}{n} \sum_{i=1}^{n} \epsilon_i \frac{\rho\left(Y_i, X_i(\tau)^T \alpha\right) - \rho\left(Y_i, X_i(\tau)^T \alpha_0\right)}{|\alpha - \alpha_0|_1} \right| \right)$$

$$\leq_{(2)} 2\sum_{j=1}^{k} \mathbb{E}\left(\sup_{\tau \in \mathcal{T}} \sup_{2^{j-1} m_{1n} \leq |\alpha - \alpha_0|_1 \leq 2^j m_{1n}} \left| \frac{1}{n} \sum_{i=1}^{n} \epsilon_i \frac{\rho\left(Y_i, X_i(\tau)^T \alpha\right) - \rho\left(Y_i, X_i(\tau)^T \alpha_0\right)}{2^{j-1} m_{1n}} \right| \right)$$

$$\leq_{(3)} 4L\sum_{j=1}^{k} \mathbb{E}\left(\sup_{\tau \in \mathcal{T}} \sup_{2^{j-1} m_{1n} \leq |\alpha - \alpha_0|_1 \leq 2^j m_{1n}} \left| \frac{1}{n} \sum_{i=1}^{n} \epsilon_i \frac{X_i(\tau)^T (\alpha - \alpha_0)}{2^{j-1} m_{1n}} \right| \right),$$

where inequality (1) is by the symmetrization theorem, inequality (2) holds for some $k \equiv \lceil \log_2(m_{2n}/m_{1n}) \rceil$, and inequality (3) follows from the contraction theorem.

Next, the identical arguments showing (C.4) yield

$$\sup_{2^{j-1}m_{1n} \leq |\alpha - \alpha_0|_1 \leq 2^j m_{1n}} \left| \frac{1}{n} \sum_{i=1}^n \epsilon_i \frac{X_i(\tau)^T (\alpha - \alpha_0)}{2^{j-1}m_{1n}} \right| \leq 2 \max_{j \leq 2p} \left| \frac{1}{n} \sum_{i=1}^n \epsilon_i X_{ij}(\tau) \right|$$

uniformly in $\tau \in \mathcal{T}$. Then, as in the proof of (C.1), Bernstein's and Markov's inequalities imply that

$$\mathbb{P} \left\{ \sup_{\tau \in \mathcal{T}} \sup_{m_{1n} \leq |\alpha - \alpha_0|_1 \leq m_{2n}} \frac{|\nu_n(\alpha, \tau) - \nu_n(\alpha_0, \tau)|}{|\alpha - \alpha_0|_1} > b_n \right\} \leq b_n^{-1} 8Lkc_{np} = \widetilde{\delta},$$

where the last equality follows by setting $L_2 = 8L$.

Proof of (C.3): As above, by the symmetrization and contraction theorems, we have that

$$\mathbb{E} \left(\sup_{\tau \in \mathcal{T}_\eta} |\nu_n(\alpha_0, \tau) - \nu_n(\alpha_0, \tau_0)| \right)$$

$$\leq 2\mathbb{E} \left(\sup_{\tau \in \mathcal{T}_\eta} \left| \frac{1}{n} \sum_{i=1}^n \epsilon_i \left[\rho\left(Y_i, X_i(\tau)^T \alpha_0 \right) - \rho\left(Y_i, X_i(\tau_0)^T \alpha_0 \right) \right] \right| \right)$$

$$\leq 4L\mathbb{E} \left(\sup_{\tau \in \mathcal{T}_\eta} \left| \frac{1}{n} \sum_{i=1}^n \epsilon_i X_i^T \delta_0 \left(1\{Q_i > \tau\} - 1\{Q_i > \tau_0\} \right) \right| \right)$$

$$\leq \frac{4LC_1(M_2 |\delta_0|_2^2 K_2 \eta)^{1/2}}{\sqrt{n}}$$

for some constant $C_1 < \infty$, where the last inequality is due to Theorem 2.14.1 of van der Vaart and Wellner (1996) with M_2 in Assumption 3.1 (v) and K_2 in Assumption 3.1 (ii). Specifically, we apply the second inequality of this theorem to the class $\mathcal{F} = \{f(\epsilon, X, Q, \tau) = \epsilon X^T \delta_0 (1\{Q > \tau\} - 1\{Q > \tau_0\}), \tau \in \mathcal{T}_\eta\}$. Note that $\mathcal{F}$ is a Vapnik-Cervonenkis class, which has a uniformly bounded entropy integral and thus $J(1, \mathcal{F})$ in their theorem is bounded, and that the L_2 norm of the envelope $|\epsilon_i X_i^T \delta_0| 1\{|Q_i - \tau_0| < \eta\}$ is proportional to the square root of the length of $\mathcal{T}_\eta$:

$$(E|\epsilon_i X_i^T \delta_0|^2 1\{|Q_i - \tau_0| < \eta\})^{1/2} \leq (2M_2 |\delta_0|_2^2 K_2 \eta)^{1/2}.$$

This implies the last inequality with C_1 being $\sqrt{2}$ times the entropy integral of the class $\mathcal{F}$. Then, by Markov's inequality, we obtain (C.3) with $L_3 = 4LC_1(M_2K_2)^{1/2}$.

C.2 Proof of Theorem 3.1

Define $D(\tau) = \mathrm{diag}(D_j(\tau) : j \leq 2p)$; and also let $D_0 = D(\tau_0)$ and $\check{D} = D(\check{\tau})$. It follows from the definition of $(\check{\alpha}, \check{\tau})$ in (2.2) that

$$\frac{1}{n}\sum_{i=1}^{n} \rho(Y_i, X_i(\check{\tau})^T \check{\alpha}) + \kappa_n |\check{D}\check{\alpha}|_1 \leq \frac{1}{n}\sum_{i=1}^{n} \rho(Y_i, X_i(\tau_0)^T \alpha_0) + \kappa_n|D_0\alpha_0|_1. \qquad (\text{C.6})$$

From (C.6) we obtain the following inequality

$$
\begin{aligned}
R(\check{\alpha}, \check{\tau}) &\leq [\nu_n(\alpha_0, \tau_0) - \nu_n(\check{\alpha}, \check{\tau})] + \kappa_n|D_0\alpha_0|_1 - \kappa_n|\check{D}\check{\alpha}|_1 \\
&= [\nu_n(\alpha_0, \check{\tau}) - \nu_n(\check{\alpha}, \check{\tau})] + [\nu_n(\alpha_0, \tau_0) - \nu_n(\alpha_0, \check{\tau})] \qquad (\text{C.7}) \\
&\quad + \kappa_n\left(|D_0\alpha_0|_1 - |\check{D}\check{\alpha}|_1\right).
\end{aligned}
$$

Note that the second component $[\nu_n(\alpha_0, \tau_0) - \nu_n(\alpha_0, \check{\tau})] = o_P\left[(s/n)^{1/2}\log n\right]$ due to (C.3) of Lemma C.1 with taking $\mathcal{T}_\eta = \mathcal{T}$ by choosing some sufficiently large $\eta > 0$. Thus, we focus on the other two terms in the following discussion. We consider two cases respectively: $|\check{\alpha} - \alpha_0|_1 \leq |\alpha_0|_1$ and $|\check{\alpha} - \alpha_0|_1 > |\alpha_0|_1$.

Suppose that $|\check{\alpha} - \alpha_0|_1 \leq |\alpha_0|_1$. Then, $\left|\check{D}\check{\alpha}\right|_1 \leq \left|\check{D}(\check{\alpha} - \alpha_0)\right|_1 + \left|\check{D}\alpha_0\right|_1 \leq 2\bar{D}|\alpha_0|_1$, and

$$\left|\kappa_n\left(|D_0\alpha_0|_1 - |\check{D}\check{\alpha}|_1\right)\right| \leq 3\kappa_n\bar{D}|\alpha_0|_1.$$

Applying (C.1) in Lemma C.1 with $m_{1n} = |\alpha_0|_1$, we obtain

$$|\nu_n(\alpha_0, \check{\tau}) - \nu_n(\check{\alpha}, \check{\tau})| \leq a_n|\alpha_0|_1 \leq \kappa_n|\alpha_0|_1 \quad \text{w.p.a.1,}$$

where the last inequality follows from the fact that $a_n \ll \kappa_n$ with κ_n satisfying (2.3). Thus,

the theorem follows in this case.

Now assume that $|\breve{\alpha} - \alpha_0|_1 > |\alpha_0|_1$. In this case, apply (C.2) of Lemma C.1 with $m_{1n} = |\alpha_0|_1$ and $m_{2n} = 2M_1 p$, where M_1 is defined in Assumption 3.1(iii), to obtain

$$\frac{|\nu_n(\alpha_0, \breve{\tau}) - \nu_n(\breve{\alpha}, \breve{\tau})|}{|\breve{\alpha} - \alpha_0|_1} \leq b_n$$

with probability arbitrarily close to one for small enough $\widetilde{\delta}$. Since $b_n \ll \underline{D}\kappa_n$, we have

$$|\nu_n(\alpha_0, \breve{\tau}) - \nu_n(\breve{\alpha}, \breve{\tau})| \leq \kappa_n \underline{D} |\breve{\alpha} - \alpha_0|_1 \leq \kappa_n \left| \breve{D}(\breve{\alpha} - \alpha_0) \right|_1 \quad \text{w.p.a.1.}$$

Therefore,

$$R(\breve{\alpha}, \breve{\tau}) + o_P\left(n^{-1/2}\log n\right) \leq \kappa_n \left(|D_0\alpha_0|_1 - |\breve{D}\breve{\alpha}|_1\right) + \kappa_n \left| \breve{D}(\breve{\alpha} - \alpha_0) \right|_1$$
$$\leq \kappa_n \left(|D_0\alpha_0|_1 - |\breve{D}\breve{\alpha}_J|_1\right) + \kappa_n \left| \breve{D}(\breve{\alpha} - \alpha_0)_J \right|_1,$$

where the last inequality follows from the fact that $\breve{\alpha} - \alpha_0 = \breve{\alpha}_{J^C} + (\breve{\alpha} - \alpha_0)_J$. Thus, the theorem follows in this case as well.

C.3 Proof of Theorem 3.2

Define

$$M^* \equiv 4 \max_{\tau \in T_n} \left(R(\alpha_0, \tau) + 2\omega_n \bar{D} |\alpha_0|_1\right)/(\omega_n \underline{D}), \tag{C.8}$$

where $T_n \subset \mathcal{T}$ will be specified below. For each τ, define

$$\widehat{\alpha}(\tau) = \operatorname{argmin}_{\alpha \in \mathcal{A}} R_n(\alpha, \tau) + \omega_n \sum_{j=1}^{2p} D_j(\tau)|\alpha_j|. \tag{C.9}$$

It follows from the definition of $\widehat{\alpha}(\tau)$ in (C.9) that

$$\frac{1}{n} \sum_{i=1}^{n} \rho(Y_i, X_i(\tau)^T \widehat{\alpha}(\tau)) + \omega_n |D(\tau)\widehat{\alpha}(\tau)|_1 \leq \frac{1}{n} \sum_{i=1}^{n} \rho(Y_i, X_i(\tau)^T \alpha_0) + \omega_n |D(\tau)\alpha_0|_1. \quad \text{(C.10)}$$

Next, let

$$t(\tau) = \frac{M^*}{M^* + |\widehat{\alpha}(\tau) - \alpha_0|_1}$$

and $\bar{\alpha}(\tau) = t(\tau)\widehat{\alpha}(\tau) + (1 - t(\tau))\alpha_0$. By construction, it follows that $|\bar{\alpha}(\tau) - \alpha_0|_1 \leq M^*$.
And also note that

$$|\bar{\alpha}(\tau) - \alpha_0|_1 \leq M^*/2 \text{ implies } |\widehat{\alpha}(\tau) - \alpha_0|_1 \leq M^* \quad \text{(C.11)}$$

since $\bar{\alpha}(\tau) - \alpha_0 = t(\tau)(\widehat{\alpha}(\tau) - \alpha_0)$.

For each τ, (C.10) and the convexity of the following map

$$\alpha \mapsto \frac{1}{n} \sum_{i=1}^{n} \rho(Y_i, X_i(\tau)^T \alpha) + \omega_n |D(\tau)\alpha|_1$$

implies that

$$\frac{1}{n} \sum_{i=1}^{n} \rho(Y_i, X_i(\tau)^T \bar{\alpha}(\tau)) + \omega_n |D(\tau)\bar{\alpha}(\tau)|_1$$

$$\leq t(\tau) \left[\frac{1}{n} \sum_{i=1}^{n} \rho(Y_i, X_i(\tau)^T \widehat{\alpha}(\tau)) + \omega_n |D(\tau)\widehat{\alpha}(\tau)|_1 \right]$$

$$+ [1 - t(\tau)] \left[\frac{1}{n} \sum_{i=1}^{n} \rho(Y_i, X_i(\tau)^T \alpha_0) + \omega_n |D(\tau)\alpha_0|_1 \right]$$

$$\leq \left[\frac{1}{n} \sum_{i=1}^{n} \rho(Y_i, X_i(\tau)^T \alpha_0) + \omega_n |D(\tau)\alpha_0|_1 \right],$$

which in turn yields the following inequality

$$R(\bar{\alpha}(\tau), \tau) + \omega_n |D(\tau)\bar{\alpha}(\tau)|_1 \leq [\nu_n(\alpha_0, \tau) - \nu_n(\bar{\alpha}(\tau), \tau)] + R(\alpha_0, \tau) + \omega_n |D(\tau)\alpha_0|_1. \quad \text{(C.12)}$$

Furthermore, by the triangle inequality, (C.12) can be written as

$$R(\bar{\alpha}(\tau), \tau) + \omega_n \underline{D} |\bar{\alpha}(\tau) - \alpha_0|_1 \le [\nu_n(\alpha_0, \tau) - \nu_n(\bar{\alpha}(\tau), \tau)] + R(\alpha_0, \tau) + 2\omega_n \overline{D} |\alpha_0|_1. \quad \text{(C.13)}$$

Now let $Z_M = \sup_{\tau \in T_n} \sup_{|\alpha - \alpha_0| \le M} |\nu_n(\alpha, \tau) - \nu_n(\alpha_0, \tau)|$ for each $M > 0$. Then, by Lemma C.1, $Z_{M^*} = o_p(\omega_n M^*)$ by the simple fact that $\log(np) \le 2\log(n \vee p)$. Thus, in view of the definition of M^* in (C.8), the following inequality holds w.p.a.1,

$$R(\bar{\alpha}(\tau), \tau) + \omega_n \underline{D} |\bar{\alpha}(\tau) - \alpha_0|_1 \le \omega_n \underline{D} M^*/2 \quad \text{(C.14)}$$

uniformly in $\tau \in T_n$.

We can repeat the same arguments for $\hat{\alpha}(\tau)$ instead of $\bar{\alpha}(\tau)$ due to (C.11) and (C.14), to obtain

$$R(\hat{\alpha}(\tau), \tau) + \omega_n \underline{D} |\hat{\alpha}(\tau) - \alpha_0|_1 \le \omega_n \underline{D} M^* = O(\omega_n s), \text{ w.p.a.1,} \quad \text{(C.15)}$$

uniformly in $\tau \in T_n$. It remains to show that there exists a set T_n such that $\hat{\tau} \in T_n$ w.p.a.1 and the corresponding $M^* = O(s)$. We split the remaining part of the proof into two cases: $\delta_0 \ne 0$ and $\delta_0 = 0$.

(Case 1: $\delta_0 \ne 0$)

Let

$$T_n = \left\{ \tau : |\tau - \tau_0| \le C n^{-1} \log \log n \right\}$$

for some constant $C > 0$. Note that we assume that if $\delta_0 \ne 0$, then

$$|\hat{\tau} - \tau_0| = O_P(n^{-1}),$$

which implies that $\widehat{\tau} \in T_n$ w.p.a.1. Furthermore, note that

$$
\begin{aligned}
R\left(\alpha_0, \tau\right) = {} & \mathbb{E}\left(\left[\rho\left(Y, X^T \theta_0\right) - \rho\left(Y, X^T \beta_0\right)\right] 1\left\{\tau < Q \leq \tau_0\right\}\right) \\
& + \mathbb{E}\left(\left[\rho\left(Y, X^T \beta_0\right) - \rho\left(Y, X^T \theta_0\right)\right] 1\left\{\tau_0 < Q \leq \tau\right\}\right).
\end{aligned} \tag{C.16}
$$

Combining the fact that the objective function is Liptschitz continuous by Assumptions B.1 (i) with Assumption 3.1, we have that

$$
\begin{aligned}
\sup_{\tau \in T_n}\left|R\left(\alpha_0, \tau\right)\right| \leq {} & L \sup_{\tau \in T_n}\left[\mathbb{E}\left(|X^T \delta_0| 1\left\{\tau < Q \leq \tau_0\right\}\right) + \mathbb{E}\left(|X^T \delta_0| 1\left\{\tau_0 < Q \leq \tau\right\}\right)\right] \\
= {} & O\left(|\delta_0|_1 \, n^{-1} \log \log n\right) \\
= {} & o\left(|\delta_0|_1 \, \omega_n^2\right).
\end{aligned}
$$

Thus, $M^* = O\left(|\alpha_0|_1\right) = O\left(s\right)$.

(**Case 2:** $\delta_0 = 0$) Redefine M^* with $T_n = \mathcal{T}$ as the maximum over the whole parameter space for τ. Note that when $\delta_0 = 0$, we have that $R\left(\alpha_0, \tau\right) = 0$ and $M^* = O\left(|\alpha_0|_1\right) = O\left(s\right)$. Therefore, the desired result follows immediately.

C.4   Proof of Theorem 5.1

Remark C.1. We first briefly provide the logic behind the proof of Theorem 5.1 here. Note that for all $\alpha \equiv (\beta^T, \delta^T)^T \in \mathbb{R}^{2p}$ and $\theta \equiv \beta + \delta$, the excess risk has the following decomposition: when $\tau_1 < \tau_0$,

$$
\begin{aligned}
R\left(\alpha, \tau_1\right) = {} & \mathbb{E}\left(\left[\rho\left(Y, X^T \beta\right) - \rho\left(Y, X^T \beta_0\right)\right] 1\left\{Q \leq \tau_1\right\}\right) \\
& + \mathbb{E}\left(\left[\rho\left(Y, X^T \theta\right) - \rho\left(Y, X^T \theta_0\right)\right] 1\left\{Q > \tau_0\right\}\right) \\
& + \mathbb{E}\left(\left[\rho\left(Y, X^T \theta\right) - \rho\left(Y, X^T \beta_0\right)\right] 1\left\{\tau_1 < Q \leq \tau_0\right\}\right),
\end{aligned} \tag{C.17}
$$

and when $\tau_2 > \tau_0$,

$$
\begin{aligned}
R\left(\alpha, \tau_2\right) = \; & \mathbb{E}\left(\left[\rho\left(Y, X^T \beta\right) - \rho\left(Y, X^T \beta_0\right)\right] 1\left\{Q \leq \tau_0\right\}\right) \\
& + \mathbb{E}\left(\left[\rho\left(Y, X^T \theta\right) - \rho\left(Y, X^T \theta_0\right)\right] 1\left\{Q > \tau_2\right\}\right) \\
& + \mathbb{E}\left(\left[\rho\left(Y, X^T \beta\right) - \rho\left(Y, X^T \theta_0\right)\right] 1\left\{\tau_0 < Q \leq \tau_2\right\}\right).
\end{aligned} \tag{C.18}
$$

The key observations are that all the six terms in the above decompositions are non-negative, and are stochastically negligible when taking $\alpha = \breve{\alpha}$, and $\tau_1 = \breve{\tau}$ if $\breve{\tau} < \tau_0$ or $\tau_2 = \breve{\tau}$ if $\breve{\tau} > \tau_0$. This follows from the risk consistency of $R(\breve{\alpha}, \breve{\tau})$. Then, the identification conditions for α_0 and τ_0 (Assumptions B.1 (ii)-(iv)), along with Assumption 4.5 (i), are useful to show that the risk consistency implies the consistency of $\breve{\tau}$.

Proof of Theorem 5.1. Recall from (C.18) that for all $\alpha = (\beta^T, \delta^T)^T \in \mathbb{R}^{2p}$ and $\theta = \beta + \delta$, the excess risk has the following decomposition: when $\tau > \tau_0$,

$$
\begin{aligned}
R\left(\alpha, \tau\right) = \; & \mathbb{E}\left(\left[\rho\left(Y, X^T \beta\right) - \rho\left(Y, X^T \beta_0\right)\right] 1\left\{Q \leq \tau_0\right\}\right) \\
& + \mathbb{E}\left(\left[\rho\left(Y, X^T \theta\right) - \rho\left(Y, X^T \theta_0\right)\right] 1\left\{Q > \tau\right\}\right) \\
& + \mathbb{E}\left(\left[\rho\left(Y, X^T \beta\right) - \rho\left(Y, X^T \theta_0\right)\right] 1\left\{\tau_0 < Q \leq \tau\right\}\right).
\end{aligned} \tag{C.19}
$$

We split the proof into four steps.

Step 1: All the three terms on the right hand side (RHS) of (C.19) are nonnegative. As a consequence, all the three terms on the RHS of (C.19) are bounded by $R(\alpha, \tau)$.

Proof of Step 1. Step 1 is implied by the condition that $\mathbb{E}[\rho(Y, X(\tau_0)^T \alpha) - \rho(Y, X(\tau_0)^T \alpha_0)|Q] \geq 0$ a.s. for all $\alpha \in \mathcal{A}$. To see this, the first two terms are nonnegative by simply multiplying $\mathbb{E}[\rho(Y, X(\tau_0)^T \alpha) - \rho(Y, X(\tau_0)^T \alpha_0)|Q] \geq 0$ with $1\{Q \leq \tau_0\}$ and $1\{Q > \tau\}$ respectively. To show that the third term is nonnegative for all $\beta \in \mathbb{R}^p$ and $\tau > \tau_0$, set $\alpha = (\beta/2, \beta/2)$ in the

inequality $1\{\tau_0 < Q \leq \tau\}\mathbb{E}[\rho(Y, X(\tau_0)^T\alpha) - \rho(Y, X(\tau_0)^T\alpha_0)|Q] \geq 0$. Then we have that

$$1\{\tau_0 < Q \leq \tau\}\mathbb{E}[\rho(Y, X^T(\beta/2 + \beta/2)) - \rho(Y, X^T\theta_0)|Q] \geq 0,$$

which yields the nonnegativeness of the third term. ∎

Step 2: Let $a \vee b = \max(a, b)$ and $a \wedge b = \min(a, b)$. Prove:

$$\mathbb{E}\left[|X^T(\beta - \beta_0)|1\{Q \leq \tau_0\}\right] \leq \frac{1}{\eta^* r^*}R(\alpha, \tau) \vee \left[\frac{1}{\eta^*}R(\alpha, \tau)\right]^{1/2}.$$

Proof of Step 2. Recall that

$$r_1(\eta) \equiv \sup_r \left\{ r : \mathbb{E}\left(\left[\rho\left(Y, X^T\beta\right) - \rho\left(Y, X^T\beta_0\right)\right]1\{Q \leq \tau_0\}\right)\right.$$

$$\left. \geq \eta\mathbb{E}[(X^T(\beta - \beta_0))^2 1\{Q \leq \tau_0\}] \text{ for all } \beta \in \mathcal{B}(\beta_0, r)\right\}.$$

For notational simplicity, write

$$\mathbb{E}[(X^T(\beta - \beta_0))^2 1\{Q \leq \tau_0\}] \equiv \|\beta - \beta_0\|_q^2,$$

and

$$F(\delta) \equiv \mathbb{E}\left(\left[\rho\left(Y, X^T(\beta_0 + \delta)\right) - \rho\left(Y, X^T\beta_0\right)\right]1\{Q \leq \tau_0\}\right).$$

Note that $F(\beta - \beta_0) = \mathbb{E}\left(\left[\rho\left(Y, X^T\beta\right) - \rho\left(Y, X^T\beta_0\right)\right]1\{Q \leq \tau_0\}\right)$, and $\beta \in \mathcal{B}(\beta_0, r)$ if and only if $\|\beta - \beta_0\|_q \leq r$.

For any β, if $\|\beta - \beta_0\|_q \leq r_1(\eta^*)$, then by the definition of $r_1(\eta^*)$, we have:

$$F(\beta - \beta_0) \geq \eta^*\mathbb{E}[(X^T(\beta - \beta_0))^2 1\{Q \leq \tau_0\}].$$

If $\|\beta - \beta_0\|_q > r_1(\eta^*)$, let $t = r_1(\eta^*)\|\beta - \beta_0\|_q^{-1} \in (0, 1)$. Since $F(\cdot)$ is convex, and $F(0) = 0$,

we have $F(\beta - \beta_0) \geq t^{-1} F(t(\beta - \beta_0))$. Moreover, define

$$\check{\beta} = \beta_0 + r_1(\eta^*) \frac{\beta - \beta_0}{\|\beta - \beta_0\|_q},$$

then $\|\check{\beta} - \beta_0\|_q = r_1(\eta^*)$ and $t(\beta - \beta_0) = \check{\beta} - \beta_0$. Hence still by the definition of $r_1(\eta^*)$,

$$F(\beta - \beta_0) \geq \frac{1}{t} F(\check{\beta} - \beta_0) \geq \frac{\eta^*}{t} \mathbb{E}[(X^T(\check{\beta} - \beta_0))^2 1\{Q \leq \tau_0\}] = \eta^* r_1(\eta^*) \|\beta - \beta_0\|_q.$$

Therefore, by Assumption B.1 (iii), and Step 1,

$$
\begin{aligned}
R(\alpha, \tau) &\geq \mathbb{E}\left(\left[\rho\left(Y, X^T\beta\right) - \rho\left(Y, X^T\beta_0\right)\right] 1\{Q \leq \tau_0\}\right) \\
&\geq \eta^* \mathbb{E}[(X^T(\beta - \beta_0))^2 1\{Q \leq \tau_0\}] \wedge \eta^* r^* \{\mathbb{E}[(X^T(\beta - \beta_0))^2 1\{Q \leq \tau_0\}]\}^{1/2} \\
&\geq \eta^* \left(\mathbb{E}\left[|X^T(\beta - \beta_0)| 1\{Q \leq \tau_0\}\right]\right)^2 \wedge \eta^* r^* \mathbb{E}\left[|X^T(\beta - \beta_0)| 1\{Q \leq \tau_0\}\right],
\end{aligned}
$$

where the last inequality follows from Jensen's inequality. ∎

Step 3: For any $\epsilon' > 0$, there is an $\varepsilon > 0$ such that for all τ and $\alpha \in \mathbb{R}^{2p}$, $R(\alpha, \tau) < \varepsilon$ implies $|\tau - \tau_0| < \epsilon'$.

Proof of Step 3. We first prove that, for any $\epsilon' > 0$, there is $\varepsilon > 0$ such that for all $\tau > \tau_0$, and $\alpha \in \mathbb{R}^{2p}$, $R(\alpha, \tau) < \varepsilon$ implies that $\tau < \tau_0 + \epsilon'$.

Suppose that $R(\alpha, \tau) < \varepsilon$. Applying the triangle inequality, for all β and $\tau > \tau_0$,

$$
\begin{aligned}
\mathbb{E}&\left[\left(\rho\left(Y, X^T\beta_0\right) - \rho\left(Y, X^T\theta_0\right)\right) 1\{\tau_0 < Q \leq \tau\}\right] \\
&\leq \left|\mathbb{E}\left[\left(\rho\left(Y, X^T\beta\right) - \rho\left(Y, X^T\theta_0\right)\right) 1\{\tau_0 < Q \leq \tau\}\right]\right| \\
&\quad + \left|\mathbb{E}\left[\left(\rho\left(Y, X^T\beta\right) - \rho\left(Y, X^T\beta_0\right)\right) 1\{\tau_0 < Q \leq \tau\}\right]\right|.
\end{aligned}
\tag{C.20}
$$

First, note that the first term on the RHS of (C.20) is the third term on the RHS of (C.19), hence is bounded by $R(\alpha, \tau) < \varepsilon$.

We now consider the second term on the RHS of (C.20). Assumption 4.5 (i) implies, with

$C_1^* = \widetilde{C}_1^{-1}\left(1 - \widetilde{C}_1\right) > 0$ and $C_2^* = \widetilde{C}_2^{-1}\left(1 - \widetilde{C}_2\right) > 0$, for all $\beta \in \mathbb{R}^p$,

$$C_2^* \mathbb{E}\left[|X^T\beta|\mathbf{1}\left\{Q > \tau_0\right\}\right] \le \mathbb{E}\left[|X^T\beta|\mathbf{1}\left\{Q \le \tau_0\right\}\right] \le C_1^* \mathbb{E}\left[|X^T\beta|\mathbf{1}\left\{Q > \tau_0\right\}\right]. \qquad \text{(C.21)}$$

It follows from the Lipschitz condition, Step 2, and (C.21) that

$$
\begin{aligned}
\left|\mathbb{E}\left[\left(\rho\left(Y, X^T\beta\right) - \rho\left(Y, X^T\beta_0\right)\right)\mathbf{1}\left\{\tau_0 < Q \le \tau\right\}\right]\right| &\le L\mathbb{E}\left[\left|X^T\left(\beta - \beta_0\right)\right|\mathbf{1}\left\{\tau_0 < Q \le \tau\right\}\right] \\
&\le L\mathbb{E}\left[\left|X^T\left(\beta - \beta_0\right)\right|\mathbf{1}\left\{\tau_0 < Q\right\}\right] \\
&\le LC_2^{*-1}\mathbb{E}\left[\left|X^T\left(\beta - \beta_0\right)\right|\mathbf{1}\left\{Q \le \tau_0\right\}\right] \\
&\le LC_2^{*-1}\left\{\varepsilon/(\eta^* r^*) \vee \sqrt{\varepsilon/\eta^*}\right\} \\
&\equiv C(\varepsilon).
\end{aligned}
$$

Thus, we have shown that (C.20) is bounded by $C(\varepsilon) + \varepsilon$.

For any $\epsilon' > 0$, it follows from Assumptions B.1 (ii), B.1 (iv) and 4.4 (iii) (see also Remark B.2) that there is a $c > 0$ such that if $\tau > \tau_0 + \epsilon'$,

$$
\begin{aligned}
c\mathbb{P}\left(\tau_0 < Q \le \tau_0 + \epsilon'\right) &\le c\mathbb{P}\left(\tau_0 < Q \le \tau\right) \\
&\le \mathbb{E}\left[\left(\rho\left(Y, X^T\beta_0\right) - \rho\left(Y, X^T\theta_0\right)\right)\mathbf{1}\left\{\tau_0 < Q \le \tau\right\}\right] \\
&\le C(\varepsilon) + \varepsilon.
\end{aligned}
$$

Since $\varepsilon \mapsto C(\varepsilon) + \varepsilon$ converges to zero as ε converges to zero, for a given $\epsilon' > 0$ choose a sufficient small $\varepsilon > 0$ such that $C(\varepsilon) + \varepsilon < c\mathbb{P}(\tau_0 < Q \le \tau_0 + \epsilon')$, so that the above inequality cannot hold. Hence we infer that for this ε, when $R(\alpha, \tau) < \varepsilon$, we must have $\tau < \tau_0 + \epsilon'$.

By the same argument, if $\tau < \tau_0$, then we must have $\tau > \tau_0 - \epsilon'$. Hence, $R(\alpha, \tau) < \varepsilon$ implies $|\tau - \tau_0| < \epsilon'$. ∎

Step 4: $\check{\tau} \overset{p}{\longrightarrow} \tau_0$.

Proof of Step 4. For the ε chosen in Step 3, consider the event $\{R(\check{\alpha}, \check{\tau}) < \varepsilon\}$, which occurs

w.p.a.1, due to Theorem 3.1. On this event, $|\breve{\tau} - \tau_0| < \epsilon'$ by Step 3. Because ϵ' is taken arbitrarily, we have proved the consistency of $\breve{\tau}$. ∎

∎

C.5   Proof of Theorem 5.2

The proof consists of several steps. First, we prove that $\breve{\beta}$ and $\breve{\theta}$ are inside the neighborhoods of β_0 and θ_0, respectively. Second, we obtain an intermediate convergence rate for $\breve{\tau}$ based on the consistency of the risk and $\breve{\tau}$. Finally, we use the compatibility condition to obtain a tighter bound.

Step 1: For any $r > 0$, w.p.a.1, $\breve{\beta} \in \mathcal{B}(\beta_0, r)$ and $\breve{\theta} \in \mathcal{G}(\theta_0, r)$.

Proof of Step 1. Suppose that $\breve{\tau} > \tau_0$. The proof of Step 2 in the proof of Theorem 5.1 implies that when $\tau > \tau_0$,

$$\mathbb{E}\left[(X^T(\beta - \beta_0))^2 1\{Q \le \tau_0\}\right] \le \frac{R(\alpha, \tau)^2}{(\eta^* r^*)^2} \vee \frac{R(\alpha, \tau)}{\eta^*}.$$

For any $r > 0$, note that $R(\breve{\alpha}, \breve{\tau}) = o_P(1)$ implies that the event $R(\breve{\alpha}, \breve{\tau}) < r^2$ holds w.p.a.1. Therefore, we have shown that $\breve{\beta} \in \mathcal{B}(\beta_0, r)$.

We now show that $\breve{\theta} \in \mathcal{G}(\theta_0, r)$. When $\tau > \tau_0$, we have that

$$
\begin{aligned}
R(\alpha, \tau) \ge_{(1)} & \ \mathbb{E}\left(\left[\rho\left(Y, X^T\theta\right) - \rho\left(Y, X^T\theta_0\right)\right] 1\left\{Q > \tau\right\}\right) \\
= & \ \mathbb{E}\left(\left[\rho\left(Y, X^T\theta\right) - \rho\left(Y, X^T\theta_0\right)\right] 1\left\{Q > \tau_0\right\}\right) \\
& - \mathbb{E}\left(\left[\rho\left(Y, X^T\theta\right) - \rho\left(Y, X^T\theta_0\right)\right] 1\left\{\tau_0 < Q \le \tau\right\}\right) \\
\ge_{(2)} & \ \eta^* \mathbb{E}\left[|X^T(\theta - \theta_0)|^2 1\{Q > \tau_0\}\right] \wedge \eta^* r^* \left(\mathbb{E}\left[|X^T(\theta - \theta_0)|^2 1\{Q > \tau_0\}\right]\right)^{1/2} \\
& - \mathbb{E}\left(\left[\rho\left(Y, X^T\theta\right) - \rho\left(Y, X^T\theta_0\right)\right] 1\left\{\tau_0 < Q \le \tau\right\}\right),
\end{aligned}
$$

where (1) is from (C.18) and (2) can be proved using arguments similar to those used in the

proof of Step 2 in the proof of Theorem 5.1. This implies that

$$\mathbb{E}\left[(X^T(\theta - \theta_0))^2 1\{Q > \tau_0\}\right] \leq \frac{\tilde{R}(\alpha, \tau)^2}{(\eta^* r^*)^2} \vee \frac{\tilde{R}(\alpha, \tau)}{\eta^*}.$$

where $\tilde{R}(\alpha, \tau) \equiv R(\alpha, \tau) + \mathbb{E}\left(\left[\rho\left(Y, X^T\theta\right) - \rho\left(Y, X^T\theta_0\right)\right] 1\{\tau_0 < Q \leq \tau\}\right)$. Thus, it suffices to show that $\tilde{R}(\breve{\alpha}, \breve{\tau}) = o_P(1)$ in order to establish that $\breve{\theta} \in \mathcal{G}(\theta_0, r)$. Note that for some constant $C > 0$,

$$\mathbb{E}\left[(\rho(Y, X^T\theta) - \rho(Y, X^T\theta_0))1\{\tau_0 < Q \leq \tau\}\right]$$

$$\leq_{(1)} L\mathbb{E}\left[|X^T(\theta - \theta_0)|1\{\tau_0 < Q \leq \tau\}\right]$$

$$\leq_{(2)} L|\theta - \theta_0|_1 \mathbb{E}\left[\max_{j \leq p}|\tilde{X}_j|1\{\tau_0 < Q \leq \tau\}\right] + L|\theta - \theta_0|_1 \mathbb{E}\left[|Q|1\{\tau_0 < Q \leq \tau\}\right]$$

$$\leq_{(3)} L|\theta - \theta_0|_1 \mathbb{E}\left[\max_{j \leq p}|\tilde{X}_j|\sup_{\tilde{x}}\mathbb{P}(\tau_0 < Q \leq \tau|\tilde{X} = \tilde{x})\right] + L|\theta - \theta_0|_1 \mathbb{E}\left[|Q|1\{\tau_0 < Q \leq \tau\}\right]$$

$$\leq_{(4)} C(\tau - \tau_0)|\theta - \theta_0|_1 \mathbb{E}\left\{\left[\max_{j \leq p}|\tilde{X}_j|\right] + 1\right\},$$

where (1) is by the Lipschitz continuity of $\rho(Y, \cdot)$, (2) is from the fact that $|X^T(\theta - \theta_0)| \leq |\theta - \theta_0|_1(\max_{j \leq p}|\tilde{X}_j| + |Q|)$, (3) is by taking the conditional probability, and (4) is from Assumption 4.4 (ii).

By the expectation-form of the Bernstein inequality (Lemma 14.12 of Bühlmann and van de Geer (2011)), $\mathbb{E}[\max_{j \leq p}|X_j|] \leq K_1 \log(p + 1) + \sqrt{2\log(p + 1)}$. By (C.27), which will be shown below, $|\breve{\theta} - \theta_0|_1 = O_P(s)$. Hence by (C.23), when $\breve{\tau} > \tau_0$,

$$|\breve{\tau} - \tau_0||\breve{\theta} - \theta_0|_1 \mathbb{E}[\max_{j \leq p}|X_j|] = O_P(\kappa_n s^2 \log p) = o_P(1).$$

Note that when $\breve{\tau} > \tau_0$, the proofs of (C.27) and (C.23) do not require $\breve{\theta} \in \mathcal{G}(\theta_0, r)$, so there is no problem of applying them here. This implies that $\tilde{R}(\breve{\alpha}, \breve{\tau}) = o_P(1)$.

The same argument yields that w.p.a.1, $\breve{\theta} \in \mathcal{G}(\theta_0, r)$ and $\breve{\beta} \in \mathcal{B}(\beta_0, r)$ when $\breve{\tau} \leq \tau_0$; hence it is omitted to avoid repetition. ∎

Step 2: Let $\bar{c}_0(\delta_0) \equiv c_0 \inf_{\tau \in \mathcal{T}_0} \mathbb{E}[(X^T \delta_0)^2 | Q = \tau]$, which is bounded away from zero and bounded above due to Assumption 4.4 (iii) . Then $\bar{c}_0(\delta_0) |\check{\tau} - \tau_0| \leq 4R(\check{\alpha}, \check{\tau})$ w.p.a.1. As a result, $|\check{\tau} - \tau_0| = O_P[\kappa_n s/\bar{c}_0(\delta_0)]$.

Proof. For any $\tau_0 < \tau$ and $\tau \in \mathcal{T}_0$, and any $\beta \in \mathcal{B}(\beta_0, r)$, $\alpha = (\beta, \delta)$ with arbitrary δ, for some $L, M > 0$ which do not depend on β and τ,

$$\left| \mathbb{E} \left(\rho \left(Y, X^T \beta \right) - \rho \left(Y, X^T \beta_0 \right) \right) 1 \{\tau_0 < Q \leq \tau\} \right|$$

$$\leq_{(1)} L\mathbb{E} \left[\left| X^T (\beta - \beta_0) \right| 1 \{\tau_0 < Q \leq \tau\} \right]$$

$$\leq_{(2)} ML(\tau - \tau_0)\mathbb{E} \left[\left| X^T (\beta - \beta_0) \right| 1 \{Q \leq \tau_0\} \right]$$

$$\leq_{(3)} ML(\tau - \tau_0) \left\{ \mathbb{E} \left[\left(X^T (\beta - \beta_0) \right)^2 1 \{Q \leq \tau_0\} \right] \right\}^{1/2}$$

$$\leq_{(4)} (ML(\tau - \tau_0))^2 / (4\eta^*) + \eta^* \mathbb{E} \left[\left(X^T (\beta - \beta_0) \right)^2 1 \{Q \leq \tau_0\} \right]$$

$$\leq_{(5)} (ML(\tau - \tau_0))^2 / (4\eta^*) + \mathbb{E} \left[\left(\rho \left(Y, X^T \beta \right) - \rho \left(Y, X^T \beta_0 \right) \right) 1 \{Q \leq \tau_0\} \right]$$

$$\leq_{(6)} (ML(\tau - \tau_0))^2 / (4\eta^*) + R(\alpha, \tau),$$

where (1) follows from the Lipschitz condition on the objective function, (2) is by Assumption 4.5 (ii), (3) is by Jensen's inequality, (4) follows from the fact that $uv \leq v^2 / (4c) + cu^2$ for any $c > 0$, (5) is from Assumption B.1 (iii), and (6) is from Step 1 in the proof of Theorem 5.1.

In addition,

$$\left| \mathbb{E} \left[\left(\rho \left(Y, X^T \beta \right) - \rho \left(Y, X^T \beta_0 \right) \right) 1 \{\tau_0 < Q \leq \tau\} \right] \right|$$

$$\geq_{(1)} \mathbb{E} \left[\left(\rho \left(Y, X^T \beta_0 \right) - \rho \left(Y, X^T \theta_0 \right) \right) 1 \{\tau_0 < Q \leq \tau\} \right]$$

$$- \left| \mathbb{E} \left[\left(\rho \left(Y, X^T \beta \right) - \rho \left(Y, X^T \theta_0 \right) \right) 1 \{\tau_0 < Q \leq \tau\} \right] \right|$$

$$\geq_{(2)} \mathbb{E} \left[\left(\rho \left(Y, X^T \beta_0 \right) - \rho \left(Y, X^T \theta_0 \right) \right) 1 \{\tau_0 < Q \leq \tau\} \right] - R(\alpha, \tau)$$

$$\geq_{(3)} c_0 \left\{ \inf_{\tau \in \mathcal{T}_0} \mathbb{E}[(X^T \delta_0)^2 | Q = \tau] \right\} (\tau - \tau_0) - R(\alpha, \tau),$$

where (1) is by the triangular inequality, (2) is from (C.18), and (3) is by Assumption B.1 (iv). Therefore, we have established that there exists a constant $\tilde{C} > 0$, independent of (α, τ), such that

$$\bar{c}_0(\delta_0)(\tau - \tau_0) \leq \tilde{C}(\tau - \tau_0)^2 + 2R(\alpha, \tau). \tag{C.22}$$

Note that when $0 < (\tau - \tau_0) < \bar{c}_0(\delta_0)(2\tilde{C})^{-1}$, (C.22) implies that

$$\bar{c}_0(\delta_0)(\tau - \tau_0) \leq \frac{\bar{c}_0(\delta_0)}{2}(\tau - \tau_0) + 2R(\alpha, \tau),$$

which in turn implies that $\tau - \tau_0 \leq \frac{4}{\bar{c}_0(\delta_0)} R(\alpha, \tau)$. By the same argument, when $-\bar{c}_0(\delta_0)(2\tilde{C})^{-1} < (\tau - \tau_0) \leq 0$, we have $\tau_0 - \tau \leq \frac{4}{\bar{c}_0(\delta_0)} R(\alpha, \tau)$ for $\alpha = (\beta, \delta)$, with any $\theta \in \mathcal{G}(\theta_0, r)$ and arbitrary β.

Hence when $\breve{\tau} > \tau_0$, on the event $\breve{\beta} \in \mathcal{B}(\beta_0, r)$, and $\breve{\tau} - \tau_0 < \bar{c}_0(\delta_0)(2\tilde{C})^{-1}$, we have

$$\breve{\tau} - \tau_0 \leq \frac{4}{\bar{c}_0(\delta_0)} R(\breve{\alpha}, \breve{\tau}). \tag{C.23}$$

When $\breve{\tau} \leq \tau_0$, on the event $\breve{\theta} \in \mathcal{G}(\theta_0, r)$, and $\tau_0 - \breve{\tau} < \bar{c}_0(\delta_0)(2\tilde{C})^{-1}$, we have $\tau_0 - \breve{\tau} \leq \frac{4}{\bar{c}_0(\delta_0)} R(\breve{\alpha}, \breve{\tau})$. Hence due to Step 1 and the consistency of $\breve{\tau}$, we have

$$|\breve{\tau} - \tau_0| \leq \frac{4}{\bar{c}_0(\delta_0)} R(\breve{\alpha}, \breve{\tau}) \quad \text{w.p.a.1.} \tag{C.24}$$

This also implies $|\breve{\tau} - \tau_0| = O_P[\kappa_n s/\bar{c}_0(\delta_0)]$ in view of the proof of Theorem 3.1. ∎

Step 3: Define $\nu_{1n}(\tau) \equiv \nu_n(\alpha_0, \tau) - \nu_n(\alpha_0, \tau_0)$ and $c_\alpha \equiv \kappa_n \left(|D_0\alpha_0|_1 - \left|\breve{D}\alpha_0\right|_1 \right) + |\nu_{1n}(\breve{\tau})|$. Then,

$$R(\breve{\alpha}, \breve{\tau}) + \frac{1}{2}\kappa_n \left|\breve{D}(\breve{\alpha} - \alpha_0)\right|_1 \leq c_\alpha + 2\kappa_n \left|\breve{D}(\breve{\alpha} - \alpha_0)_J\right|_1 \quad \text{w.p.a.1.} \tag{C.25}$$

Proof. Recall the following basic inequality in (C.7):

$$R(\check{\alpha}, \check{\tau}) \leq [\nu_n(\alpha_0, \check{\tau}) - \nu_n(\check{\alpha}, \check{\tau})] - \nu_{1n}(\check{\tau}) + \kappa_n \left(|D_0\alpha_0|_1 - |\check{D}\check{\alpha}|_1 \right). \tag{C.26}$$

Now applying Lemma C.1 to $[\nu_n(\alpha_0, \check{\tau}) - \nu_n(\check{\alpha}, \check{\tau})]$ with a_n and b_n replaced by $a_n/2$ and $b_n/2$, we can rewrite the basic inequality in (C.26) by

$$\kappa_n |D_0\alpha_0|_1 \geq R(\check{\alpha}, \check{\tau}) + \kappa_n \left| \check{D}\check{\alpha} \right|_1 - \frac{1}{2}\kappa_n \left| \check{D}(\check{\alpha} - \alpha_0) \right|_1 - |\nu_{1n}(\check{\tau})| \quad \text{w.p.a.1.}$$

Now adding $\kappa_n \left| \check{D}(\check{\alpha} - \alpha_0) \right|_1$ on both sides of the inequality above and using the fact that $|\alpha_{0j}|_1 - |\check{\alpha}_j|_1 + |(\check{\alpha}_j - \alpha_{0j})|_1 = 0$ for $j \notin J$, we have that

$$\kappa_n \left(|D_0\alpha_0|_1 - \left| \check{D}\alpha_0 \right|_1 \right) + |\nu_{1n}(\check{\tau})| + 2\kappa_n \left| \check{D}(\check{\alpha} - \alpha_0)_J \right|_1$$

$$\geq R(\check{\alpha}, \check{\tau}) + \frac{1}{2}\kappa_n \left| \check{D}(\check{\alpha} - \alpha_0) \right|_1 \quad \text{w.p.a.1.}$$

Therefore, we have proved Step 3. ∎

We prove the remaining part of the steps by considering two cases: (i) $\kappa_n \left| \check{D}(\check{\alpha} - \alpha_0)_J \right|_1 \leq c_\alpha$; (ii) $\kappa_n \left| \check{D}(\check{\alpha} - \alpha_0)_J \right|_1 > c_\alpha$. We first consider Case (ii).

Step 4: Suppose that $\kappa_n \left| \check{D}(\check{\alpha} - \alpha_0)_J \right|_1 > c_\alpha$. Then

$$|\check{\tau} - \tau_0| = O_P \left[\kappa_n^2 s / \bar{c}_0(\delta_0) \right] \quad \text{and} \quad |\check{\alpha} - \alpha_0| = O_P(\kappa_n s).$$

Proof. By $\kappa_n \left| \check{D}(\check{\alpha} - \alpha_0)_J \right|_1 > c_\alpha$ and the basic inequality (C.25) in Step 3,

$$6 \left| \check{D}(\check{\alpha} - \alpha_0)_J \right|_1 \geq \left| \check{D}(\check{\alpha} - \alpha_0) \right|_1 = \left| \check{D}(\check{\alpha} - \alpha_0)_J \right|_1 + \left| \check{D}(\check{\alpha} - \alpha_0)_{J^c} \right|_1, \tag{C.27}$$

which enables us to apply the compatibility condition in Assumption 4.2.

Recall that $\|Z\|_2 = (EZ^2)^{1/2}$ for a random variable Z. Note that for $s = |J(\alpha_0)|_0$,

$$
\begin{aligned}
& R\left(\breve{\alpha}, \breve{\tau}\right) + \frac{1}{2}\kappa_n \left|\breve{D}\left(\breve{\alpha} - \alpha_0\right)\right|_1 \\
& \leq_{(1)} 3\kappa_n \left|\breve{D}\left(\breve{\alpha} - \alpha_0\right)_J\right|_1 \\
& \leq_{(2)} 3\kappa_n \bar{D} \left\|X(\breve{\tau})^T(\breve{\alpha} - \alpha_0)\right\|_2 \sqrt{s}/\phi \\
& \leq_{(3)} \frac{9\kappa_n^2 \bar{D}^2 s}{2\tilde{c}\phi^2} + \frac{\tilde{c}}{2} \left\|X(\breve{\tau})^T(\breve{\alpha} - \alpha_0)\right\|_2^2,
\end{aligned} \tag{C.28}
$$

where (1) is from the basic inequality (C.25) in Step 3, (2) is by the compatibility condition (Assumption 4.2), and (3) is from the inequality that $uv \leq v^2/(2\tilde{c}) + \tilde{c}u^2/2$ for any $\tilde{c} > 0$.

We will show below in Step 5 that there is a constant $C_0 > 0$ such that

$$
\left\|X(\breve{\tau})^T(\breve{\alpha} - \alpha_0)\right\|_2^2 \leq C_0 R(\breve{\alpha}, \breve{\tau}) + C_0 \bar{c}_0(\delta_0)|\breve{\tau} - \tau_0|, \text{ w.p.a.1.} \tag{C.29}
$$

Recall that by (C.24), $\bar{c}_0(\delta_0) |\breve{\tau} - \tau_0| \leq 4R\left(\breve{\alpha}, \breve{\tau}\right)$. Hence, (C.28) with $\tilde{c} = (5C_0)^{-1}$ implies that

$$
R\left(\breve{\alpha}, \breve{\tau}\right) + \kappa_n \left|\breve{D}\left(\breve{\alpha} - \alpha_0\right)\right|_1 \leq \frac{9\kappa_n^2 \bar{D}^2 s}{\tilde{c}\phi^2}. \tag{C.30}
$$

By (C.30) and (C.24), $|\breve{\tau} - \tau_0| = O_P\left[\kappa_n^2 s/\bar{c}_0(\delta_0)\right]$. Also, by (C.30), $|\breve{\alpha} - \alpha_0| = O_P\left(\kappa_n s\right)$ since $D(\breve{\tau}) \geq \underline{D}$ w.p.a.1 by Assumption 3.1 (iv). ∎

Step 5: There is a constant $C_0 > 0$ such that $\left\|X(\breve{\tau})^T(\breve{\alpha} - \alpha_0)\right\|_2^2 \leq C_0 R(\breve{\alpha}, \breve{\tau}) + C_0\bar{c}_0(\delta_0)|\breve{\tau} - \tau_0|$, w.p.a.1.

Proof. Note that

$$
\begin{aligned}
\left\|X(\tau)^T(\alpha - \alpha_0)\right\|_2^2 &\leq 2 \left\|X(\tau)^T\alpha - X(\tau_0)^T\alpha\right\|_2^2 \\
&\quad + 4 \left\|X(\tau_0)^T\alpha - X(\tau_0)^T\alpha_0\right\|_2^2 + 4 \left\|X(\tau_0)^T\alpha_0 - X(\tau)^T\alpha_0\right\|_2^2.
\end{aligned} \tag{C.31}
$$

We bound the three terms on the right hand side of (C.31). When $\tau > \tau_0$, there is a constant

$C_1 > 0$ such that

$$\left\| X(\tau)^T \alpha - X(\tau_0)^T \alpha \right\|_2^2$$

$$= \mathbb{E}\left[(X^T \delta)^2 1\{\tau_0 \le Q < \tau\} \right]$$

$$= \int_{\tau_0}^{\tau} \mathbb{E}\left[(X^T \delta)^2 \big| Q = t \right] dF_Q(t)$$

$$\le 2 \int_{\tau_0}^{\tau} \mathbb{E}\left[(X^T \delta_0)^2 \big| Q = t \right] dF_Q(t) + 2 \int_{\tau_0}^{\tau} \mathbb{E}\left[(X^T (\delta - \delta_0))^2 \big| Q = t \right] dF_Q(t)$$

$$\le C_1 \bar{c}_0(\delta_0)(\tau - \tau_0),$$

where the last inequality is by Assumptions 3.1, 4.4 (ii), 4.4 (iii), and 4.5 (ii).

Similarly, $\left\| X(\tau_0)^T \alpha_0 - X(\tau)^T \alpha_0 \right\|_2^2 = \mathbb{E}\left[(X^T \delta_0)^2 1\{\tau_0 \le Q < \tau\} \right] \le C_1 \bar{c}_0(\delta_0)(\tau - \tau_0)$.

Hence, the first and third terms of the right hand side of of (C.31) are bounded by $6C_1 \bar{c}_0(\delta_0)(\tau - \tau_0)$.

To bound the second term, note that there exists a constant $C_2 > 0$ such that

$$\left\| X(\tau_0)^T \alpha - X(\tau_0)^T \alpha_0 \right\|_2^2$$

$$=_{(1)} \mathbb{E}\left[(X^T (\theta - \theta_0))^2 1\{Q > \tau_0\} \right] + \mathbb{E}\left[(X^T (\beta - \beta_0))^2 1\{Q \le \tau_0\} \right]$$

$$\le_{(2)} (\eta^*)^{-1} \mathbb{E}\left[\left(\rho\left(Y, X^T \theta\right) - \rho\left(Y, X^T \theta_0\right) \right) 1\{Q > \tau_0\} \right]$$

$$+ (\eta^*)^{-1} \mathbb{E}\left[\left(\rho\left(Y, X^T \beta\right) - \rho\left(Y, X^T \beta_0\right) \right) 1\{Q \le \tau_0\} \right]$$

$$\le_{(3)} (\eta^*)^{-1} R(\alpha, \tau) + (\eta^*)^{-1} \mathbb{E}\left[\left(\rho\left(Y, X^T \theta\right) - \rho\left(Y, X^T \theta_0\right) \right) 1\{\tau_0 < Q \le \tau\} \right]$$

$$\le_{(4)} (\eta^*)^{-1} R(\alpha, \tau) + (\eta^*)^{-1} L \mathbb{E}\left[|X^T (\theta - \theta_0)| 1\{\tau_0 < Q \le \tau\} \right]$$

$$=_{(5)} (\eta^*)^{-1} R(\alpha, \tau) + (\eta^*)^{-1} L \int_{\tau_0}^{\tau} \mathbb{E}\left[|X^T (\theta - \theta_0)| \big| Q = t \right] dF_Q(t)$$

$$\le_{(6)} (\eta^*)^{-1} R(\alpha, \tau) + C_3 (\tau - \tau_0),$$

where (1) is simply an identity, (2) from Assumption B.1 (iii), (3) is due to (C.19): namely,

$$\mathbb{E}\left[\left(\rho\left(Y, X^T \theta\right) - \rho\left(Y, X^T \theta_0\right) \right) 1\{Q > \tau\} \right] + \mathbb{E}\left[\left(\rho\left(Y, X^T \beta\right) - \rho\left(Y, X^T \beta_0\right) \right) 1\{Q \le \tau_0\} \right] \le R(\alpha, \tau),$$

(4) is by the Lipschitz continuity of $\rho(Y, \cdot)$, (5) is by rewriting the expectation term, and (6) is by Assumptions 3.1 (ii) and 4.5 (ii). Therefore, we have shown that $\left\|X(\tau)^T(\alpha - \alpha_0)\right\|_2^2 \leq C_0 R(\alpha, \tau) + C_0 \bar{c}_0(\delta_0)(\tau - \tau_0)$ for some constant $C_0 > 0$. The case of $\tau \leq \tau_0$ can be proved using the same argument. Hence, setting $\tau = \check{\tau}$, and $\alpha = \check{\alpha}$, we obtain the desired result. ∎

Step 6: We now consider Case (i). Suppose that $\kappa_n \left|\check{D}(\check{\alpha} - \alpha_0)_J\right|_1 \leq c_\alpha$. Then

$$|\check{\tau} - \tau_0| = O_P\left[\kappa_n^2 s/\bar{c}_0(\delta_0)\right] \quad \text{and} \quad |\check{\alpha} - \alpha_0| = O_P\left(\kappa_n s\right).$$

Proof. Recall that X_{ij} is the jth element of X_i, where $i \leq n, j \leq p$. By Assumption 3.1 and Step 2,

$$\sup_{1 \leq j \leq p} \frac{1}{n} \sum_{i=1}^n |X_{ij}|^2 |1(Q_i < \check{\tau}) - 1(Q_i < \tau_0)| = O_P\left[\kappa_n s/\bar{c}_0(\delta_0)\right].$$

By the mean value theorem,

$$\kappa_n \left||D_0\alpha_0|_1 - \left|\check{D}\alpha_0\right|_1\right|$$

$$\leq \kappa_n \sum_{j=1}^p \left(\frac{4}{n} \sum_{i=1}^n |X_{ij}1\{Q_i > \bar{\tau}\}|^2\right)^{-1/2} \left|\delta_0^{(j)}\right| \frac{1}{n} \sum_{i=1}^n |X_{ij}|^2 |1\{Q_i > \check{\tau}\} - 1\{Q_i > \tau_0\}|$$

$$= O_P\left[\kappa_n^2 s |J(\delta_0)|_0/\bar{c}_0(\delta_0)\right]. \tag{C.32}$$

Here, recall that $\bar{\tau}$ is the right-end point of $\mathcal{T}$ and $|J(\delta_0)|_0$ is the dimension of nonzero elements of δ_0.

Due to Step 2 and (C.3) in Lemma C.1,

$$|\nu_{1n}(\check{\tau})| = O_P\left[\frac{|\delta_0|_2}{\sqrt{\bar{c}_0(\delta_0)}}(\kappa_n s/n)^{1/2}\right]. \tag{C.33}$$

Thus, under Case (i), we have that, by (C.24), (C.25), (C.32), and (C.33),

$$
\begin{aligned}
\frac{\bar{c}_0(\delta_0)}{4} \left| \check{\tau} - \tau_0 \right| &\leq \frac{\kappa_n}{2} \left| \check{D} \left(\check{\alpha} - \alpha_0 \right) \right|_1 + R \left(\check{\alpha}, \check{\tau} \right) \\
&\leq 3\kappa_n \left(\left| D_0 \alpha_0 \right|_1 - \left| \check{D} \alpha_0 \right|_1 \right) + 3 \left| \nu_{1n} \left(\check{\tau} \right) \right| \\
&= O_P \left(\kappa_n^2 s^2 \right) + O_P \left[s^{1/2} \left(\kappa_n s / n \right)^{1/2} \right],
\end{aligned} \tag{C.34}
$$

where the last equality uses the fact that $|J(\delta_0)|_0 / \bar{c}_0(\delta_0) = O(s)$ and $|\delta_0|_2 / \sqrt{\bar{c}_0(\delta_0)} = O(s^{1/2})$ at most (both could be bounded in some cases).

Therefore, we now have an improved rate of convergence in probability for $\check{\tau}$ from $r_{n0,\tau} \equiv \kappa_n s$ to $r_{n1,\tau} \equiv [\kappa_n^2 s^2 + s^{1/2}(\kappa_n s/n)^{1/2}]$. Repeating the arguments identical to those to prove (C.32) and (C.33) yields that

$$
\kappa_n \left| \left| D_0 \alpha_0 \right|_1 - \left| \check{D} \alpha_0 \right|_1 \right| = O_P \left[r_{n1,\tau} \kappa_n s \right] \quad \text{and} \quad \left| \nu_{1n} \left(\check{\tau} \right) \right| = O_P \left[s^{1/2} \left(r_{n1,\tau} / n \right)^{1/2} \right].
$$

Plugging these improved rates into (C.34) gives

$$
\begin{aligned}
\bar{c}_0(\delta_0) \left| \check{\tau} - \tau_0 \right| &= O_P \left(\kappa_n^3 s^3 \right) + O_P \left[s^{1/2} (\kappa_n s)^{3/2} / n^{1/2} \right] + O_P \left(\kappa_n s^{3/2} / n^{1/2} \right) + O_P \left[s^{3/4} (\kappa_n s)^{1/4} / n^{3/4} \right] \\
&= O_P \left(\kappa_n^2 s^{3/2} \right) + O_P \left[s^{3/4} (\kappa_n s)^{1/4} / n^{3/4} \right] \\
&\equiv O_P(r_{n2,\tau}),
\end{aligned}
$$

where the second equality comes from the fact that the first three terms are $O_P \left(\kappa_n^2 s^{3/2} \right)$ since $\kappa_n s^{3/2} = o(1)$, $\kappa_n n / s \to \infty$, and $\kappa_n \sqrt{n} \to \infty$ in view of the assumption that $\kappa_n s^2 \log p = o(1)$. Repeating the same arguments again with the further improved rate $r_{n2,\tau}$, we have that

$$
\left| \check{\tau} - \tau_0 \right| = O_P \left(\kappa_n^2 s^{5/4} \right) + O_P \left[s^{7/8} (\kappa_n s)^{1/8} / n^{7/8} \right] \equiv O_P(r_{n3,\tau}).
$$

Thus, repeating the same arguments k times yields

$$\bar{c}_0(\delta_0) \left| \breve{\tau} - \tau_0 \right| = O_P \left(\kappa_n^2 s^{1+2^{-k}} \right) + O_P \left[s^{(2^k-1)/2^k} (\kappa_n s)^{1/2^k} / n^{(2^k-1)/2^k} \right] \equiv O_P(r_{nk,\tau}).$$

Then letting $k \to \infty$ gives the desired result that $\bar{c}_0(\delta_0) \left| \breve{\tau} - \tau_0 \right| = O_P \left(\kappa_n^2 s \right)$. Finally, the same iteration based on (C.34) gives $\left| \breve{D} \left(\breve{\alpha} - \alpha_0 \right) \right| = o_P \left(\kappa_n s \right)$, which proves the desired result since $D(\breve{\tau}) \geq \underline{D}$ w.p.a.1 by Assumption 3.1 (iv). ∎

C.6   Proof of Theorem 5.3

Proof of Theorem 5.3. The asymptotic property of $\widetilde{\tau}$ is well-known in the literature (see Lemma C.3 below for its asymptotic distribution). Specifically, we can apply Theorem 3.4.1 of van der Vaart and Wellner (1996) (by defining the criterion $\mathbb{M}_n \left(\cdot \right) \equiv R_n^* \left(\cdot \right)$, $M_n \left(\cdot \right) \equiv \mathbb{E} R_n^* \left(\cdot \right) = R(\alpha_0, \tau)$, the distance function $d \left(\tau, \tau_0 \right) \equiv \left| \tau - \tau_0 \right|^{1/2}$, and $\phi_n \left(\delta \right) \equiv \delta$) to characterize the convergence rate of $\widetilde{\tau}$, which results in the super-consistency in the sense that $\widetilde{\tau} - \tau_0 = O_p(n^{-1})$. See e.g. Section 14.5 of Kosorok (2008).

Furthermore, it is worth noting that the same theorem also implies that if

$$[R_n^* \left(\widehat{\tau} \right) - R_n^* \left(\tau_0 \right)] - [R_n \left(\breve{\alpha}, \widehat{\tau} \right) - R_n \left(\breve{\alpha}, \tau_0 \right)] = O_p(r_n^{-2}) \tag{C.35}$$

for some sequence r_n satisfying $r_n^2 \phi_n \left(r_n^{-1} \right) \leq \sqrt{n}$, then

$$r_n d \left(\widehat{\tau}, \tau_0 \right) = O_p \left(1 \right).$$

This is because

$$R_n^* \left(\widehat{\tau} \right) = R_n^* \left(\widehat{\tau} \right) - \left[R_n \left(\breve{\alpha}, \widehat{\tau} \right) - R_n \left(\breve{\alpha}, \tau_0 \right) + R_n^* \left(\tau_0 \right) \right] + \left[R_n \left(\breve{\alpha}, \widehat{\tau} \right) - R_n \left(\breve{\alpha}, \tau_0 \right) + R_n^* \left(\tau_0 \right) \right]$$

$$\leq_{(1)} R_n^* \left(\widehat{\tau} \right) - \left[R_n \left(\breve{\alpha}, \widehat{\tau} \right) - R_n \left(\breve{\alpha}, \tau_0 \right) + R_n^* \left(\tau_0 \right) \right] + \left[R_n \left(\breve{\alpha}, \tau_0 \right) - R_n \left(\breve{\alpha}, \tau_0 \right) + R_n^* \left(\tau_0 \right) \right]$$

$$=_{(2)} \left\{ \left[R_n^* \left(\widehat{\tau} \right) - R_n^* \left(\tau_0 \right) \right] - \left[R_n \left(\breve{\alpha}, \widehat{\tau} \right) - R_n \left(\breve{\alpha}, \tau_0 \right) \right] \right\} + R_n^* \left(\tau_0 \right)$$

$$=_{(3)} O_p \left(r_n^{-2} \right) + R_n^* \left(\tau_0 \right),$$

where inequality (1) uses the fact that $\widehat{\tau}$ is a minimizer of $R_n \left(\breve{\alpha}, \tau \right)$, equality (2) follows since $R_n \left(\breve{\alpha}, \tau_0 \right) - R_n \left(\breve{\alpha}, \tau_0 \right) + R_n^* \left(\tau_0 \right) = R_n^* \left(\tau_0 \right)$, and equality (3) comes from (C.35).

Then, note that we can set $r_n^{-2} = a_n s_n \log(np)$ with $s_n = 1$ and $a_n = \kappa_n s \log n$ due to Lemma C.2 and the rate of convergence $\breve{\alpha} - \alpha_0 = O_p \left(\kappa_n s \right)$ given by Theorem 5.2. Next, we will apply a chaining argument to obtain the convergence rate of $\widehat{\tau}$ by repeatedly verifying the condition $R_n^*(\widehat{\tau}) \leq R_n^*(\tau_0) + O_p(r_n^{-2})$, with an iteratively improved rate r_n. Applying Theorem 3.4.1 of van der Vaart and Wellner (1996) with $r_n = (a_n \log(np))^{-1/2}$, we have

$$\widehat{\tau} - \tau_0 = O_p \left(a_n \log(np) \right) = O_p \left(\kappa_n s \log n \log(np) \right).$$

Next, we reset $s_n = \kappa_n s \left(\log n \right)^2 \log(np)$ and $a_n = \kappa_n s \log n$ to apply Lemma C.2 again and then Theorem 3.4.1 of van der Vaart and Wellner (1996) with $r_n = (s_n a_n \log(np))^{-1/2}$. It follows that

$$\widehat{\tau} - \tau_0 = O_p \left([\kappa_n s]^2 \left(\log n \right)^3 \left(\log(np) \right)^2 \right).$$

In the next step, we set $r_n = \sqrt{n}$ since it should satisfy the constraint that $r_n^2 \phi_n \left(r_n^{-1} \right) \leq \sqrt{n}$ as well. Then, we conclude that $\widehat{\tau} = \tau_0 + O_p \left(n^{-1} \right)$. Furthermore, in view of Lemma C.2, $\widehat{\tau} = \tau_0 + O_p \left(n^{-1} \right)$ implies that the asymptotic distribution of $n \left(\widehat{\tau} - \tau_0 \right)$ is identical to $n \left(\widetilde{\tau} - \tau_0 \right)$ since each of them is characterized by the minimizer of the weak limit of $n \left(R_n \left(\alpha, \tau_0 + tn^{-1} \right) - R_n \left(\alpha, \tau_0 \right) \right)$ with $\alpha = \breve{\alpha}$ and $\alpha = \alpha_0$, respectively. That is, the weak limits of the processes are identical due to Lemma C.2. Therefore, we have proved the first

conclusion of the theorem. Lemma C.3 establishes the second conclusion. ∎

Lemma C.2. *Suppose that $\alpha \in \mathcal{A}_n \equiv \left\{ \alpha = \left(\beta^T, \delta^T \right)^T : |\alpha - \alpha_0|_1 \leq K a_n \right\}$ and $\tau \in \mathcal{T}_n \equiv \left\{ |\tau - \tau_0| \leq K s_n \right\}$ for some $K < \infty$ and for some sequences a_n and s_n as $n \to \infty$. Then,*

$$\sup_{\alpha \in \mathcal{A}_n, \tau \in \mathcal{T}_n} \left| \left\{ R_n \left(\alpha, \tau \right) - R_n \left(\alpha, \tau_0 \right) \right\} - \left\{ R_n \left(\alpha_0, \tau \right) - R_n \left(\alpha_0, \tau_0 \right) \right\} \right| = O_p \left[a_n s_n \log(np) \right].$$

Proof of Lemma C.2. Noting that

$$\rho \left(Y_i, X_i^T \beta + X_i^T \delta 1 \left\{ Q_i > \tau \right\} \right) = \rho \left(Y_i, X_i^T \beta \right) 1 \left\{ Q_i \leq \tau \right\} + \rho \left(Y_i, X_i^T \beta + X_i^T \delta \right) 1 \left\{ Q_i > \tau \right\},$$

we have, for $\tau > \tau_0$,

$$\begin{aligned}
D_n \left(\alpha, \tau \right) &:= \left\{ R_n \left(\alpha, \tau \right) - R_n \left(\alpha, \tau_0 \right) \right\} - \left\{ R_n \left(\alpha_0, \tau \right) - R_n \left(\alpha_0, \tau_0 \right) \right\} \\
&= \frac{1}{n} \sum_{i=1}^n \left[\rho \left(Y_i, X_i^T \beta \right) - \rho \left(Y_i, X_i^T \beta_0 \right) \right] 1 \left\{ \tau_0 < Q_i \leq \tau \right\} \\
&\quad - \frac{1}{n} \sum_{i=1}^n \left[\rho \left(Y_i, X_i^T \theta \right) - \rho \left(Y_i, X_i^T \theta_0 \right) \right] 1 \left\{ \tau_0 < Q_i \leq \tau \right\} \\
&=: D_{n1} \left(\alpha, \tau \right) - D_{n2} \left(\alpha, \tau \right).
\end{aligned}$$

However, the Lipschiz property of ρ yields that

$$\begin{aligned}
|D_{n1} \left(\alpha, \tau \right)| &= \left| \frac{1}{n} \sum_{i=1}^n \left[\rho \left(Y_i, X_i^T \beta \right) - \rho \left(Y_i, X_i^T \beta_0 \right) \right] 1 \left\{ \tau_0 < Q_i \leq \tau \right\} \right| \\
&\leq L \max_{i,j} |X_{ij}| \, |\beta - \beta_0|_1 \frac{1}{n} \sum_{i=1}^n 1 \left\{ \tau_0 < Q_i \leq \tau \right\} \\
&= O_p \left[\log(np) \cdot a_n \cdot s_n \right] \quad \text{uniformly in } (\alpha, \tau) \in \mathcal{A}_n \times \mathcal{T}_n,
\end{aligned}$$

where $\log(np)$ term comes from the Bernstein inequality and the s_n term follows from the fact that $\mathbb{E} \left| \frac{1}{n} \sum_{i=1}^n 1 \left\{ \tau_0 < Q_i \leq \tau \right\} \right| = \mathbb{E} 1 \left\{ \tau_0 < Q_i \leq \tau \right\} \leq C \cdot K s_n$ due to the boundedness of the density of Q_i around τ_0. The other term $D_{n2} \left(\alpha, \tau \right)$ can be bounded similarly. The

case of $\tau < \tau_0$ can be treated analogously and hence details are omitted. ∎

Lemma C.3. *We have that $n\left(\widetilde{\tau} - \tau_0\right)$ converges in distribution to the smallest minimizer of a compound Poisson process defined in Theorem 5.3.*

Proof of Lemma C.3. The convergence rate of $\widetilde{\tau}$ is standard as commented in the beginning of the proof of Theorem 5.3 and thus details are omitted here. We present the characterization of the asymptotic distribution for the given convergence rate n.

Recall that $\rho\left(t, s\right) = \dot{\rho}\left(t - s\right)$, where $\dot{\rho}\left(t\right) = t\left(\gamma - 1\left\{t \leq 0\right\}\right)$. Note that

$$
\begin{aligned}
& nR_n^*\left(\tau\right) \\
= \ & \sum_{i=1}^{n} \dot{\rho}\left(Y_i - X_i^T \beta_0 - X_i^T \delta_0 1\left\{Q_i > \tau\right\}\right) - \dot{\rho}\left(Y_i - X_i^T \beta_0 - X_i^T \delta_0 1\left\{Q_i > \tau_0\right\}\right) \\
= \ & \sum_{i=1}^{n} \left[\dot{\rho}\left(U_i - X_i^T \delta_0 \left(1\left\{Q_i > \tau\right\} - 1\left\{Q_i > \tau_0\right\}\right)\right) - \dot{\rho}\left(U_i\right)\right]\left(1\left\{\tau < Q_i \leq \tau_0\right\} + 1\left\{\tau_0 < Q_i \leq \tau\right\}\right) \\
= \ & \sum_{i=1}^{n} \left[\dot{\rho}\left(U_i - X_i^T \delta_0\right) - \dot{\rho}\left(U_i\right)\right] 1\left\{\tau < Q_i \leq \tau_0\right\} \\
& + \sum_{i=1}^{n} \left[\dot{\rho}\left(U_i + X_i^T \delta_0\right) - \dot{\rho}\left(U_i\right)\right] 1\left\{\tau_0 < Q_i \leq \tau\right\}.
\end{aligned}
$$

Thus, the asymptotic distribution of $n\left(\widetilde{\tau} - \tau_0\right)$ is characterized by the smallest minimizer of the weak limit of

$$
M_n\left(h\right) = \sum_{i=1}^{n} \dot{\rho}_{1i} 1\left\{\tau_0 + \frac{h}{n} < Q_i \leq \tau_0\right\} + \sum_{i=1}^{n} \dot{\rho}_{2i} 1\left\{\tau_0 < Q_i \leq \tau_0 + \frac{h}{n}\right\}
$$

for $|h| \leq K$ for some large K, where $\dot{\rho}_{1i} = \dot{\rho}\left(U_i - X_i^T \delta_0\right) - \dot{\rho}\left(U_i\right)$ and $\dot{\rho}_{2i} = \dot{\rho}\left(U_i + X_i^T \delta_0\right) - \dot{\rho}\left(U_i\right)$. The weak limit of the empirical process $M_n\left(\cdot\right)$ is well developed in the literature, (see e.g. Pons (2003); Kosorok and Song (2007); Lee and Seo (2008)) and the argmax continuous mapping theorem by Seijo and Sen (2011b) yields the asymptotic distribution, namely the smallest minimizer of a compound Poisson process, which is defined in Theorem 5.3. ∎

C.7 Proof of Theorem 5.4

Let $\widehat{D} \equiv D(\widehat{\tau})$. It follows from the definition of $\widehat{\alpha}$ in (2.5) that

$$\frac{1}{n}\sum_{i=1}^{n}\rho(Y_i, X_i(\widehat{\tau})^T\widehat{\alpha}) + \omega_n|\widehat{D}\widehat{\alpha}|_1 \leq \frac{1}{n}\sum_{i=1}^{n}\rho(Y_i, X_i(\widehat{\tau})^T\alpha_0) + \omega_n|\widehat{D}\alpha_0|_1.$$

From this, we obtain the following inequality

$$R(\widehat{\alpha}, \widehat{\tau}) \leq [\nu_n(\alpha_0, \widehat{\tau}) - \nu_n(\widehat{\alpha}, \widehat{\tau})] + R(\alpha_0, \widehat{\tau}) + \omega_n|\widehat{D}\alpha_0|_1 - \omega_n|\widehat{D}\widehat{\alpha}|_1. \tag{C.36}$$

Now applying Lemma C.1 to $[\nu_n(\alpha_0, \widehat{\tau}) - \nu_n(\widehat{\alpha}, \widehat{\tau})]$, we rewrite the basic inequality in (C.36) by

$$\omega_n\left|\widehat{D}\alpha_0\right|_1 \geq R(\widehat{\alpha}, \widehat{\tau}) + \omega_n\left|\widehat{D}\widehat{\alpha}\right|_1 - \frac{1}{2}\omega_n\left|\widehat{D}(\widehat{\alpha} - \alpha_0)\right|_1 - |R(\alpha_0, \widehat{\tau})| \quad \text{w.p.a.1.}$$

As before, adding $\omega_n\left|\widehat{D}(\widehat{\alpha} - \alpha_0)\right|_1$ on both sides of the inequality above and using the fact that $|\alpha_{0j}|_1 - |\widehat{\alpha}_j|_1 + |(\widehat{\alpha}_j - \alpha_{0j})|_1 = 0$ for $j \notin J$, we have that

$$R(\widehat{\alpha}, \widehat{\tau}) + \frac{1}{2}\omega_n\left|\widehat{D}(\widehat{\alpha} - \alpha_0)\right|_1 \leq |R(\alpha_0, \widehat{\tau})| + 2\omega_n\left|\widehat{D}(\widehat{\alpha} - \alpha_0)_J\right|_1 \quad \text{w.p.a.1.} \tag{C.37}$$

As in the proof of Theorem 5.2, we consider two cases: (i) $\omega_n\left|\widehat{D}(\widehat{\alpha} - \alpha_0)_J\right|_1 \leq |R(\alpha_0, \widehat{\tau})|$; (ii) $\omega_n\left|\widehat{D}(\widehat{\alpha} - \alpha_0)_J\right|_1 > |R(\alpha_0, \widehat{\tau})|$. We first consider case (ii). Recall that $\|Z\|_2 = (EZ^2)^{1/2}$ for a random variable Z. It follows from the compatibility condition (Assumption 4.2) and the same arguments as in (C.28) that

$$\begin{aligned}
\omega_n\left|\widehat{D}(\widehat{\alpha} - \alpha_0)_J\right|_1 &\leq \omega_n\bar{D}\left\|X(\widehat{\tau})^T(\widehat{\alpha} - \alpha_0)\right\|_2 \sqrt{s}/\phi \\
&\leq \frac{\omega_n^2\bar{D}^2 s}{2\tilde{c}\phi^2} + \frac{\tilde{c}}{2}\left\|X(\widehat{\tau})^T(\widehat{\alpha} - \alpha_0)\right\|_2^2
\end{aligned} \tag{C.38}$$

for any $\tilde{c} > 0$. Recall that $\bar{c}_0(\delta_0) \equiv c_0 \inf_{\tau \in \mathcal{T}_0} \mathbb{E}[(X^T\delta_0)^2|Q = \tau]$. As in Step 5 of the proof of

Theorem 5.2, there is a constant $C_0 > 0$ such that

$$\left\| X(\widehat{\tau})^T (\widehat{\alpha} - \alpha_0) \right\|_2^2 \le C_0 R(\widehat{\alpha}, \widehat{\tau}) + C_0 \bar{c}_0(\delta_0) |\widehat{\tau} - \tau_0|, \tag{C.39}$$

w.p.a.1. Combining (C.37)-(C.39) with a sufficiently small $\tilde{c}$ yields

$$R\left(\widehat{\alpha}, \widehat{\tau}\right) + \omega_n \left| \widehat{D}\left(\widehat{\alpha} - \alpha_0\right) \right|_1 \le C \left(\omega_n^2 s + |\widehat{\tau} - \tau_0| \right) \tag{C.40}$$

for some finite constant $C > 0$. Since $|\widehat{\tau} - \tau_0| = O_P(n^{-1})$ by Theorem 5.3, the desired results follow (C.40) immediately.

Now we consider case (i). In this case,

$$R\left(\widehat{\alpha}, \widehat{\tau}\right) + \frac{1}{2}\omega_n \left| \widehat{D}\left(\widehat{\alpha} - \alpha_0\right) \right|_1 \le 3 \left| R(\alpha_0, \widehat{\tau}) \right|. \tag{C.41}$$

As shown in the proof of Theorem 3.2, we have that

$$\left| R\left(\alpha_0, \widehat{\tau}\right) \right| = O_P \left(|\delta_0|_1 \, n^{-1} \log n \right) = O_P \left(\omega_n^2 s \right). \tag{C.42}$$

Therefore, we obtain the desired results in case (i) as well by combining (C.42) with (C.41).

C.8   Proof of Theorems 5.5

We write α_J be a subvector of α whose components' indices are in $J(\alpha_0)$. Define $\bar{Q}_n(\alpha_J) \equiv \widetilde{S}_n((\alpha_J, 0))$, so that

$$\bar{Q}_n(\alpha_J) = \frac{1}{n} \sum_{i=1}^n \rho(Y_i, X_{iJ}(\widehat{\tau})^T \alpha_J) + \mu_n \sum_{j \in J(\alpha_0)} w_j \widehat{D}_j |\alpha_j|.$$

For notational simplicity, here we write $\widehat{D}_j \equiv D_j(\widehat{\tau})$. When τ_0 is identifiable, our argument is conditional on

$$\widehat{\tau} \in \mathcal{T}_n = \left\{ |\tau - \tau_0| \le n^{-1} \log n \right\}, \tag{C.43}$$

whose probability goes to 1 due to Theorem 5.3.

We first prove the following two lemmas. Define

$$\bar{\alpha}_J \equiv \operatorname*{argmin}_{\alpha_J} \bar{Q}_n(\alpha_J). \tag{C.44}$$

Lemma C.4. *Suppose that* $M_n^2 (\log n)^2/(s \log s) = o(n)$, $s^4 \log s = o(n)$, $s^2 \log n / \log s = o(n)$ *and* $\widehat{\tau} \in \mathcal{T}_n$ *if* $\delta_0 \ne 0$; *suppose that* $s^4 \log s = o(n)$ *and* $\widehat{\tau}$ *is any value in* $\mathcal{T}$ *if* $\delta_0 = 0$. *Then*

$$|\bar{\alpha}_J - \alpha_{0J}|_2 = O_P \left(\sqrt{\frac{s \log s}{n}} \right).$$

Proof of Lemma C.4. Let $k_n = \sqrt{\frac{s \log s}{n}}$. We first prove that for any $\epsilon > 0$, there is $C_\epsilon > 0$, with probability at least $1 - \epsilon$,

$$\inf_{|\alpha_J - \alpha_{0J}|_2 = C_\epsilon k_n} \bar{Q}_n(\alpha_J) > \bar{Q}_n(\alpha_{0J}) \tag{C.45}$$

Once this is proved, then by the continuity of $\bar{Q}_n$, there is a local minimizer of $\bar{Q}_n(\alpha_J)$ inside $B(\alpha_{0J}, C_\epsilon k_n) \equiv \{\alpha_J \in \mathbb{R}^s : |\alpha_{0J} - \alpha_J|_2 \le C_\epsilon k_n\}$. Due to the convexity of $\bar{Q}_n$, such a local minimizer is also global. We now prove (C.45).

Write

$$l_J(\alpha_J) = \frac{1}{n} \sum_{i=1}^n \rho(Y_i, X_{iJ}(\widehat{\tau})^T \alpha_J), \quad L_J(\alpha_J, \tau) = \mathbb{E}[\rho(Y, X_J(\tau)^T \alpha_J)].$$

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Then for all $|\alpha_J - \alpha_{0J}|_2 = C_\epsilon k_n$,

$$
\begin{aligned}
&\bar{Q}_n(\alpha_J) - \bar{Q}_n(\alpha_{0J}) \\
=\ & l_J(\alpha_J) - l_J(\alpha_{0J}) + \sum_{j \in J(\alpha_0)} w_j \mu_n \widehat{D}_j(|\alpha_j| - |\alpha_{0j}|) \\
\geq\ & \underbrace{L_J(\alpha_J, \widehat{\tau}) - L_J(\alpha_{0J}, \widehat{\tau})}_{(1)} - \underbrace{\sup_{|\alpha_J - \alpha_{0J}|_2 \leq C_\delta k_n} |\nu_n(\alpha_J, \widehat{\tau}) - \nu_n(\alpha_{0J}, \widehat{\tau})|}_{(2)} + \underbrace{\sum_{j \in J(\alpha_0)} \mu_n \widehat{D}_j w_j(|\alpha_j| - |\alpha_{0j}|)}_{(3)}.
\end{aligned}
$$

To analyze (1), note that $|\alpha_J - \alpha_{0J}|_2 = C_\epsilon k_n$ and $m_J(\tau_0, \alpha_0) = 0$ and when $\delta_0 = 0$, $m_J(\tau, \alpha_{0J})$ is free of τ. Then there is $c_3 > 0$,

$$
\begin{aligned}
&L_J(\alpha_J, \widehat{\tau}) - L_J(\alpha_{0J}, \widehat{\tau}) \\
\geq\ & m_J(\tau_0, \alpha_{0J})^T(\alpha_J - \alpha_{0J}) + (\alpha_J - \alpha_{0J})^T \frac{\partial^2 \mathbb{E}[\rho(Y, X_J(\widehat{\tau})^T \alpha_{0J})]}{\partial \alpha_J \partial \alpha_J^T}(\alpha_J - \alpha_{0J}) \\
& - |m_J(\tau_0, \alpha_{0J}) - m_J(\widehat{\tau}, \alpha_{0J})|_2 |\alpha_J - \alpha_{0J}|_2 - c_3 |\alpha_{0J} - \alpha_J|_1^3 \\
\geq\ & \lambda_{\min}\left(\frac{\partial^2 \mathbb{E}[\rho(Y, X_J(\widehat{\tau})^T \alpha_{0J})]}{\partial \alpha_J \partial \alpha_J^T} \right) |\alpha_J - \alpha_{0J}|_2^2 \\
& - (|m_J(\tau_0, \alpha_{0J}) - m_J(\widehat{\tau}, \alpha_{0J})|_2)|\alpha_J - \alpha_{0J}|_2 - c_3 s^{3/2} |\alpha_{0J} - \alpha_J|_2^3 \\
\geq\ & c_1 C_\epsilon^2 k_n^2 - (|m_J(\tau_0, \alpha_{0J}) - m_J(\widehat{\tau}, \alpha_{0J})|_2) C_\epsilon k_n - c_3 s^{3/2} C_\delta^3 k_n^3 \\
\geq\ & C_\epsilon k_n(c_1 C_\epsilon k_n - M_n n^{-1} \log n - c_3 s^{3/2} C_\epsilon^2 k_n^2) \geq c_1 C_\delta^2 k_n^2/3,
\end{aligned}
$$

where the last inequality follows from $M_n n^{-1} \log n < 1/3 c_1 C_\epsilon k_n$ and $c_3 s^{3/2} C_\epsilon^2 k_n^2 < 1/3 c_1 C_\epsilon k_n$. These follow from the conditions $M_n^2 (\log n)^2/(s \log s) = o(n)$ and $s^4 \log s = o(n)$.

To analyze (2), by the symmetrization theorem and the contraction theorem (see, for example, Theorems 14.3 and 14.4 of Bühlmann and van de Geer (2011)), there is a Rademacher sequence $\epsilon_1, ..., \epsilon_n$ independent of $\{Y_i, X_i, Q_i\}_{i \leq n}$ such that (note that when $\delta_0 = 0$, $\alpha_J = \beta_J$,

$$
\nu_n(\alpha_J, \tau) \equiv \frac{1}{n} \sum_{i=1}^n \left[\rho\left(Y_i, X_{J(\beta_0)i}^T \beta_J\right) - \mathbb{E}\rho\left(Y, X_{J(\beta_0)}^T \beta_J\right) \right],
$$

which is free of τ)

$$
\begin{aligned}
V_n &= \mathbb{E}\left(\sup_{\tau \in \mathcal{T}_n} \sup_{|\alpha_J - \alpha_{0J}|_2 \leq C_\epsilon k_n} |\nu_n(\alpha_J, \tau) - \nu_n(\alpha_{0J}, \tau)|\right) \\
&\leq 2\mathbb{E}\left(\sup_{\tau \in \mathcal{T}_n} \sup_{|\alpha_J - \alpha_{0J}|_2 \leq C_\epsilon k_n} \left|\frac{1}{n}\sum_{i=1}^n \epsilon_i [\rho(Y_i, X_{iJ}(\tau)^T \alpha_J) - \rho(Y_i, X_{iJ}(\tau)^T \alpha_{0J})]\right|\right) \\
&\leq 4L\mathbb{E}\left(\sup_{\tau \in \mathcal{T}_n} \sup_{|\alpha_J - \alpha_{0J}|_2 \leq C_\epsilon k_n} \left|\frac{1}{n}\sum_{i=1}^n \epsilon_i (X_{iJ}(\tau)^T (\alpha_J - \alpha_{0J}))\right|\right),
\end{aligned}
$$

which is bounded by the sum of the following two terms, $V_{1n} + V_{2n}$, due to the triangle inequality and the fact that $|\alpha_J - \alpha_{0J}|_1 \leq |\alpha_J - \alpha_{0J}|_2 \sqrt{s}$: when $\delta_0 \neq 0$ and τ_0 is identifiable,

$$
\begin{aligned}
V_{1n} &= 4L\mathbb{E}\left(\sup_{\tau \in \mathcal{T}_n} \sup_{|\alpha_J - \alpha_{0J}|_1 \leq C_\epsilon k_n \sqrt{s}} \left|\frac{1}{n}\sum_{i=1}^n \epsilon_i (X_{iJ}(\tau) - X_{iJ}(\tau_0))^T (\alpha_J - \alpha_{0J})\right|\right) \\
&\leq 4L\mathbb{E}\left(\sup_{\tau \in \mathcal{T}_n} \sup_{|\delta_J - \delta_{0J}|_1 \leq C_\epsilon k_n \sqrt{s}} \left|\frac{1}{n}\sum_{i=1}^n \epsilon_i X_{iJ(\delta_0)}^T (1\{Q_i > \tau\} - 1\{Q_i > \tau_0\})(\delta_J - \delta_{0J})\right|\right) \\
&\leq 4LC_\epsilon k_n \sqrt{s}\mathbb{E}\left(\sup_{\tau \in \mathcal{T}_n} \max_{j \in J(\delta_0)} \left|\frac{1}{n}\sum_{i=1}^n \epsilon_i X_{ij} (1\{Q_i > \tau\} - 1\{Q_i > \tau_0\})\right|\right) \\
&\leq 4LC_\epsilon k_n \sqrt{s} C_1 |J(\delta_0)|_0 \sqrt{\frac{\log n}{n^2}},
\end{aligned}
$$

due to the maximal inequality (for VC class indexed by τ and j); when $\delta_0 = 0$, $V_{1n} \equiv 0$.

$$
\begin{aligned}
V_{2n} &= 4L\mathbb{E}\left(\sup_{|\alpha_J - \alpha_{0J}|_1 \leq C_\epsilon k_n \sqrt{s}} \left|\frac{1}{n}\sum_{i=1}^n \epsilon_i X_{iJ}(\tau_0)^T (\alpha_J - \alpha_{0J})\right|\right) \\
&\leq 4LC_\epsilon k_n \sqrt{s}\mathbb{E}\left(\max_{j \in J(\alpha_0)} \left|\frac{1}{n}\sum_{i=1}^n \epsilon_i X_{ij}(\tau_0)\right|\right) \leq 4LC_\epsilon C_2 k_n^2,
\end{aligned}
$$

due to the Bernstein's moment inequality (Lemma 14.12 of Bühlmann and van de Geer (2011) for some $C_2 > 0$. Therefore,

$$
V_n \leq 4LC_\epsilon k_n \sqrt{s} C_1 |J(\delta_0)|_0 \sqrt{\frac{\log n}{n^2}} + 4LC_\epsilon C_2 k_n^2 < 5LC_\epsilon C_2 k_n^2,
$$

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where the last inequality is due to $s^2 \log n / \log s = o(n)$. Therefore, conditioning on the event $\widehat{\tau} \in \mathcal{T}_n$ when $\delta_0 \neq 0$, or for $\widehat{\tau} \in \mathcal{T}$ when $\delta_0 = 0$, with probability at least $1 - \epsilon$, $(2) \leq \frac{1}{\epsilon} 5 L C_2 C_\epsilon k_n^2$.

In addition, note that $P(\max_{j \in J(\alpha_0)} |w_j| = 0) = 1$, so $(3) = 0$ with probability approaching one. Hence

$$\inf_{|\alpha_J - \alpha_{0J}|_2 = C_\epsilon k_n} \bar{Q}_n(\alpha_J) - \bar{Q}_n(\alpha_{0J}) \geq \frac{c_1 C_\epsilon^2 k_n^2}{3} - \frac{1}{\epsilon} 5 L C_2 C_\epsilon k_n^2 > 0.$$

The last inequality holds for $C_\epsilon > \frac{15 L C_2}{c_1 \epsilon}$. By the continuity of $\bar{Q}_n$, there is a local minimizer of $\bar{Q}_n(\alpha_J)$ inside $\{\alpha_J \in \mathbb{R}^s : |\alpha_{0J} - \alpha_J|_2 \leq C_\epsilon k_n\}$, which is also a global minimizer due to the convexity. ∎

On $\mathbb{R}^{2p}$, recall that

$$R_n(\tau, \alpha) = \frac{1}{n} \sum_{i=1}^{n} \rho(Y_i, X_i(\tau)^T \alpha).$$

For $\bar{\alpha}_J = (\bar{\beta}_{J(\beta_0)}, \bar{\delta}_{J(\delta_0)}) \equiv (\bar{\beta}_J, \bar{\delta}_J)$ in the previous lemma, define

$$\bar{\alpha} = (\bar{\beta}_J^T, 0^T, \bar{\delta}_J^T, 0^T)^T.$$

Without introducing confusions, we also write $\bar{\alpha} = (\bar{\alpha}_J, 0)$ for notational simplicity. This notation indicates that $\bar{\alpha}$ has zero entries on the indices outside the oracle index set $J(\alpha_0)$. We prove the following lemma.

Lemma C.5. *With probability approaching one, there is a random neighborhood of $\bar{\alpha}$ in $\mathbb{R}^{2p}$, denoted by $\mathcal{H}$, so that $\forall \alpha = (\alpha_J, \alpha_{J^c}) \in \mathcal{H}$, if $\alpha_{J^c} \neq 0$, we have $\widetilde{S}_n(\alpha_J, 0) < \widetilde{Q}_n(\alpha)$.*

Proof of Lemma C.5. Define an l_2-ball, for $r_n \equiv \mu_n / \log n$,

$$\mathcal{H} = \{\alpha \in \mathbb{R}^{2p} : |\alpha - \bar{\alpha}|_2 < r_n/(2p)\}.$$

Then $\sup_{\alpha \in \mathcal{H}} |\alpha - \bar{\alpha}|_1 = \sup_{\alpha \in \mathcal{H}} \sum_{l \leq 2p} |\alpha_l - \bar{\alpha}_l| < r_n$. Consider any $\tau \in \mathcal{T}_n$. For any $\alpha =$

$(\alpha_J, \alpha_{J^c}) \in \mathcal{H}$, write

$$R_n(\tau, \alpha_J, 0) - R_n(\tau, \alpha)$$

$$= R_n(\tau, \alpha_J, 0) - \mathbb{E}R_n(\tau, \alpha_J, 0) + \mathbb{E}R_n(\tau, \alpha_J, 0) - R_n(\tau, \alpha) + \mathbb{E}R_n(\tau, \alpha) - \mathbb{E}R_n(\tau, \alpha)$$

$$\leq \mathbb{E}R_n(\tau, \alpha_J, 0) - \mathbb{E}R_n(\tau, \alpha) + |R_n(\tau, \alpha_J, 0) - \mathbb{E}R_n(\tau, \alpha_J, 0) + \mathbb{E}R_n(\tau, \alpha) - R_n(\tau, \alpha)|$$

$$\leq \mathbb{E}R_n(\tau, \alpha_J, 0) - \mathbb{E}R_n(\tau, \alpha) + |\nu_n(\alpha_J, 0, \tau) - \nu_n(\alpha, \tau)|.$$

Note that $|(\alpha_J, 0) - \bar{\alpha}|_2^2 = |\alpha_J - \bar{\alpha}_J|_2^2 \leq |\alpha_J - \bar{\alpha}_J|_2^2 + |\alpha_{J^c} - 0|_2^2 = |\alpha - \bar{\alpha}|_2^2$. Hence $\alpha \in \mathcal{H}$ implies $(\alpha_J, 0) \in \mathcal{H}$. In addition, by definition of $\bar{\alpha} = (\bar{\alpha}_J, 0)$ and $|\bar{\alpha}_J - \alpha_{0J}|_2 = O_P(\sqrt{\frac{s \log s}{n}})$ (Lemma C.4), we have $|\bar{\alpha} - \alpha_0|_1 = O_P(s\sqrt{\frac{\log s}{n}})$, which also implies

$$\sup_{\alpha \in \mathcal{H}} |\alpha - \alpha_0|_1 = O_P\left(s\sqrt{\frac{\log s}{n}}\right) + r_n,$$

where the randomness in $\sup_{\alpha \in \mathcal{H}} |\alpha - \alpha_0|_1$ comes from that of $\mathcal{H}$.

By the mean value theorem, there is h in the segment between α and $(\alpha_J, 0)$,

$$\mathbb{E}R_n(\tau, \alpha_J, 0) - \mathbb{E}R_n(\tau, \alpha) = \mathbb{E}\rho(Y, X_J(\tau)^T \alpha_J) - \mathbb{E}\rho(Y, X_J(\tau)^T \alpha_J + X_{J^c}(\tau)^T \alpha_{J^c})$$

$$= -\sum_{j \notin J(\alpha_0)} \frac{\partial \mathbb{E}\rho(Y, X(\tau)^T h)}{\partial \alpha_j} \alpha_j \equiv \sum_{j \notin J(\alpha_0)} m_j(\tau, h) \alpha_j$$

where $m_j(\tau, h) = -\frac{\partial \mathbb{E}\rho(Y, X(\tau)^T h)}{\partial \alpha_j}$. Hence, $\mathbb{E}R_n(\tau, \alpha_J, 0) - \mathbb{E}R_n(\tau, \alpha) \leq \sum_{j \notin J} |m_j(\tau, h)||\alpha_j|$.

Because h is on the segment between α and $(\alpha_J, 0)$, so $h \in \mathcal{H}$. So for all $j \notin J(\alpha_0)$,

$$|m_j(\tau, h)| \leq \sup_{\alpha \in \mathcal{H}} |m_j(\tau, \alpha)| \leq \sup_{\alpha \in \mathcal{H}} |m_j(\tau, \alpha) - m_j(\tau, \alpha_0)| + |m_j(\tau, \alpha_0) - m_j(\tau_0, \alpha_0)|.$$

We now argue that we can apply Assumption B.2 (ii). Let

$$c_n \equiv s\sqrt{(\log s)/n} + r_n.$$

For any $\epsilon > 0$, there is $C_\epsilon > 0$, with probability at last $1-\epsilon$, $\sup_{\alpha \in \mathcal{H}} |\alpha - \alpha_0|_1 \leq C_\epsilon c_n$. $\forall \alpha \in \mathcal{H}$, write $\alpha = (\beta, \delta)$ and $\theta = \beta + \delta$. On the event $|\alpha - \alpha_0|_1 \leq C_\epsilon c_n$, we have $|\beta - \beta_0|_1 \leq C_\epsilon c_n$ and $|\theta - \theta_0|_1 \leq C_\epsilon c_n$. Hence $\mathbb{E}[(X^T(\beta - \beta_0))^2 1\{Q \leq \tau_0\}] \leq |\beta - \beta_0|_1^2 \max_{i,j \leq p} E|X_i X_j| < r^2$, yielding $\beta \in \mathcal{B}(\beta_0, r)$. Similarly, $\theta \in \mathcal{G}(\theta_0, r)$. Therefore, by Assumption B.2 (ii), with probability at least $1 - \epsilon$, (note that neither C_ϵ, L nor c_n depend on α)

$$\max_{j \notin J(\alpha_0)} \sup_{\tau \in \mathcal{T}_n} \sup_{\alpha \in \mathcal{H}} |m_j(\tau, \alpha) - m_j(\tau, \alpha_0)| \leq L \sup_{\alpha \in \mathcal{H}} |\alpha - \alpha_0|_1 \leq L(C_\epsilon c_n),$$

$$\max_{j \leq 2p} \sup_{\tau \in \mathcal{T}_n} |m_j(\tau, \alpha_0) - m_j(\tau_0, \alpha_0)| \leq M_n n^{-1} \log n.$$

In particular, when $\delta_0 = 0$, $m_j(\tau, \alpha_0) = 0$ for all τ. Therefore, when $\delta_0 \neq 0$,

$$\sup_{j \notin J(\alpha_0)} \sup_{\tau \in \mathcal{T}_n} |m_j(\tau, h)| = O_P(c_n + M_n n^{-1} \log n) = o_P(\mu_n);$$

when $\delta_0 = 0$, $\sup_{j \notin J(\alpha_0)} \sup_{\tau \in \mathcal{T}} |m_j(\tau, h)| = O_P(c_n) = o_P(\mu_n)$.

Let $\epsilon_1, ..., \epsilon_n$ be a Rademacher sequence independent of $\{Y_i, X_i, Q_i\}_{i \leq n}$. Then by the symmetrization and contraction theorems,

$$\mathbb{E}\left(\sup_{\tau \in \mathcal{T}} |\nu_n(\alpha_J, 0, \tau) - \nu_n(\alpha, \tau)|\right)$$

$$\leq 2\mathbb{E}\left(\sup_{\tau \in \mathcal{T}} \left|\frac{1}{n} \sum_{i=1}^n \epsilon_i [\rho(Y_i, X_{iJ}(\tau)^T \alpha_J) - \rho(Y_i, X_i(\tau)^T \alpha)]\right|\right)$$

$$\leq 4L\mathbb{E}\left(\sup_{\tau \in \mathcal{T}} \left|\frac{1}{n} \sum_{i=1}^n \epsilon_i [X_{iJ}(\tau)^T \alpha_J - X_i(\tau)^T \alpha]\right|\right)$$

$$\leq 4L\mathbb{E}\left(\sup_{\tau \in \mathcal{T}} \left\|\frac{1}{n} \sum_{i=1}^n \epsilon_i X_i(\tau)\right\|_{\max}\right) \sum_{j \notin J(\alpha_0)} |\alpha_j| \leq 2\omega_n \sum_{j \notin J(\alpha_0)} |\alpha_j|,$$

where the last equality follows from (C.5).

Thus uniformly over $\alpha \in \mathcal{H}$, $R_n(\tau, \alpha_J, 0) - R_n(\tau, \alpha) = o_P(\mu_n) \sum_{j \notin J(\alpha_0)} |\alpha_j|$. On the other hand,

$$\sum_{j \in J(\alpha_0)} w_j \mu_n \widehat{D}_j |\alpha_j| - \sum_j w_j \mu_n \widehat{D}_j |\alpha_j| = \sum_{j \notin J(\alpha_0)} \mu_n w_j \widehat{D}_j |\alpha_j|.$$

Also, w.p.a.1, $w_j = 1$ and $\widehat{D}_j \geq \overline{D}$ for all $j \notin J(\alpha_0)$. Hence with probability approaching one, $\widetilde{Q}_n(\alpha_J, 0) - \widetilde{Q}_n(\alpha)$ equals

$$R_n(\widehat{\tau}, \alpha_J, 0) + \sum_{j \in J(\alpha_0)} \widehat{D}_j w_j \lambda_n |\alpha_j| - R_n(\widehat{\tau}, \alpha) - \sum_{j \leq 2p} \widehat{D}_j w_j \omega_n |\alpha_j| \leq -\underline{D} \frac{\mu_n}{2} \sum_{j \notin J(\alpha_0)} |\alpha_j| < 0. \quad \blacksquare$$

Proof of Theorem 5.5. Conditions in Lemmas C.4 and C.5 are expressed in terms of M_n. By Lemma B.1, we verify that in quantile regression models, $M_n = C s^{1/2}$ for some $C > 0$. Then all the required conditions in Lemmas C.4 and C.5 are satisfied by the conditions imposed in Theorem 5.5.

By Lemmas C.4 and C.5, w.p.a.1, for any $\alpha = (\alpha_J, \alpha_{J^c}) \in \mathcal{H}$,

$$\widetilde{S}_n(\bar{\alpha}_J, 0) = \bar{Q}_n(\bar{\alpha}_J) \leq \bar{Q}_n(\alpha_J) = \widetilde{S}_n(\alpha_J, 0) \leq \widetilde{S}_n(\alpha).$$

Hence $(\bar{\alpha}_J, 0)$ is a local minimizer of $\widetilde{S}_n$, which is also a global minimizer due to the convexity. This implies that w.p.a.1, $\widetilde{\alpha} = (\widetilde{\alpha}_J, \widetilde{\alpha}_{J^c})$ satisfies: $\widetilde{\alpha}_{J^c} = 0$, and $\widetilde{\alpha}_J = \bar{\alpha}_J$, so

$$|\widetilde{\alpha}_J - \alpha_{0J}|_2 = O_P\left(\sqrt{\frac{s \log s}{n}}\right), \quad |\widetilde{\alpha}_J - \alpha_{0J}|_1 = O_P\left(s\sqrt{\frac{\log s}{n}}\right).$$

$\blacksquare$

C.9   Proof of Theorem 5.6

Recall that by Theorems 5.3 and 5.5, we have

$$|\widetilde{\alpha}_J - \alpha_{0J}|_2 = O_P\left(\sqrt{\frac{s \log s}{n}}\right) \quad \text{and} \quad |\widehat{\tau} - \tau_0| = O_P(n^{-1}), \qquad \text{(C.46)}$$

and the set of regressors with nonzero coefficients is recovered w.p.a.1. Hence we can restrict ourselves on the oracle space $J(\alpha_0)$. In view of (C.46), define $r_n \equiv \sqrt{n^{-1}s\log s}$ and s_n. Let

$$R_n^* (\alpha_J, \tau) \equiv \frac{1}{n} \sum_{i=1}^{n} \rho \left(Y_i, X_{iJ}(\tau)^T \alpha_J \right),$$

where $\alpha_J \in \mathcal{A}_n \equiv \{\alpha_J : |\alpha_J - \alpha_{0J}|_2 \leq Kr_n\} \subset \mathbb{R}^s$ and $\tau \in \mathcal{T}_n \equiv \{\tau : |\tau - \tau_0| \leq Ks_n\}$ for some $K < \infty$, where K is a generic finite constant.

The following lemma is useful to establish that α_0 can be estimated as if τ_0 were known.

Lemma C.6 (Asymptotic Equivalence). *Assume that $\frac{\partial}{\partial \alpha} E\left[\rho \left(Y, X^T \alpha \right) | Q = t \right]$ exists for all t in a neighborhood of τ_0 and all its elements are continuous and bounded. Suppose that $s^3(\log s)(\log n) = o(n)$. Then*

$$\sup_{\alpha_J \in \mathcal{A}_n, \tau \in \mathcal{T}_n} |\{R_n^* (\alpha_J, \tau) - R_n^* (\alpha_J, \tau_0)\} - \{R_n^* (\alpha_{0J}, \tau) - R_n^* (\alpha_{0J}, \tau_0)\}| = o_P \left(n^{-1} \right).$$

This lemma implies that the asymptotic distribution of $\operatorname{argmin}_{\alpha_J} R_n^* (\alpha_J, \widehat{\tau})$ can be characterized by $\widehat{\alpha}_J^* \equiv \operatorname{argmin}_{\alpha_J} R_n^* (\alpha_J, \tau_0)$. It then follows immediately from the variable selection consistency that the asymptotic distribution of $\widetilde{\alpha}_J$ is equivalent to that of $\widehat{\alpha}_J^*$. Therefore, we have proved the theorem.

Proof of Lemma C.6. Noting that

$$\rho \left(Y_i, X_i^T \beta + X_i^T \delta 1 \{Q_i > \tau\} \right) = \rho \left(Y_i, X_i^T \beta \right) 1 \{Q_i \leq \tau\} + \rho \left(Y_i, X_i^T \beta + X_i^T \delta \right) 1 \{Q_i > \tau\},$$

we have, for $\tau > \tau_0$,

$$D_n(\alpha, \tau)$$

$$\equiv \{R_n(\alpha, \tau) - R_n(\alpha, \tau_0)\} - \{R_n(\alpha_0, \tau) - R_n(\alpha_0, \tau_0)\}$$

$$= \frac{1}{n} \sum_{i=1}^n \left[\rho\left(Y_i, X_i^T \beta\right) - \rho\left(Y_i, X_i^T \beta_0\right) \right] 1\{\tau_0 < Q_i \leq \tau\}$$

$$- \frac{1}{n} \sum_{i=1}^n \left[\rho\left(Y_i, X_i^T \beta + X_i^T \delta\right) - \rho\left(Y_i, X_i^T \beta_0 + X_i^T \delta_0\right) \right] 1\{\tau_0 < Q_i \leq \tau\}$$

$$=: D_{n1}(\alpha, \tau) - D_{n2}(\alpha, \tau).$$

To prove this lemma, we consider empirical processes

$$\mathbb{G}_{nj}(\alpha_J, \tau) \equiv \sqrt{n}\left(D_{nj}(\alpha_J, \tau) - \mathbb{E}D_{nj}(\alpha_J, \tau)\right), \quad (j = 1, 2),$$

and apply the maximal inequality in Theorem 2.14.2 of van der Vaart and Wellner (1996).

First, for $\mathbb{G}_{n1}(\alpha_J, \tau)$, we consider the following class of functions indexed by (β_J, τ):

$$\mathcal{F}_n \equiv \{\left(\rho\left(Y_i, X_{iJ}^T \beta_J\right) - \rho\left(Y_i, X_{iJ}^T \beta_{0J}\right)\right) 1\left(\tau_0 < Q_i \leq \tau\right) : |\beta_J - \beta_{0J}|_2 \leq Kr_n \text{ and } |\tau - \tau_0| \leq Ks_n\}.$$

Note that the Lipschitz property of ρ yields that

$$\left|\rho\left(Y_i, X_{iJ}^T \beta_J\right) - \rho\left(Y_i, X_{iJ}^T \beta_{0J}\right)\right| 1\{\tau_0 < Q_i \leq \tau\} \leq \left|X_{iJ}^T\right|_2 |\beta_J - \beta_{0J}|_2 1\{|Q_i - \tau_0| \leq Ks_n\}.$$

Thus, we let the envelope function be $F_n(X_{iJ}, Q_i) \equiv |X_{iJ}|_2 Kr_n 1\{|Q_i - \tau_0| \leq Ks_n\}$ and note that its L_2 norm is $O\left(\sqrt{s}r_n\sqrt{s_n}\right)$.

To compute the bracketing integral

$$J_{[]}(1, \mathcal{F}_n, L_2) \equiv \int_0^1 \sqrt{1 + \log N_{[]}\left(\varepsilon\|F_n\|_{L_2}, \mathcal{F}_n, L_2\right)} d\varepsilon,$$

note that its 2ε bracketing number is bounded by the product of the ε bracketing numbers of two classes $\mathcal{F}_{n1} \equiv \left\{ \rho\left(Y_i, X_{iJ}^T \beta_J\right) - \rho\left(Y_i, X_{iJ}^T \beta_0\right) : |\beta_J - \beta_{0J}|_2 \le K r_n \right\}$ and $\mathcal{F}_{n2} \equiv \left\{ 1\left(\tau_0 < Q_i \le \tau\right) : |\tau - \tau_0| \le K s_n \right\}$ by Lemma 9.25 of Kosorok (2008) since both classes are bounded w.p.a.1 (note that w.p.a.1, $|X_{iJ}|_2 K r_n < C < \infty$ for some constant C). That is,

$$N_{[]}\left(2\varepsilon\|F_n\|_{L_2}, \mathcal{F}_n, L_2\right) \le N_{[]}\left(\varepsilon\|F_n\|_{L_2}, \mathcal{F}_{n1}, L_2\right) N_{[]}\left(\varepsilon\|F_n\|_{L_2}, \mathcal{F}_{n2}, L_2\right).$$

Let $F_{n1}(X_{iJ}) \equiv |X_{iJ}|_2 K r_n$ and $l_n(X_{iJ}) \equiv |X_{iJ}|_2$. Note that by Theorem 2.7.11 of van der Vaart and Wellner (1996), the Lipschitz property of ρ implies that

$$N_{[]}\left(2\varepsilon\|l_n\|_{L_2}, \mathcal{F}_{n1}, L_2\right) \le N(\varepsilon, \{\beta_J : |\beta_J - \beta_{0J}|_2 \le K r_n\}, |\cdot|_2),$$

which in turn implies that, for some constant C,

$$
\begin{aligned}
N_{[]}\left(\varepsilon\|F_n\|_{L_2}, \mathcal{F}_{n1}, L_2\right) &\le N\left(\frac{\varepsilon\|F_n\|_{L_2}}{2\|l_n\|_{L_2}}, \{\beta_J : |\beta_J - \beta_{0J}|_2 \le K r_n\}, |\cdot|_2\right) \\
&\le C\left(\frac{\sqrt{s}}{\varepsilon\sqrt{s_n}}\right)^s = C\left(\frac{\sqrt{ns}}{\varepsilon}\right)^s,
\end{aligned}
$$

where the last inequality holds since a ε-ball contains a hypercube with side length $\varepsilon/\sqrt{s}$ in the s-dimensional Euclidean space. On the other hand, for the second class of functions $\mathcal{F}_{n2}$ with the envelope function $F_{n2}(Q_i) \equiv 1\{|Q_i - \tau_0| \le K s_n\}$, we have that

$$N_{[]}\left(\varepsilon\|F_n\|_{L_2}, \mathcal{F}_{n2}, L_2\right) \le C\frac{\sqrt{s_n}}{\varepsilon\|F_n\|_{L_2}} = \frac{C}{\varepsilon\sqrt{s}r_n} = \frac{C\sqrt{n}}{\varepsilon s\sqrt{\log s}},$$

for some constant C. Combining these results together yields that

$$N_{[]}\left(\varepsilon\|F_n\|_{L_2}, \mathcal{F}_n, L_2\right) \le \frac{C^2\sqrt{n}}{\varepsilon s\sqrt{\log s}}\left(\frac{\sqrt{ns}}{\varepsilon}\right)^s \le C^2\varepsilon^{-s-1}n^{(s+1)/2}$$

for all sufficiently large n. Then we have that

$$J_{[]}\left(1, \mathcal{F}_n, L_2\right) \leq C^2(\sqrt{s \log n} + \sqrt{s})$$

for all sufficiently large n. Thus, by the maximal inequality in Theorem 2.14.2 of van der Vaart and Wellner (1996),

$$
\begin{aligned}
n^{-1/2} \, \mathbb{E} \sup_{\mathcal{A}_n \times \mathcal{T}_n} |\mathbb{G}_{n1}\left(\alpha_J, \tau\right)| &\leq O\left[n^{-1/2}\sqrt{s}r_n\sqrt{s_n}(\sqrt{s \log n} + \sqrt{s})\right] \\
&= O\left[\frac{s}{n^{3/2}}\sqrt{\log s}(\sqrt{s \log n} + \sqrt{s})\right] \\
&= o\left(n^{-1}\right),
\end{aligned}
$$

where the last equality follows from the restriction that $s^3(\log s)(\log n) = o(n)$. Identical arguments also apply to $\mathbb{G}_{n2}\left(\alpha_J, \tau\right)$.

Turning to $\mathbb{E}D_n\left(\alpha, \tau\right)$, note that by the condition that $\frac{\partial}{\partial \alpha}E\left[\rho\left(Y, X^T\alpha\right)|Q = t\right]$ exists for all t in a neighborhood of τ_0 and all its elements are continuous and bounded, we have that for some mean value $\tilde{\beta}_J$ between β_J and β_{0J},

$$
\begin{aligned}
&\left|\mathbb{E}\left(\rho\left(Y, X_J^T\beta_J\right) - \rho\left(Y, X_J^T\beta_{0J}\right)\right)\mathbf{1}\left\{\tau_0 < Q \leq \tau\right\}\right| \\
&= \left|\mathbb{E}\left[\frac{\partial}{\partial \beta}\mathbb{E}\left[\rho\left(Y, X^T\tilde{\beta}_J\right)|Q\right]\mathbf{1}\left\{\tau_0 < Q \leq \tau\right\}\right](\beta - \beta_0)\right| \\
&= O\left(sr_n s_n\right) \\
&= O\left[\frac{s^{3/2}}{n^{3/2}}\sqrt{\log s}\right] \\
&= o\left(n^{-1}\right),
\end{aligned}
$$

where the last equality follows from the restriction that $s^3(\log s) = o(n)$. Since the same holds for the other term in $\mathbb{E}D_n$, $\sup |\mathbb{E}D_n\left(\alpha, \tau\right)| = o\left(n^{-1}\right)$ as desired. ∎

C.10 Proof of Theorem 5.7

By definition,

$$\frac{1}{n}\sum_{i=1}^{n}\rho(Y_i, X_i(\widehat{\tau})^T\widetilde{\alpha}) + \mu_n|W\widehat{D}\widetilde{\alpha}|_1 \leq \frac{1}{n}\sum_{i=1}^{n}\rho(Y_i, X_i(\widehat{\tau})^T\alpha_0) + \mu_n|W\widehat{D}\alpha_0|_1.$$

where $W = \text{diag}\{w_1, ..., w_{2p}\}$. From this, we obtain the following inequality

$$R(\widetilde{\alpha}, \widehat{\tau}) + \mu_n|W\widehat{D}\widetilde{\alpha}|_1 \leq |\nu_n(\alpha_0, \widehat{\tau}) - \nu_n(\widetilde{\alpha}, \widehat{\tau})| + R(\alpha_0, \widehat{\tau}) + \mu_n|W\widehat{D}\alpha_0|_1.$$

Now applying Lemma C.1 yields, when $\sqrt{\log(np)/n} = o(\mu_n)$ (which is true under the assumption that $\omega_n \ll \mu_n$), we have that w.p.a.1, $|\nu_n(\alpha_0, \hat{\tau}) - \nu_n(\widetilde{\alpha}, \widehat{\tau})| \leq \frac{1}{2}\mu_n|\widehat{D}(\alpha_0 - \widetilde{\alpha})|_1$. Hence on this event,

$$R(\widetilde{\alpha}, \widehat{\tau}) + \mu_n|W\widehat{D}\widetilde{\alpha}|_1 \leq \frac{1}{2}\mu_n|\widehat{D}(\alpha_0 - \widetilde{\alpha})|_1 + R(\alpha_0, \widehat{\tau}) + \mu_n|W\widehat{D}\alpha_0|_1.$$

Note that $\max_j w_j \leq 1$, so for $\Delta := \widetilde{\alpha} - \alpha_0$,

$$R(\widetilde{\alpha}, \widehat{\tau}) + \mu_n|(W\widehat{D}\Delta)_{J^c}|_1 \leq \frac{3}{2}\mu_n|\widehat{D}\Delta_J|_1 + \frac{1}{2}\mu_n|\widehat{D}\Delta_{J^c}|_1 + R(\alpha_0, \widehat{\tau}).$$

By Theorem 5.2, $\max_{j\notin J}|\widehat{\alpha}_j| = O_P(\omega_n s)$. Hence for any $\epsilon > 0$, there is $C > 0$, $\max_{j\notin J}|\widehat{\alpha}_j| \leq C\omega_n s < \mu_n$ with probability at least $1 - \epsilon$. On the event $\max_{j\notin J}|\widehat{\alpha}_j| \leq C\omega_n s < \mu_n$, by definition, $w_j = 1 \ \forall j \notin J$. Hence on this event,

$$R(\widetilde{\alpha}, \widehat{\tau}) + \frac{1}{2}\mu_n|(\widehat{D}\Delta)_{J^c}|_1 \leq \frac{3}{2}\mu_n|\widehat{D}\Delta_J|_1 + R(\alpha_0, \widehat{\tau}).$$

We now consider two cases: (i) $\frac{3}{2}\mu_n|\widehat{D}\Delta_J|_1 \leq R(\alpha_0, \widehat{\tau})$; (ii) $\frac{3}{2}\mu_n|\widehat{D}\Delta_J|_1 > R(\alpha_0, \widehat{\tau})$.

case 1: $\frac{3}{2}\mu_n|\widehat{D}\Delta_J|_1 \leq R(\alpha_0, \widehat{\tau})$

We have: for $C = 14\underline{D}^{-1}/3$, $\mu_n|\Delta|_1 \leq CR(\alpha_0, \widehat{\tau})$. If $\widehat{\tau} > \tau_0$, for $\tau = \widehat{\tau}$ in the inequalities

below,

$$R(\alpha_0, \widehat{\tau}) = \mathbb{E}(\rho(Y, X^T\beta_0) - \rho(X^T\theta_0))1\{\tau_0 < Q < \tau\} \le L\mathbb{E}|X^T\delta_0|1\{\tau_0 < Q < \tau\}$$
$$\le L|\delta_0|_1 \max_{j \le p} E|X_j|1\{\tau_0 < Q < \tau\} \le L|\delta_0|_1 \max_{j \le p} \sup_q E(|X_j||Q = q)P(\tau_0 < Q < \tau)$$
$$\le Cs(\tau - \tau_0).$$

The case for $\tau \le \tau_0$ follows from the same argument. Hence $\mu_n|\Delta|_1 \le C|\widehat{\tau} - \tau_0|s$.

case 2: $\frac{3}{2}\mu_n|\widehat{D}\Delta_J|_1 > R(\alpha_0, \widehat{\tau})$

Then by the compatibility property,

$$R(\widetilde{\alpha}, \widehat{\tau}) + \frac{1}{2}\mu_n|(\widehat{D}\Delta)_{J^c}|_1 \le 3\mu_n|\widehat{D}\Delta_J|_1 \le 3\mu_n\bar{D}\sqrt{s}\|X(\tau_0)\Delta\|_2/\sqrt{\phi}.$$

The same argument as that of Step 5 in the proof of Theorem 5.2 yields

$$\|X(\tau_0)\Delta\|_2^2 \le CR(\widetilde{\alpha}, \widehat{\tau}) + C|\widehat{\tau} - \tau_0|$$

for some generic constant $C > 0$. This implies, for some generic constant $C > 0$,

$$R(\widetilde{\alpha}, \widehat{\tau})^2 \le \mu_n^2 sC(R(\widetilde{\alpha}, \widehat{\tau}) + |\widehat{\tau} - \tau_0|).$$

It follows that $R(\widetilde{\alpha}, \widehat{\tau}) \le C(\mu_n^2 s + |\widehat{\tau} - \tau_0|)$, and $\|X(\tau_0)\Delta\|_2^2 \le C(\mu_n^2 s + |\widehat{\tau} - \tau_0|)$. Hence

$$|\Delta|_1^2 \le Cs\|X(\tau_0)\Delta\|_2^2 \le C(\mu_n^2 s^2 + |\widehat{\tau} - \tau_0|s).$$

Combining both cases, we reach:

$$|\widetilde{\alpha} - \alpha_0|_1^2 \le C(\mu_n^2 s^2 + |\widehat{\tau} - \tau_0|s + \frac{1}{\mu_n^2}|\widehat{\tau} - \tau_0|^2 s^2),$$

which gives the desired result since the first term $\mu_n^2 s^2$ dominates the other two terms.

C.11 Proof of Theorem 6.1

If $\delta_0 = 0$, τ_0 is non-identifiable. In this case, we decompose the excess risk in the following way:

$$
\begin{aligned}
R\left(\alpha, \tau\right) &= \mathbb{E}\left(\left[\rho\left(Y, X^T\beta\right) - \rho\left(Y, X^T\beta_0\right)\right] 1\left\{Q \leq \tau\right\}\right) \\
&\quad + \mathbb{E}\left(\left[\rho\left(Y, X^T\theta\right) - \rho\left(Y, X^T\beta_0\right)\right] 1\left\{Q > \tau\right\}\right).
\end{aligned}
\tag{C.47}
$$

We split the proof into three steps.

Step 1: For any $r > 0$, we have that w.p.a.1, $\breve{\beta} \in \tilde{\mathcal{B}}(\beta_0, r, \breve{\tau})$ and $\breve{\theta} \in \tilde{\mathcal{G}}(\beta_0, r, \breve{\tau})$.

Proof of Step 1. As in the proof of Step 1 in the proof of Theorem 5.2, Assumption B.3 (iv) implies that

$$
\mathbb{E}\left[(X^T(\beta - \beta_0))^2 1\{Q \leq \tau\}\right] \leq \frac{R(\alpha, \tau)^2}{(\eta^* r^*)^2} \vee \frac{R(\alpha, \tau)}{\eta^*}.
$$

For any $r > 0$, note that $R(\breve{\alpha}, \breve{\tau}) = o_P(1)$ implies that the event $R(\breve{\alpha}, \breve{\tau}) < r^2$ holds w.p.a.1. Therefore, we have shown that $\breve{\beta} \in \tilde{\mathcal{B}}(\beta_0, r, \breve{\tau})$. The other case can be proved similarly. ∎

Step 2 : Suppose that $\delta_0 = 0$. Then

$$
R\left(\breve{\alpha}, \breve{\tau}\right) + \frac{1}{2}\kappa_n \left|\breve{D}\left(\breve{\alpha} - \alpha_0\right)\right|_1 \leq 2\kappa_n \left|\breve{D}\left(\breve{\alpha} - \alpha_0\right)_J\right|_1 \quad \text{w.p.a.1.}
\tag{C.48}
$$

Proof. The proof of this step is similar to that of Step 3 in the proof of Theorem 5.2. Since $(\breve{\alpha}, \breve{\tau})$ minimizes the ℓ_1-penalized objective function in (2.2), we have that

$$
\frac{1}{n}\sum_{i=1}^{n} \rho(Y_i, X_i(\breve{\tau})^T\breve{\alpha}) + \kappa_n|\breve{D}\breve{\alpha}|_1 \leq \frac{1}{n}\sum_{i=1}^{n} \rho(Y_i, X_i(\breve{\tau})^T\alpha_0) + \kappa_n|\breve{D}\alpha_0|_1.
\tag{C.49}
$$

When $\delta_0 = 0$, $\rho(Y, X(\breve{\tau})^T\alpha_0) = \rho(Y, X(\tau_0)^T\alpha_0)$. Using this fact and (C.49), we obtain the

following inequality

$$R(\check{\alpha}, \check{\tau}) \leq [\nu_n(\alpha_0, \check{\tau}) - \nu_n(\check{\alpha}, \check{\tau})] + \kappa_n|\check{D}\alpha_0|_1 - \kappa_n|\check{D}\check{\alpha}|_1. \tag{C.50}$$

As in Step 3 in the proof of Theorem 5.2, we apply Lemma C.1 to $[\nu_n(\alpha_0, \check{\tau}) - \nu_n(\check{\alpha}, \check{\tau})]$ with a_n and b_n replaced by $a_n/2$ and $b_n/2$. Then we can rewrite the basic inequality in (C.50) by

$$\kappa_n \left|\check{D}\alpha_0\right|_1 \geq R(\check{\alpha}, \check{\tau}) + \kappa_n \left|\check{D}\check{\alpha}\right|_1 - \frac{1}{2}\kappa_n \left|\check{D}(\check{\alpha} - \alpha_0)\right|_1 \quad \text{w.p.a.1.}$$

Now adding $\kappa_n \left|\check{D}(\check{\alpha} - \alpha_0)\right|_1$ on both sides of the inequality above and using the fact that $|\alpha_{0j}|_1 - |\check{\alpha}_j|_1 + |(\check{\alpha}_j - \alpha_{0j})|_1 = 0$ for $j \notin J$, we have that w.p.a.1,

$$2\kappa_n \left|\check{D}(\check{\alpha} - \alpha_0)_J\right|_1 \geq R(\check{\alpha}, \check{\tau}) + \frac{1}{2}\kappa_n \left|\check{D}(\check{\alpha} - \alpha_0)\right|_1.$$

Therefore, we have obtained the desired result. ∎

Step 3 : Suppose that $\delta_0 = 0$. Then

$$R(\check{\alpha}, \check{\tau}) = O_P(\kappa_n^2 s) \quad \text{and} \quad |\check{\alpha} - \alpha_0| = O_P(\kappa_n s).$$

Proof. By Step 2,

$$4\left|\check{D}(\check{\alpha} - \alpha_0)_J\right|_1 \geq \left|\check{D}(\check{\alpha} - \alpha_0)\right|_1 = \left|\check{D}(\check{\alpha} - \alpha_0)_J\right|_1 + \left|\check{D}(\check{\alpha} - \alpha_0)_{J^c}\right|_1, \tag{C.51}$$

which enables us to apply the compatibility condition in Assumption 4.2.

Recall that $\|Z\|_2 = (EZ^2)^{1/2}$ for a random variable Z. Note that for $s = |J(\alpha_0)|_0$,

$$
\begin{aligned}
&R\left(\breve{\alpha}, \breve{\tau}\right) + \frac{1}{2}\kappa_n \left|\breve{D}\left(\breve{\alpha} - \alpha_0\right)\right|_1 \\
&\leq_{(1)} 2\kappa_n \left|\breve{D}\left(\breve{\alpha} - \alpha_0\right)_J\right|_1 \\
&\leq_{(2)} 2\kappa_n \bar{D} \left\|X(\breve{\tau})^T(\breve{\alpha} - \alpha_0)\right\|_2 \sqrt{s}/\phi \\
&\leq_{(3)} \frac{4\kappa_n^2 \bar{D}^2 s}{2\tilde{c}\phi^2} + \frac{\tilde{c}}{2} \left\|X(\breve{\tau})^T(\breve{\alpha} - \alpha_0)\right\|_2^2,
\end{aligned}
\tag{C.52}
$$

where (1) is from the basic inequality (C.48) in Step 2, (2) is by the compatibility condition (Assumption 4.2), and (3) is from the inequality that $uv \leq v^2/(2\tilde{c}) + \tilde{c}u^2/2$ for any $\tilde{c} > 0$.

Note that

$$
\begin{aligned}
&\left\|X(\tau)^T\alpha - X(\tau)^T\alpha_0\right\|_2^2 \\
&=_{(1)} \mathbb{E}\left[(X^T(\theta - \beta_0))^2 1\{Q > \tau\}\right] + \mathbb{E}\left[(X^T(\beta - \beta_0))^2 1\{Q \leq \tau\}\right] \\
&\leq_{(2)} (\eta^*)^{-1}\mathbb{E}\left[\left(\rho\left(Y, X^T\theta\right) - \rho\left(Y, X^T\beta_0\right)\right) 1\{Q > \tau\}\right] \\
&\quad + (\eta^*)^{-1}\mathbb{E}\left[\left(\rho\left(Y, X^T\beta\right) - \rho\left(Y, X^T\beta_0\right)\right) 1\{Q \leq \tau\}\right] \\
&\leq_{(3)} (\eta^*)^{-1}R(\alpha, \tau),
\end{aligned}
$$

where (1) is simply an identity, (2) from Assumption B.3 (iv) , and (3) is due to (C.47). Hence, (C.52) with $\tilde{c} = \eta^*$ implies that

$$
R\left(\breve{\alpha}, \breve{\tau}\right) + \kappa_n \left|\breve{D}\left(\breve{\alpha} - \alpha_0\right)\right|_1 \leq \frac{4\kappa_n^2 \bar{D}^2 s}{\eta^*\phi^2}.
\tag{C.53}
$$

Therefore, $R\left(\breve{\alpha}, \breve{\tau}\right) = O_P(\kappa_n^2 s)$. Also, $|\breve{\alpha} - \alpha_0| = O_P\left(\kappa_n s\right)$ since $D(\breve{\tau}) \geq \underline{D}$ w.p.a.1 by Assumption 3.1 (iv). ∎

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C.12   Proof of Theorem 6.2

When τ_0 is not identifiable ($\delta_0 = 0$), $\widehat{\tau}$ obtained in the second-step estimation can be any value in $\mathcal{T}$. Note that Lemmas C.4 and C.5 are stated and proved for this case as well. Similar to the proof of Theorem 5.5, by Lemma B.1, in quantile regression models, $M_n = Cs^{1/2}$ for some $C > 0$. Hence all the required conditions in Lemmas C.4 and C.5 are satisfied by the conditions imposed in Theorem 6.2. Then by Lemmas C.4 and C.5, w.p.a.1, for any $\alpha = (\alpha_J, \alpha_{J^c}) \in \mathcal{H}$,

$$\widetilde{S}_n(\bar{\alpha}_J, 0) = \bar{Q}_n(\bar{\alpha}_J) \leq \bar{Q}_n(\alpha_J) = \widetilde{S}_n(\alpha_J, 0) \leq \widetilde{S}_n(\alpha).$$

Hence $(\bar{\alpha}_J, 0)$ is a local minimizer of $\widetilde{S}_n$, which is also a global minimizer due to the convexity. This implies that w.p.a.1, $\widetilde{\alpha} = (\widetilde{\alpha}_J, \widetilde{\alpha}_{J^c})$ satisfies: $\widetilde{\alpha}_{J^c} = 0$, and $\widetilde{\alpha}_J = \bar{\alpha}_J$, so

$$|\widetilde{\alpha}_J - \alpha_{0J}|_2 = O_P\left(\sqrt{\frac{s \log s}{n}}\right), \quad |\widetilde{\alpha}_J - \alpha_{0J}|_1 = O_P\left(s\sqrt{\frac{\log s}{n}}\right).$$

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