# Efficiency in Search and Matching Models: A Generalized Hosios Condition*

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**Abstract**

This paper generalizes the well-known Hosios (1990) efficiency condition to dynamic search and matching environments where the expected match output depends on the market tightness. Such environments give rise to a novel externality – the *output externality* – which may be either positive or negative. The generalized Hosios condition is simple: entry is constrained efficient when buyers' surplus share equals the matching elasticity plus the *surplus elasticity* (i.e. the elasticity of the expected joint match surplus with respect to buyers). This intuitive condition captures both the standard externalities generated by the frictional matching process and the output externality. *JEL Codes:* C78, D83, E24, J64

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# 1 Introduction

The well-known Hosios rule specifies a precise condition under which markets featuring search and matching frictions are *constrained efficient*. The original version of the rule introduced in Hosios (1990) states that an equilibrium allocation is constrained efficient only when buyers' share of the total joint surplus equals the elasticity of the *matching* function with respect to buyers. This condition has proven to be widely applicable across a broad range of search-theoretic models. However, it does not apply in settings where the expected match output is *endogenous* in the sense that it depends on the market tightness.

This paper generalizes the Hosios rule to environments where the expected match output is endogenous.[1] Endogeneity of the expected match output can arise naturally in markets where either buyers (or sellers) are *heterogeneous* prior to matching.[2] We identify two distinct channels. With one-on-one or bilateral meetings, the expected match output may depend on market tightness when there is a participation decision by heterogeneous buyers (or sellers) and the market composition is endogenous. We call this the *composition channel*. With many-on-one or multilateral meetings, there is an additional channel: the expected match output may depend on market tightness when sellers face a choice regarding buyers.[3] We call this the *selection channel*.

When the standard Hosios condition holds, decentralized markets internalize the search externalities that arise through the frictional matching process. However, when the expected match output depends on the market tightness, a novel externality arises. We call this the *output externality* and it can arise through either the composition or the selection channel. Depending on the specific environment, the expected match output may be either increasing or decreasing in the buyer/seller ratio and therefore the externality may be either

[1] We use the term "match output" because our examples focus on labor markets, but the term *output* can be interpreted more broadly to cover any trade or productive activity.

[2] We focus on one-sided heterogeneity and do not consider search and matching environments with two-sided heterogeneity and assortative matching such as Shimer and Smith (2000, 2001), Shi (2001), and Eeckhout and Kircher (2010a).

[3] With bilateral meetings, the heterogeneity must be *ex ante*. However, with multilateral meetings, buyers and sellers need not be ex ante heterogeneous: they can be identical prior to *meetings* provided there is some heterogeneity prior to *matching*.

positive or negative. The standard Hosios condition does not internalize this new externality and it may therefore result in either over-entry or under-entry relative to the social optimum. We provide examples of both possibilities.

Consider an environment with buyer entry. An equilibrium allocation is constrained efficient when buyers are paid their marginal contribution to the social surplus. If the expected output per match is exogenous, buyers need only be paid for their effect on the total *number* of matches and the standard Hosios condition applies: entry is constrained efficient only when buyers' surplus share equals the matching elasticity. If the expected match output is endogenous, however, buyers must be compensated for their effect on both the total number of matches and the expected *value* of the joint surplus created by each match.

Our main result is the *generalized Hosios condition*: entry is constrained efficient only when buyers' surplus share equals the matching elasticity plus the *surplus elasticity* (i.e. the elasticity of the expected match surplus with respect to buyers). When this condition holds, both the standard search externalities and the output externality are fully internalized by a decentralized market. Like the original version, the generalized Hosios condition is highly intuitive. Moreover, the simple rule that arises in a static environment carries over directly to dynamic settings with enduring matches (as found in the labor market).

In a way similar to the standard Hosios condition, the generalized Hosios condition unifies a number of seemingly unrelated efficiency results throughout the search and matching literature. As in Hosios (1990), our main contribution is to offer a simple but general approach to determining efficiency and to show that many existing results can be understood through this lens. To the best of our knowledge, this general condition is new to the literature.

As a guiding principle, Hosios (1990) suggests that when we want to determine the efficiency properties of a particular model, the question we need to ask is "whether the unattached agents who participate in the corresponding matching process receive more or less than their social marginal product" (p. 296). This guiding principle remains true. However, Hosios states that *all* we need to do to answer this question is determine the equilibrium surplus-sharing rule, and the matching technology, and then simply apply what is now known as the “Hosios rule”. Our paper shows that when the expected match output is endogenous, this rule must be generalized. In addition to considering the surplus shares and the matching technology, we must also consider the *output technology* which determines how changes in the market tightness affect the expected match output. That is, we need the generalized Hosios condition.

**Outline.** This paper proceeds as follows. In Section 2, we present our key result: the generalized Hosios condition. We first discuss a static economy and then derive the main result for a dynamic economy with enduring matches. Section 3 provides a number of examples that apply the generalized Hosios condition. We discuss the relevant literature throughout the paper.

# 2 Generalized Hosios Condition

To build intuition, we first consider a static environment and then show that the simple, intuitive efficiency condition that applies to the static economy extends directly to a dynamic economy with enduring matches.

## 2.1 Static economy

There is a measure $B$ of risk-neutral buyers and measure $S$ of risk-neutral sellers. The market tightness, or buyer/seller ratio, is denoted by $\theta \equiv B/S$. Buyers and sellers are matched according to a constant-returns-to-scale matching function. The matching probabilities for sellers and buyers are denoted respectively by $m(\theta)$ and $m(\theta)/\theta$. We call the function $m(.)$ the *matching technology* and it satisfies the following standard properties.

**Assumption 1.** *The function* $m(.)$ *has the following properties: (i)* $m'(\theta) > 0$ *and* $m''(\theta) < 0$ *for all* $\theta \in \mathbb{R}_+$*, (ii)* $\lim_{\theta\to 0} m(\theta) = 0$*, (iii)* $\lim_{\theta\to 0} m'(\theta) = 1$*, (iv)* $\lim_{\theta\to\infty} m(\theta) = 1$*, (v)* $\lim_{\theta\to\infty} m'(\theta) = 0$*; (vi)* $m(\theta)/\theta$ *is strictly decreasing in* $\theta$ *for all* $\theta \in \mathbb{R}_+$*; and (vii)* $\eta_m(\theta)$ *is weakly decreasing in* $\theta$ *for all* $\theta \in \mathbb{R}_+$*.*

Let $p(\theta)$ denote the *expected match output*. When the expected match output is exogenous, we have $p(\theta) = p$ for all $\theta \in \mathbb{R}_+$. In general, the expected match output $p(\theta)$ is *endogenous*: it depends directly on the market tightness. We call the function $p(.)$ the *output technology.*

There is free entry of buyers, each paying a cost $c > 0$ to enter, and $b \geq 0$ is the outside option of sellers.[4] For simplicity, we assume that $p(\theta) > b$ for all $\theta \in \mathbb{R}_+$. The expected joint match surplus created by each match is $s(\theta) \equiv p(\theta) - b$. The total expected joint surplus per seller is $x(\theta) \equiv m(\theta)s(\theta)$.

**Assumption 2.** *The function $x(.)$ defined by $x(\theta) \equiv m(\theta)s(\theta)$ has the following properties:* $\lim_{\theta\to 0} x'(\theta) > c$, $\lim_{\theta\to\infty} x'(\theta) \leq 0$, *and* $x''(\theta) < 0$ *for all* $\theta \in \mathbb{R}_+$.

Suppose the social planner is constrained by both the matching technology $m(.)$ *and* the output technology $p(.)$. That is, the social planner takes both functions $m(.)$ and $p(.)$ as given. Let $\Omega(\theta)$ denote the social surplus per seller:

$$\Omega(\theta) = x(\theta) + b - c\theta. \tag{1}$$

The social planner chooses a market tightness $\theta$ that maximizes (1). The first-order condition for the social planner's problem is

$$\Omega'(\theta) = x'(\theta) - c = 0. \tag{2}$$

Applying the intermediate value theorem, Assumption 2 ensures the existence of a unique social optimum $\theta^P > 0$.

In Section 3, we present some examples of decentralized markets. In this section, we simply denote the equilibrium market tightness by $\theta^*$.[5] If there exists a unique social optimum $\theta^P$, we say that a decentralized equilibrium allocation is *constrained efficient* if and only if $\theta^* = \theta^P$. Here, "constrained" means that the social planner is constrained both in terms of the matching technology and the output technology, which are taken as given.

Let $\eta_x(\theta) \equiv x'(\theta)\theta/x(\theta)$, the elasticity of the total expected joint surplus (per seller), with respect to buyers. Rearranging the first-order condition (2),

[4] We focus on buyer entry, but similar efficiency results and an analogous generalized Hosios condition hold when there is *seller* entry instead.

[5] In the examples we consider in Section 3, the equilibrium market tightness $\theta^*$ is unique.

the social planner's solution $\theta^P$ satisfies the following:

$$\eta_x(\theta) = \frac{c\theta}{x(\theta)}. \tag{3}$$

Since there is free entry of buyers, the expected payoff per buyer equals the cost of entry $c$ and the term $c\theta/x(\theta)$ equals buyers' *surplus share.* Condition (3) says that the social planner chooses the market tightness $\theta^P$ that equates buyers' surplus share and the elasticity of the total expected joint surplus (per seller) with respect to buyers.

Now let $\eta_m(\theta) \equiv m'(\theta)\theta/m(\theta)$, the elasticity of the matching probability $m(\theta)$ with respect to $\theta$.[6] We call this the *matching elasticity.* Let $\eta_s(\theta) \equiv s'(\theta)\theta/s(\theta)$, the elasticity of the expected joint match surplus, $s(\theta)$. We call this the *surplus elasticity.* Since $x(\theta) \equiv m(\theta)s(\theta)$, we have $\eta_x(\theta) = \eta_m(\theta) + \eta_s(\theta)$. Substituting into (3), the social optimum $\theta^P$ is the unique solution to

$$\underbrace{\eta_m(\theta)}_{\text{matching elasticity}} + \underbrace{\eta_s(\theta)}_{\text{surplus elasticity}} = \underbrace{\frac{c\theta}{x(\theta)}}_{\text{buyers' surplus share}}. \tag{4}$$

Since $\theta^P$ is unique, we have constrained efficiency if and only if $\theta^*$ also satisfies condition (4). We call this the *generalized Hosios condition* because it generalizes the standard Hosios condition to static environments with both matching frictions and an expected match output that depends directly on the market tightness. When the expected match output is *exogenous*, $\eta_s(\theta^*) = 0$ and we recover the standard Hosios condition: the matching elasticity with respect to buyers must equal their surplus share. In general, if the expected match surplus depends on the market tightness, buyers' surplus share must equal the *matching elasticity* plus the *surplus elasticity.*

**Discussion.** In search and matching models with free entry, there are two standard externalities related to the frictional matching process: the congestion and thick market externalities. The former is a negative externality that

[6] Note that $\eta_m(\theta) < 1$ follows from our assumption that $m(\theta)/\theta$ is strictly decreasing.

arises because a higher buyer/seller ratio reduces the matching probability of each buyer. The latter is a positive externality that arises because a higher buyer/seller ratio increases the matching probability of each seller. In general, these search externalities are fully captured by the standard Hosios condition through the matching elasticity.

In environments where the expected match output depends on market tightness, a novel externality arises. Depending on the specific environment, a higher buyer/seller ratio may either increase or decrease the expected match output. We call this the *output externality* and it may be either positive or negative. Under the standard Hosios condition, buyers' entry decisions fail to internalize the output externality and entry is not constrained efficient. To ensure that entry is efficient, we need the generalized Hosios condition. When this condition is satisfied, buyers' entry decisions internalize both the search externalities *and* the output externality. The standard externalities are captured by the matching elasticity, while the output externality is reflected in the surplus elasticity.

## 2.2 Dynamic economy

Consider a continuous-time dynamic environment that extends the above setting in a straightforward manner. In period $t$, there is a measure $v_t$ of risk-neutral buyers and measure one of risk-neutral sellers. There is a measure $u_t$ of unmatched sellers in period $t$ and the market tightness is defined by $\theta_t \equiv v_t/u_t$. There is free entry of buyers who pay a cost $c > 0$ each period.

The matching probabilities for sellers and buyers respectively are $m(\theta_t)$ and $m(\theta_t)/\theta_t$ where $m(.)$ satisfies Assumption 1. The expected match output for a match *created* in period $t$ is $p(\theta_t)$. The average match output across *all* active matches during period $t$ is $p_t$. The flow payoff for unmatched sellers is $b \geq 0$ is where $p(\theta_t) > b$ for all $\theta_t \in \mathbb{R}_+$. The expected flow value of the transfer paid to sellers by buyers for matches created in period $t$ is $w(\theta_t)$. At the start of each period, buyer-seller matches are destroyed at an exogenous rate $\delta \in (0, 1]$. Future payoffs are discounted at a rate $r > 0$.

Both unemployment $u_t$ and average match output $p_t$ follow laws of motion. For $u_t$, it is standard. To understand the law of motion for $p_t$, it is easier to think in discrete time. The average match output $p_{t+1}$ at time $t+1$ is a weighted average of the expected match output $p(\theta_t)$ for newly created matches and the average match output $p_t$ at time $t$. The weight on $p(\theta_t)$ is the measure of new matches created in period $t$, divided by $1-u_{t+1}$, the measure of active matches in period $t+1$. The weight on $p_t$ is the measure of old matches that survive match destruction, divided by $1-u_{t+1}$. In the proof of Proposition 1 found in the Appendix, we derive the continuous time law of motion $\dot{p}_t$.

Now let $\Omega$ denote the social surplus, given by

$$\Omega = \int_0^\infty e^{-rt}((1-u_t)p_t + bu_t - c\theta_t u_t)dt. \tag{5}$$

Given initial conditions $u_0$ and $p_0$, the social planner chooses $\theta_t$ for all $t \in \mathbb{R}_+$ to maximize (5) subject to the following constraints:

$$\dot{u}_t = \delta(1-u_t) - m(\theta_t)u_t \tag{6}$$

and

$$\dot{p}_t = \frac{m(\theta_t)u_t(p(\theta_t)-p_t)}{1-u_t}. \tag{7}$$

In the proof of Proposition 1, we solve the current value Hamiltonian for this problem. We focus on steady state solutions where $\dot{u}_t = \dot{p}_t = 0$ and $\dot{\theta}_t = 0$.

Before presenting Proposition 1, we first determine the steady state expected joint match surplus, $s(\theta)$. Let $V_S$ and $V_B$ denote the steady state asset values for matched sellers and buyers respectively, and let $U_S$ and $U_B$ denote the steady state asset values for unmatched sellers and buyers respectively. In steady state, the expected joint match surplus is

$$s(\theta) \equiv V_B + V_S - U_B - U_S. \tag{8}$$

Using the Bellman equations, and the fact that $U_B = 0$ with free entry, Lemma 1 provides a useful expression for the expected match surplus $s(\theta)$ in the dynamic economy.[7] We can also define $x(\theta) \equiv m(\theta)s(\theta)$, the total expected joint surplus per unmatched seller. We assume $x(.)$ satisfies Assumption 2.

**Lemma 1.** *In steady state, the expected joint match surplus $s(\theta)$ is*

$$s(\theta) = \frac{p(\theta) - b + c\theta}{r + \delta + m(\theta)}. \tag{9}$$

*Proof.* See Appendix. □

We are now in a position to present a necessary condition for efficiency.

**Proposition 1.** *Any steady state social optimum $\theta^P$ must satisfy*

$$\eta_m(\theta) + \frac{p'(\theta)\theta}{(r+\delta)s(\theta)} = \frac{c\theta}{x(\theta)}. \tag{10}$$

*Proof.* See Appendix. □

In its current form, it is unclear how to reconcile condition (10) with the intuitive condition (4) that we found in the static economy. In fact, condition (10) turns out to be *equivalent* to the generalized Hosios condition (4).

Using expression (9) for $s(\theta)$, we can write $\eta_s(\theta) \equiv s'(\theta)\theta/s(\theta)$ as the elasticity of the numerator minus the elasticity of the denominator:

$$\eta_s(\theta) = \frac{(p'(\theta) + c)\theta}{p(\theta) - b + c\theta} - \frac{m'(\theta)\theta}{r + \delta + m(\theta)}. \tag{11}$$

Using (11) and (9), it can be shown that condition (10) is equivalent to

$$\eta_m(\theta) + \eta_s(\theta) = \frac{c\theta}{x(\theta)}. \tag{12}$$

Further, using $x(\theta) = m(\theta)s(\theta)$, condition (12) is also equivalent to

$$\eta_x(\theta) = \frac{c\theta}{x(\theta)}, \tag{13}$$

[7] Note that in the static economy, $V_B + V_S = p(\theta)$ and $U_S = b$, so we have $s(\theta) = p(\theta) - b$.

and rearranging, using $\eta_x(\theta) \equiv x'(\theta)\theta/x(\theta)$, condition (13) is equivalent to

$$x'(\theta) = c, \tag{14}$$

which is exactly the first-order condition for the static economy. As before, it follows immediately from Assumption 2 that there exists a unique $\theta^P$ that satisfies the necessary condition (14). In the Appendix, we use Arrow's sufficiency theorem to prove that it is indeed a global maximum for $\Omega$.

**Lemma 2.** *There exists a unique social optimum $\theta^P > 0$.*

*Proof.* See Appendix. □

Proposition 2 generalizes the standard Hosios condition to dynamic environments with both matching frictions and an endogenous match output that depends on the market tightness. To achieve constrained efficiency, buyers' surplus share must equal the *matching elasticity* plus the *surplus elasticity.* When this condition holds, buyers' entry decisions fully internalize both the standard externalities due to the matching process and the output externality. The matching elasticity captures the standard matching externalities, while the output externality is reflected in the surplus elasticity.

**Proposition 2** (**Generalized Hosios Condition**). *A steady state equilibrium allocation is constrained efficient if and only if*

$$\underbrace{\eta_m(\theta^*)}_{\text{matching elasticity}} + \underbrace{\eta_s(\theta^*)}_{\text{surplus elasticity}} = \underbrace{\frac{c\theta^*}{x(\theta^*)}}_{\text{buyers' surplus share}}. \tag{15}$$

*Proof.* Using Proposition 1, together with Lemma 2, we know that there exists a unique social optimum $\theta^P > 0$ that satisfies condition (12) and therefore we have constrained efficiency ($\theta^* = \theta^P$) if and only if $\theta^*$ also satisfies (12). □

Depending on the specific environment, the surplus elasticity may be either positive or negative. This means that simply applying the standard Hosios condition may result in either over-entry or under-entry of buyers relative to the social optimum. Corollary 1 tells us that the direction of the inefficiency depends *only* on the output technology $p(\theta)$. In particular, the direction of the inefficiency depends on whether the expected match output $p(\theta)$ is increasing or decreasing in the buyer/seller ratio at the equilibrium $\theta^*$.

**Corollary 1.** *There is under-entry (over-entry) of buyers under the standard Hosios condition if and only if* $p'(\theta^*) > (<) \; 0$.

*Proof.* See Appendix. □

When $p'(\theta^*) > 0$, the output externality arising from buyer entry is *positive* and the standard Hosios condition results in under-entry. Alternatively, if $p'(\theta^*) < 0$, the output externality is *negative* and the standard Hosios condition results in over-entry. If $p'(\theta^*) = 0$, there is no output externality and entry is constrained efficient under the standard Hosios condition.

# 3 Examples

In this section, we discuss a number of examples of different search and matching environments to illustrate the usefulness of the generalized Hosios condition. We focus mainly on labor market environments in which sellers and buyers are unemployed workers and firms (or vacancies), but the results hold more generally for any kinds of buyers and sellers. For simplicity, we focus on static environments for these examples but the efficiency results extend to dynamic environments as shown in Section 2.

## 3.1 Nash bargaining with endogenous match output

Consider a static Diamond-Mortensen-Pissarides (DMP) style environment where meetings are bilateral and wages are determined by generalized Nash bargaining.[8] The measure of vacancies or firms is $V$, the measure of unemployed workers is $U$, and the labor market tightness is $\theta \equiv V/U$. While the environment is otherwise standard, the expected output per match – or average labor productivity – is *endogenous* in the sense that it depends directly on the market

[8]The classic references are Mortensen and Pissarides (1994) and Pissarides (2000).

tightness $\theta$. For now, we simply *assume* that output per match $p(\theta)$ is a function of market tightness. In the following sections, we will see how dependence of expected match output on market tightness can arise naturally.

There is free entry of firms or vacancies at a cost $c > 0$. The matching probabilities for workers and firms are $m(\theta)$ and $m(\theta)/\theta$ respectively where $m(.)$ satisfies Assumption 1. Workers' bargaining parameter is $\beta$ and the value of non-market activity is $b$ where $p(\theta) > b$ for all $\theta \in \mathbb{R}_+$.

The expected match surplus is $s(\theta) = p(\theta) - b$ and $x(\theta) = m(\theta)s(\theta)$. If $\eta_x(\theta) < 1$ for all $\theta \in \mathbb{R}_+$,[9] $\lim_{\theta\to\infty} x(\theta)/\theta = 0$, and $c < (1-\beta)\lim_{\theta\to 0} x(\theta)/\theta$, there exists a unique equilibrium $\theta^* > 0$ that satisfies

$$\frac{m(\theta)}{\theta}(1-\beta)(p(\theta)-b) = c \tag{16}$$

or equivalently, the equilibrium $\theta^* > 0$ satisfies

$$1-\beta = \frac{c\theta}{x(\theta)}. \tag{17}$$

If Assumption 2 is satisfied, there exists a unique social optimum $\theta^P > 0$. Applying the generalized Hosios condition in Proposition 2, and using (17), the economy is constrained efficient if and only if $\theta^*$ satisfies

$$\underbrace{\eta_m(\theta)}_{\text{matching elasticity}} + \underbrace{\eta_s(\theta)}_{\text{surplus elasticity}} = \underbrace{1-\beta}_{\text{firms' bargaining power}}. \tag{18}$$

Intuitively, the economy is efficient only when firms are paid for their contribution to both the *number* of matches and the *value* of the expected match surplus. Corollary 1 says that the standard Hosios condition may result in either *under-entry* or *over-entry* of firms, depending on whether the expected match output is increasing or decreasing in the market tightness, i.e. depending on whether the output externality is positive or negative, i.e. $p'(\theta^*) > 0$ or $p'(\theta^*) < 0$.

[9] Note that $\eta_x(\theta) < 1$ if and only if $x(\theta)/\theta$ is strictly decreasing.

**Example 3.1.1**

Consider the special case where the expected match output is *exogenous*, i.e. $p(\theta) = p$ for all $\theta \in \mathbb{R}_+$. Output per match may be either constant or stochastic provided that the *expected* match output does not depend on the market tightness $\theta$.[10] In this case, we recover a standard DMP style model. According to (18), we have constrained efficiency only when the equilibrium $\theta^*$ satisfies the following well-known condition:

$$\eta_m(\theta) = 1 - \beta. \tag{19}$$

Clearly, the standard Hosios condition is a special case of (18). When the expected match output is exogenous, we have constrained efficiency only when the matching elasticity $\eta_m(\theta)$ equals firms' bargaining power at the equilibrium $\theta^*$. As is well-known, constrained efficiency does not generally obtain. For example, if $\eta_m(\theta) = \eta$, entry is efficient only in the knife-edge case where $\eta = 1 - \beta$. Often, this condition is simply *imposed* in DMP style search models.

**Example 3.1.2**

Suppose that $p(\theta) = A(\theta)y(k)$ where $A(\theta) = \theta^\gamma$ and $\gamma \in [0, 1)$, $b = 0$, and $\eta_m(\theta) = \eta$ where $\eta + \gamma < 1$. All firms are endowed with $k$ units of capital and $y(k)$ is a neoclassical production function.[11] We can think of $A(\theta)$ as total factor productivity (TFP).[12] Clearly, the expected match output $p(\theta)$ is increasing in the market tightness and the surplus elasticity is $\eta_s(\theta) = \gamma$.

In this example, the generalized Hosios condition (18) is particularly simple. We have constrained efficiency if and only if

$$\eta + \gamma = 1 - \beta. \tag{20}$$

[10] For example, we could have match-specific productivities $y$ drawn from an exogenous distribution $F$ with $E_F(y) = p$.

[11] In Section 3.6 we develop a detailed example with ex ante investment in capital.

[12] Lagos (2006) provides a model of TFP in a DMP style environment with Nash bargaining. In that paper, TFP is endogenous and it depends on the market tightness and other labor market variables. We present a highly stylized example here and simply assume $A(\theta)$.

In general, there is no compelling reason why this condition would hold since both the matching elasticity $\eta$ and the parameter $\gamma$ governing the output technology are independent of both the Nash bargaining parameter $\beta$ and each other. To ensure constrained efficiency, we must impose the generalized Hosios condition by setting firms' bargaining power, $1-\beta$, equal to $\eta+\gamma$, the matching elasticity plus the surplus elasticity.

If we impose the standard Hosios condition, this fails to capture the output externality. To see this, suppose that the standard Hosios condition is true, i.e. $\eta = 1 - \beta$. Since firms are paid only for their role in creating new matches, not for the positive effect of greater firm entry on the expected match output, there will be *under-entry* of firms relative to the social optimum. Equivalently, the unemployment rate will be inefficiently high under the standard Hosios condition. Since $p'(\theta^*) > 0$, this is consistent with Corollary 1.

## 3.2 Nash bargaining with ex ante firm heterogeneity

When there is *ex ante* heterogeneity among buyers or sellers, dependence of the expected match output on market tightness can arise naturally through market composition. If the market tightness influences the individual entry decisions of buyers or sellers that are ex ante heterogeneous with respect to characteristics that affect match output, then average output per match will depend on market tightness. We call this the *composition channel.*

Albrecht, Navarro, and Vroman (2010) consider an environment where workers are ex ante heterogeneous with respect to their market productivity and there is both firm entry and a labor force participation decision.[13] The authors show that such an environment can violate the standard Hosios rule: when workers' bargaining parameter satisfies the standard Hosios condition, there is *over-entry* of firms relative to the social optimum. To illustrate the use of the generalized Hosios condition, we consider a related but simpler environment that features ex ante *firm* heterogeneity instead of worker heterogeneity.

Suppose there is a measure $U$ of unemployed workers and a measure $M$ of

[13] Related literature following Albrecht et al. (2010) includes Gavrel (2011), Charlot, Malherbet, and Ulus (2013), and Masters (2015). See also Albrecht, Navarro, and Vroman (2009).

firms that may choose to search. Firms' productivities $y$ are distributed according to a twice differentiable distribution with cdf $F$ and density $f$ where $F(0) = 0$ and $f(y) > 0$ for all $y \in [0, 1]$. Firms learn their own productivity before deciding whether to pay the entry cost $c > 0$ and search. Wages are determined by generalized Nash bargaining where workers' bargaining parameter is $\beta$ and the value of non-market activity is zero. We assume that $c < 1 - \beta$.

Let $V$ be the measure of *searching* firms and define $\theta = V/U$. Meetings are bilateral and the probabilities of matching for workers and firms are $m(\theta)$ and $m(\theta)/\theta$ respectively. A firm with productivity $y$ chooses to pay the cost $c$ to search for a worker if and only if

$$\frac{m(\theta)}{\theta}(1-\beta)y > c \tag{21}$$

and therefore the cut-off productivity for firm entry is

$$y^* = \frac{c\theta}{(1-\beta)m(\theta)} \tag{22}$$

and expected output per match is $p(\theta) = E(y|y \geq y^*)$, which is given by

$$p(\theta) = \int_{y^*}^{1} \frac{yf(y)}{1 - F(y^*)} dy. \tag{23}$$

The cut-off productivity $y^*$ is increasing in $\theta$ since $m(\theta)/\theta$ is decreasing. This is intuitive: as the market tightness increases, the probability of finding a worker is lower so only high productivity firms choose to pay the cost $c$ and search. At the same time, the average match output $p(\theta)$ is increasing in the cut-off productivity $y^*$ and therefore $p'(\theta) > 0$ for all $\theta \in \mathbb{R}_+$.

The equilibrium $\theta^*$ satisfies

$$\theta = (1 - F(y^*))\frac{M}{U} \tag{24}$$

where $y^*$ is given by (22). Defining $R(\theta) \equiv 1 - F(y^*)$, the proportion of firms that choose to search, the equilibrium condition (24) is equivalent to

$$\frac{R(\theta)}{\theta} = \frac{U}{M}. \tag{25}$$

Using (22) and Assumption 1, we have $\lim_{\theta\to 0} R(\theta)/\theta = \infty$ and $\lim_{\theta\to\infty} R(\theta)/\theta = 0$. Also, $R'(\theta) < 0$ and therefore there exists a unique equilibrium $\theta^* > 0$.

The expected match surplus is $s(\theta) = p(\theta)$ since $b = 0$, and $x(\theta) = m(\theta)s(\theta)$. If Assumption 2 is satisfied, there exists a unique social optimum $\theta^P$ and we can apply the generalized Hosios condition in Proposition 2. We have constrained efficiency if and only if $\theta^*$ satisfies

$$\underbrace{\eta_m(\theta)}_{\text{matching elasticity}} + \underbrace{\eta_s(\theta)}_{\text{surplus elasticity}} = \frac{c\theta}{\underbrace{x(\theta)}_{\text{firms' surplus share}}}. \tag{26}$$

Unlike the previous example, firms' surplus share $c\theta/x(\theta)$ does not equal $1-\beta$ here. Instead, using (22), the right-hand side of (26) equals $(1-\beta)y^*/p(\theta)$.

In this environment, the *composition channel* gives rise to endogenous match output that depends on the market tightness. The threshold $y^*$ is increasing in $\theta$, leading to a positive output externality from firm entry, i.e. $p'(\theta) > 0$. Since $p'(\theta^*) > 0$, Corollary 1 implies that there is under-entry of firms under the standard Hosios condition. While it has been known at least since Shimer and Smith (2001) that the standard Hosios condition does not apply in environments with *ex ante* heterogeneity, Proposition 2 provides us with a generalized version of the Hosios condition that does apply in such environments.

## 3.3 Competitive search with endogenous match output

Unlike DMP style models with generalized Nash bargaining, models with directed or competitive search are typically constrained efficient (Shimer (1996); Moen (1997)). In such models, firms internalize the standard externalities arising from the matching process and the standard Hosios condition typically holds endogenously.[14] Early papers on directed or competitive search include Montgomery (1991), Peters (1991), Acemoglu and Shimer (1999b,a), Julien, Kennes, and King (2000), Burdett, Shi, and Wright (2001), Shi (2001, 2002).[15]

Consider a simple competitive search model in the spirit of Moen (1997). There is a continuum of submarkets indexed by $i \in [0, 1]$ and free entry of firms at a cost $c > 0$. Workers in submarket $i$ post the same wage $w_i$ and face the same market tightness $\theta_i$, the ratio of firms to workers in that submarket. Firms' search is *directed* by observing the posted wages and deciding which submarkets to enter. Within each submarket $i$, the matching probabilities for workers and firms are $m(\theta_i)$ and $m(\theta_i)/\theta_i$ respectively, where $m(.)$ satisfies Assumption 1.

Suppose that the expected match output $p(\theta_i)$ in submarket $i$ depends on the market tightness $\theta_i$ in that submarket.[16] The value of non-market activity is $b$ where $p(\theta_i) > b$ for all $\theta_i \in \mathbb{R}_+$. The expected match surplus in submarket $i$ is $s(\theta_i) = p(\theta_i) - b$ and $x(\theta_i) = m(\theta_i)s(\theta_i)$ where $x(.)$ satisfies Assumption 2.

For simplicity, we focus on symmetric equilibria in which firms are indifferent across submarkets and all workers post the same wage.

The expected payoff for firms operating in submarket $i$ with wage $w_i$ and market tightness $\theta_i$ is given by

$$\Pi(\theta_i, w_i) = \frac{m(\theta_i)}{\theta_i}(p(\theta_i) - w_i), \tag{27}$$

and the expected payoff for workers in submarket $i$ with market tightness $\theta_i$ is

$$V(\theta_i, w_i) = m(\theta_i)w_i + (1 - m(\theta_i))b. \tag{28}$$

[14] Guerrieri (2008) develops a model of competitive search with informational asymmetries and identifies a new externality that means the decentralized equilibrium is not always constrained efficient. Guerrieri, Shimer, and Wright (2010), Moen and Rosen (2011), and Julien and Roger (2015) also consider competitive search environments with informational frictions.

[15] Hosios (1990) also considers an example based on Peters (1984) that is similar to directed search and is constrained efficient.

[16] In Section 3.4, we will see how an output technology $p(\theta)$ can arise endogenously in an environment with multilateral meetings. For now, we simply assume the function $p(.)$.

Workers in submarket $i$ choose a wage $w_i^*$ and market tightness $\theta_i^*$ that solve

$$\max_{w_i, \theta_i \in \mathbb{R}_+} (m(\theta_i) w_i + (1 - m(\theta_i)) b) \tag{29}$$

subject to $\Pi(\theta_i, w_i) \leq c$ and $\theta_i \geq 0$ with complementary slackness. To induce participation by firms in submarket $i$, i.e. $\theta_i > 0$, the constraint $\Pi(\theta_i, w_i) \leq c$ is binding and we therefore have

$$\frac{m(\theta_i)}{\theta_i}(p(\theta_i) - w_i) = c. \tag{30}$$

Solving for $w_i$ using (30), we obtain

$$w(\theta_i) = p(\theta_i) - \frac{c\theta_i}{m(\theta_i)}. \tag{31}$$

Choosing a wage $w_i^*$ is thus equivalent to choosing a market tightness $\theta_i^*$ where

$$\theta_i^* = \arg\max_{\theta_i \in \mathbb{R}_+} (m(\theta_i) w(\theta_i) + (1 - m(\theta_i)) b) \tag{32}$$

and using (31), this is equivalent to

$$\theta_i^* = \arg\max_{\theta_i \in \mathbb{R}_+} (m(\theta_i) p(\theta_i) - c\theta_i + (1 - m(\theta_i)) b) \,. \tag{33}$$

Using $x(\theta_i) = m(\theta_i)(p(\theta_i) - b)$, (33) is equivalent to

$$\theta_i^* = \arg\max_{\theta_i \in \mathbb{R}_+} (x(\theta_i) + b - c\theta_i) \tag{34}$$

and $\theta_i^*$ is the unique solution to the first-order condition

$$x'(\theta_i) = c, \tag{35}$$

which can be rearranged to give

$$\underbrace{\eta_m(\theta_i)}_{\text{matching elasticity}} + \underbrace{\eta_s(\theta_i)}_{\text{surplus elasticity}} = \underbrace{\frac{c\theta_i}{x(\theta_i)}}_{\text{firms' surplus share}} . \tag{36}$$

That is, the generalized Hosios condition holds *within* each submarket. In symmetric equilibrium, $\theta_i^* = \theta^*$ for all submarkets $i$ and Proposition 2 tells us that firm entry is constrained efficient.

In this example, as in Section 3.1, we have simply assumed the output technology $p(.)$. The output externality from buyer entry may therefore be either positive or negative, i.e. $p'(\theta^*) > 0$ or $p'(\theta^*) < 0$, and the standard Hosios condition may result in either under-entry or over-entry, depending on the specific environment. In Section 3.2, we saw that an output technology $p(.)$ with $p'(\theta^*) > 0$ can arise endogenously via the *composition channel.* In Sections 3.4 and 3.5, we demonstrate how an output technology $p(.)$ (with either $p'(\theta^*) > 0$ or $p'(\theta^*) < 0$) can arise endogenously through the *selection channel.*

## 3.4 Competing auctions with endogenous match output

Competing auctions models with buyer heterogeneity are similar to directed or competitive search models where the expected match output is endogenous. Importantly, unlike the previous example, both the fact that the expected match output $p(\theta)$ depends directly on the market tightness $\theta$, and the specific properties of the function $p(.)$, are not assumptions: the function $p(.)$ and its properties arise *endogenously.* In particular, the fact that meetings can be many-on-one or multilateral is crucial: such meetings give rise to the possibility of choice among a number of potential trading partners. Through the auction mechanism, the *selection channel* endogenizes the expected match output.

In a competing auctions environment, a large number of sellers compete for buyers by posting second-price auctions with reserve prices equal to their own valuations. McAfee (1993) showed that this is an optimal mechanism for sellers.[17] Following the seminal work of Peters and Severinov (1997), recent pa-

[17]More recently, Lester, Visschers, and Wolthoff (2015) show that it is crucial that the

pers that use competing auctions include Albrecht, Gautier, and Vroman (2012, 2014, 2016); Kim and Kircher (2015); Lester, Visschers, and Wolthoff (2015); and Mangin (2015). It is well-known that buyer entry in a static competing auctions environment is constrained efficient.[18] Since the expected match output is *endogenous*, however, we cannot simply apply the Hosios rule in its traditional form. In fact, the constrained efficiency of entry in competing auctions environments is a direct application of Proposition 2.

Consider the labor market environment in Mangin (2015). Workers are identical sellers who auction their labor using second-price auctions with posted reservation wages. Firms are ex ante identical buyers who pay a cost $c > 0$ to enter and approach a single worker at random.[19] Meetings can be multilateral: more than one firm may approach a worker simultaneously. Firms' valuations $y$ of workers' labor are match-specific productivity draws that are private information for the firms. Valuations are drawn *ex post* (i.e. after approaching the worker) independently from a distribution with cdf $G$ that is twice differentiable with density $g = G' > 0$, a finite mean, and support $[y_0, \infty)$ where $y_0 \geq 0$.

The labor market tightness is $\theta \equiv V/U$, the ratio of vacancies or firms to unemployed workers, and the matching probability for workers is $m(\theta) = 1 - e^{-\theta}$, which satisfies Assumption 1. In equilibrium, workers' reservation wage is equal to the value of non-market activity, $b \in [0, y_0]$.[20] There exists a unique equilibrium $\theta^* \in \mathbb{R}_+$, and if $c < E_G(y) - b$ then $\theta^* > 0$ satisfies

$$\int_{y_0}^{\infty} e^{-\theta(1-G(y))}(1 - G(y))dy + e^{-\theta}(y_0 - b) = c. \tag{37}$$

underlying meeting technology is *invariant*. Lester et al. (2015) show that invariance implies *non-rivalry* as defined in Eeckhout and Kircher (2010b), and Cai, Gautier, and Wolthoff (2016) show that it also implies *joint concavity* as defined in that paper. In this example, the meeting technology is Poisson and therefore invariant.

[18] As we discuss below, Albrecht et al. (2014) establishes the efficiency of *seller* entry in a competing auctions enivronment.

[19] If firms approach workers at random, it is a standard result that the number of firms approaching each worker is a Poisson random variable with parameter $\theta$.

[20] In a labor market setting, it is important that we relax the standard "no gap" assumption found in Peters and Severinov (1997) and Albrecht et al. (2014) by allowing $b < y_0$ (i.e. sellers' valuation can be strictly less than the minimum buyers' valuation). This enables us to nest the directed search model in Example 3.4.1 as a special case.

Since $m(\theta) = 1 - e^{-\theta}$, it can be shown that expected output per match is

$$(38) \qquad p(\theta) = \frac{\int_{y_0}^{\infty} \theta e^{-\theta(1-G(y))} y dG(y)}{1 - e^{-\theta}}.$$

The expected match surplus is $s(\theta) = p(\theta) - b$ and $x(\theta) = m(\theta)s(\theta)$. To establish constrained efficiency, it is easier to work directly with expected *output per worker*, $f(\theta) \equiv m(\theta)p(\theta)$, given by

$$(39) \qquad f(\theta) = \int_{y_0}^{\infty} \theta e^{-\theta(1-G(y))} y dG(y).$$

We summarize some general properties of the function $f(.)$ in Lemma 3.

**Lemma 3.** *For any distribution $G$, the function $f(.)$ has the following properties: (i) $f'(\theta) > 0$ and $f''(\theta) < 0$ for all $\theta \in \mathbb{R}_+$, (ii) $\lim_{\theta \to 0} f(\theta) = 0$, (iii) $\lim_{\theta \to 0} f'(\theta) = E_G(y)$, (iv) $\lim_{\theta \to \infty} f(\theta) = +\infty$, (v) $\lim_{\theta \to \infty} f'(\theta) = 0$, and (vi) $f(\theta)/\theta$ is strictly decreasing in $\theta$ for all $\theta \in \mathbb{R}_+$.*

*Proof.* See Proposition 1 in Mangin (2015). □

Let $\eta_f(\theta) \equiv f'(\theta)\theta/f(\theta)$, the elasticity of $f(\theta)$ with respect to $\theta$. Lemma 4 provides an alternative version of the generalized Hosios condition that is useful in environments where it is simpler to work directly with the function $f(.)$.

**Lemma 4.** *The generalized Hosios condition is equivalent to*

$$(40) \qquad \eta_f(\theta) = \frac{c\theta}{f(\theta)} + \frac{\eta_m(\theta)b}{p(\theta)}.$$

*Proof.* See Appendix. □

Differentiating (39) and simplifying yields

$$(41) \qquad f'(\theta) = \int_{y_0}^{\infty} e^{-\theta(1-G(y))}(1 - G(y))dy + y_0 e^{-\theta}.$$

Since $f''(\theta) - bm''(\theta) < 0$ for all $\theta \in \mathbb{R}_+$, if $c < E_G(y) - b$ then Assumption 2 is satisfied and there exists a unique social optimum $\theta^P > 0$. Using (41) and (37), condition (40) clearly holds at $\theta^*$. Since this is equivalent to the generalized Hosios condition (15), we have constrained efficiency.

To apply Corollary 1, we need to show that $p'(\theta^*) > 0$. Lemma 5 uses the property that $G$ is *well-behaved.* This is a very mild condition that is satisfied by almost all standard distributions. It is weaker than the increasing hazard rate condition and weaker than log-concavity.[21] To introduce this condition, we first define $\varepsilon_G(y)$, the *generalized hazard rate* of $G$, as follows:

$$\varepsilon_G(y) \equiv \frac{yg(y)}{1 - G(y)}. \tag{42}$$

We say that $G$ is *well-behaved* if and only if $\varepsilon'_G(y) \geq 0$ for all $y \in [y_0, \infty)$.

**Lemma 5.** *If $G$ is well-behaved, we have $p'(\theta) > 0$ for all $\theta \in \mathbb{R}_+$.*

*Proof.* See Proposition 4 in Mangin (2015). □

In this environment, the *selection channel* gives rise to a positive output externality from firm entry. This is because the auction mechanism selects the most productive firm at each meeting. If firms were simply chosen at random, the selection channel would be shut down and the expected match output would not depend on the market tightness. In this way, the nature of the output technology $p(.)$, which transforms the market tightness into expected match output, depends on features of the decentralized market which the social planner takes as given when determining the efficient level of firm entry.

Since $p'(\theta^*) > 0$ in this environment, Corollary 1 tells us that applying the standard Hosios rule would result in *under-entry* of firms or, equivalently, inefficiently high unemployment. Intuitively, this is because firms do not internalize the fact that greater entry of firms leads not only to more matches for workers, but a higher expected match output. Under the generalized Hosios condition, however, firms internalize both the standard search externalities and the positive output externality that arises here.

[21] Banciu and Mirchandani (2013) provides a detailed list of distributions that satisfy this condition. Examples include the Uniform, Exponential, Normal, Logistic, Laplace, Gumbel, Weibull, Gamma, Beta, Pareto, Chi, Lognormal, Cauchy, and F distributions.

**Discussion.** Albrecht et al. (2014) considers entry of *sellers* in a competing auctions environment and establishes that seller entry is constrained efficient.[22] We consider *buyer* entry in this example because it is simpler, but analogous results hold for seller entry. The output externality that arises here also appears in Albrecht et al. (2014), but the discussion is framed differently. In Albrecht et al. (2014), we also have $p'(\theta) > 0$ where $\theta$ is the buyer/seller ratio. However, this is a *negative* externality with regard to seller entry, since the expected match output is decreasing in the number of sellers per buyer, $1/\theta$. Albrecht et al. (2014) call this the "business-stealing" effect.

One might expect that the "business-stealing" effect would lead to over-entry of sellers relative to the social optimum. However, Albrecht et al. (2014) shows that this is exactly offset by the "informational rents" that buyers extract from sellers through the auction mechanism. The level of seller entry is therefore efficient. Although Albrecht et al. (2014) do not explicitly identify it, the generalized Hosios condition applies in this setting. It is this condition that ensures that both the standard search externalities and the output externality – or "business-stealing" effect – are fully internalized by the decentralized market.

**Example 3.4.1**

In the special case where the distribution $G$ is degenerate, we recover the large economy version of the directed search model found in Julien et al. (2000) where workers post second-price auctions. All firms have the same productivity $y_0 = p$ and pay a cost $c > 0$ to search. The matching probability for workers is $m(\theta) = 1 - e^{-\theta}$. In equilibrium, workers set reserve prices equal to their outside option $b$ and there exists a unique equilibrium market tightness $\theta^*$. If $c < p - b$, then $\theta^* > 0$ satisfies

$$e^{-\theta}(p - b) = c. \tag{43}$$

[22] Albrecht et al. (2014) considers both ex ante and ex post buyer heterogeneity, and also allows for seller heterogeneity. For simplicity, we only consider homogeneous sellers and ex post buyer heterogeneity (where *ex post* means after *meetings* occur).

We can easily recover the constrained efficiency of directed search models such as Julien et al. (2000) by applying condition (15). In this case, it is just the standard Hosios condition: entry is efficient if and only if $\theta^*$ satisfies

$$\frac{\theta e^{-\theta}}{1-e^{-\theta}} = \frac{c\theta}{x(\theta)} \tag{44}$$

since $\eta_m(\theta) = \theta e^{-\theta}/(1-e^{-\theta})$. Substituting $x(\theta) = m(\theta)(p-b)$ into (44) and rearranging, we have constrained efficiency since $\theta^*$ satisfies (43).

**Example 3.4.2**

Suppose that $G$ is the Pareto distribution, $G(y) = 1 - y^{-1/\lambda}$ for $y \in [1, \infty)$ and zero otherwise, where $\lambda \in (0,1)$. To enter and search for a worker, firms must hire one unit of capital at cost $c > 0$. For simplicity, let $b = 0$. Intuitively, the parameter $\lambda$ measures the degree of dispersion of the match-specific productivity distribution $G$. A higher value of $\lambda$ implies greater dispersion.[23]

Mangin (2015) shows that a "frictionless" limit of this economy delivers a familiar benchmark: a Cobb-Douglas aggregate production function with constant factor shares. In general, we obtain an aggregate production function that directly incorporates matching frictions. Letting $f(\theta)$ be output per capita,

$$f(\theta) = \theta^{\lambda}\gamma(1-\lambda, \theta) \tag{45}$$

where $\gamma(1-\lambda,\theta) \equiv \int_0^{\theta} t^{-\lambda}e^{-t}$.[24] Observe that $f(\theta) \sim A\theta^{\lambda}$ where $A = \Gamma(1-\lambda)$ in the limit as $\theta \to \infty$ and $f(\theta) = 1 - e^{-\theta}$ if $G$ is degenerate.

Substituting into (37), the equilibrium $\theta^* > 0$ satisfies

$$\lambda\theta^{\lambda-1}\gamma(1-\lambda,\theta) + e^{-\theta} = c \tag{46}$$

[23] While an increase in $\lambda$ is not a mean-preserving spread in the distribution $G$, an increase in $\lambda$ does lead to an increase in the coefficient of variation.

[24] See Fact 1 in Mangin (2015) for a derivation and some properties of this function, which is a generalization of the Gamma function in the sense that $\lim_{\theta\to\infty}\gamma(1-\lambda,\theta) = \Gamma(1-\lambda)$.

if $c < 1/(1-\lambda)$. Expected output per match $p(\theta) \equiv f(\theta)/m(\theta)$ is

$$p(\theta) = \frac{\theta^{\lambda}\gamma(1-\lambda,\theta)}{1-e^{-\theta}}. \tag{47}$$

Since $G$ is well-behaved, $p'(\theta) > 0$ for all $\theta \in \mathbb{R}_+$. Using the fact that $b = 0$ and $f'(\theta) = \lambda\theta^{\lambda-1}\gamma(1-\lambda,\theta) + e^{-\theta}$, it is easy to show that (40) holds and we therefore have constrained efficiency.

## 3.5 Applicant ranking with endogenous match output

In environments with competing auctions, the expected match output $p(\theta)$ depends on the market tightness. However, private information is not necessary: what is essential is that buyers participate in many-on-one or multilateral meetings in which sellers can *choose* with whom to trade. This gives rise to the *selection channel*. For example, Shimer (2005) presents a model with full information where ex ante heterogeneous workers who face coordination frictions apply for jobs and firms choose to hire the most productive applicant. In that setting, the expected match output for a vacancy depends on the queue length (or expected number of applicants) for each *type* of worker.[25]

We present a simple example of applicant ranking found in Gavrel (2012). There is free entry of firms or vacancies. Workers apply to firms and firms rank applicants according to the degree of (match-specific) *mismatch* between the worker and the firm. The worker with the least "mismatch" is hired by the firm. As in Marimon and Zilibotti (1999), the degree of mismatch is measured by the distance on a circle between a worker and a firm.

Gavrel (2012) derives an expression for the expected output per match $p(\theta)$ where $\theta \equiv V/U$, the ratio of firms to unemployed workers. Let $x$ be the degree of mismatch between the worker and the firm, and let $y(x)$ be the match output given $x$. The expected match output is

$$p(\theta) = \int_0^{1/2} y(x)\rho(x,\theta)dx \tag{48}$$

[25] In Shimer (2005), there is two-sided ex ante heterogeneity and a finite number of types.

where $\rho(x, \theta)$ is the density of mismatch among filled vacancies.

In this environment, the selection channel gives rise to a *negative* output externality from firm entry via the applicant ranking mechanism. Gavrel (2012) proves that $p'(\theta) < 0$ for all $\theta \in \mathbb{R}_+$. Intuitively, a greater number of firms per unemployed worker (higher $\theta$) implies *fewer* applicants per vacancy (lower $1/\theta$), which increases the expected degree of mismatch between the best applicant and the firm, and thereby lowers output per match. As $\theta$ increases, firms are less selective and the greater resulting mismatch between workers and firms means that expected output per match falls.

Gavrel's key result is that the presence of applicant ranking leads to an *over-entry* of vacancies (i.e. job creation is inefficiently high) when wages are determined by Nash bargaining and the standard Hosios condition is satisfied.[26] That is, the unemployment rate is inefficiently *low* under the standard Hosios condition. Since $p'(\theta^*) < 0$, the fact that there is over-entry of firms under the standard Hosios condition is an immediate application of Corollary 1. To obtain constrained efficiency, what is needed is the generalized Hosios condition found in Proposition 2, not the standard Hosios condition.

## 3.6 Ex ante capital investment with endogenous TFP

Consider a simple model with *ex ante* capital investment and post-match bargaining based on Acemoglu and Shimer (1999b).[27] To illustrate the generalized Hosios condition, we incorporate a novel feature: the endogenous match output depends directly on *both* capital and the labor market tightness.

We do this partly to highlight the differences between these two channels. If the expected match output is endogenous only in the sense that it depends on capital, as in Acemoglu and Shimer (1999b), the standard Hosios condition applies. However, if the expected match output is endogenous in the sense that it depends also on the labor market tightness, the generalized Hosios condition

[26] On the other hand, Gavrel shows that competitive search through wage posting à la Moen (1997) restores constrained efficiency.

[27] Masters (1998, 2011) examines a frictional labor market model with two-sided ex ante investment in both physical and human capital. Mailath, Postlewaite, and Samuelson (2013) considers two-sided ex ante investment in a matching model with two-sided heterogeneity.

is necessary for efficient entry. In both cases, efficiency of both entry and capital intensity is impossible when wages are determined ex post by Nash bargaining.

To align the results with the previous examples, we incorporate a cost $c > 0$ of vacancy creation. There exists a perfect capital market where capital can be rented at price $r$. Importantly, capital $k$ is an ex ante investment that is made by firms *prior* to the matching process. Wages are determined ex post by generalized Nash bargaining where workers have bargaining power $\beta \in [0, 1]$ and the value of non-market activity $b$ is zero.

Let the output from a worker-firm match be $g(k, \theta)$ where $\theta \equiv V/U$, the ratio of vacancies to unemployed workers. We assume that $g(k, \theta) = A(\theta)y(k)$ where $A(\theta)$ is total factor productivity (TFP) and $y(k)$ is a standard neoclassical production function with capital-output elasticity $\varepsilon_y(k) \equiv y'(k)k/y(k)$ where $\varepsilon_y(k) < 1$. Since endogenizing $A(\theta)$ is not the focus of this example, we abstract from details here and simply assume $A(\theta)$ is given.[28]

While we assume that $g(k, \theta) = A(\theta)y(k)$ in this example, we set up the model in terms of the general function $g(k, \theta)$. The bargaining problem takes $k$ as given and the wage is given by

$$w(k, \theta) = \arg\max_{w \in \mathbb{R}_+} (g(k, \theta) - w)^{1-\beta} w^{\beta}, \tag{49}$$

with well-known solution $w(k, \theta) = \beta g(k, \theta)$. Firms take the wage $w(k, \theta)$ as given and choose capital intensity $k$ by solving

$$k(\theta) = \arg\max_{k \in \mathbb{R}_+} \left( \frac{m(\theta)}{\theta} (g(k, \theta) - w(k, \theta)) - rk \right). \tag{50}$$

Substituting in the bargained wage,

$$k(\theta) = \arg\max_{k \in \mathbb{R}_+} \left( \frac{m(\theta)}{\theta} (1 - \beta) g(k, \theta) - rk \right). \tag{51}$$

[28] It would be straightforward to endogenize $A(\theta)$ by using multilateral meetings and a distribution of match-specific productivities similar to that described in Section 3.4 or 3.5.

The first-order condition is

$$\frac{m(\theta)}{\theta}(1-\beta)g_k(k,\theta) = r. \tag{52}$$

Under free entry of vacancies, we also have

$$\frac{m(\theta)}{\theta}(1-\beta)g(k,\theta) - rk = c. \tag{53}$$

An equilibrium $(k^*, \theta^*)$ solves equations (52) and (53).

The social planner chooses capital $k$ for each firm opening a vacancy, and the labor market tightness $\theta$, to maximize the social surplus per worker,

$$\Omega(k,\theta) = m(\theta)g(k,\theta) - rk\theta - c\theta. \tag{54}$$

The first-order conditions are:

$$\Omega_k(k,\theta) = m(\theta)g_k(k,\theta) - r\theta = 0 \tag{55}$$

and

$$\Omega_\theta(k,\theta) = \ m'(\theta)g(k,\theta) + m(\theta)g_\theta(k,\theta) - rk - c = 0. \tag{56}$$

Any social planner's solution must satisfy (55) and (56).

Now let $g(k,\theta) = A(\theta)y(k)$. To prove the following two propositions, we assume the function $f(.)$ defined by $f(\theta) \equiv m(\theta)A(\theta)$ has the following properties: *(i)* $f'(\theta) > 0$ and $f''(\theta) < 0$ for all $\theta \in \mathbb{R}_+$, *(ii)* $\lim_{\theta\to 0} f(\theta) = 0$, *(iii)* $\lim_{\theta\to 0} f'(\theta) \geq 1$, *(iv)* $\lim_{\theta\to\infty} f(\theta) = +\infty$, *(v)* $\lim_{\theta\to\infty} f'(\theta) = 0$, and *(vi)* $f(\theta)/\theta$ is strictly decreasing in $\theta$ for all $\theta \in \mathbb{R}_+$.[29]

**Proposition 3.** *If $\varepsilon_y'(k) \leq 0$ for all $k \in \mathbb{R}_+$, there exists a unique equilibrium $(k^*, \theta^*)$.*

[29] While we simply assume these properties in this example, they arise *endogenously* in an environment with multilateral meetings and a distribution of match-specific productivities similar to that described in Section 3.4. As shown in Mangin (2015), the property that $\sigma_f(\theta) \leq 1$ (used in Proposition 4) also arises in such an environment.

*Proof.* See Appendix. □

Let $\sigma_f(\theta)$ be the elasticity of substitution between vacancies and unemployed workers for the function $f(.)$. The condition that $\sigma_f(\theta) \leq 1$ is equivalent to the condition that $\eta_f'(\theta) \leq 0$ where $\eta_f(\theta) \equiv f'(\theta)\theta/f(\theta)$.[30]

**Proposition 4.** *If $\varepsilon_y'(k) \leq 0$ for all $k \in \mathbb{R}_+$ and $\sigma_f(\theta) \leq 1$ for all $\theta \in \mathbb{R}_+$, there exists a unique social optimum $(k^P, \theta^P)$.*

*Proof.* See Appendix. □

Using (55), we have efficiency of capital intensity $k$ only when $(k^*, \theta^*)$ satisfies

$$\frac{m(\theta)}{\theta} g_k(k, \theta) = r. \tag{57}$$

The joint match surplus is $s(k, \theta) = g(k, \theta)$ and $\eta_s(k, \theta) \equiv g_\theta(k, \theta)\theta/g(k, \theta)$. Rearranging (56), we have constrained efficiency of *entry* only when the decentralized equilibrium $(k^*, \theta^*)$ satisfies

$$\underbrace{\eta_m(\theta)}_{\text{matching elasticity}} + \underbrace{\eta_s(k, \theta)}_{\text{surplus elasticity}} = \underbrace{\frac{(rk + c)\theta}{x(k, \theta)}}_{\text{surplus share of firms + capital}}. \tag{58}$$

where $x(k, \theta) = m(\theta)s(k, \theta)$. This is just the generalized Hosios condition found in Proposition 2, where the *effective* cost of entry is $rk + c$.

Using both equilibrium conditions (52) and (53) above, $(k^*, \theta^*)$ satisfies

$$1 - \beta = \frac{(rk + c)\theta}{x(k, \theta)}. \tag{59}$$

Using $g(k, \theta) = A(\theta)y(k)$ and comparing (59) above with condition (58), we have constrained efficiency of entry if and only if $\theta^*$ satisfies

$$\underbrace{\eta_m(\theta)}_{\text{matching elasticity}} + \underbrace{\eta_A(\theta)}_{\text{TFP elasticity}} = \underbrace{1 - \beta}_{\text{firms' bargaining power}} \tag{60}$$

[30] In the special case where $A(\theta) = A$, we recover the standard regularity condition that requires the elasticity of the matching function to be weakly decreasing, i.e. $\eta_m'(\theta) \leq 0$.

where $\eta_A(\theta) \equiv A'(\theta)\theta/A(\theta)$. Efficiency of entry requires that firms are compensated for their effect on both the *number* of matches created and the *value* of the expected match output. The value of the expected match output is directly influenced by firm entry through the endogenous TFP term $A(\theta)$. Consistent with Corollary 1, if $A'(\theta) > 0$ there is a positive output externality that would result in under-entry of vacancies under the standard Hosios condition.

For efficiency of capital intensity $k$, the equilibrium $(k^*, \theta^*)$ must satisfy (57). Comparing with (52), this implies $\beta = 0$. Due to the "hold up" problem created by their ex ante investment in capital, firms require all the bargaining power. As discussed in detail in Acemoglu and Shimer (1999b), this means that efficiency of both entry and capital investment is not possible with ex post bargaining.[31] This example extends the result of Acemoglu and Shimer (1999b) to environments where the expected match output depends directly on the market tightness. In such environments, efficiency of entry requires the generalized Hosios condition.

While this example uses Nash bargaining, Acemoglu and Shimer (1999b) show that efficiency of both entry and capital investment is indeed possible in a competitive search environment where firms post capital and workers direct their search. We expect the generalized Hosios condition to hold endogenously in such a setting when match output depends directly on the market tightness.

# 4 Conclusion

This paper presents a generalized version of the well-known Hosios rule that determines the conditions under which entry in search and matching models is constrained efficient. We extend this simple rule to environments where the expected match output depends on the market tightness. Such environments give rise to a novel externality that we call the *output externality*. This externality is not captured by the standard Hosios condition, which internalizes only the search externalities arising from the frictional matching process.

[31] In the special case where $g(k, \theta) = Ay(k)$ and $c = 0$, we recover a static version of the results in Section 4 of Acemoglu and Shimer (1999b). Constrained efficiency of both entry and capital intensity would require both $\eta_m(\theta^*) = 1 - \beta$ and $\beta = 0$, which is impossible.

To ensure constrained efficiency, decentralized markets must internalize the effect of entry on both the *number* of matches created and the average *value* of a match. We show that this occurs only when buyers' surplus share equals the *matching elasticity* plus the *surplus elasticity.* We call this simple, intuitive condition the *generalized Hosios condition.* Like the standard Hosios condition, the simplicity of this general rule carries over directly to dynamic environments with enduring matches. When this condition holds, both the matching externalities and the output externality are internalized.

In this paper, we consider the efficiency of entry and assume that the social planner is constrained by both the matching technology and the *output technology,* i.e. the "technology" which transforms the market tightness into the expected match output. However, the nature of the output technology arises directly from specific features of the decentralized market, such as the underlying meeting technology and the trading mechanism. One possible direction for future research would be to integrate our general result regarding the efficiency of entry with the literature that examines the optimality of the trading mechanism itself in environments with search frictions.

Another potential direction for further research would be to consider search and matching environments with two-sided heterogeneity. In such environments, as Eeckhout and Kircher (2010a) point out, the social planner also cares about both the number of matches created and the average value of a match – which depends on the types of agents that form matches. In future research, it would be interesting to integrate results regarding the efficiency of frictional environments with two-sided heterogeneity with the general result presented here.

# For Online Publication: Appendix

## Proof of Lemma 1

In steady state, we have the following Bellman equations:

$$rU_B = -c + \frac{m(\theta)}{\theta}(V_B - U_B), \tag{61}$$

$$rV_B = p(\theta) - w(\theta) + \delta(U_B - V_B), \tag{62}$$

$$rU_S = b + m(\theta)(V_S - U_S), \tag{63}$$

$$rV_S = w(\theta) + \delta(U_S - V_S), \tag{64}$$

With free entry, $U_B = 0$ and $s(\theta) = V_B + V_S - U_S$, so we have

$$V_B + V_S = \frac{p(\theta) - \delta s(\theta)}{r}. \tag{65}$$

Substituting back into $s(\theta) = V_B + V_S - U_S$, and rearranging yields

$$s(\theta) = \frac{p(\theta) - rU_S}{r + \delta}. \tag{66}$$

Next, using (63) and (64), we find that

$$U_S = \frac{b(r+\delta) + m(\theta)w(\theta)}{r(r + \delta + m(\theta))}, \tag{67}$$

and, substituting into (66), we obtain

$$s(\theta) = \left( \frac{p(\theta) - b + m(\theta)\left(\frac{p(\theta)-w(\theta)}{r+\delta}\right)}{r + \delta + m(\theta)} \right). \tag{68}$$

Now (61) implies $V_B = c\theta/m(\theta)$ when $U_B = 0$. Substituting into (62), we have

$$\frac{p(\theta) - w(\theta)}{r + \delta} = \frac{c\theta}{m(\theta)}, \tag{69}$$

and, substituting (69) into $s(\theta)$, we obtain

$$s(\theta) = \frac{p(\theta) - b + c\theta}{r + \delta + m(\theta)}. \tag{70}$$

## Proof of Proposition 1

In discrete time, the law of motion for the unemployment rate $u_t$ is

$$u_{t+1} - u_t = \delta(1 - u_t) - m(\theta_t)u_t \tag{71}$$

and the law of motion for average match output $p_t$ is given by

$$p_{t+1} = \frac{(1 - \delta)(1 - u_t)p_t + m(\theta_t)u_t p(\theta_t)}{1 - u_{t+1}}. \tag{72}$$

Defining $x_t \equiv (1 - u_t)p_t$, we have

$$x_{t+1} - x_t = -\delta x_t + m(\theta_t)u_t p(\theta_t). \tag{73}$$

In continuous time, the laws of motion for $u_t$ and $x_t$ are

$$\dot{u}_t = \delta(1 - u_t) - m(\theta_t)u_t \tag{74}$$

and

$$\dot{x}_t = -(\delta x_t - m(\theta_t)u_t p(\theta_t)). \tag{75}$$

Also, since $x_t \equiv (1 - u_t)p_t$, we have

$$\dot{x}_t = -\dot{u}_t p_t + (1 - u_t)\dot{p}_t \tag{76}$$

and, rearranging, we have

$$\dot{p}_t = \frac{\dot{x}_t + \dot{u}_t p_t}{1-u_t}. \tag{77}$$

Substituting in $\dot{x}_t$ and $\dot{u}_t$ from (75) and (74) and simplifying,

$$\dot{p}_t = \frac{m(\theta_t)u_t\left(p(\theta_t) - p_t\right)}{1-u_t}. \tag{78}$$

The social planner chooses $\theta_t$ for all $t \in \mathbb{R}_+$ to maximize the following:

$$\int_0^\infty e^{-rt}((1-u_t)p_t + bu_t - c\theta_t u_t)dt \tag{79}$$

subject to

$$\dot{u}_t = \delta(1-u_t) - m(\theta_t)u_t \tag{80}$$

and

$$\dot{p}_t = \frac{m(\theta_t)u_t\left(p(\theta_t) - p_t\right)}{1-u_t}. \tag{81}$$

The current value Hamiltonian is

(82)

$$H = ((1-u_t)p_t + bu_t - c\theta_t u_t) + \lambda_t(\delta(1-u_t) - m(\theta_t)u_t) + \mu_t\left(\frac{m(\theta_t)u_t\left(p(\theta_t) - p_t\right)}{1-u_t}\right).$$

The first-order necessary conditions are

$$\frac{\partial H}{\partial \theta_t} = -cu_t - \lambda_t m'(\theta_t)u_t + \mu_t\left(\frac{m'(\theta_t)u_t\left(p(\theta_t) - p_t\right) + m(\theta_t)u_t p'(\theta_t)}{1-u_t}\right) = 0 \tag{83}$$

$$\begin{aligned} \frac{dH}{du_t} &= -(p_t - b + c\theta_t) - \lambda_t(\delta + m(\theta_t)) \\ &\quad + \mu_t\left(\frac{(1-u_t)\,m(\theta_t)\left(p(\theta_t) - p_t\right) + u_t m(\theta_t)\left(p(\theta_t) - p_t\right)}{(1-u_t)^2}\right) \\ &= -\dot{\lambda}_t + r\lambda_t \end{aligned} \tag{84}$$

(85)
$$\frac{\partial H}{\partial p_t} = 1 - u_t - \mu_t \left( \frac{m(\theta_t) u_t}{1 - u_t} \right) = -\dot{\mu}_t + r\mu_t$$

(86)
$$\frac{\partial H}{\partial \lambda_t} = \delta(1 - u_t) - m(\theta_t) u_t = \dot{u}_t$$

(87)
$$\frac{\partial H}{\partial \mu_t} = \frac{m(\theta_t) u_t \left( p(\theta_t) - p_t \right)}{1 - u_t} = \dot{p}_t$$

and the transversality conditions are

(88)
$$\lim_{t \to \infty} e^{-rt} \lambda_t u_t = 0,$$

(89)
$$\lim_{t \to \infty} e^{-rt} \mu_t p_t = 0.$$

Now, in steady state, we have $\dot{u}_t = 0$ and $\dot{p}_t = 0$ and therefore $p(\theta_t) = p_t = p(\theta)$. Also, in steady state, $\dot{\mu}_t = 0$ and $\dot{\lambda}_t = 0$. Substituting into the above first-order conditions,

(90)
$$-cu - \lambda m'(\theta) u + \mu \left( \frac{m(\theta) u p'(\theta)}{1 - u} \right) = 0,$$

(91)
$$-(p(\theta) - b + c\theta) - \lambda(\delta + m(\theta)) = r\lambda,$$

(92)
$$1 - u - \mu \left( \frac{m(\theta) u}{1 - u} \right) = r\mu.$$

Using the fact that $\delta(1 - u) = m(\theta) u$ in steady state, we have

(93)
$$-\lambda m'(\theta) u + \mu \delta p'(\theta) = cu,$$

(94)
$$\lambda = -\frac{p(\theta) - b + c\theta}{r + \delta + m(\theta)},$$

$$\mu = \frac{1-u}{r+\delta}. \tag{95}$$

It is clear that the transversality conditions are satisfied by $\lambda$ and $\mu$. Substituting $\lambda$ and $\mu$ into the first equation, we have

$$\frac{p(\theta) - b + c\theta}{r + \delta + m(\theta)} m'(\theta) u + \frac{(1-u)\delta p'(\theta)}{r+\delta} = cu. \tag{96}$$

Again using $\delta(1-u) = m(\theta)u$ and simplifying,

$$\frac{p(\theta) - b + c\theta}{r + \delta + m(\theta)} m'(\theta) + \frac{m(\theta)p'(\theta)}{r+\delta} = c. \tag{97}$$

Defining $s(\theta)$ as in (9), letting $x(\theta) = m(\theta)s(\theta)$, and multiplying by $\theta/x(\theta)$,

$$\frac{m'(\theta)\theta}{m(\theta)} + \frac{p'(\theta)\theta}{(r+\delta)s(\theta)} = \frac{c\theta}{x(\theta)}. \tag{98}$$

That is,

$$\eta_m(\theta) + \frac{p'(\theta)\theta}{(r+\delta)s(\theta)} = \frac{c\theta}{x(\theta)}, \tag{99}$$

where $\eta_m(\theta) \equiv m'(\theta)\theta/m(\theta)$. Any social optimum must satisfy (99).

## Proof of Lemma 2

As stated in the main text, it follows immediately from Assumption 2 that there exists a unique $\theta^P > 0$ that satisfies the necessary condition (10). We now prove that the steady state solution $\theta^P$ given by (10) is indeed a global maximum using Arrow's Sufficiency Theorem. To show this, it is simpler to formulate the current value Hamiltonian in terms of the state variable $x_t$. Using (74) and (75), the current value Hamiltonian as a function of state and control variables is

$$H(x,u,\theta) = (x + bu - c\theta u) + \lambda_1(\delta(1-u) - m(\theta)u) + \mu_1(-(\delta x - m(\theta)up(\theta))). \tag{100}$$

First, we define the maximized Hamiltonian as follows:

$$M_H(x,u) \equiv \max_{\theta\in\mathbb{R}_+}[(x+bu-c\theta u)+\lambda_1(\delta(1-u)-m(\theta)u)+\mu_1(-(\delta x-m(\theta)up(\theta))].$$

We now apply Arrow's Sufficiency Theorem.[32] To prove that the solution $\theta^P$ to (99) is a global maximum, it is sufficient to show that *(i)* the maximized Hamiltonian $M_H(x,u)$ is jointly weakly concave in $u$ and $x$; and *(ii)* there exists a unique solution $\theta^P$ that satisfies the necessary condition (99). Since we know that part *(ii)* holds, it remains only to prove *(i)*. To find $\theta^* \equiv \arg\max_{\theta\in\mathbb{R}_+} H(x,u,\theta)$, we set

$$\frac{\partial H}{\partial \theta} = -cu - \lambda_1 m'(\theta)u + \mu_1 u(m'(\theta)p(\theta) + m(\theta)p'(\theta)) = 0. \tag{101}$$

Also, we have

$$\frac{\partial^2 H}{\partial \theta^2} = -\lambda_1 m''(\theta)u + \mu_1 u(m''(\theta)p(\theta) + 2m'(\theta)p'(\theta) + m(\theta)p''(\theta)) < 0, \tag{102}$$

provided that $m''(\theta)p(\theta) + 2m'(\theta)p'(\theta) + m(\theta)p''(\theta) < 0$ and $m''(\theta) < 0$ since $\lambda_1 < 0$ and $\mu_1 > 0$. Assumption 1 states that $m''(\theta) < 0$ for all $\theta \in \mathbb{R}_+$ and Assumption 2 says that $x''(\theta) < 0$ for all $\theta \in \mathbb{R}_+$ where $x(\theta) \equiv m(\theta)s(\theta)$. In the special case where $b = 0$ in the static economy, we have $x(\theta) \equiv m(\theta)p(\theta)$ and therefore $x''(\theta) < 0$ implies that $m''(\theta)p(\theta) + 2m'(\theta)p'(\theta) + m(\theta)p''(\theta) < 0$.

So $\theta^*$ is indeed a maximum and the maximized Hamiltonian is

$$M_H(x,u) = (x+bu-c\theta^* u)+\lambda_1(\delta(1-u)-m(\theta^*)u)+\mu_1(-(\delta x-m(\theta^*)up(\theta^*))). \tag{103}$$

Since the $u$ cancels out in (101) and $x$ does not appear in that equation, $\theta^*$ does not depend directly on $u$ or $x$. Also, it can be verified that neither $\lambda_1$ nor $\mu_1$ depends on either $u$ or $x$.[33] The function $M_H(x,u)$ is linear in both $x$ and $u$

[32] Arrow's Sufficiency Theorem generalizes Mangasarian's sufficiency conditions. See Kamien and Schwartz (1991), p. 221-222.

[33] Note that the co-state variables $\lambda_1$ and $\mu_1$ for the current value Hamiltonian with state variables $u_t$ and $x_t$ are different to the co-state variables $\lambda$ and $\mu$ for the current value Hamiltonian with state variables $u_t$ and $p_t$.

and it is therefore weakly concave. Since there exists a unique solution $\theta^P$ that satisfies the necessary condition (99), this solution is the global maximum.

## Proof of Corollary 1

Assume that the standard Hosios condition holds, namely

$$\frac{m'(\theta^*)\theta^*}{m(\theta^*)} = \frac{c\theta^*}{x(\theta^*)}. \tag{104}$$

We prove the result in two parts. First, we show that there is under-entry (over-entry) of buyers if and only if $s'(\theta^*) > (<)0$. Second, we show that $s'(\theta^*) > 0$ if and only if $p'(\theta^*) > 0$. Using $x(\theta) = m(\theta)s(\theta)$ and simplifying (104), we have $m'(\theta^*)s(\theta^*) = c$. By (14), we have $x'(\theta^P) = c$ and therefore $x'(\theta^P) = m'(\theta^*)s(\theta^*)$. Now $m'(\theta^*)s(\theta^*) = x'(\theta^*) - m(\theta^*)s'(\theta^*)$ and thus

$$x'(\theta^P) = x'(\theta^*) - m(\theta^*)s'(\theta^*). \tag{105}$$

If $s'(\theta^*) > 0$, then $x'(\theta^P) < x'(\theta^*)$. If $x''(\theta) < 0$ for all $\theta$ then $x'(\theta^P) < x'(\theta^*)$ implies that $\theta^* < \theta^P$ and there is *under-entry* of buyers. Similarly, if $s'(\theta^*) < 0$, there is *over-entry* of buyers, $\theta^* > \theta^P$. Differentiating $s(\theta)$ using (9),

$$s'(\theta) = \frac{(p'(\theta) + c)(r + \delta + m(\theta)) - m'(\theta)(p(\theta) - b + c\theta)}{(r + \delta + m(\theta))^2}. \tag{106}$$

Using expression (9) for $s(\theta)$ and rearranging, $s'(\theta^*) > 0$ if and only if

$$p'(\theta^*) > m'(\theta^*)s(\theta^*) - c, \tag{107}$$

and since $m'(\theta^*)s(\theta^*) = c$, we have $s'(\theta^*) > 0$ if and only if $p'(\theta^*) > 0$.

## Proof of Lemma 4

Starting with the generalized Hosios condition found in Proposition 2, and using the fact that $s(\theta) = p(\theta) - b$, we have

$$\eta_m(\theta) + \frac{p'(\theta)\theta}{p(\theta) - b} = \frac{c\theta}{x(\theta)}. \tag{108}$$

Expressing (108) in terms of $\eta_p(\theta) \equiv p'(\theta)\theta/p(\theta)$, we have

$$\eta_m(\theta) + \frac{\eta_p(\theta)p(\theta)}{p(\theta) - b} = \frac{c\theta}{x(\theta)}. \tag{109}$$

Since $p(\theta) = f(\theta)/m(\theta)$, we have $\eta_p(\theta) = \eta_f(\theta) - \eta_m(\theta)$. Substituting $\eta_p(\theta)$ into (109) and simplifying, we obtain

$$\eta_f(\theta) = \frac{c\theta}{f(\theta)} - \frac{\eta_m(\theta)b}{p(\theta)}. \tag{110}$$

## Proof of Proposition 3

An equilibrium $(k^*, \theta^*)$ solves the following two equations:

$$\frac{m(\theta)}{\theta}(1-\beta)g_k(k,\theta) = r \tag{111}$$

and

$$\frac{m(\theta)}{\theta}(1-\beta)g(k,\theta) - rk = c. \tag{112}$$

Letting $g(k,\theta) = A(\theta)y(k)$ and $f(\theta) = m(\theta)A(\theta)$, these are equivalent to

$$\frac{f(\theta)}{\theta}(1-\beta)y'(k) = r \tag{113}$$

and

$$\frac{f(\theta)}{\theta}(1-\beta)y(k) - rk = c. \tag{114}$$

Combining (113) and (114), an equilibrium $k$ satisfies

$$\frac{r}{y'(k)} = \frac{c + rk}{y(k)}. \tag{115}$$

Rearranging and simplifying, (115) is equivalent to

$$k\left(\frac{y(k)}{y'(k)k} - 1\right) = \frac{c}{r}. \tag{116}$$

Substituting $\varepsilon_y(k) \equiv y'(k)k/y(k)$ into (116), an equilibrium $k$ satisfies

$$k\left(\frac{1}{\varepsilon_y(k)} - 1\right) = \frac{c}{r}. \tag{117}$$

Now suppose that $\varepsilon_y'(k) \leq 0$ for all $k$. Then $\frac{d}{dk}\left(\frac{1}{\varepsilon_y(k)}\right) \geq 0$, so $k\left(\frac{1}{\varepsilon_y(k)} - 1\right)$ is strictly increasing in $k$. Also, we have

$$\lim_{k\to 0} k\left(\frac{1}{\varepsilon_y(k)} - 1\right) = \lim_{k\to 0}\left(\frac{y(k)}{y'(k)} - k\right) = 0 \tag{118}$$

since $\lim_{k\to 0} y(k) = 0$ and $\lim_{k\to 0} y'(k) = \infty$. Finally, since $\varepsilon_y'(k) \leq 0$ and $\varepsilon_y(k) \in [0,1)$ by assumption, we have $\lim_{k\to\infty} \varepsilon_y(k) \in [0,1)$ and $\lim_{k\to\infty} k\left(\frac{1}{\varepsilon_y(k)} - 1\right) = +\infty$. So there exists a unique solution $k^* > 0$ to (117).

Given $k^*$, an equilibrium $\theta$ must satisfy (113), which is equivalent to

$$\frac{f(\theta)}{\theta} = \frac{r}{(1-\beta)y'(k^*)}. \tag{119}$$

By assumption, $f(\theta)/\theta$ is strictly decreasing in $\theta$, so any solution $\theta^*$ is unique. Also, we have $\lim_{\theta\to 0} f(\theta)/\theta = \lim_{\theta\to 0} f'(\theta)$ by L'Hopital's rule, and $\lim_{\theta\to 0} f'(\theta) \geq 1$ by assumption. Finally, $\lim_{\theta\to\infty} f(\theta)/\theta = \lim_{\theta\to\infty} f'(\theta)$ by L'Hopital's rule, and $\lim_{\theta\to\infty} f'(\theta) = 0$ by assumption. So there exists a unique equilibrium $(k^*, \theta^*)$ where $\theta^* > 0$ if $r < (1-\beta)y'(k^*)$ where $k^*$ is the unique solution to (117). If $r \geq (1-\beta)y'(k^*)$, then $\theta^* = 0$.

## Proof of Proposition 4

Suppose that $\sigma_f(\theta) \leq 1$ for all $\theta \in \mathbb{R}_+$. We show that there exists a unique social optimum $(k^P, \theta^P)$. The first-order conditions for the social planner's problem are:

$$\Omega_k(k,\theta) = m(\theta)g_k(k,\theta) - r\theta = 0 \tag{120}$$

and

$$\Omega_\theta(k,\theta) = m'(\theta)g(k,\theta) + m(\theta)g_\theta(k,\theta) - rk - c = 0. \tag{121}$$

Letting $g(k,\theta) = A(\theta)y(k)$ and $f(\theta) = m(\theta)A(\theta)$, these are equivalent to

$$\Omega_k(k,\theta) = f(\theta)y'(k) - r\theta = 0 \tag{122}$$

and

$$\Omega_\theta(k,\theta) = \ f'(\theta)y(k) - rk - c = 0. \tag{123}$$

We can rewrite (122) as

$$y'(k) = \frac{r\theta}{f(\theta)} \tag{124}$$

Since $f(\theta)/\theta$ is strictly decreasing by assumption, the right-hand side of (124) is strictly increasing in $\theta$. So for any given $\theta \in \mathbb{R}_+$, there is a unique $y'(k)$ that satisfies (124). Since $y''(k) < 0$ for all $k \in \mathbb{R}_+$ by assumption, $k$ must also be unique and we can therefore write $k(\theta)$.

Using (124), we have

$$\lim_{\theta\to 0} y'(k(\theta)) = \lim_{\theta\to 0} \frac{r\theta}{f(\theta)} = \lim_{\theta\to 0} \frac{r}{f'(\theta)} \leq r \tag{125}$$

since $f'(0) \geq 1$. By assumption, $\lim_{k\to 0} y'(k) = \infty$, so $\lim_{\theta\to 0} k(\theta) > 0$. Also,

$$\lim_{\theta\to\infty} y'(k(\theta)) = \lim_{\theta\to\infty} \frac{r\theta}{f(\theta)} = \lim_{\theta\to\infty} \frac{r}{f'(\theta)} = \infty \tag{126}$$

since $f'(\infty) = \infty$. Since $\lim_{k\to 0} y'(k) = \infty$ and $y''(k) < 0$, this implies $\lim_{\theta\to\infty} k(\theta) = 0$. So we have $k(0) > 0$ and $k(\infty) = 0$.

We can explicitly derive $k'(\theta)$ using the implicit function theorem:

$$k'(\theta) = \frac{-(1-\eta_f(\theta))y'(k)}{\theta y''(k)} \tag{127}$$

where $\eta_f(\theta) \equiv f'(\theta)\theta/f(\theta)$. We have $k'(\theta) < 0$ for all $\theta \in (0,\infty)$ because $f(\theta)/\theta$ is strictly decreasing by assumption and therefore $\eta_f(\theta) < 1$.

**Uniqueness.** First, we prove the uniqueness of any $\theta^P$ that satisfies the social planner's first-order conditions. Rewriting (122) as $r = f(\theta)y'(k)/\theta$ and substituting this and $k(\theta)$ into (123), we obtain

$$L(\theta) \equiv \; f'(\theta)y(k(\theta)) - \frac{f(\theta)}{\theta}y'(k(\theta))k(\theta) = c. \tag{128}$$

To prove uniqueness, we show that $L'(\theta) < 0$. Differentiating (128), we obtain

$$\begin{aligned} L'(\theta) &= f''(\theta)y(k) + f'(\theta)y'(k)k'(\theta) - \left(\frac{f'(\theta)\theta - f(\theta)}{\theta^2}\right) y'(k)k \\ &\quad - \frac{f(\theta)}{\theta}\left(y''(k)k'(\theta)k + y'(k)k'(\theta)\right). \end{aligned} \tag{129}$$

Substituting in $\eta_f(\theta) \equiv f'(\theta)\theta/f(\theta)$ and $k'(\theta)$ using (127) yields

$$-\left(\frac{f'(\theta)\theta - f(\theta)}{\theta^2}\right) y'(k)k - \frac{f(\theta)}{\theta}y''(k)k'(\theta)k = 0 \tag{130}$$

so (129) is equivalent to

$$L'(\theta) = f''(\theta)y(k) + \left(f'(\theta) - \frac{f(\theta)}{\theta}\right) y'(k)k'(\theta). \tag{131}$$

Now, the elasticity of substitution $\sigma_f(\theta)$ is given by

$$\sigma_f(\theta) = \frac{-f'(\theta)\left(f(\theta) - f'(\theta)\theta\right)}{f''(\theta)f(\theta)\theta}. \tag{132}$$

Substituting the definitions of $\sigma_f(\theta)$ and $\eta_f(\theta)$ into (131), we find

$$L'(\theta) = \frac{f''(\theta)(y'(k))^2}{y''(k)} \frac{1-\eta_f(\theta)}{\eta_f(\theta)} \left( \frac{y(k)y''(k)}{(y'(k))^2} \frac{\eta_f(\theta)}{1-\eta_f(\theta)} + \sigma_f(\theta) \right). \tag{133}$$

Since $f''(\theta) < 0$ and $y''(k) < 0$ by assumption, and $\eta_f(\theta) < 1$, we have $L'(\theta) < 0$ if and only if

$$\sigma_f(\theta) < \frac{-y(k)y''(k)}{(y'(k))^2} \frac{\eta_f(\theta)}{1-\eta_f(\theta)}. \tag{134}$$

Using both first-order conditions (122) and (123), we have

$$\frac{rk+c}{r\theta} = \frac{f'(\theta)y(k)}{f(\theta)y'(k)} \tag{135}$$

and, rearranging, this is equivalent to

$$\eta_f(\theta) = \frac{y'(k)k}{y(k)} \left( \frac{rk+c}{rk} \right). \tag{136}$$

Since $c > 0$, we have

$$\eta_f(\theta) > \varepsilon_y(k) \tag{137}$$

at any point satisfying the first-order conditions. Since $\eta_f(\theta) > \varepsilon_y(k)$,

$$\frac{\eta_f(\theta)}{1-\eta_f(\theta)} = \frac{1}{\frac{1}{\eta_f(\theta)} - 1} > \frac{1}{\frac{y(k)}{y'(k)k} - 1}, \tag{138}$$

or, equivalently,

$$\frac{\eta_f(\theta)}{1-\eta_f(\theta)} > \frac{y'(k)k}{y(k) - y'(k)k} \tag{139}$$

To prove (134), it therefore suffices to show that

$$\sigma_f(\theta) \leq \frac{-y''(k)y(k)k}{y'(k)(y(k)-y'(k)k)}. \tag{140}$$

But the right-hand side is just the reciprocal of the elasticity of substitution, $\sigma_y(k)$, of the function $y(.)$. So the condition we require is

$$\sigma_f(\theta)\sigma_y(k) \leq 1, \tag{141}$$

which is true. We have $\sigma_f(\theta) \leq 1$ by assumption and $\sigma_y(k) \leq 1$ since we assume that $\varepsilon_y'(k) \leq 0$ and $\sigma_y(k) \leq 1$ is equivalent to $\varepsilon_y'(k) \leq 0$. We therefore have uniqueness of the social planner's solution $(k^P, \theta^P)$ where $k^P = k(\theta^P)$.

**Existence.** Next, we establish existence of $(k^P, \theta^P)$. Rearranging (128),

$$L(\theta) = \frac{y(k(\theta))f(\theta)}{\theta}\left(\frac{f'(\theta)\theta}{f(\theta)} - \frac{y'(k(\theta))k(\theta)}{y(k(\theta))}\right) \tag{142}$$

Or, equivalently,

$$L(\theta) = \frac{y(k(\theta))f(\theta)}{\theta}\big(\eta_f(\theta) - \varepsilon_y(k)\big) \tag{143}$$

From (137), we know that $\eta_f(\theta) > \varepsilon_y(k)$. We also know that $\eta_f(\theta) \in (0,1)$ and $\varepsilon_y(k) \in (0,1)$, so $\eta_f(\theta) - \varepsilon_y(k) \in (0,1)$. We therefore have

$$\lim_{\theta\mapsto\infty} L(\theta) = \lim_{\theta\mapsto\infty} \frac{y(k(\theta))f(\theta)}{\theta}\big(\eta_f(\theta) - \varepsilon_y(k)\big) = 0 \tag{144}$$

using the fact that $\lim_{\theta\mapsto\infty} \frac{y(k(\theta))f(\theta)}{\theta} = \lim_{\theta\mapsto\infty} y(k(\theta))f'(\theta) = 0$ since $\lim_{\theta\mapsto\infty} f'(\theta) = 0$ and $\lim_{\theta\mapsto\infty} y(k(\theta)) = \lim_{k\mapsto 0} y(k) = 0$ since $k(\infty) = 0$. We also have

$$\lim_{\theta\mapsto 0} L(\theta) = \lim_{\theta\mapsto 0} \frac{y(k(\theta))f(\theta)}{\theta}\big(\eta_f(\theta) - \varepsilon_y(k)\big) > \big(\eta_f(\theta) - \varepsilon_y(k)\big)\, y(k(0)) \tag{145}$$

where $k(0) > 0$, using the fact that $\lim_{\theta\mapsto 0} \frac{f(\theta)}{\theta} = \lim_{\theta\mapsto 0} f'(\theta)$ by L'Hopital's rule and $\lim_{\theta\mapsto 0} f'(\theta) \geq 1$ by assumption. Therefore, if $c < \lim_{\theta\mapsto 0} \big(\eta_f(\theta) - \varepsilon_y(k)\big)\, y(k(\theta))$, there exists a unique $\theta^P > 0$ and $k^P = k(\theta^P) > 0$. Otherwise, $\theta^P = 0$ and $k^P = k(\theta^P) = k(0) > 0$.

It remains only to prove that the unique solution $(k^P, \theta^P)$ is a global maximizer for $\Omega(k, \theta)$. Consider the Hessian matrix $H$ of partial derivatives of $\Omega(k, \theta)$. If $\det H > 0$ and $\Omega_{\theta\theta}(k, \theta) < 0$ at $(k^P, \theta^P)$, then it is a unique local minimum and we need only check the boundaries to ensure it is also a global maximum. Substituting into (54), we find that $\Omega(k, 0) = 0$ and $\Omega(0, \theta) = -c\theta < 0$. Since $\Omega(k^P, \theta^P) > 0$ when $\theta^P > 0$ and $k^P > 0$, $(k^P, \theta^P)$ is the unique global maximizer for $\Omega(.)$ provided that it is the unique local minimum.

Using (122) and (123), the partial derivatives of $\Omega(k, \theta)$ are

$$\begin{aligned} \Omega_{kk}(k, \theta) &= f(\theta)y''(k) \\ \Omega_{\theta\theta}(k, \theta) &= f''(\theta)y(k) \\ \Omega_{k\theta}(k, \theta) &= f'(\theta)y'(k) - r \end{aligned} \tag{146}$$

Clearly, $\Omega_{\theta\theta}(k, \theta) < 0$ if $k > 0$ since $f''(\theta) < 0$ by assumption. We have $\det H > 0$ if and only if

$$\Omega_{\theta\theta}(k, \theta)\Omega_{kk}(k, \theta) - (\Omega_{k\theta}(k, \theta))^2 > 0. \tag{147}$$

Substituting in the partial derivatives (146), $\det H > 0$ if and only if

$$f''(\theta)y(k)f(\theta)y''(k) > (f'(\theta)y'(k) - r)^2. \tag{148}$$

Using the first-order condition (122), we have $r = f(\theta)y'(k)/\theta$, so we require

$$f''(\theta)y(k)f(\theta)y''(k) > \left(f'(\theta)y'(k) - \frac{f(\theta)y'(k)}{\theta}\right)^2. \tag{149}$$

With some algebra, this is equivalent to

$$\sigma_f(\theta) < \frac{-y(k)y''(k)}{(y'(k))^2}\frac{\eta_f(\theta)}{1 - \eta_f(\theta)} \tag{150}$$

which is identical to inequality (134) that is proven above.

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