Low Real Interest Rates and the Zero Lower Bound

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Abstract

How do low real interest rates constrain monetary policy? Is the zero lower bound optimal if the real interest rate is sufficiently low? A model is constructed that incorporates sticky price frictions, along with an array of assets rich enough to capture how monetary policy works in practice. The model has neo-Fisherian properties. Forward guidance in a liquidity trap works through the promise of higher future inflation, generated by a higher future nominal interest rate. With binding collateral constraints, the real interest rate is low, but the optimal nominal interest rate is positive.

1 Introduction

The purpose of this paper is to develop a simple macroeconomic model with some key frictions that can address the following questions: (i) Why are real rates of interest low, and how does that matter for monetary policy? (ii) If low real interest rates imply that monetary policy is constrained by the zero lower bound (ZLB) on the nominal interest rate, is there a role for central bank forward guidance, and what is it? (iii) How do frictions that give rise to a role for assets in exchange matter relative to sticky price frictions in formulating monetary policy at the ZLB?

By any measure, real rates of interest on government debt have been declining in the world since the early 1980s. For example, Figure 1 shows a short-term real interest rate for the United States, measured as the three-month Treasury bill rate minus the 12-month rate of increase in the personal consumption deflator, for the period 1980-2016. By this measure, the real interest rate has decreased on trend since the 1981-1982 recession, and has been particularly low following the 2008-2009 recession.

*The views expressed are those of the author and do not necessarily reflect official positions of the Federal Reserve Bank of St. Louis, the Federal Reserve System, or the Board of Governors.

There is now an extensive New Keynesian literature that analyzes monetary policy at the ZLB – a "liquidity trap." Two key (and closely-related) papers in this literature are Eggertsson and Woodford (2003) and Werning (2011). These authors use sticky price models to capture a low-real-interest-rate environment that results from a temporary fall in the subjective rate of time preference – a fall in the "natural rate of interest." The key findings are:

1. If the central banker in the model cannot commit, this creates an inefficiency in the low-real-interest-rate liquidity trap state. With the nominal interest rate at zero, the extent of the inefficiency at the beginning of the liquidity trap period rises as prices become *less* sticky, and as the length of the liquidity trap period increases.

2. If the central banker in the model can commit, then forward guidance – promises concerning the path for the nominal interest rate after the liquidity trap period ends – is effective.

3. Optimal forward guidance takes the form of promises to keep the nominal interest rate low after the liquidity trap period ends. This forward guidance produces higher inflation and output than what the central banker would choose if he or she could not commit.

While Eggertsson and Woodford (2003) and Werning (2011) focus on the use of forward guidance when the ZLB is a binding constraint for the central banker, other solutions to the ZLB problem have been suggested, and some of these have been implemented in practice. Such solutions include: (i) raising the central bank's inflation target; (ii) quantitative easing (QE); (iii) negative nominal interest rates; (iv) helicopter money. It is certainly important to understand the effects of QE, negative nominal interest rates, and helicopter money, but this paper will focus on forward guidance and inflation targeting. The effects of QE and large central bank balance sheets are studied in detail in Williamson (2014, 2015, 2016).

A primary goal in this paper is to address these monetary policy issues in a framework that is amenable to standard discrete-time analytical techniques, without the need for numerical simulation. To this end, we start with a simple representative-household sticky-price model that permits a straightforward analysis of optimal policy. The model has standard New Keynesian features, in that nominal government bonds are priced in the usual way, yielding what New Keynesians call an "IS" relationship. There is also Phillips curve tradeoff. There is a sticky price inefficiency, and zero inflation is optimal, provided that monetary policy is unconstrained by the zero lower bound.

Part of what makes the model simple is quasilinear preferences, which does away with wealth effects. The model then becomes starkly Neo-Fisherian, in that the current nominal interest rate determines the expected inflation rate. This highlights a feature of all New Keynesian models, as pointed out in particular by Cochrane (2016) and Rupert and Sustek (2016). That is, inflation dynamics in these models is essentially Fisherian – higher nominal interest rates tend to increase inflation.

The first step in the analysis is to subject the model to "natural real interest rate" shocks of the type studied by Eggertsson and Woodford (2003) and Werning (2011). In this model, we get a similar characterization of optimal policy whether the real interest rate is low because the discount factor is temporarily high (as in the New Keynesian literature), or because productivity growth is anticipated to be temporarily low. Much as in the related literature, the allocation is inefficient in the liquidity trap state if the central banker cannot commit, and welfare increases if the central banker can commit to higher inflation once the economy reverts to the non-liquidity trap state. However, in the absence of commitment, actual inflation exceeds optimal inflation in the liquidity trap state, and if the central banker provides credible forward guidance, this takes the form of a higher promised nominal interest rate in the post-liquidity trap state than would hold without commitment. These results have a neo-Fisherian tone, and are the opposite of what we find in Eggertsson and Woodford (2003) and Werning (2011), where the liquidity trap problem is too-low inflation, and forward guidance is about promising low nominal interest rates in the future non-liquidity trap state. Further, in contrast to what Werning (2011) obtains, the liquidity trap problem is not at its worst when prices are close to perfectly flexible.

New Keynesian models have been criticized for not being explicit about how monetary policy works (e.g. Williamson and Wright 2010, 2011). For this problem, this seems particularly important, as it may matter why the real interest rate is low. It certainly seems unsatisfactory to model the temporarily low real interest rate as a discount factor shock, given what is currently known about the phenomenon in practice. In particular, Krishnamurthy and Vissing-Jorgensen (2012), Andolfatto and Williamson (2015), and Caballero et al. (2016) lend empirical and theoretical support to the idea that low real interest rates on government debt can be explained by a shortage of safe collateral. To do justice to the problem at hand, it seems important to capture this, along with an explicit account of the monetary policy mechanism, in a model with a sufficiently rich set of assets.

So that we can understand what is going on, it helps to develop the model by adding complications one at a time. We first take the basic model with sticky prices, which is a Woodford-type "cashless" economy, and assume that all transactions are conducted using secured credit. For convenience, the only available collateral is government debt. If the real quantity of government debt is sufficiently small, collateral constraints bind and the real interest rate is low. This then leads to a similar liquidity trap problem to what occurs in the baseline model, but there are two sources of inefficiency instead of just the sticky price friction. While the sticky price friction causes an inefficiency only in the market for sticky-price goods, a binding collateral constraint also results in inefficiency in the market for flexible-price goods. Further, the problem is complicated by the existence of multiple equilibria. Given a particular monetary policy, if the collateral constraint binds in the temporary liquidity trap state, there can be two equilibria – one with a high inflation rate, a tighter collateral constraint, and a lower real interest rate, and one with a lower inflation rate, a less-tight collateral constraint, and a not-so-low real interest rate. Thus, if we are explicit about the source of the low real interest rate, our results do not look like the ones we get by taking a reduced form approach (e.g. modeling the liquidity trap state as arising from a high discount factor).

The final version of the model includes money, government debt, and secured credit, which allows us to be explicit about the source of the low real interest rate, *and* about open market operations. Now, we have three inefficiencies to worry about: (i) the sticky price friction, common to the first two setups; (ii) scarce collateral, as in the second setup; (iii) a Friedman-rule type inefficiency, i.e. scarcity of cash. Here, it helps to consider inefficiencies (ii) and (iii), and then add the sticky prices. In this context, open market operations are non-neutral – an open market purchase of government bonds will in general lower the real interest rate permanently, as this tightens the collateral constraint.

With flexible prices, we suppose that monetary policy is conducted optimally, treating suboptimal fiscal policy (the scarcity of government debt) as given. This turns basic New Keynesian results on their head. If collateral constraints did not bind, then a Friedman rule would be optimal – the optimal nominal interest rate would be zero. But, a shortage of government debt, reflected in a low real interest rate, implies that the nominal interest rate should be greater than zero. This implies an optimal tradeoff between the Friedman rule inefficiency and the scarce collateral inefficiency.

Finally, if we introduce sticky prices into this model with money and bonds, these results do not go away. We can show that tight collateral constraints typically imply that the ZLB is suboptimal.

The paper proceeds as follows. In the first section, the baseline model is constructed, and its properties are analyzed. Then, in the second section, secured credit is added to the model, followed by the third section, which adds money and open market operations.

2 Baseline Model

There is a continuum of households with unit mass, with each maximizing

$$E_0 \sum_{t=0}^{\infty} \prod_{s=0}^{t} (\beta_s) \left[u(c_t^f) + u(c_t^s) - (n_t^f + n_t^s) \right] \tag{1}$$

Here, β_t is the discount factor for period $t+1$ utility relative to period t utility. As well, c_t^f is consumption by the household of the flexible-price good, c_t^s is consumption of the sticky-price good, and n_t^f and n_t^s denote, respectively, labor the household supplies to produce the flexible-price and sticky-price good. A household cannot consume its own output. It purchases and sells goods on competitive markets.

As in mainstream New Keynesian cashless models, goods are denominated in terms of money, and money does not serve as a medium of exchange, only as a unit of account. Let P_t denote the price of flexible-price goods in units of money. The spot market in flexible-price goods clears every period, but households are technologically constrained to sell sticky-price goods at the price P_{t-1}, and must satisfy whatever demand for these goods arises at that price. Demand is assumed to be distributed uniformly among households in the sticky-price goods market. This setup is equivalent to a world in which there are two physically distinct goods. For a given good, the price remains the same for two periods, and that price is competitively determined on the spot market in the first period. Also, there is staggered price-setting, in that the prices for one good are set on competitive markets in even periods, while the prices for the other good are set in odd periods. This yields the setup we have specified, with this period's flexible-price good being next period's sticky-price good.

The household produces goods using a linear technology, identical for the two goods. Output of flexible-price and fixed-price goods is $\gamma_t n_t^f$ and $\gamma_t n_t^s$, respectively, where productivity γ_t follows a first-order Markov process. The household's period t budget constraint is

$$q_t B_{t+1} + P_t c_t^f + P_{t-1} c_t^s = P_t \gamma_t n_t^f + P_{t-1} \gamma_t n_t^s + B_t, \tag{2}$$

where B_{t+1} denotes the quantity of one-period bonds acquired in period t at price q_t. Each of these bonds is a promise to deliver one unit of money in period $t+1$, so B_t denotes the payoffs from bonds acquired in period $t-1$.

In period t, the household observes γ_t and market prices, and then chooses c_t^f, c_t^s, n_t^f, and B_{t+1}. The quantity n_t^s is determined by the demand for the sticky-price good at market prices. From the first-order conditions for an optimum and market clearing in the bond market (on which the supply of bonds is zero) and in the market for flexible-price goods,

$$u'(c_t^f) = \frac{1}{\gamma_t}, \tag{3}$$

$$u'(c_t^f) = \pi_t u'(c_t^s), \tag{4}$$

$$q_t u'(c_t^f) = \beta_t E_t \left[\frac{u'(c_{t+1}^f)}{\pi_{t+1}} \right]. \tag{5}$$

Here, $\pi_t = \frac{P_t}{P_{t-1}}$ is the relative price of the two goods, and π_t is also the gross rate of increase in the the price of the flexible price good. Though π_t is the economically interesting relative price in the model, we can also calculate the measured gross inflation rate, which is

$$\mu_t = \frac{P_t c_t^f + P_{t-1} c_t^s}{P_{t-1} c_{t-1}^f + P_{t-2} c_{t-1}^s}. \tag{6}$$

Equation (3) states that exchange in the market for the flexible-price good is efficient, (4) states that the marginal rate of substitution of flexible price goods for sticky-price goods is equal to their relative price, and (5) is a standard Euler equation that prices a nominal bond. Following the New Keynesian literature, we might call (5) the IS curve, and from (3) and (4) we get

$$\frac{1}{\gamma_t} = \pi_t u'(c_t^s), \tag{7}$$

Then, from (3), output in the flexible-price sector is tied down by fundamentals (technology and preferences), and (7) specifies a Phillips curve relationship – a positive relationship between π_t and consumption in the sticky-price goods sector.

In standard New Keynesian fashion, assume that the central bank can determine q_t, with $R_t = \frac{1}{q_t} - 1$ denoting the one-period nominal interest rate.

2.1 Equilibrium

From (3) and (5) we obtain

$$q_t = \beta_t E_t \left[\frac{\gamma_t}{\gamma_{t+1}\pi_{t+1}} \right] \tag{8}$$

Equation (8) determines the stochastic process $\{\pi_{t+1}\}_{t=0}^{\infty}$ given a central bank policy specifying a rule for q_t. Then we can determine consumption of the sticky-price good from (7). As is typical in New Keynesian cashless models, π_0 is indeterminate, but we could use arguments from the fiscal theory of the price level literature (e.g. Leeper 1991) to determine π_0.

From equation (8), the current nominal interest rate, and anticipated productivity growth, determine anticipated future inflation. To make this clearer, if we take a linear approximation to (8), we obtain

$$E_t i_{t+1} = R_t - \rho_t - E_t g_{t+1},$$

where i_t is the inflation rate, R_t is the nominal interest rate, ρ_t is the subjective discount rate ($\frac{1}{1+\rho_t} = \beta_t$), and g_t is productivity growth. Thus, this economy fundamentally behaves in a neo-Fisherian fashion. In a world in which the monetary policy instrument is a market nominal interest rate, nominal interest rates cause future inflation.

2.2 Optimal Policy When the Zero Lower Bound is not Binding

We will first characterize an optimal monetary policy rule in the case where the zero lower bound (ZLB) constraint on the nominal interest rate does not bind, as a benchmark case, before we go on to examine cases where the ZLB matters.

A social planner seeking to maximize household utility in this economy need only solve a sequence of period-by-period static problems, i.e.

$$\max_{c_t^f, c_t^s} \left[u(c_t^f) + u(c_t^s) - \frac{(n_t^f + n_t^s)}{\gamma_t} \right],$$

and the first-order conditions for an optimum are

$$u(c_t^i) = \frac{1}{\gamma_t}, \text{ for } i = f, s. \tag{9}$$

What is an optimal monetary policy rule? Any such rule must satisfy the zero lower bound (ZLB) constraint

$$q_t \leq 1 \tag{10}$$

for all t. If we ignore the ZLB constraint, then it is straightforward to determine an optimal policy rule. From (3) and (4), if

$$\pi_t = 1$$

for all t, then this will support the social planner's optimum in equilibrium. From (8), if we substitute $\pi_t = 1$ for all t, then

$$q_t = \beta_t E_t \left[\frac{\gamma_t}{\gamma_{t+1}} \right] \tag{11}$$

is an optimal policy rule, provided that this rule satisfies the ZLB constraint, or

$$\beta_t E_t \left[\frac{\gamma_t}{\gamma_{t+1}} \right] \leq 1.$$

The optimal policy rule (11) should be familiar from the New Keynesian literature, as it states that the nominal interest rate is equal to the "natural real rate of interest."

There is a problem, though. Given the policy rule (11), from (8),

$$E_t \left[\frac{1}{\gamma_{t+1}} \right] = E_t \left[\frac{1}{\gamma_{t+1}\pi_{t+1}} \right] \tag{12}$$

Then, given the policy rule, an equilibrium is a stochastic process for $\{\pi_t\}$ satisfying (12). Clearly, $\pi_t = 1$ for all t satisfies (12), but there may be other equilibria, given the policy rule. For example, suppose that γ_t is i.i.d., and we consider equilibria for which π_t is i.i.d. and independent of γ_t. Then, from (12) π_t must satisfy

$$E_t \left[\frac{1}{\pi_{t+1}} \right] = 1.$$

Thus, in the case of i.i.d. productivity shocks, there are many equilibria, and any equilibrium for which the distribution of $\frac{1}{\pi_t}$ is not degenerate is suboptimal. Thus, the policy rule (11) is optimal, in that it always supports an efficient equilibrium, provided the ZLB constraint is not violated. But there are cases for which there exist other suboptimal equilibria, given this rule. In specific cases, we can solve this indeterminacy problem by designing a more sophisticated policy rule, as we will show.

2.2.1 Technology Shocks

Next, consider a special case. Assume that the discount factor is $\beta_t = \beta$, a constant, for all t, and that productivity is currently high, and is expected to revert to a low level forever sometime in the future. Therefore, in this setup current expected productivity growth is low, so that the real interest rate is low, but productivity growth is expected to revert to zero sometime in the future, with a correspondingly higher real interest rate. To be more specific, there are two states for productivity, γ^h and γ^l with $\gamma^h > \gamma^l$. Assume that the initial state is $\gamma_0 = \gamma^h$, and that

$$\Pr[\gamma_{t+1} = \gamma^h \mid \gamma_t = \gamma^h] = \rho,$$

where $0 < \rho < 1$, and that the low productivity state is an absorbing state, i.e.

$$\Pr \Pr[\gamma_{t+1} = \gamma^l \mid \gamma_t = \gamma^l] = 1.$$

This stochastic process was chosen specifically to capture, in a straightforward way, a temporary state of the world in which the central bank could be faced with a low real interest rate and the possibility that the ZLB constraint could bind. First, though, we want to characterize an optimal policy when the central bank does not encounter the ZLB.

Solving for relative prices in the high-productivity and low-productivity states, π^h and π^l, respectively, from (8) we obtain:

$$\pi^h = \frac{\beta\rho}{q^h - (1-\rho)q^l \frac{\gamma^h}{\gamma^l}}, \tag{13}$$

$$\pi^l = \frac{\beta}{q^l}, \tag{14}$$

where q^i, $i = h, l$, denotes the policy choice of the bond price in high and low productivity states, respectively. As in the general case, if policy can achieve an equilibrium in which $\pi^h = \pi^l = 1$, then such a policy is optimal, provided that the policy satisfies the ZLB constraints

$$q^i \leq 1 \text{ for } i = h, l. \tag{15}$$

From (13) and (14) we can solve for the optimal policy, which is

$$q^h = \beta \left[\rho + (1-\rho)\frac{\gamma^h}{\gamma^l} \right] = q^{h*} \tag{16}$$

$$q^l = \beta = q^{l*} \tag{17}$$

From (13), (14), (16), and (17), we do not have an indeterminacy problem here. That is, the optimal policy described by (16) and (17) supports a unique efficient equilibrium, so long as the ZLB constraints (15) hold. Since $\gamma^h > \gamma^l$, the ZLB constraints hold if and only if

$$\frac{\gamma^h}{\gamma^l} \leq \frac{1-\beta\rho}{\beta\,(1-\rho)}. \tag{18}$$

Thus, the ZLB constraint does not bind if and only if the drop in productivity when reversion takes place is sufficiently small. So, assuming that (18) holds, the nominal interest rate is low while productivity is high, and then reverts to a higher level. Basically, the inflation rate is zero at the optimum, so the optimal nominal interest rate must track the real interest rate, which is low when productivity is high. That is, the "natural" interest rate is given by $\frac{1}{s_t}-1$, where s_t is the price a household would pay currently for a claim to one unit of consumption in the next period, if prices were flexible. Then, if s^i, $i=h,l$, denotes the price of such a claim in the high-productivity and low-productivity states, respectively, then

$$s^h = \beta\left[\rho + (1-\rho)\frac{\gamma^h}{\gamma^l}\right],$$

and $s^l = \beta$. Therefore, the natural rate is low in the high-productivity state, and then reverts to a higher value.

2.2.2 Discount Factor Shocks

The low natural rate could also arise because of a preference shock, i.e. a high discount factor, as in some of the New Keynesian literature. Suppose for example that $\gamma_t = \gamma$, a constant, for all t, and that, rather than a high-productivity state that reverts to a low-productivity state, there is a high-discount-factor state with $\beta_t = \beta^h$ that ultimately reverts to a low-discount-factor state with $\beta_t = \beta^l$, where $\beta^h > \beta^l$. Then, in a similar fashion to the solution with productivity shocks, the solution for relative prices in each state is:

$$\pi^h = \frac{\beta^h\rho}{q^h - (1-\rho)q^l\frac{\beta^h}{\beta^l}}, \tag{19}$$

$$\pi^l = \frac{\beta^l}{q^l}, \tag{20}$$

In this case, an optimal monetary policy when the ZLB constraints (15) do not bind is

$$q^i = \beta^i, \text{ for } i=h,l, \tag{21}$$

so the ZLB constraints do not bind if and only if

$$\beta^h \leq 1. \tag{22}$$

As with technology shocks, the optimal policy is a low nominal interest rate when the natural rate is low.

So far this is straightforward, and in line with typical New Keynesian models. Price stability is optimal, as this implies that there are no relative price distortions arising from sticky prices. But, in the context of aggregate shocks, active monetary policy is necessary to induce price stability. Further, for either productivity shocks or preference shocks, the optimal policy implies that the nominal interest rate is low when the natural real interest rate is low. Finally, note that the optimal monetary policy is time consistent when the ZLB constraint does not bind. That is, once the natural interest rate reverts to its long-run higher value, the central bank has no incentive to deviate from its promise to increase the nominal interest rate when the natural rate goes up.

2.3 Optimal Policy with a Binding ZLB Constraint

In this section, we will determine optimal monetary policies with productivity shocks or preference shocks, for cases in which the ZLB constraint binds. First, suppose that $\beta_t = \beta$, and there are productivity shocks as specified in the previous section. Also assume that (18) does not hold, so that the ZLB constraint binds in the high-productivity state, i.e. $q^h = 1$.

First, consider the case in which the central bank cannot commit. Then, when productivity reverts to the low state, $\gamma_t = \gamma^l$, it is optimal for the central bank to choose $q^l = \beta$, and $q^l = 1$, so from (13) and (14), $\pi^l = 1$ and

$$\pi^h = \frac{\beta\rho}{1 - (1-\rho)\beta\frac{\gamma^h}{\gamma^l}}. \tag{23}$$

Since (18) does not hold, we have $\pi^h > 1$, i.e. the binding ZLB constraint and lack of commitment implies that the inflation rate is higher than it would be at the optimum if the ZLB constraint did not bind. This is quite different from Werning (2011), who argues that a binding ZLB constraint in related circumstances will lead to inflation below the central bank's target.

Next, assume that the central bank can commit in period 0 to a policy q^l when productivity reverts to its lower value. We can interpret this as a form of forward guidance, which will work so long as the central bank's promises are credible. The central bank's problem is to commit to a policy that maximizes the welfare of the household at in the high-productivity state. That is, the central bank solves

$$\max_{q^l} \left\{ (1-\beta)\left[u(c^h) - \frac{c^h}{\gamma^h}\right] + \beta(1-\rho)\left[u(c^l) - \frac{c^l}{\gamma^l}\right] \right\}, \tag{24}$$

where c^h and c^l denote, respectively, consumption of sticky-price goods in the high and low-productivity states (note that flexible price consumption is always efficient). From (7), (13), (14), and (16), c^h and c^l solve

$$1 = \beta\left[\rho\gamma^h u'(c^h) + (1-\rho)\frac{\gamma^h q^l}{\gamma^l \beta}\right], \tag{25}$$

$$u'(c^l) = \frac{q^l}{\gamma^l \beta}. \tag{26}$$

Thus, the central bank's problem is to solve (24) subject to (25) and (26). From (25) and (26) it is clear that c^h is an increasing function of q^l, while c^l is a decreasing function of q^l. If we let $\hat{q} \equiv \frac{\gamma^l(1-\beta\rho)}{\gamma^h(1-\rho)}$, then from (18), (**??**), (25), and (26), equilibrium welfare is strictly increasing in q^l for $q^l \in (0, \hat{q}]$, and welfare is strictly decreasing for $q^l \in [\beta, 1]$. Further, note that, since (18) does not hold, therefore $\hat{q} < \beta$. Thus, welfare is maximized for $q^l \in (\hat{q}, \beta)$.

The optimal policy therefore implies, from (3) and (4), that $\pi^h > 1$ and $\pi^l > 1$, so the best policy at the zero lower bound is a promise of high inflation in the future when productivity reverts to its lower value. Note that this higher inflation is achieved with a higher nominal interest rate than the central bank would choose if it could not commit. That is, the optimal policy for the central bank, without commitment, once productivity reverts to its low value, is $q^l = \beta$, but the policy the central bank wants to commit to when in the high-productivity state is $q^l < \beta$. Thus, the optimal policy for the central bank is achieved with a neo-Fisherian commitment.

An issue that arises, for example in Werning (2011), is what happens as we vary price flexibility. In the ZLB problem that Werning (2011) specifies, as price flexibility increases, the zero lower bound problem gets worse. That is, with only a small amount of price flexibility, the welfare loss from the distortion that results at the ZLB is larger than if prices were more sticky.

The degree of price flexibility in this model is determined by the length of the period, but in adjusting the length of the period, we have to make appropriate adjustments in the discount factor and the probability of reversion to the low-productivity state. That is, shortening the period implies that the discount factor should increase and the probability of reversion to the low-productivity state should decrease. Letting Δ denote the length of a period, we let $\beta = e^{-r\Delta}$, where r is the discount rate per unit time. As well, if reversion to the low-productivity state is a Poisson arrival with arrival rate α, then $\rho = e^{-\alpha\Delta}$. From (18) the zero lower bound on the nominal interest rate is a binding constraint for the central banker if and only if

$$1 < \beta \left[\rho + (1-\rho)\frac{\gamma^h}{\gamma^l} \right],$$

or

$$1 \leq \frac{\gamma^h}{\gamma^l} e^{-r\Delta} - \left(\frac{\gamma^h}{\gamma^l} - 1 \right) e^{-(r+\alpha)\Delta} \tag{27}$$

Then, let

$$\phi(\Delta) = \frac{\gamma^h}{\gamma^l} e^{-r\Delta} - \left(\frac{\gamma^h}{\gamma^l} - 1 \right) e^{-(r+\alpha)\Delta}$$

So,

$$\phi(0) = 1$$

$$\lim_{\Delta \to \infty} \phi(\Delta) = 0$$

$$\phi'(\Delta) = -r\frac{\gamma^h}{\gamma^l}e^{-r\Delta} + (r+\alpha)\left(\frac{\gamma^h}{\gamma^l}-1\right)e^{-(r+\alpha)\Delta} \tag{28}$$
$$= e^{-r\Delta}\left[-r\frac{\gamma^h}{\gamma^l} + (r+\alpha)\left(\frac{\gamma^h}{\gamma^l}-1\right)e^{-\alpha\Delta}\right]$$

So,

$$\phi'(0) = \left[-r\frac{\gamma^h}{\gamma^l} + (r+\alpha)\left(\frac{\gamma^h}{\gamma^l}-1\right)\right]$$
$$= \alpha\left(\frac{\gamma^h}{\gamma^l}-1\right) - r$$

So, if

$$\alpha\left(\frac{\gamma^h}{\gamma^l}-1\right) - r \leq 0,$$

then (27) does not hold for any Δ, and the ZLB is not a problem. However, if

$$\alpha\left(\frac{\gamma^h}{\gamma^l}-1\right) - r > 0,$$

or

$$\frac{\alpha}{r} - \frac{\gamma^l}{\gamma^h - \gamma^l} > 0$$

then $\phi'(\Delta) > 0$ for $\Delta \in [0, \Delta^*)$, and $\phi'(\Delta) < 0$ for $\Delta > \Delta^*$, and (27) holds for $\Delta \in (0, \bar{\Delta})$, where $\phi(\bar{\Delta}) = 1$. Further, price stickiness is worst for $\Delta = \Delta^*$, where $\phi'(\Delta^*) = 0$, so from (28),

$$-r\frac{\gamma^h}{\gamma^l} + (r+\alpha)\left(\frac{\gamma^h}{\gamma^l}-1\right)e^{-\alpha\Delta^*} = 0$$

or

$$e^{-\alpha\Delta^*} = \frac{r\frac{\gamma^h}{\gamma^l}}{(r+\alpha)\left(\frac{\gamma^h}{\gamma^l}-1\right)}$$
$$= \left(\frac{r}{r+\alpha}\right)\left(\frac{\gamma^h}{\gamma^h-\gamma^l}\right)$$

or

$$e^{\alpha\Delta^*} = \left(1+\frac{\alpha}{r}\right)\left(1-\frac{\gamma^l}{\gamma^h}\right)$$

or

$$\Delta^* = \frac{\ln\left(1+\frac{\alpha}{r}\right) + \ln\left(1-\frac{\gamma^l}{\gamma^h}\right)}{\alpha}$$

And there is no discontinuity at zero, which is very different from Werning's model.

Next, consider the case in which $\gamma_t = \gamma$, a constant, for all t, and there is a high discount factor instead of high productivity in the low natural real interest rate state. In this case, we write the central bank's problem as

$$\max_{q^l} \left\{ (1-\beta^l) \left[u(c^h) - \frac{c^h}{\gamma^h} \right] + \beta^h (1-\rho) \left[u(c^l) - \frac{c^l}{\gamma^l} \right] \right\}, \tag{29}$$

Then, suppose (22) does not hold, so that $q^h = 1$ at the optimum (the ZLB constraint binds in the high state). Then, from (19), (20), (7), and (8), the two equations

$$1 = \beta^h \left[\rho\gamma u'(c^h) + \frac{(1-\rho)\, q^l}{\beta^l} \right], \tag{30}$$

$$u'(c^l) = \frac{q^l}{\gamma\beta^l}, \tag{31}$$

solve for c^h and c^l given a monetary policy q^l. We can then obtain a result that is qualitatively identical to that with productivity shocks. In particular, if (22) does not hold, this implies that the optimal policy is $q^l \in (\tilde{q}, \beta^l)$, where

$$\tilde{q} = \frac{(1-\beta^h\rho)\beta^l}{(1-\rho)\beta^h},$$

and optimal policy implies $\pi^h > 1$, $\pi^l > 1$, and a commitment that the nominal interest rate will be higher when the natural interest rate rises than would be the case without commitment.

On some dimensions, our conclusions in this subsection are in line with standard New Keynesian cashless reduced-form models, in particular Werning (2011). In particular, if the real interest rate is temporarily low due to some exogenous factor – for example a temporary high discount factor – and the ZLB binds, then this implies that an optimal forward guidance policy is to promise that inflation will be higher in the future than would be the case if the central bank could not commit. A key difference here is that high inflation is achieved in the future with a nominal interest rate setting that is higher than it would be in the absence of commitment. As well, the commitment problem is not at its worst with a high degree of flexibility in prices.

3 Credit, Collateral, and Low Real Interest Rates

In the previous section, we included exogenous shocks that would imply a binding ZLB constraint, due to a temporarily low real interest rate. But, what if we are more explicit about the reasons for a low real interest rate? One potential explanation for the low real interest rates that have been observed recently in the world is that there is a low supply of safe assets relative to the demand for such assets (see Andolfatto and Williamson 2015 and Caballero et al. 2016). We can model this safe asset shortage by including an explicit role for government debt in the model. For now, we will retain the assumption that money is in zero supply – the economy is cashless. However, all transactions in the goods market are assumed to be secured credit transactions, in which government debt serves as collateral. An interpretation of this arrangement is that there are banks which issue deposits and hold government debt as assets, and bank deposits are used in transactions.

Assume that the representative household receives a lump-sum transfer τ_t from the government in period t, so we can write the government's budget constraints as

$$q_0 b_0 = \tau_0.$$

$$q_t b_t = \frac{b_{t-1}}{\pi_t} + \tau_t, \text{ for } t = 1, 2, ... \tag{32}$$

Also assume that the fiscal authority sets exogenously the total value of government debt, V_t, so

$$V_t = q_t b_t, \tag{33}$$

which implies that transfers are endogenous in periods $t = 1, 2, ...$. Re-write the household's budget constraint to incorporate the transfer:

$$q_t b_{t+1} + c_t^f + \frac{c_t^s}{\pi_t} = \gamma_t n_t^f + \frac{\gamma_t n_t^s}{\pi_t} + \frac{b_t}{\pi_t} + \tau_t \tag{34}$$

Assume that, within the period, goods must be purchased with credit. For example, suppose that each household is a buyer/seller pair. At the beginning of the period, the buyer in the household purchases goods with IOUs, while the seller exchanges goods for IOUs. Then, within-period debts are settled at the end of the period. Also assume that households cannot commit, and that there is no memory. In particular, no records can be kept of past defaults. This implies that there can be no unsecured credit. But, there exists a technology which allows households to post government debt as collateral. Then, the following incentive constraint must be satisfied

$$c_t^f + \frac{c_t^s}{\pi_t} \leq \hat{q}_t b_{t+1}, \tag{35}$$

where $\hat{q}_t$ denotes the price of government debt at the end of the period. The inequality (35) states that the value of purchases of consumption goods (in units of the flexible-price good) cannot exceed the value of the collateral posted by the household. Here the value of the collateral is assessed as the value to the household at the end of the period. In other words, the household must post sufficient collateral that it has the incentive to pay off its debts at the end of the period rather than absconding.

For simplicity, assume that $\gamma_t = 1$ and $\beta_t = \beta$, a constant, for all t. Then, letting μ_t and λ_t denote, respectively, the multipliers associated with (34) and (35), the following must be satisfied:

$$u'(c_t^f) - \mu_t - \lambda_t = 0, \tag{36}$$

$$u'(c_t^s) - \frac{(\mu_t + \lambda_t)}{\pi_t} = 0, \tag{37}$$

$$-1 + \mu_t = 0, \tag{38}$$

$$-q_t\mu_t + \lambda_t\hat{q}_t + \beta E_t\left[\frac{\mu_{t+1}}{\pi_{t+1}}\right] = 0. \tag{39}$$

The value of government bonds at the end of the period is

$$\hat{q}_t = \beta E_t\left[\frac{\mu_{t+1}}{\pi_{t+1}}\right]. \tag{40}$$

Then, (38), (39) and (40) give

$$q_t = (1 + \lambda_t)\,\hat{q}_t, \tag{41}$$

so the price of government debt at the beginning of the period exceeds its value at the end of the period if and only if the collateral constraint binds ($\lambda_t > 0$).

From (33), (35), (36)-(39), and (41), if the collateral constraint (35) does not bind in period t, then

$$u'(c_t^f) = 1, \tag{42}$$

$$u'(c_t^s) = \frac{1}{\pi_t} \tag{43}$$

$$-q_t + \beta E_t\left[\frac{1}{\pi_{t+1}}\right] = 0 \tag{44}$$

$$c_t^f + \frac{c_t^s}{\pi_t} \leq V \tag{45}$$

However, if the collateral constraint binds in period t, then

$$u'(c_t^f) - \pi_t u'(c_t^s) = 0, \tag{46}$$

$$-q_t + u'(c_t^f)\beta E_t\left[\frac{1}{\pi_{t+1}}\right] = 0 \tag{47}$$

$$c_t^f + \frac{c_t^s}{\pi_t} = \frac{V}{u'(c_t^f)} \tag{48}$$

$$u'(c_t^f) - 1 \geq 0 \tag{49}$$

Note, in (47), that the price of government debt reflects a liquidity premium, which increases with the inefficiency wedge in the market for the flexible price good. The inefficiency wedge is $u'(c_t^f) - 1 = \lambda_t$, the multiplier on the household's collateral constraint. Thus, the tighter is the collateral constraint, the larger is the inefficiency wedge, and the higher is the liquidity premium on government debt.

3.1 Optimal Monetary Policy When the Collateral Constraint is Tight

Since this model potentially has quite different implications from the one in the previous section, we will start with the simplest case.
Suppose that V is constant for all t, and $q_t = q$ for all t. Look for an equilibrium in which all quantities and the relative price of sticky-price and flexible price goods are constant for all t, and suppose that V is sufficiently small that the collateral constraint always binds. Then, from (46)-(49), an equilibrium consists of c^f, c^s, and π satisfying

$$u'(c^f) - \pi u'(c^s) = 0, \tag{50}$$

$$-q + \frac{u'(c^f)\beta}{\pi} = 0, \tag{51}$$

$$c^f + \frac{c^s}{\pi} = \frac{V}{u'(c^f)}, \tag{52}$$

$$u'(c^f) - 1 \geq 0, \tag{53}$$

given monetary policy q. Simplifying, from (50)-(52), the consumption allocation (c^f, c^s) solves

$$q = \beta u'(c^s), \tag{54}$$

$$c^f u'(c^f) + c^s u'(c^s) = V. \tag{55}$$

Next, restrict attention to constant-relative-risk-aversion utility, where α denotes the coefficient of relative risk aversion. Further, assume that $0 < \alpha < 1$. Roughly, this assumption implies that the aggregate demand for collateral is strictly increasing with consumption. Then, from (54) and (55), we can write the monetary policy problem as

$$\max_q \left[\frac{\left(c^f\right)^{1-\alpha}}{1-\alpha} - c^f + \frac{(c^s)^{1-\alpha}}{1-\alpha} - c^s \right]$$

subject to

$$q = \beta (c^s)^{-\alpha}, \tag{56}$$

$$q \leq 1 \tag{57}$$

$$\left(c^f\right)^{1-\alpha} + (c^s)^{1-\alpha} = V \tag{58}$$

subject to (56) and (58).

If the ZLB constraint (56) does not bind, then the solution is

$$c^s = c^f = \left(\frac{V}{2}\right)^{\frac{1}{1-\alpha}},$$

$$q = \beta \left(\frac{2}{V}\right)^{\frac{\alpha}{1-\alpha}}, \tag{59}$$

$$\pi = 1,$$

Then, from (59), the ZLB constraint does not bind if and only if

$$V \geq 2\beta^{\frac{1-\alpha}{\alpha}},$$

Also, note that, from (49), the collateral constraint binds if and only if

$$V < 2,$$

so the ZLB does not bind in this constrained equilibrium if and only if

$$2\beta^{\frac{1-\alpha}{\alpha}} \leq V < 2.$$

But if

$$V < 2\beta^{\frac{1-\alpha}{\alpha}}$$

then the zero lower bound constraint binds, and optimal policy is given by $q = 1$, with

$$c^s = \beta^{\frac{1}{\alpha}},$$

$$c^f = \left(V - \beta^{\frac{1-\alpha}{\alpha}}\right)^{\frac{1}{1-\alpha}},$$

$$\pi = \beta\left(V - \beta^{\frac{1-\alpha}{\alpha}}\right)^{\frac{-\alpha}{1-\alpha}} > 1$$

Therefore, if V is sufficiently large, then zero inflation is optimal, though there is an inefficiency wedge associated with the binding collateral constraint, i.e. $u'(c^s) = u'(c^f) > 1$. But if V is small, which implies a large inefficiency wedge and a large liquidity premium on government debt, the ZLB constraint binds, and $\pi > 1$ at the optimum. Further, a decline in V when the ZLB constraint binds at the optimum implies the inflation rate rises. That is, a tighter collateral constraint raises inflation at the optimum.

Next, to give us a scenario like the one we considered with the previous version of the model, suppose that there are two states, $V_t = V^l$ and $V_t = V^h$, with $V^l < V^h$. Assume that the economy is initially in the state with low V, and assume that the state evolves as in the previous sections. That is

$$\Pr[V_{t+1} = V^l \mid V_t = V^l] = \rho,$$

$$\Pr[V_{t+1} = V^h \mid V_t = V^h] = 1.$$

Suppose first that, given optimal monetary policy, the collateral constraint does not bind in the high-V state, but that it binds in the low-V state. Let c^{hf} and c^{hs} denote, respectively, consumption of flexible-price and fixed-price goods in the high-V state, while c^{lf} and c^{ls} are the corresponding quantities in the low-V state. Also let $\pi_t = \pi^i$, and $q_t = q^i$, where $i = h, l$ denote the high and low-V states, respectively. We will continue to assume that the utility function has a constant coefficient of relative risk aversion α.

First, if the central bank cannot commit, then it will choose $q^h = \beta$, implying $\pi^h = 1$ and $c^{hf} = c^{hs} = 1$. Then, from (46)-(48), we obtain

$$q^l = \beta\rho(c^{ls})^{-\alpha} + \beta(1-\rho)(c^{lf})^{-\alpha} \tag{60}$$

$$V^l = (c^{ls})^{1-\alpha} + (c^{lf})^{1-\alpha} \tag{61}$$

Equations (60) and (61) solve for consumption in the low-V state, (c^{lf}, c^{ls}), given monetary policy in the low-V state, q^l. Similar to the case with constant V and a binding collateral constraint, if

$$2\beta^{\frac{1-\alpha}{\alpha}} \leq V^l < 2,$$

then there is an optimal monetary policy in the low-V state given by

$$q^l = \beta \left(\frac{V}{2}\right)^{-\frac{\alpha}{1-\alpha}}$$

which yields an equilibrium

$$c^{lf} = c^{ls} = \left(\frac{V^l}{2}\right)^{\frac{1}{1-\alpha}}$$

In this case, the collateral constraint binds in the low-V state, but a policy supporting an optimal allocation (given a lack of commitment) does not imply a binding ZLB constraint. However, the policy that supports such an equilibrium may also imply the existence of another equilibrium, which is suboptimal.

Next, suppose that the central bank can commit to a policy $\left(q^l, q^h\right)$. Then, from (42)-(44), in the high-V state,

$$c^{hf} = 1 \tag{62}$$

$$c^{hs} = \left(\frac{\beta}{q^h}\right)^{\frac{1}{\alpha}} \tag{63}$$

$$\pi^h = \frac{\beta}{q^h} \tag{64}$$

As well, from (45), assume that

$$1 + \left(\frac{\beta}{q^h}\right)^{\frac{1}{\alpha}-1} \leq V^h,$$

which implies that the collateral constraint does not bind in the high-V state. In the low-V state, from (46)-(48), c^{lf} and c^{ls} solve

$$-q^l + \beta\rho(c^{ls})^{-\alpha} + (1-\rho)\, q^h (c^{lf})^{-\alpha} = 0, \tag{65}$$

$$\left(c^{lf}\right)^{1-\alpha} + \left(c^{ls}\right)^{1-\alpha} = V^l \tag{66}$$

We will say that a monetary policy (q^l, q^h) is feasible if there exists a solution to (65) and (66) satisfying $c^{lf} < 1$. But, if a monetary policy is feasible, then in general there are two equilibria – a general case of what occurs when the central bank cannot commit.

What is an optimal policy? Assuming commitment is feasible, the central bank solves the problem

$$\max_{q^l, q^h} \left\{ (1-\beta) \left[\frac{\left(c^{lf}\right)^{1-\alpha}}{1-\alpha} - c^{lf} + \frac{\left(c^{ls}\right)^{1-\alpha}}{1-\alpha} - c^{ls} \right] + \beta(1-\rho) \left[\frac{\left(c^{hs}\right)^{1-\alpha}}{1-\alpha} - c^{hs} \right] \right\} \tag{67}$$

subject to (63), (65), (66),

$$q^l \leq 1 \tag{68}$$

and

$$q^h \leq 1. \tag{69}$$

Suppose that

$$V^l < 2\beta^{\frac{1}{\alpha}-1},$$

so that it is not possible to support the no-commitment policy with a nonbinding ZLB constraint in the low-V state. Then, if

$$V \geq 2\left(\beta\rho\right)^{\frac{1}{\alpha}-1},$$

then the optimal policy is $q^h \in (\hat{q}, \beta)$, where

$$\hat{q} = \frac{\left(\frac{V}{2}\right)^{\frac{\alpha}{1-\alpha}} - \beta\rho}{1-\rho}$$

Thus $q^h < \beta$ so, just as in the previous analysis, if the central bank can commit when the ZLB binds in the low-V state, it should commit to a higher nominal interest rate and higher inflation in the high-V state than if it cannot commit. If

$$\frac{\beta\rho}{(1-\beta)\,\hat{q}} < 1, \tag{70}$$

then $\pi^l > 1$, so there is more inflation in the low-V state than would be the case if the central bank were not concerned about households' welfare in the high-V state. However, if

$$\frac{\beta\rho}{(1-\beta)\,\hat{q}} > 1, \tag{71}$$

then $\pi^l < 1$ at the optimum, and there is deflation in the low-V state at the optimum.

A key difference between this example and the two examples with, respectively, state-dependent productivity and a state-dependent discount factor, is that the real interest rate is inefficiently low in this example. Indeed, the low real interest rate results from a low supply of government debt, and fiscal policy could raise the real interest rate to its efficient level by supplying more government debt. This would then eliminate the central bank's ZLB problem. Further, even if we treat the inefficiency caused by fiscal policy as given, we do not get the same conclusions about optimal monetary policy. In particular, multiple equilibria present a policy problem for the central bank when the collateral constraint binds temporarily.

4 Money, Collateral, and Credit

The next step is to analyze a full-blown model that includes the full set of assets that is important for monetary policy, and that can also explain why the real interest rate can be low in equilibrium. Here, we add monetary exchange to our model, along with secured credit, captured in the same way as in the last section. There will now potentially be three distortions to be concerned with: (i) a standard Friedman-rule distortion under which there is a suboptimally low quantity of currency, in real terms; (ii) a shortage of interest-bearing debt, reflected in a low real rate of interest; (iii) a sticky price friction. To understand how this version of the model works, it will help to first consider a setup with flexible prices, which includes only the first two distortions, followed by the sticky price case, which includes all three distortions.

4.1 Flexible Prices

This case will work in a manner similar to Andolfatto and Williamson (2015), though a key difference is in the role that government debt plays in the model. In particular, in this model government debt serves as collateral rather than being traded directly, as in Andolfatto and Williamson (2015).

We want to be explicit about how exchange works. Assume that a household consists of a continuum of consumers with unit mass, and a producer. Each consumer in the household has a period utility function $u(c_t)$, and there are two markets on which goods are sold. In the *cash-only market*, sellers of goods accept only money, as there is no technology available to verify collateral if the consumer attempts to make a credit transaction. In the *cash-and-credit market*, sellers are able to verify the ownership of government debt that is posted as collateral in a credit transaction, and sellers will also accept money. Each consumer in a household receives a shock which determines the market he or she participates in. With probability θ the consumer goes to the cash-only market, and with probability $1-\theta$, he or she goes to the cash-and-credit market. The household allocates assets to each consumer in the household – money and and any government debt to be posted as collateral – and consumers consume on the spot in the markets they go to. That is, consumption cannot be shared within the household.

The producer in the household supplies labor n_t, and can produce one unit of output for each unit of labor input. Output is perfectly divisible and can be sold on either the cash-only market or the cash-and-credit market, or both.

The preferences of each household are given by

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[\theta u(c_t^m) + (1-\theta)u(c_t^b) - n_t,\right]$$

where c_t^m denotes the consumption of each consumer who goes to the cash-only market, while c_t^b is consumption of each consumer in the cash-and-credit market.

At the beginning of the period, the household trades on the asset market and faces the constraint

$$q_t b_{t+1} + \theta c_t^m + m_t' \leq \frac{m_t + b_t}{\pi_t} + \tau_t. \tag{72}$$

On the right-hand side of inequality (72), the household has wealth at the beginning of the period consisting of the payoffs on money and bonds held over from the previous period and the lump-sum transfer from the fiscal authority. Here, m_t denotes beginning-of-period money balances in units of the period $t-1$ cash market consumption good. The left-hand side of (72) includes purchases of one-period nominal government bonds, currency (in units of the period t cash market good) the household requires for cash market goods purchases, and money, m_t', that is sent with consumers to the cash-and-credit market.

In the cash-and-credit market, consumers from the household can purchase goods with cash m_t', or with credit secured by government debt, so the following constraint must hold:

$$(1-\theta)c_t^b \leq b_{t+1} + m_t'. \tag{73}$$

In inequality (73), note that the IOUs issued by the household (by way of consumers in the household) are settled at the end of the period, at which time the bonds the household acquired at the beginning of the period are worth b_{t+1}. That is, at the end of the period, government bonds which pay off at the beginning of the subsequent period are equivalent to cash. Inequality (73) states that, for cash-and-credit purchases in excess of what is paid for with cash, the household will prefer to pay its debt at the end of the period rather than enduring seizure of the bonds posted as collateral.

Finally, the household must satisfy its budget constraint

$$\theta c_t^m + (1-\theta)c_t^b + q_t b_{t+1} + m_{t+1} \leq n_t + \frac{m_t + b_t}{\pi_t} + \tau_t. \tag{74}$$

Then, the first order conditions for an optimum are

$$u'(c_t^m) = \lambda_t^1 + \mu_t$$

$$u'(c_t^b) = \lambda_t^2 + \mu_t$$

$$-1 + \mu_t = 1$$

$$-q_t\left(\lambda_t^1 + \mu_t\right) + \lambda_t^2 + \beta E_t\left(\frac{\lambda_{t+1}^1 + \mu_{t+1}}{\pi_{t+1}}\right) = 0$$

$$-\mu_t + \beta E_t \left(\frac{\lambda^1_{t+1} + \mu_{t+1}}{\pi_{t+1}} \right) = 0$$

So,

$$u'(c^b_t) = q_t u'(c^m_t) \tag{75}$$

$$1 = \beta E_t \left[\frac{u'(c^m_{t+1})}{\pi_{t+1}} \right] \tag{76}$$

$$q_t = \frac{u'(c^b_t) - 1}{u'(c^m_t)} + \beta E_t \left[\frac{u'(c^m_{t+1})}{u'(c^m_t)\pi_{t+1}} \right] \tag{77}$$

The consolidated government budget constraints are:

$$m_1 + q_0 b_1 = \tau_0, \tag{78}$$

$$m_{t+1} + q_t b_{t+1} - \frac{m_t + b_t}{\pi_t} = \tau_t, \tag{79}$$

where m_t denotes the real quantity of currency outstanding at the beginning of period t, before government intervention occurs. Here, we will assume that that the fiscal authority fixes exogenously the path for the real value of the consolidated government debt, i.e.

$$v_t = m_{t+1} + q_t b_{t+1}, \tag{80}$$

where v_t is exogenous. Then, solving for an equilibrium, in any period t, (75) and (76) hold and either

$$u'(c^b_t) = 1$$

and

$$\theta c^m_t + (1-\theta) q_t c^b_t \leq V_t$$

or

$$u'(c^b_t) > 1$$

and

$$\theta c^m_t + (1-\theta) q_t c^b_t = V_t$$

Thus, in period t either exchange is efficient in the cash-and-credit market and the collateral constraint does not bind, or exchange is inefficient in the cash-and-credit market and the collateral constraint binds.

4.1.1 Optimality

Note that the model solves period-by-period for c^m_t and c^b_t, and thus for labor supply, as

$$n_t = \theta c^m_t + (1-\theta) c^b_t$$

Letting c^* denote the solution to $u'(c^*) = 1$, if

$$V \geq c^*,$$

then $q_t = 1$ at the optimum, and $c_t^m = c_t^b = c^*$. This is essentially a Friedman rule result. If the collateral constraint does not bind, then exchange will be efficient in the cash-and-credit market. Therefore, if $q_t = 1$, and the collateral constraint does not bind, exchange is efficient in both markets in period t.

However, if

$$V < c^*,$$

then

$$\theta c^m + (1-\theta)qc^b = V$$
$$u'(c^b) = qu'(c^m)$$

or

$$\theta c^m u'(c^m) + (1-\theta)c^b u'(c^b) - Vu'(c^m) = 0$$
$$u'(c^b) - qu'(c^m) = 0$$

So, if I totally differentiate for $q = 1$, I get

$$\{\theta\,[u' + cu''] - Vu''\}\,dc^m + (1-\theta)[u' + cu'']dc^b = 0$$
$$-dc^m + dc^b = \frac{u'}{u''}dq$$

Then,

$$\begin{aligned}
\nabla &= \theta\,[u' + cu''] - Vu'' + (1-\theta)[u' + cu''] \\
&= u' + cu'' - Vu'' \\
&= u' > 0 \\
\frac{dc^m}{dq} &= \frac{-(1-\theta)[u' + cu'']}{u''} \\
\frac{dc^b}{dq} &= \frac{\{\theta\,[u' + cu''] - Vu''\}}{u''}
\end{aligned}$$

So, if our welfare measure is

$$W = \theta\,[u(c^m) - c^m] + (1-\theta)\left[u(c^b) - c^b\right]$$

Then, evaluate the derivative for $q = 1:$

$$\begin{aligned}
\frac{\partial W}{\partial q} &= \left[\theta\frac{dc^m}{dq} + (1-\theta)\frac{dc^b}{dq}\right][u'(V) - V] \\
&= -(1-\theta)V\,[u'(V) - V] < 0
\end{aligned}$$

So, if the collateral constraint binds, a zero nominal interest rate is suboptimal. This then reverses the implications of the sticky price model we started with. With sticky prices, shocks that lower the real interest rate can make a zero nominal interest rate optimal, in which case forward guidance in the form of commitments to high future inflation (and high nominal interest rates) are also part of optimal policy. But here, forward guidance does not play a role, and the nominal interest rate is zero when the collateral constraint is not binding and the real interest rate is high. As well, the nominal interest rate should be greater than zero in states of the world in which the collateral constraint binds and the real interest rate is low.

4.2 Sticky Prices

Next, we will extend this model of money and credit to include sticky prices, as in the baseline model. Assume, as in the previous subsection, that there exists a continuum of consumers in each household. Each period, an individual consumer receives a shock that determines whether he or she receives utility from flexible-price or sticky-price goods. With probability $\frac{1}{2}$ the consumer gets utility only from the flexible price good, and with probability $\frac{1}{2}$ the consumer receives utility only from the sticky price good. As well, goods are sold in the cash-only market, and the cash-and-credit market. Each consumer in a household receives a shock each period determining which market they participate in. With probability θ the consumer goes to the cash-only market, and with probability $1-\theta$, he or she goes to the cash-and-credit market. Further, the preference shock and the shock determining market participation are independent of each other and are also independent across consumers.

On the production side, households can choose the quantities of flexible price goods to supply in each market. However, as before, the demand for sticky price goods is distributed uniformly among households, which must then supply the quantity of sticky price goods demanded at market prices. Assume in this section that there are no technology shocks – one unit of labor input produces one unit of any good.

Preferences of the household are therefore given by

$$E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \theta \left[u(c_t^{mf}) + u(c_t^{ms}) \right] + (1-\theta) \left[u(c_t^{bf}) + u(c_t^{bs}) \right] - (n_t^f + n_t^s) \right\}. \tag{81}$$

Thus, there are now four different goods: c^{mf} (c^{ms}) denotes consumption of flexible-price (fixed-price) goods that can be purchased only with money, while c^{bf} (c^{bs}) denotes consumption of flexible-price (fixed-price) goods that can be purchased with secured credit or money. At the beginning of the period, the household faces a financing constraint

$$q_t b_{t+1} + \theta \left[c_t^{mf} + \frac{c_t^{ms}}{\pi_t} \right] + m_t' \leq \frac{m_t + b_t}{\pi_t} + \tau_t. \tag{82}$$

In the constraint (82), on the right-hand side, m_t and b_t denote the money and bonds, respectively, that the households carries over from the previous period, both expressed in units of the period $t-1$ flexible-price good. On the left-hand side, the household spends on bonds b_{t+1} that pay off in period $t+1$, and on consumption goods that purchased in the cash-only market. Finally, m_t' denotes the quantity of money allocated by the household to the purchase of goods in the cash-and-credit market. Then, household consumption goods in the cash-and-credit market is constrained by

$$(1-\theta) \left[c_t^{bf} + \frac{c_t^{bs}}{\pi_t} \right] \leq m_t' + b_{t+1}. \tag{83}$$

On the right-hand side of the constraint, b_{t+1} appears because the bonds acquired at the beginning of the period can be pledged as collateral that secures credit used in purchases of the goods on the left-hand side of the constraint. Note that bonds have the same value as money at the end of the period when debts are repaid. Finally, the household's budget constraint is

$$q_t b_{t+1} + \theta \left[c_t^{mf} + \frac{c_t^{ms}}{\pi_t} \right] + (1-\theta) \left[c_t^{bf} + \frac{c_t^{bs}}{\pi_t} \right] + m_{t+1} \leq \frac{m_t + b_t}{\pi_t} + \tau_t + n_t^f + \frac{n_t^s}{\pi_t} \tag{84}$$

The government's budget constraints are the same as in the flexible-price version of the model, i.e. (78) and (79) hold. As well, the fiscal authority follows the rule (80), i.e. the real value of the consolidated government debt is set exogenously at v_t in period t.

Given optimization and market clearing, we can characterize an equilibrium as follows. In each period, the following hold:

$$1 = \beta E_t \left[\frac{u'(c_{t+1}^{mf})}{\pi_{t+1}} \right], \tag{85}$$

$$\pi_t u'(c_t^{ms}) = u'(c_t^{mf}), \tag{86}$$

$$\frac{u'(c_t^{bf})}{q_t} = u'(c_t^{mf}), \tag{87}$$

$$\frac{\pi_t}{q_t} u'(c_t^{bs}) = u'(c_t^{mf}), \tag{88}$$

As well, either

$$q_t = \beta E_t \left[\frac{u'(c_{t+1}^{mf})}{u'(c_t^{mf}) \pi_{t+1}} \right] \tag{89}$$

and

$$\theta \left[c_t^{mf} + \frac{c_t^{ms}}{\pi_t} \right] + (1-\theta) q_t \left[c_t^{bf} + \frac{c_t^{bs}}{\pi_t} \right] \leq V_t \tag{90}$$

or

$$q_t = \underbrace{\frac{u'(c_t^{bf}) - 1}{u'(c_t^{mf})}}_{\text{liquidity premium}} + \underbrace{\beta E_t \left[\frac{u'(c_{t+1}^{mf})}{u'(c_t^{mf}) \pi_{t+1}} \right]}_{\text{fundamental}}, \tag{91}$$

$$u'(c_t^{bf}) > 1 \tag{92}$$

$$\theta \left[c_t^{mf} + \frac{c_t^{ms}}{\pi_t} \right] + (1-\theta) q_t \left[c_t^{bf} + \frac{c_t^{bs}}{\pi_t} \right] = V_t \tag{93}$$

Thus, in period t, the collateral constraint (83) may not bind, in which case (89) holds – government debt sells at its fundamental price, the appropriately discounted value of the payoff stream on the asset – and (90) holds in equilibrium, i.e. the value of the consolidated government debt is large enough to finance all consumption purchases. Alternatively, (83) binds, so that there is a liquidity premium on government debt, reflected in the tight collateral constraint and the resulting inefficiency in the market for flexible price goods in the cash-and-credit market (inequality (92)). As well, in (93), the value of consolidated government debt is just sufficient to purchase all goods.

4.2.1 Unconstrained Equilibrium

First, the case where $V_t = V$ for all t, and V is sufficiently large that (83) does not bind. Then, solving for a stationary equilibrium from (85)-(89),

$$\pi = \frac{\beta}{q}, \tag{94}$$

$$u'(c^{mf}) = \frac{1}{q}, \tag{95}$$

$$u'(c^{ms}) = \frac{1}{\beta}, \tag{96}$$

$$u'(c^{bf}) = 1, \tag{97}$$

$$u'(c^{bs}) = \frac{q}{\beta}. \tag{98}$$

Then, the period utility of the household is given by

$$W = \theta\left[u(c_t^{mf}) - c_t^{mf}\right] + \theta\left[u(c_t^{ms}) - c_t^{ms}\right] + (1-\theta)\left[u(c_t^{bf}) - c_t^{bf}\right] + (1-\theta)\left[u(c_t^{bs}) - c_t^{bs}\right].$$

The central bank's problem in this equilibrium is then to choose q to maximize welfare in equilibrium. From the equilibrium solution, it is straightforward to show that welfare is strictly increasing in q for $q \leq \beta$, and strictly decreasing in q when $q = 1$. Therefore, the optimal monetary policy satisfies $q \in (\beta, 1)$. When the collateral constraint does not bind in this model, there are in general two inefficiencies at work. The first is a standard monetary friction, which is corrected if the nominal interest rate is zero, i.e. $q = 1$ or a Friedman rule. The second is the sticky price friction, which is corrected when $q = \beta$, which implies $\pi = 1$. The optimal monetary policy then trades off these two frictions. A zero nominal interest rate is not optimal, and neither is price stability, as $\pi < 1$ at the optimum.

4.2.2 Constrained Equilibrium

The purpose of this subsection is to analyze an equilibrium and optimal policy in a situation analogous to what was considered earlier in the paper in cashless economies. Assume that, in the current state, $V_t = V^l$, and that the state will revert permanently to the state $V_{t+1} = V^h$ with probability $1-\rho$. Here, $V^l < V^h$, and the collateral constraint binds (at least for some monetary policies) in the low-V state and does not bind in the high-V state.

From (85)-(88), (93), and (94)-(98), an equilibrium consists of consumption quantities c^{mf}, c^{ms}, c^{bf}, and c^{bs}, and relative price π in the low-V state solving

$$1 = \beta \left[\frac{\rho u'(c^{mf})}{\pi} + \frac{(1-\rho)}{\beta} \right], \tag{99}$$

$$\pi u'(c^{ms}) = u'(c^{mf}), \tag{100}$$

$$\frac{u'(c^{bf})}{q} = u'(c^{mf}), \tag{101}$$

$$\frac{\pi}{q} u'(c^{bs}) = u'(c^{mf}), \tag{102}$$

$$\theta \left[c^{mf} + \frac{c^{ms}}{\pi} \right] + (1-\theta) q \left[c^{bf} + \frac{c^{bs}}{\pi} \right] = V^l, \tag{103}$$

given monetary policy q. Note, in (99)-(103), that monetary policy in the future state in which reversion to the high-V state occurs has no bearing on the determination of quantities in current period. That is, in this monetary model forward guidance is irrelevant.

This section is incomplete, but we can show that there are conditions under which the ZLB is suboptimal, no matter how tight the collateral constraint is. Thus, even with a very low real interest rate, ZLB policies need not be optimal.

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Figure 1: Real Interest Rate
3-Month Treasury Bill: Secondary Market Rate-Personal Consumption Expenditures: Chain-type Price Index
%-% Chg. from Yr. Ago
7.5
5.0
2.5
0.0
-2.5
-5.0
1980
1982
1984
1986
1988
1990
1992
1994
1996
1998
2000
2002
2004
2006
2008
2010
2012
2014
2016
fred.stlouisfed.org
myf.red/g/7LeA