Money and Capital in a Persistent Liquidity Trap[*]

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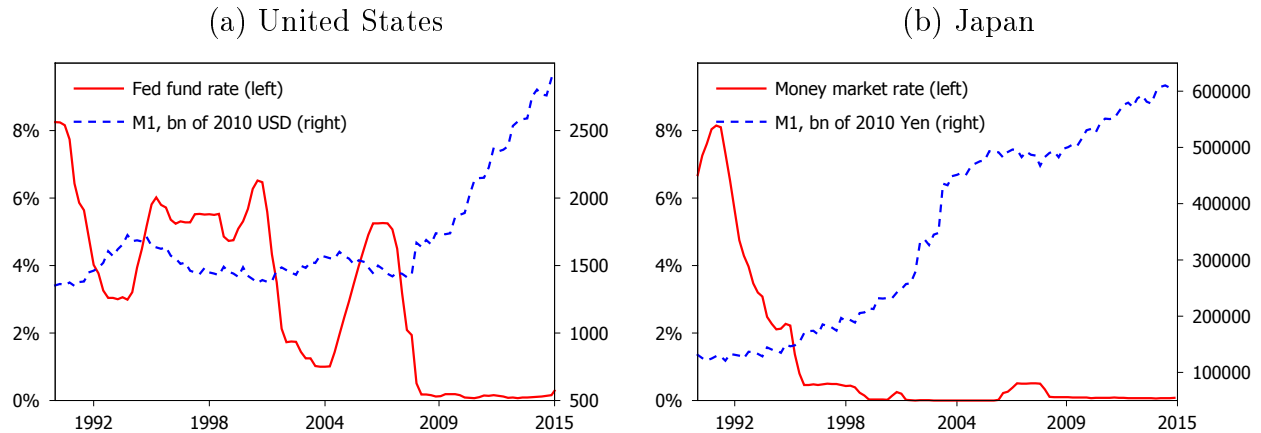Banque de France

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Abstract

In this paper we analyze the implications of a persistent liquidity trap in a monetary model with asset scarcity and price flexibility. We show that a liquidity trap leads to an increase in cash holdings and may be associated with a long-term output decline. This long-term impact is a supply-side effect that may arise when agents are heterogeneous. It occurs in particular with a persistent deleveraging shock, leading investors to hold cash yielding a low return. Policy implications differ from shorter-run analyses. Quantitative easing leads to a deeper liquidity trap. Exiting the trap by increasing expected inflation or applying negative interest rates does not solve the asset scarcity problem.

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JEL Classification Numbers: E40, E22, E58.

(a) United States (b) Japan

Source: International Financial Statistics. M1 is deflated by the CPI.

Figure 1: Policy rates and M1 in the US and Japan.

1 Introduction

Periods of persistent liquidity traps typically coincide with substantial increases in cash holdings, as illustrated in Figure 1 for the U.S. and Japan. Moreover, these periods are associated with disappointing levels of investment and of output growth.[1] Can increased money holdings crowd out physical investment and contribute to lower growth? Most macroeconomic models would give a negative answer to this question, since money typically has not long-run effect. In this paper, we argue that in a liquidity trap investment can be negatively related to money holdings. We consider a monetary model with flexible prices, where money is only held for transaction purposes in normal times, with no impact on the allocation of resources. In a liquidity trap, however, investors' allocate part of their saving to money holdings that have a low return. With agents heterogeneity, this lower return may then hamper the investment capacity of the economy and have a long-lasting impact on output. This mechanism implies that a liquidity trap may have supply-side effects that contribute to a slower recovery. The policy implications of these supply-side effects differ from shorter-run analyses.

We focus on a liquidity trap generated by a deleveraging shock. Due to this shock, desired aggregate saving exceeds desired investment and the nominal interest rate cannot adjust downward because of the Zero Lower Bound (ZLB). In that case, excess desired saving materializes

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[1]E.g., see International Monetary Fund (2015) for the recent period. Persistently low investment has led to downward revisions of estimates of potential output and fueled speculation as to whether the world economy might be suffering from "secular stagnation". See Teulings and Baldwin (2015) for an interesting collection of essays on secular stagnation.

as higher real cash holdings. We assume a persistent deleveraging shock, so that the liquidity trap can be persistent, even in the long run after prices have adjusted. This contrasts with the literature focusing on long-run demand effects and modeling long-lasting liquidity traps by assuming persistent nominal rigidities. Instead, we make the conventional assumption that prices are flexible in the long run and analyze the long-term implications of the liquidity trap.

More precisely, we introduce money in a model with scarce (liquid) assets due to the lack of income pledgeability, in the spirit of Woodford (1990) and Holmström and Tirole (1998).[2] Investors find investment opportunities every other period, so that they alternate between investing phases and saving phases. In their investing phase, they use their past liquid saving and borrow to invest, but this borrowing is limited by credit constraints. Agents can save in two liquid assets, real bonds and money. As long as the nominal interest rate is positive, money is dominated as an asset and is held only for transaction purposes. At the ZLB, bonds and money become substitutes and money can be held for saving purposes as well. In this framework, we consider a persistent deleveraging shock, modeled as in Eggertsson and Krugman (2012) by a tightening of the investors' borrowing constraints.[3] This shock generates a decrease in the interest rate until the nominal rate eventually hits the ZLB. This creates a gap between the effective real interest rate and the shadow real rate that would prevail without the ZLB. In our model, the fall in the shadow interest rate lasts as long as the deleveraging shock. If this shock is permanent, then the ZLB persists in the steady state. The lack of liquid assets in the economy indeed prevents investors from moving away from their credit constraints through saving.

We show that the consequences of a deleveraging shock are very different outside the ZLB and at the ZLB. Outside the ZLB, a deleveraging shock has no effect on long-run capital accumulation and output (in our benchmark specification) as the interest rate can adjust downward and offset the tighter borrowing constraint. However, large deleveraging shocks that bring the economy to the ZLB have a negative effect on capital and output. Since the deleveraging shock reduces the investors' supply of assets, their excess saving is allocated to money in the absence

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[2]See also Farhi and Tirole (2012) and Bacchetta and Benhima (2015) for more recent contributions.

[3]With nominal rigidities, the literature has already shown that such a deleveraging shock can lead to low levels of output and employment in the short run, due to lower demand. Eggertsson and Krugman (2012), Werning (2012), Benigno et al. (2014), or Caballero and Farhi (2015) show this in New-Keynesian models.

of interest rate adjustment. Money has then two effects on capital accumulation. First, saving in money rather than bonds means that fewer funds are channelled to investment—a negative crowding-out effect. Subsequently, however, money provides funds as it can be liquidated to finance investment—a positive liquidity effect. But since money has a low return, it is a poor source of liquidity, so the crowding-out effect dominates and investment decreases in the long run.

The long-run investment slow-down is due to an increase in *investors'* demand for cash, so that it is crucial that the deleveraging shock affects *investors*.[4] Indeed, tighter credit constraints among investors increase their net saving. This extra demand for saving is satisfied by money at the ZLB, and their capacity to finance investment is then directly affected by the low return on money. On the contrary, a deleveraging shock affecting only workers has no long-run effects in the liquidity trap, because it does not alter the investors' demand for saving.[5] Besides, other types of shocks, such as an increase in the discount rate or a decrease in the growth rate of productivity, do not have a negative long-term effect on the investment rate. In these cases, the crowding-out of investment by cash is compensated by an increase in the aggregate saving rate. Our results therefore suggest that investors' deleveraging is an important factor of growth slowdowns in persistent liquidity traps.

The negative effect of cash in the liquidity trap mainly comes from a Pigou-Patinkin real balance effect, which leads to higher consumption as a share of output and therefore less investment. While real balance effects cannot arise in a Ricardian world, they are present in our framework due to credit constraints and to agent heterogeneity. In addition, for positive inflation rates, the inflation tax also redistributes part of investors' resources to other agents (here, workers), which further hurts investment.

Our framework has different policy implications than traditional shorter-run analyses. In a liquidity trap, typical policies are quantitative easing (QE), negative interest rates, or an increase in expected inflation. These policies may have their merits in the short run, but they

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[4]The online Appendix decomposes the rise in cash holdings in the US and shows that it comes from the less constrained firms and households, which would correspond to investors in the model.

[5]The empirical literature shows that all sectors of the private economy suffer from deleveraging in the Great Recession. See Mian and Sufi (2010) and Mian and Sufi (2012) for the evidence on households' deleveraging. See Chodorow-Reich (2014), Giroud and Mueller (2015), Bentolila et al. (2009) for the evidence on firms' deleveraging.

have serious drawbacks in the long run. QE operations, by taking public bonds away from the markets, decrease the shadow real interest rate and generate a deeper liquidity trap. Negative nominal interest rates or an increase in expected inflation help to exit the liquidity trap by lowering the effective real interest rate. However, these policies do not solve the asset scarcity problem but deteriorate the allocation of resources across time by further lowering the real interest rate. Instead, improving the supply of liquidity helps exiting the liquidity trap by increasing the shadow interest rate. This can be done both through credit easing or through a higher supply of government debt.[6] However, while a higher supply of liquidity improves the allocation of resources across time, it can have undesirable redistributive effects by reducing wages. This occurs especially if investors are net debtors, so a higher interest rate generates costs that limit their investment capacities. If instead investors become net creditors, a higher interest rate generates more resources for investment.

Our asset-scarce environment is characterized by a low interest rate, so it is prone to rational bubbles. When we allow for bubbles that can be held by savers, we show that they play a role similar to money, generating crowding-out and liquidity effects. By sustaining a higher interest rate, the emergence of a bubble rules out money and brings the economy out of the ZLB.

Related literature The paper is related to the recent literature on persistent ZLB equilibria. In the existing literature, liquidity traps usually arise when the natural rate of interest falls enough to make the nominal rate hit the ZLB (Krugman, 1998; Eggertsson and Krugman, 2012; Werning, 2012). In standard models with an unconstrained infinitely-lived representative agent, this cannot be a persistent equilibrium since the natural rate is tightly linked to time preference through the Euler equation. A steady state can only be at the ZLB if inflation is far below target, leaving the real rate and the allocation of resources unchanged, as in the self-fulfilling equilibrium of Benhabib et al. (2001). Schmitt-Grohé and Uribe (2013) add permanent nominal rigidities (a non-vertical long-run Phillips curve) to this framework to get a lower output at the ZLB. Benigno and Fornaro (2015) introduce endogenous growth along permanent nominal rigidities and get a self-fulfilling ZLB steady state with low output, low growth, and a low real interest rate. Moving away from the representative agent framework,

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[6]Such policies are also discussed in policy circles, e.g., Kocherlakota (2015).

Eggertsson and Mehrotra (2014) and Caballero and Farhi (2015) use a non-Ricardian OLG framework with financial frictions, where the equilibrium real rate of interest can be arbitrarily low. Michau (2015) gets a similar feature with wealth in the utility function of an otherwise standard representative agent. Assuming the interest rate is stuck at the ZLB, adjustment in these three papers is supposed to come from a persistently negative output gap, which again requires long-run nominal rigidities. Hence, in the existing literature, stagnation in a persistent liquidity trap remains a demand-side phenomenon.

Like us, Buera and Nicolini (2016), Guerrieri and Lorenzoni (2015) and Ragot (2016) examine the effects of a deleveraging shock at the ZLB in the absence of nominal rigidities. Guerrieri and Lorenzoni (2015) focus on consumer spending in a model where households face borrowing limits, and Ragot (2016) studies optimal monetary policy in a model where money has redistributive effects due to limited participation. In both models, there is no capital accumulation. Closer to our approach, Buera and Nicolini (2016) consider a monetary model where producers need external funds to buy capital. While we focus on the negative relationship between cash holdings and capital, they study the reallocative effects of low real interest rates on total factor productivity and capital, and assume a moneyless economy in most of their paper. Like us, they discuss the trade-offs associated to the inflation policy but do not consider increases in public debt large enough to exit the liquidity trap by raising the shadow interest rate.

The crowding-out and liquidity effects of money we emphasize are reminiscent of the effects of external liquidity in other models where investors' income is not fully pledgeable, such as Woodford (1990), Holmström and Tirole (1998), and more recently Covas (2006), Angeletos and Panousi (2009), Kiyotaki and Moore (2012), Kocherlakota (2009) and Farhi and Tirole (2012). A short-term crowding-out effect is also present in Andolfatto (2015). The role of money as a saving instrument is also evocative of the literature on the value of fiat money (Samuelson (1958), Townsend (1980)). In our paper, transactions are not constrained by demography or spatial separation, but by the lack of income pledgeability.

Our paper is also related to the literature on bubbles, which are an alternative saving instrument in asset-scarce environments with a low interest rate. In Samuelson (1958) and Tirole (1985), bubble-prone asset-scarcity is due to the OLG structure of their economies. In Martin and Ventura (2012) and Farhi and Tirole (2012), it is due both to the OLG structure

and to financial frictions. Asriyan et al. (2016) introduce bubbles in a monetary environment. They also analyse liquidity traps and some of their policy analysis is similar to ours.

The real balance effect that underlies the adjustment mechanism present in our model has been originally studied by Pigou (1943) and Patinkin (1956). More recently, Weil (1991), Ireland (2005), Bénassy (2008) and Devereux (2011) have analyzed real balance effects in OLG models.

The rest of the paper is organized as follows. Section 2 presents the basic model with infinitely-lived entrepreneurs and workers. Section 3 describes the steady state with flexible prices and the long-run effect of deleveraging shocks. Section 4 examines policy options. Section 5 studies several extensions of the benchmark model: bubbles, preference and growth shocks, financial intermediation, inefficient saving technology, idiosyncratic uncertainty, partial capital depreciation, nominal government bonds, and nominal rigidities. Section 6 concludes.

2   A Model with Scarce Assets and Money

We consider a heterogenous-agents, non-Ricardian monetary model where the supply of bonds and the distribution of money holdings matters. Prices are flexible as we focus on the long run. In normal times, bonds dominate money and the real interest rate adjusts to balance the supply and demand for bonds. In a liquidity trap, however, bonds and money become perfect substitutes. The supply and demand of assets are then balanced by an adjustment in real money holdings (coming from either prices or money supply). These two adjustment mechanisms, through interest rates or money holdings, have different implications for investment and output, and therefore for policy. We show that in a liquidity trap real money holdings by investors tend to increase, which may have a negative impact on capital and output in the long-run. This is in particular the case for a deleveraging shock, which we analyze in Section 3. In this section, we describe the model and the equilibrium.

2.1   The Setup

We model a monetary economy with heterogeneous investors, workers, and firms. There are three types of assets: bonds, money, and capital. We assume that bonds are real bonds, that is,

promises to pay one unit of final good in the next period.[7] Denote by r_{t+1} their gross real rate of return expressed in units of final goods: a bond issued in period t is traded against $1/r_{t+1}$ units of final goods. The gross nominal return expressed in units of currency is $i_{t+1} = r_{t+1}E_t P_{t+1}/P_t$, where P_t is the price of the final good in units of currency in period t and E_t denotes the expectation as of time t. While r_{t+1} represents the effective interest rate, at the ZLB we will also consider the *shadow* interest rate r_{t+1}^s, which is the real interest interest rate that would prevail if the ZLB were not binding.

Money bears no interest rate; that is, it pays a gross nominal return equal to 1. While bond holdings can be both positive or negative, money holdings are non-negative. In addition, money provides transaction services by relaxing a cash-in-advance constraint faced by workers. In normal times, when the gross nominal return i is strictly larger than 1, money is strictly dominated by bonds as a saving instrument. Then, only workers hold money, for transaction purposes. However, when $i = 1$, a situation that will obtain in a liquidity trap, money becomes as good a saving instrument as bonds and investors start holding money as well.

Investors   Following Woodford (1990), investors find investment opportunities every other period, so that they alternate between a saving period and an investment period. This simple alternating approach is a convenient limit case allowing to capture idiosyncratic shocks in a very tractable way. Section 5 examines the more general case with idiosyncratic uncertainty on the occurrence of an investment opportunity and shows that the analysis is similar. Consequently, at each point in time there are two groups of investors, assumed of equal size one, investing and saving every other period. We call investors in their saving phase S-investors, or simply savers, and denote them by S, while investors in their investment phase are called I-investors and are denoted by I. Each group is of measure 1. We assume logarithmic utility in order to get closed-form solutions. An individual investor i maximizes

$$U_t^i = E_t \sum_{s=0}^{\infty} \beta^s \log(c_{t+s}^i)$$

where c_t^i refers to her consumption in period t.

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[7]The case of nominal bonds is considered in Section 5.

In period t, I-investors start with wealth $a_t + \frac{M_t^S}{P_t}$ where a_t and M_t^S are respectively real bond holdings and nominal money holdings inherited from their preceding saving phase. They get an investment opportunity, which consists in a match with a firm. I-investors consume c_t^I, issue b_{t+1} bonds, and invest k_{t+1} in the firm. We abstract from money demand by I-investors, as it is always zero in equilibrium. Their budget constraint is

$$\frac{b_{t+1}}{r_{t+1}} + a_t + \frac{M_t^S}{P_t} = c_t^I + k_{t+1}. \tag{1}$$

In period t, S-investors start with equity k_t and outstanding debt b_t inherited from their preceding investment phase. They receive a dividend $\rho_t k_t$. Then, they consume c_t^S, buy a_{t+1} real bonds and save M_{t+1}^S in money. Their budget constraint is

$$\rho_t k_t = c_t^S + b_t + \frac{a_{t+1}}{r_{t+1}} + \frac{M_{t+1}^S}{P_t}. \tag{2}$$

In general, the return on capital is larger than r_t. Thus, I-investors choose to leverage up when they receive an investment opportunity. But they face a borrowing constraint as they can only pledge a fraction ϕ_t of dividends so that

$$b_{t+1} \leq \phi_t \rho_{t+1} k_{t+1}. \tag{3}$$

In this framework, where investment opportunities are lumpy and investors cannot fully pledge their future income, there is an asynchronicity between the investors' access to and their need for resources. This creates a demand for assets for liquidity purposes in the investors' saving phase.[8] Both bonds and money can satisfy this demand for liquidity, or demand for assets (we will use these two terms interchangeably). Capital, on the other hand, is illiquid, since it cannot be fully pledged.

Firms There is a unit measure of one-period-lived firms, who are each matched with an I-investor. Firms use their investor's funds to buy capital k_t and produce output y_t with capital and labor through a Cobb-Douglas production function so that $y_t = F(k_t, h_t) = k_t^\alpha h_t^{1-\alpha} +$

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[8]We use the term liquidity in the same spirit as Woodford (1990) and Holmström and Tirole (1998).

$(1 - \delta)k_t$. Labor h_t is paid at the real wage w_t and all profits are distributed to I-investors as dividends, i.e., $\Pi_t = y_t - w_t h_t$. As the labor market is competitive, these profits are linear in k and can be rewritten as $\Pi_t = \rho_t k_t$, where ρ is the equilibrium return per unit of capital.[9] For expositional clarity, we assume full depreciation. The case of partial depreciation $\delta < 1$ is deferred to the online Appendix. In equilibrium, profits are then simply $\rho_t k_t = \alpha y_t$.

Workers There is a unit measure of workers who maximize

$$U_t^w = E_t \sum_{s=0}^{\infty} \beta^s \log(c_{t+s}^w)$$

where c_t^w refers to workers' consumption. They have a fixed unitary labor supply, so that $h_t = 1$ in equilibrium. Their budget constraint is:

$$c_t^w + \frac{M_{t+1}^w}{P_t} + l_t^w = w_t + \frac{T_t^w}{P_t} + \frac{M_t^w}{P_t} + \frac{l_{t+1}^w}{r_{t+1}}, \tag{4}$$

where l^w is the amount of real bonds issued, M^w money holdings, and T^w a monetary transfer from the government.

Workers are subject to a cash-in-advance (CIA) constraint: they cannot consume more than their real money holdings. Assuming the bond market opens before the market for goods, these holdings are the sum of money carried over from the previous period, monetary transfers from the government, and money borrowed on the bond market (net of debt repayment):

$$c_t^w \leq \frac{M_t^w + T_t^w}{P_t} + \frac{l_{t+1}^w}{r_{t+1}} - l_t^w. \tag{5}$$

Workers also face a borrowing constraint

$$l_{t+1}^w \leq \bar{l}_t^w y_{t+1}. \tag{6}$$

We assume that the borrowing limit is linear in the wage bill and therefore proportional to output (since the equilibrium wage bill is a fraction $1 - \alpha$ of output). We allow for the case

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[9]ρ is given by $\rho = F(1, 1/k(w)) + 1 - \delta - w/k(w)$ where $k(w)$ is the equilibrium capital-labor ratio defined by $w = F_h(k(w), 1)$.

$\bar{l}^w < 0$, which represents forced saving by workers.

When $\beta r < 1$, which we will assume throughout the analysis, workers would prefer to dissave and always hold the minimum amount of money, so that the CIA is always binding. Together with their budget constraint (4), this implies that their money holdings are simply equal to the wage bill: $M_{t+1}^w / P_t = w_t$. Since the wage bill is equal to $(1 - \alpha) y_t$ in equilibrium, money demand by workers is given by:

$$M_{t+1}^w = (1 - \alpha) P_t y_t. \tag{7}$$

Money supply and government policy Denote by M_t the money supply at the beginning of period t. In period t, the government can finance transfers to agents by creating additional money $M_{t+1} - M_t$ and by issuing real bonds l_{t+1}^g. For simplicity, we assume that the government only makes transfers to workers. The budget constraint of the government is:

$$\frac{M_{t+1}}{P_t} + \frac{l_{t+1}^g}{r_{t+1}} = \frac{M_t}{P_t} + \frac{T_t^w}{P_t} + l_t^g. \tag{8}$$

Several fiscal and monetary policies can be considered. As a benchmark case, we assume that the fiscal authority provides a real supply of bonds that is proportional to output $l_{t+1}^g = \bar{l}_t^g y_{t+1}$ and that the monetary authority controls the growth of money

$$M_{t+1}/M_t = \theta_{t+1}. \tag{9}$$

Transfers to households then adjust to satisfy the budget constraint (8). We assume that money growth is constant in the long run and equal to θ, which enables us to pin down steady-state inflation easily, as it will be equal to θ.

We make the following parametric assumption:

Assumption 1 $\theta > \beta$.

Assumption 1 implies that the economy can only hit the zero lower bound in a steady state where $\beta r < 1$, that is with binding borrowing constraints. Indeed, in the steady state, the nominal gross interest rate is $i = r\theta$. With assumption 1, $i = 1$ implies $\beta r = \beta/\theta < 1$. This assumption is naturally satisfied as long as $\theta \geq 1$, that is with a non-negative steady-state inflation.

Market clearing for bonds and money   The market for bonds clears so that

$$b_{t+1} + l^w_{t+1} + l^g_{t+1} = a_{t+1}. \tag{10}$$

Similarly, equilibrium on the money market is given by:

$$M^S_{t+1} + M^w_{t+1} = M_{t+1}. \tag{11}$$

Sequences of leverage   We assume that the sequences of leverage $\{\phi_t, \bar{l}^w_t, \bar{l}^g_t\}$ are exogenous and deterministic. As a consequence, investors have perfect foresight, which will enable us to derive closed-form solutions.

2.2   Equilibrium

In an asset-scarce environment, the dynamics of the economy can be summarized by four key equations: a complementary slackness condition that determines whether the economy is in a liquidity trap or not, the Euler equation for savers, the investors' aggregate budget constraint and the equilibrium on the money market.

Asset scarcity and binding borrowing constraints   We focus on equilibria where strong borrowing constraints prevent borrowers from supplying the saving instruments needed by savers. In such an "asset-scarce" economy, we will have $\beta r < 1$ in the long run, so the borrowing constraints are binding for workers and I-investors at the vicinity of the steady state, which we assume throughout.

A binding borrowing constraint for workers sets their supply of assets to $l^w_{t+1} = \bar{l}^w_t y_{t+1}$. We define the supply of bonds to investors by the rest of the economy, which includes workers and the government, by

$$l_{t+1} = l^w_{t+1} + l^g_{t+1} = \bar{l}_t y_{t+1} \tag{12}$$

where $\bar{l} = \bar{l}^g + \bar{l}^w$. In equilibrium, l_{t+1} is also the net position of investors.

The zero lower bound and money demand   The portfolio choice of S-investors can be summarized by the following complementary slackness condition:

$$M_{t+1}^S \left(r_{t+1} - \frac{P_t}{P_{t+1}} \right) = 0. \tag{13}$$

As long as $i > 1$, money has a strictly lower expected return than bonds and investors hold the minimum amount of money, which is zero. Then, we have $M^S = 0$. We refer to periods where $i > 1$ and investors hold no money as "cashless" periods. We use the term "cashless" only in reference to investors since workers always hold money, regardless of the nominal interest rate.

When $i = 1$, that is $r_{t+1} = P_t/P_{t+1}$, bonds and money become perfect substitutes for savers, and they start holding money for saving purposes, so $M^S \geq 0$. We refer to periods where $i = 1$ and S-investors hold money as "liquidity trap" periods.

Euler equation of savers   S-investors are typically unconstrained, so their Euler equation is satisfied: $1/c_t^S = \beta r_{t+1}/c_{t+1}^I$. With log-utility, consumption is a fraction $1 - \beta$ of wealth for both types of investors.[10] Then, $c_{t+1}^I = (1 - \beta)(a_{t+1} + M_{t+1}^S/P_{t+1})$ and $c_t^S = (1 - \beta)(\rho_t k_t - b_t) = (1 - \beta)(\alpha y_t - b_t)$. Assuming binding borrowing constraints (3) and (6), and using the market clearing condition for bonds (10), the Euler equation of S-investors can be rewritten

$$\beta \alpha (1 - \phi_{t-1}) y_t = \frac{1}{r_{t+1}} \left[(\phi_t \alpha + \bar{l}_t) y_{t+1} + m_{t+1}^S \right]. \tag{14}$$

where $m_{t+1}^S = M_{t+1}^S/P_{t+1}$ are the real money holdings of S-investors. This Euler equation can also be interpreted as an equilibrium condition for saving instruments. The left-hand side (LHS) is the demand for saving instruments by S-investors, which depends on current income. The right-hand-side (RHS) is the supply of saving instruments. The first term is the supply of bonds, which depends on future pledgeable income. It depends on ϕ, the leverage ratio of I-investors, and on $\bar{l}$, the leverage ratio of workers and the government. Finally, the last term on the RHS corresponds to money used by S-investors as a saving instrument.

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[10]The proof of this property is available upon request. The case of log-utility is a realistic one when it comes to modeling the saving behavior of agents, as a unitary elasticity of intertemporal substitution is well within the estimated ranges.

Aggregate budget constraint Replacing consumption, the budget constraints of I-investors and S-investors (1) and (2) become respectively $\beta(a_t + M_t^S/P_t) = k_{t+1} - b_{t+1}/r_{t+1}$ and $\beta(\alpha y_t - b_t) = M_{t+1}^S/P_t + a_{t+1}/r_{t+1}$. Aggregating these two constraints and using the bond market clearing condition (10), we find

$$k_{t+1} + \pi_{t+1}m_{t+1}^S + \frac{1}{r_{t+1}}\bar{l}_t y_{t+1} = \beta\left[(\alpha + \bar{l}_{t-1})y_t + m_t^S\right] \tag{15}$$

where $\pi_{t+1} = P_{t+1}/P_t$ is the gross rate of inflation. This equation represents the aggregate resource constraint of S- and I-investors, and describes capital accumulation. Aggregate savings (on the RHS) must be equal to aggregate investment in capital, bonds and money (on the LHS).

Consider how money affects capital accumulation. First, on the LHS, an increase in desired money holdings by S-investors decreases the capital stock, because other things equal the corresponding funds are not channeled to I-investors. This is the *crowding-out effect of money*. For a given level of future real money holdings that the S-investors want to secure, the crowding-out effect is stronger if inflation, which is the price of (real) money, is larger. Second, on the RHS, past savings in money of current I-investors increase the capital stock, because they can be liquidated to finance investment. This is the *liquidity effect of money*. This liquidity effect is stronger if β is larger, because then I-investors use a higher share of their wealth to invest. Note that the bond's external position of investors has similar effects, except that the price of liquidity in the case of bonds is not inflation but $1/r_{t+1}$.

Importantly, this equation shows that I-investors' leverage ϕ does not matter outside its potential equilibrium effect on the interest rate, out of the liquidity trap, or through its effect on the demand for money in the liquidity trap. This is because the net position of investors as a whole, $a - b$, ultimately depends on the net supply of bonds by the rest of the economy $l = \bar{l}y$, as $a - b = l$ from (10). This is an important result that greatly simplifies the analysis. To understand the dynamics of capital, it is enough to study the crowding-out and liquidity effects of $\bar{l}$ and m^S.

Money market Substituting (7) into (11), we get

$$\frac{M_{t+1}}{P_t} = (1 - \alpha)y_t + \pi_{t+1}m_{t+1}^S. \tag{16}$$

Money supply has to be equal to the demand for money for transaction purposes plus the demand for money for saving purposes. With perfectly flexible prices, this equation ensures that any real demand for money can be met through a price adjustment, even with a predetermined money supply M_{t+1}.

Equilibrium The Euler equation (14), the aggregate resource constraint (15), and the money market equilibrium (16) describe a constrained equilibrium, which can be formally defined in the following way:

Definition 1 (Constrained equilibrium) *Consider an exogenous sequence of leverage* $\{\phi_t, \bar{l}_t^w\}_{t \geq 0}$, *a policy* $\{\theta_{t+1}, T_t^w, \bar{l}_t^g\}_{t \geq 0}$ *satisfying (8), and initial assets* $\{k_0, M_0, M_0^S, M_0^w\}$. *The associated constrained equilibrium is an allocation* $\{y_t, k_{t+1}, M_{t+1}, M_{t+1}^w, M_{t+1}^S, m_{t+1}^s, \bar{l}_{t+1}\}_{t \geq 0}$ *and a price vector* $\{i_{t+1}, r_{t+1}, w_t, P_t, \pi_{t+1}\}_{t \geq 0}$ *satisfying* $\pi_{t+1} = P_{t+1}/P_t$, $i_{t+1} = r_{t+1}\pi_{t+1}$, $\bar{l}_{t+1} = \bar{l}_{t+1}^w + \bar{l}_{t+1}^g$, $m_t^S = M_t^S/P$, $y_t = k_t^\alpha$, $w_t = (1-\alpha)y_t$, (7), (9), (13), (14), (15), and (16).

In the next section, we will focus on steady state equilibria. It will be useful to distinguish between cashless and liquidity-trap steady states. The definition of these steady states is made formally in the following definition:

Definition 2 (Cashless and liquidity-trap steady states) *A constrained steady state is a constrained equilibrium where* $\{\phi, \bar{l}^w, \theta, \tau^w, \bar{l}^g, y, k, m, m^w, m^S, \bar{l}, i, r, w, \pi\}$ *are constant, where* $\tau^w = T^w/P$, $m = M/P$, *and* $m^w = M^w/P$. *A cashless steady state is a constrained steady state satisfying* $i > 1$ *and* $m^S = 0$. *A liquidity-trap steady state is a constrained steady state satisfying* $i = 1$ *and* $m^S > 0$.

Note that there are non-zero money holdings in both equilibria. However, in the cashless equilibrium, money is only held by workers, not by investors.

3   The Long-term Impact of Deleveraging

This section studies the long-term effects of deleveraging. In our setting, a deleveraging shock on investors can be modeled by a drop in ϕ. Likewise, a deleveraging shock on workers can be modeled by a drop in $\bar{l}$ (coming from a drop in $\bar{l}^w$). We consider permanent shocks, which allows us to analyze changes in steady states.

3.1 The Effect of Investors' Deleveraging

A deleveraging shock leads to an excess net demand for saving instruments by investors. The equilibrium implications of this excess demand are very different depending on whether the economy is in a cashless equilibrium or in a liquidity trap. In cashless equilibria, adjustment comes from a lower equilibrium interest rate which helps restore a higher supply of bonds. In the liquidity trap, as the interest rate cannot adjust, the higher net demand for saving instruments by investors takes the form of higher money holdings. As we will see, this diverts resources away from investment and leads to lower capital and output in the long-run.

We first study the effect of a deleveraging shock affecting investors, that is, a drop in ϕ. We consider first the simpler case with $\bar{l} = 0$, where investors are in autarky: S-investors lend to I-investors. Afterwards, we examine the case with $\bar{l} < 0$, where investors have a net debt vis-à-vis the rest of the economy, as this case is more realistic.

Autarkic investors   As capital accumulation does not depend directly on ϕ, neither does the long-run capital stock. When $\bar{l} = 0$, the cashless dynamics of capital accumulation, given by (15) with $m^S = 0$, is also independent of the real interest rate r, so the capital stock does not depend at all on ϕ, neither directly nor indirectly through its effect on the interest rate. The capital stock is indeed given by:

$$k = \beta \alpha y = \beta \alpha k^\alpha. \tag{17}$$

A deleveraging shock on investors (a decrease in ϕ) affects the distribution of wealth between S- and I-investors, but not their aggregate saving, so it leaves the capital stock unchanged. This requires a change in the interest rate as an equilibrating mechanism. Indeed, for a given interest rate, the shock generates a decrease in the bond supply b by I-investors. Besides, as S-investors start the period with less debt, it increases their wealth and hence their demand for bonds a. Since the net supply of bonds by the rest of the economy remains unchanged at zero, the adjustment takes place through a decrease in the interest rate, which enables I-investors to borrow more. This is clear from the Euler equation (14), which defines r in the cashless steady state as

$$r = \frac{\phi}{\beta(1 - \phi)}. \tag{18}$$

Notice that a decrease in r implies a proportional decrease in $i = r\theta$ for a given steady-state inflation rate θ. Therefore, a strong contraction of credit may lead to the ZLB. This is the case when $\phi/[\beta(1 - \phi)] \leq 1/\theta$. Similarly, a high enough ϕ brings the equilibrium interest rate at $1/\beta$. Beyond this, the credit constraint is not biding anymore.

If i hits the ZLB at $i = 1$, the equilibrium becomes a liquidity trap. The effective real interest rate is simply $1/\theta$. We define the shadow real interest rate r^s as the interest rate that would prevail if the ZLB were not binding. It is given by the right-hand side of (18), i.e., $r^s = \phi/[\beta(1 - \phi)]$.[11] We then define the interest rate gap as the difference between the effective and the shadow interest rates:

$$\Delta \equiv r - r^s = \frac{1}{\theta} - \frac{\phi}{\beta(1 - \phi)}$$

We could think of the magnitude of this gap as the depth of the liquidity trap.

In a liquidity trap steady state, the Euler equation (14) becomes:

$$m^S = \alpha \left[(1 - \phi)\frac{\beta}{\theta} - \phi \right] y. \tag{19}$$

m^S/y is decreasing in ϕ: an increase in investors' net demand for saving instruments triggered by a deleveraging shock is now accommodated by an increase in their real money holdings m^S. Indeed, at the ZLB, bonds and money have the same return and money becomes a saving instrument. It is also interesting to notice that m^S is proportional to the interest rate gap Δ:

$$m^S = \kappa \Delta y \tag{20}$$

where $\kappa = \alpha\beta(1 - \phi)$. The magnitude of investors' real money demand is therefore also a measure of the depth of the liquidity trap.

This switch to money takes out resources from investment, as suggested by (15), which becomes in a steady state

$$k = \beta\alpha y - (\theta - \beta)m^S. \tag{21}$$

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[11]The shadow rate goes to 0 when ϕ goes to 0. This is an extreme situation where savers, absent money, would have no instruments to trade intertemporally. Section 5 introduces an alternative inefficient saving technology, which puts a strictly positive lower bound on the shadow rate.

From Assumption 1, we have $\theta > \beta$ and holding additional money entails a net resource cost that decreases the long-run stock of capital. Indeed, in the steady state, the cost of saving in money for S-investors, $\pi_{t+1} = \theta$, is then larger than the I-investors' propensity to use money holdings for investment β. The reduction in the funds coming from S-investors is therefore not compensated by the liquidity service of money to I-investors. In other words, the crowding-out effect of money overcomes its liquidity effect.

Notice that asset scarcity is crucial here. First, it generates a persistent drop in interest rate, making the liquidity trap persistent. Second, asset scarcity means that the return on bonds, and hence the return on money in the liquidity trap, is below $1/\beta$, so bond or money accumulation in the liquidity trap is costly.

The net resource cost for investors arises because of a real balance effect together with an inflation tax, as can be seen by rewriting Equation (21):

$$k = \beta \alpha y - \underbrace{(\theta - 1)m^S}_{\text{Inflation tax}} - \underbrace{(1 - \beta)m^S}_{\text{Extra consumption}} .$$

Because cash is considered as net wealth by investors (a consequence of the non-Ricardian structure of the model), they consume a fraction $1 - \beta$ of it. In addition, a fraction $\theta - 1$ of cash is lost as an inflation tax, which is redistributed to workers through transfers.[12]

How does the adjustment in investors' real money holdings m^S take place? From (16) taken in the steady state, we have $m = M/P = (1 - \alpha)y/\theta + m^S$. Since workers' money holdings always equal their wage bill, the supply of total real money holdings m has to increase. For a given path of money supply, given by (9), this implies a downward shift in the path of prices P_t. At the ZLB, a deleveraging shock is disinflationary, which endogenously increases real money holdings to accommodate the higher net demand for saving instruments by investors.

Using this analysis, we establish the following Proposition:

Proposition 1 (Steady state with autarkic investors) *Define $\phi_T = \beta/(\theta+\beta)$ and $\phi_{max} = 1/2$. If $0 < \phi < \phi_{max}$, then there exists a locally constrained steady state with $r < 1/\beta$.*

(i) *If, additionally, $\phi \geq \phi_T$, then the steady state is cashless.*

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[12]This second effect would be lower if investors also received transfers from the Government.
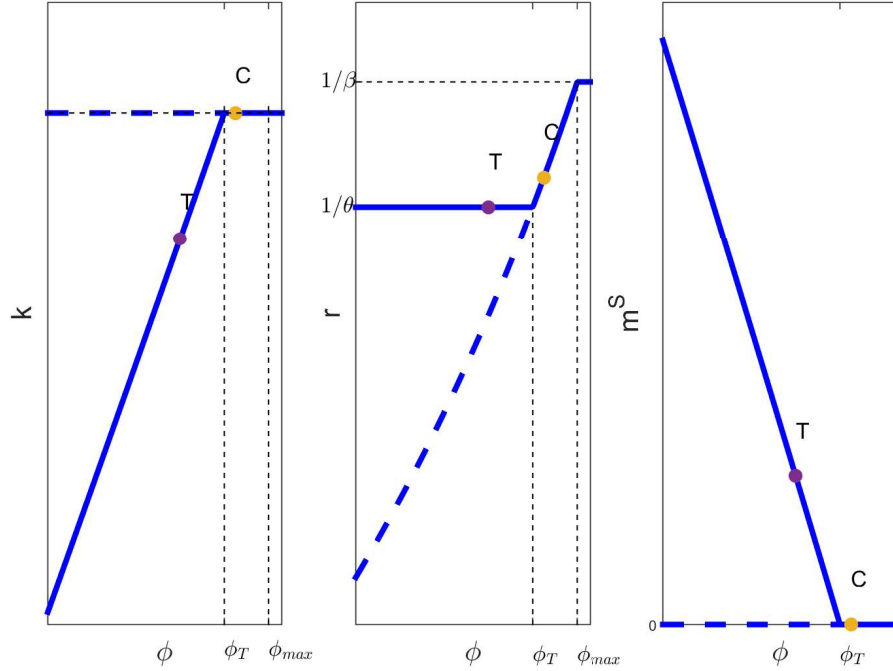
Figure 2: Steady states - Comparative statics w.r.t. ϕ, with $\bar{l} = 0$

(ii) *If $\phi < \phi_T$, then the steady state is a liquidity trap.*

(iii) *In the cashless steady state, the real interest rate r and the nominal interest rate i are increasing in ϕ, $m^S = 0$ and k is invariant in ϕ.*

(iv) *In the liquidity-trap steady state, the real interest rate r is invariant in ϕ, m^S/y is decreasing in ϕ and k is increasing in ϕ.*

Proof. See Appendix A. ∎

This Proposition establishes under which condition on ϕ the steady state is cashless or a liquidity trap. It is illustrated in Figure 2. The solid lines show the levels of k, r, and m^S as a function of ϕ, while the broken lines show the levels of the shadow rate r^s and of k and m^S if the ZLB were not binding. For intermediate values of ϕ (between ϕ_T and ϕ_{max}), the cashless real interest rate r is higher than $1/\theta$, and the steady state is cashless as the nominal interest rate i is above the ZLB, as is illustrated by equilibrium C. When ϕ falls below ϕ_T, the steady state becomes a liquidity trap where the effective interest rate is $r = 1/\theta$ and is larger than the shadow rate r^s. It is characterized by positive real money holdings among investors, for saving purposes, as illustrated by point T.

As long as the economy is in the cashless steady state (when $\phi > \phi_T$), a permanent deleveraging shock on investors (a decrease in ϕ) has no effect on capital, but it has a negative effect on the real interest rate r, as illustrated by Figure 2. But a deleveraging shock large enough to make the economy fall into a liquidity trap (by bringing ϕ below ϕ_T), has negative long-run effects on capital and output. A permanent deleveraging shock, as the one that brings the economy from C to T in the figure, is then consistent with a lower output. The effects come from the disinvestment due to the resource cost of money, thus from the supply side of the economy, and hold in the absence of any nominal rigidity. This contrasts with the recent literature, where long-run stagnation is driven by a fall in consumption demand in the presence of persistent nominal rigidities.

The fact that higher money holdings lead to lower capital and output in the long run does not imply that investors would be better off if money did not exist. By putting a lower bound on the real rate of interest, money helps investors better smooth consumption across time. Under a mild assumption on the degree of decreasing returns to scale to capital, α, this can be shown to make both groups of investors better off in a liquidity trap steady state than they would be in the corresponding cashless steady state, despite the lower capital stock (see the online Appendix). Workers may however be hurt by lower wages.

Investors are net debtors   Whereas the case where investors are in autarky is a useful simplification, the case where investors are net debtors is more realistic ($\bar{l} < 0$). Indeed, using Flow-of-Fund data and the Survey of Consumer Finances, we establish that firms and households owning a business or participating to the stock market have a negative net position in interest-bearing assets in the US. The online Appendix gives the details of our analysis.[13]

In that case, changes in the interest rate have a redistributive effect between investors and workers. The steady-state cashless capital accumulation equation now becomes:

$$k = \beta \alpha y - \left(\frac{1}{r} - \beta \right) \bar{l} y. \tag{22}$$

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[13]Note that in the presence of positive government debt ($\bar{l}^g > 0$), $\bar{l} < 0$ implies that workers have a positive net position ($\bar{l}^w < 0$). This is consistent with a high proportion of wealthy hand-to-mouth households, that is, households who own sizeable amounts of illiquid assets (like retirement accounts) but hold little liquid assets, as documented by Kaplan et al. (2014).

Since the economy is liquidity-scarce, the price of liquidity—here $1/r$—is still larger than the propensity to save β. With a lower interest rate, the price of liquidity increases even further, but now investors are *net suppliers* of liquidity ($\bar{l} < 0$), so asset scarcity generates net resources that increase the capital stock. Besides, as shown by the cashless steady-state Euler equation:

$$r = \frac{\phi + \bar{l}/\alpha}{\beta(1 - \phi)}, \tag{23}$$

the interest rate falls after a deleveraging shock in the cashless economy as before. Therefore, a deleveraging shock should *increase* the long-run capital stock in the cashless economy.[14]

In a liquidity trap however, a deleveraging shock still has a negative long-run effect on capital. In that case, as money and bonds are perfect substitutes, capital accumulation is not affected by the net supply of bonds $\bar{l}$ per se, but by the total amount of net liquidity $s = m^S + \bar{l}y$:

$$k = \beta\alpha y - (\theta - \beta)s. \tag{24}$$

where s is determined by the steady-state Euler equation taken in a liquidity trap, independently from the net supply of bonds $\bar{l}$:

$$s = \alpha \left[(1 - \phi)\frac{\beta}{\theta} - \phi \right] y. \tag{25}$$

This equation is similar to (19), with net liquidity s replacing cash holdings m^S. After a deleveraging shock on investors, the price of liquidity remains fixed at θ, whereas liquidity s increases. Since s has the same price as money in a liquidity trap, an increase in s takes resources away from investment as in the case of autarkic investors. Notice that we still have $m^S = \kappa\Delta y$, where the shadow rate is now defined by the right-hand side of (23). We therefore refer to $\bar{l}y$ as the shadow liquidity, as $s = \bar{l}y$ when $\Delta = 0$.

The main results are summarized in the following Proposition:

Proposition 2 (Steady state when entrepreneurs are net debtors) *Define $\phi_{min}(\bar{l}) = -\bar{l}/\alpha$, $\phi_{max}(\bar{l}) = (1 - \bar{l}/\alpha)/2$ and $\phi_T(\bar{l}) = (\beta - \theta\bar{l}/\alpha)/(\theta + \beta)$. If $\phi_{min} < \phi < \phi_{max}(\bar{l})$, then there exists*

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[14]The positive effects on capital accumulation of financial frictions is not an uncommon result: uninsurable risk and credit constraints in Bewley-Aiyagari models notoriously leads to an overaccumulation of capital. See Aiyagari (1994), Krusell and Smith (1997) Covas (2006) and Dávila et al. (2012).
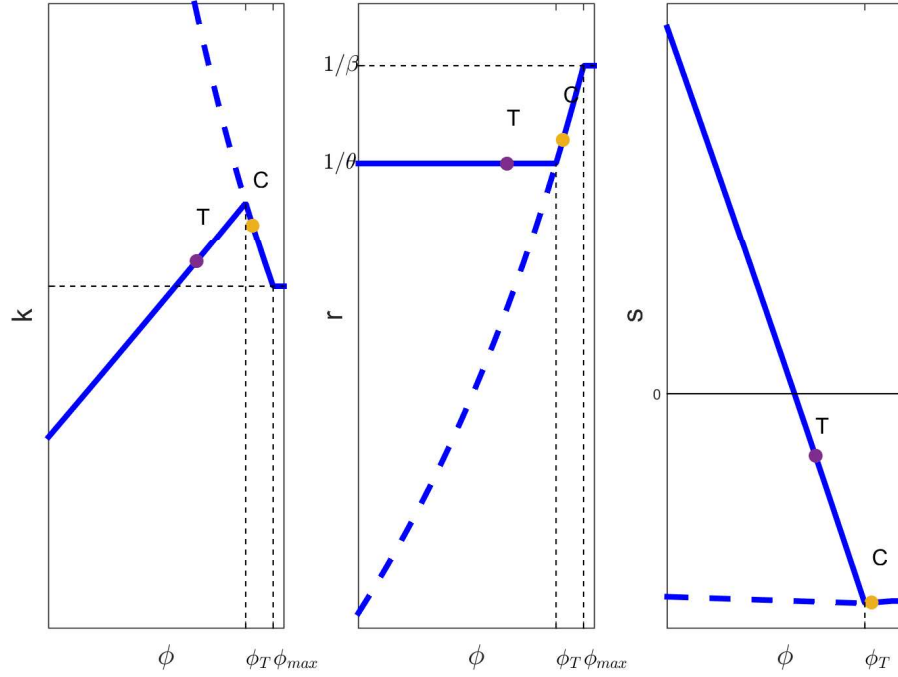
Figure 3: Steady states - Comparative statics w.r.t. ϕ, with $\bar{l} < 0$

a locally constrained steady state with $r < 1/\beta$.

(i) *If, additionally, $\phi \geq \phi_T(\bar{l})$, then the steady state is cashless.*

(ii) *If $\phi < \phi_T(\bar{l})$, then the steady state is a liquidity trap.*

(iii) *In the cashless steady state, the real interest rate r and the nominal interest rate i are increasing in ϕ, $m^S = 0$ and if $\bar{l} < 0$ $(\bar{l} > 0)$, then k is decreasing (increasing) in ϕ.*

(iv) *In the liquidity-trap steady state, the real interest rate r and the nominal interest rate i are invariant in ϕ, m^S/y is decreasing in ϕ and k is increasing in ϕ.*

Proof. See Appendix A. ∎

Figure 3 represents the effect of ϕ on the steady state with a net supply of bonds from the rest of the economy $(\bar{l} < 0)$. The solid lines show the effective values of k, r, and s as a function of ϕ, while the broken lines show their values if the ZLB were not binding. When ϕ is above ϕ_T, the steady state is cashless, so $s = \bar{l}y$. When ϕ decreases while staying above ϕ_T, the equilibrium interest rate decreases. Since investors are net debtors, this has a positive effect on the investors' income, which increases the long-run capital stock. When ϕ falls below ϕ_T, then

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the steady state state is a liquidity trap. As a result, the interest rate does not fall as a response to a deleveraging shock, thus not reestablishing the financing capacities of investors. Instead, investors start increasing their liquidity s by holding money, which has a negative effect on capital accumulation.[15] As a result, an economy that experiences a drop in ϕ that brings the equilibrium from C to T as in Figure 3 has less capital in the long run.

3.2 Workers' Deleveraging

Consider a deleveraging shock on workers, that is, a fall in $\bar{l}$ through a fall in $\bar{l}^w$. As apparent from Equation (23), the effect on r is similar to a deleveraging shock on investors, because a deleveraging shock on workers limits the economy's supply of assets. Workers' deleveraging can therefore also lead to the zero lower bound. This is illustrated in the right panel of Figure 4, where a decrease in $\bar{l}$ makes the economy switch from C, a cashless steady state, to T, a liquidity trap, through a fall in r.

However, once the economy is in a liquidity trap, changes in $\bar{l}$ have no effect. Indeed, since the interest rate cannot adjust in a liquidity trap, the net demand for assets s is constant, so any decrease in the supply of assets to investors through $\bar{l}$ is matched by an increase through m^S. As before, higher real holdings of money obtain through a downward shift in the path of prices. The key difference between a deleveraging shock on workers and on investors is that the former affects the supply of assets to investors, while the latter affects their net *demand* for assets. Both are fully accommodated by an adjustment in real money holdings, but only a change in demand actually changes the asset holdings of investors, which is the source of disinvestment. This is established in the following Proposition:

Proposition 3 (Effect of $\bar{l}$) *Define $\bar{l}_0(\phi) = \alpha\sqrt{\phi}(\sqrt{1-\phi} - \sqrt{\phi})$, $\bar{l}_{min}(\phi) = -\alpha\phi$, $\bar{l}_{max}(\phi) = \alpha(1-2\phi)$ and $\bar{l}_T(\phi) = \alpha\beta(1-\phi)/\theta - \alpha\phi$. We have $\bar{l}_{min} < \bar{l}_0 < \bar{l}_{max}$ iif $0 < \phi < 1/2$. For $\bar{l}_{min}(\phi) < \bar{l} < \bar{l}_{max}(\phi)$, then there exists a locally constrained steady state with $r < 1/\beta$.*

(i) *If, additionally, $\bar{l}_T(\phi) \leq \bar{l}$, the steady state is cashless.*

(ii) *If $\bar{l} < \bar{l}_T(\phi)$, the steady state is a liquidity trap.*

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[15]When investors are net creditors $(\bar{l} > 0)$, the capital stock decreases in ϕ both in the cashless and liquidity trap steady state. However, this case is less realistic.

(iii) *In the cashless steady state, the real interest rate r and the nominal interest rate i are increasing in $\bar{l}$, $m^S = 0$ and k is decreasing (increasing) in $\bar{l}$ for $\bar{l} < \bar{l}_0$ ($\bar{l} > \bar{l}_0$).*

(iv) *In the liquidity-trap state, the real interest rate r and the nominal interest rate i are invariant in $\bar{l}$, m^S/y is decreasing one for one in $\bar{l}$ and k is invariant in $\bar{l}$.*

(v) *if $\phi > \beta/(\beta + \theta)$, then $\bar{l}_T < 0$, so there exists cashless steady states with $\bar{l} < 0$. In that case, $\bar{l}_0 > \bar{l}_T$ so k is decreasing in $\bar{l}$ in the right neighborhood of $\bar{l}_T$.*

Proof. See Appendix A. ∎

Result (v) implies that, when investors are net debtors ($\bar{l} < 0$), a workers' deleveraging shock has a positive effect on capital outside the liquidity trap.[16] This is illustrated in Figure 4. When switching from the cashless steady state C to the liquidity trap T, the economy experiences an increase in the capital stock. However, workers' deleveraging does not affect the long-run capital stock in the liquidity trap.

The broken lines in Figure 4 also show the shadow variables if the ZLB were not binding. Besides the shadow interest rate r^s, we see the shadow liquidity, which is equal to $\bar{l}y$. As we will see in the next section, the shadow liquidity is important as it is the liquidity available when the economy leaves the ZLB.

4 Policy

We examine the policy implications of the model in exiting the liquidity trap. We consider standard policies: public debt issuance (including implications for quantitative and credit easing), negative interest rate on money, inflation and fiscal policy. Previous studies focus on short-term effects in the presence of nominal rigidities and hence demand-side policies are paramount. Our analysis instead highlights long-run effects that arise independently from nominal rigidities and therefore put emphasis on supply-side effects. How the effects of such policies translate to a welfare analysis is of course not straightforward in our framework with heterogeneous agents.

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[16]In fact, in the cashless steady state, as shown by Equations (22) and (23), changing the net liquidity position $\bar{l}$ has two effects. A lower $\bar{l}$ has a positive effect on investment as liquidity has net cost $1/r - \beta > 0$. It also increases the price of liquidity, which either decreases the resources of investors if they are net creditors ($\bar{l} > 0$), or increases them if they are net debtors ($\bar{l} < 0$). Since we assume that investors are net debtors, both effects are positive.
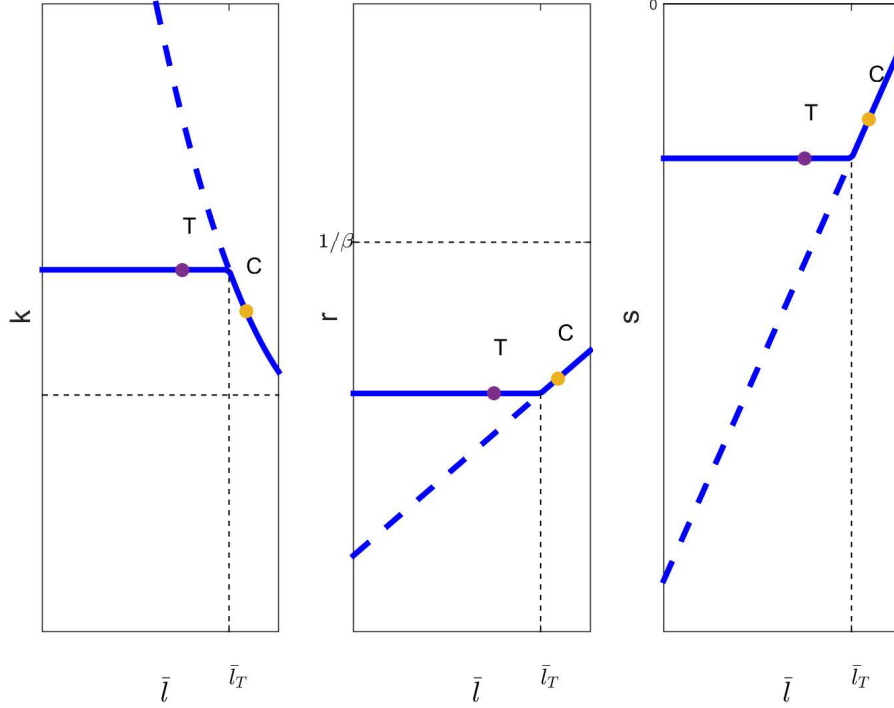
Figure 4: Steady states - Comparative statics w.r.t. $\bar{l}$, with $\bar{l} < 0$

Nevertheless, the supply of liquidity is a key factor outside of the ZLB. The online Appendix shows that an adequate supply of liquidity enables the economy to reach a Pareto-*efficient* equilibrium. Indeed, by raising the real interest rate, this enables optimal consumption smoothing by all agents as well as the optimal level of capital (see Proposition 1 of the online Appendix).[17] However, since this equilibrium might have less capital and therefore a lower wage, it is not necessarily Pareto-*improving*.

Exiting from a liquidity trap implies driving the interest rate gap to zero. The authorities can eliminate the interest rate gap either by decreasing the effective rate or by increasing the shadow rate. We have:

$$\Delta = \frac{i}{\theta} - \frac{\phi + \bar{l}/\alpha}{\beta(1 - \phi)}$$

While a strict ZLB implies $i = 1$, we can allow $i < 1$ to analyze the impact of negative interest rates.

In this section, we examine the various policies that can eliminate the interest rate gap,

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[17]The proposition shows that the efficient level of capital is given by $k = \beta \alpha y$. From Equation (22), this level obtains when $r = 1/\beta$, which also corresponds to perfect consumption smoothing, and requires a high enough public debt $\bar{l} = \bar{l}_{max}(\phi)$. When investors are net debtors out of the ZLB, capital is too high compared to a Pareto-efficient allocation and a higher public debt crowds out this inefficiently high capital stock.

assuming a constant ϕ. The government can choose the growth rate of money θ, its debt/GDP ratio $\bar{l}^g$ or its primary deficit τ^w (equal to lump-sum transfers on workers). However, these three variables cannot be chosen independently as they are linked by the government budget constraint:

$$(\theta - 1)m + \bar{l}^g y \left(\frac{1}{r} - 1\right) = \tau^w \tag{26}$$

The first term on the left-hand side is seigniorage and the second term is related to debt service. The real value of money m and output y are determined by the private economy. We first characterize policy by $(\theta, \bar{l}^g)$ and let τ^w adjust to balance the budget constraint. At the end of the section we examine the constraints on fiscal policy τ^w.

4.1   Enhancing Shadow Liquidity

In an environment with scarce assets, the public supply of liquidity plays a crucial role. At the ZLB though, public debt only affects the shadow interest rate, as money also plays the role of liquidity. However, by increasing public debt, which is shadow liquidity at the ZLB, the government can increase the shadow interest rate and help the economy exit the ZLB.

Public Debt and the ZLB   An increase in the supply of government bonds, by increasing $\bar{l}$, can obviously bring the economy out of the liquidity trap by increasing the shadow interest rate and shadow liquidity. However, marginal changes in $\bar{l}$ only affect shadow values as long as the economy remains in the liquidity trap, consistently with the "irrelevance result" highlighted in the literature. Indeed, the private demand for liquidity s is fixed at the ZLB, as shown by Equation (25). Within liquid assets, money and bonds are substitutes, so an increase in the supply of bonds is matched by a lower demand for real money balances. To accommodate for lower real money balances, prices increase, unless the central bank intervenes to stabilize prices by decreasing money supply. Only a massive increase in public debt, that fully compensates for the private deleveraging shock, can bring the economy out of a liquidity trap.

Quantitative Easing   The above analysis implies that QE has no effect per se in the liquidity trap steady state. QE consists in creating money through open market operations, i.e., increasing M by decreasing Pl^g. Since money and government bonds are perfect substitutes, this has

no effect in our setting.[18] However, QE entails a decrease in the available amount of government bonds l^g, which decreases shadow liquidity and the shadow interest rate. QE therefore leads to a deeper liquidity trap. It is thus important to time the exit from QE appropriately so that the economy does not linger in a liquidity trap.

Public Debt and Capital   Getting out of the liquidity trap through a higher public supply of liquidity, while leading to better consumption smoothing thanks to a higher interest rate, might have either a negative or a positive effect on capital accumulation and output. This depends on the level of liquidity $\bar{l}$ that prevails at the exit of the liquidity trap. Indeed, out of the ZLB, the interest rate starts to increase, and the net position of investors $\bar{l}$ determines the effect of a higher r on capital accumulation. If investors become net creditors due to the liquidity injection, then a higher r has a positive effect on capital accumulation. If investors remain net debtors, then a higher interest rate has a negative effect. Proposition 3 defines the corresponding threshold $\bar{l}_0$, as well as the level of liquidity $\bar{l}_T$ necessary to get out of the ZLB.

Hence, if $\bar{l}_T$ is lower than $\bar{l}_0$, then exiting the liquidity trap through a higher public debt would have a negative effect on capital. We can show that this happens if the deleveraging shock is not too large, leaving $\phi > \beta^2/(\beta^2 + \theta^2)$. Indeed, in that case, the level of liquidity necessary to get out of the ZLB is low. If on the opposite the deleveraging shock is large, so that $\phi < \beta^2/(\beta^2 + \theta^2)$, then $\bar{l}_T$ is higher than $\bar{l}_0$, leading to a positive effect on capital. This is illustrated in Figure 5. The left panel consider the case with a high ϕ, where at the exit of the ZLB capital starts to decrease. The right panel considers the case with a low ϕ, where at the exit of the ZLB capital starts to increase.

Welfare and Pareto Efficiency   While leading to a Pareto-efficient equilibrium, an increase in liquidity may not Pareto-*improve* on the initial equilibrium as workers may be hurt by lower output and wages, if the increase in interest rate leads to a lower capital stock. Moreover, while the economy would *converge* to a Pareto-efficient equilibrium, the whole equilibrium including transition dynamics would not be a Pareto equilibrium. The higher interest rate would initially

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[18]Note that we abstract from some potential channels of QE. In particular, the perfect substitutability of money and bonds means that there is no broad portfolio balance channel that could lower term or risk premia. Similarly, there is no signalling effect on future rates in our model since the liquidity trap is a steady state. See Borio and Disyatat (2009) for a detailed description of the channels of QE.
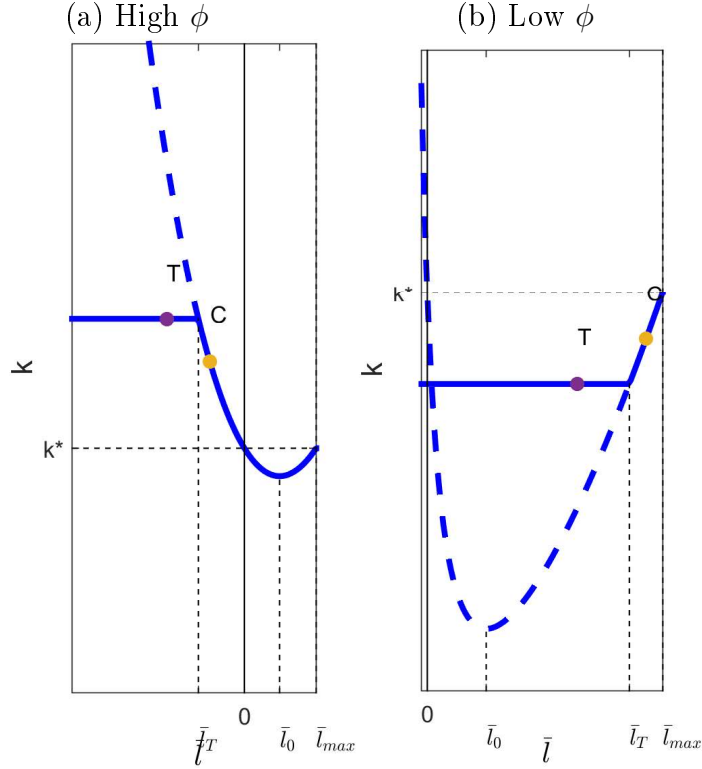
Figure 5: Steady states - Comparative statics w.r.t. $\bar{l}$, with high and low ϕ

hurt borrowers and initially decrease investment even lower than its liquidity trap level.[19]

Addressing these two problems requires many additional policy instruments. The online Appendix shows how three additional taxes/subsidies make it possible for the policy maker to implement a Pareto-efficient equilibrium path (including the transitory dynamics) that Pareto-improves on the initial liquidity trap.

Credit Easing Our model does not account for the fact that QE sometimes goes hand-in-hand with credit easing aimed at improving credit conditions for the private sector, which can alleviate the effect of deleveraging. Credit easing would consist in the government issuing new debt dl^g to lend an amount $(d\phi)\alpha y$ to I-investors above the limit of their borrowing constraint, effectively relaxing this constraint. As $dl^g = (d\phi)\alpha y$, the government net debt does not change and stays equal to l^g. Credit easing can therefore be effective in getting out of the liquidity trap, as it helps re-leveraging investors after a deleveraging shock. Similarly to public debt

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[19]A potential third issue is that reducing the capital stock can be undesirable in its own respect if there are external growth spillovers for example.

issuance, the interest rate gap is closed by increasing the shadow interest rate.

4.2   Lowering the Effective Real Interest Rate

Increasing liquidity closes the interest rate gap by increasing the shadow interest rate. The alternative is to decrease the effective rate. This could be done by increasing expected inflation through an increase in θ. This is a natural solution mentioned in the literature on the liquidity trap (e.g., Krugman (1998)). Alternatively, there could be a negative nominal interest rate. Suppose that cash is replaced by Central Bank digital money, on which a negative interest rate can be charged. There would then be no ZLB on the nominal interest rate and we could have $i < 1$.

A lower effective rate would sustain capital in the long run. As with liquidity policies, workers might be better off thanks to higher wages, but the lower interest rate would impair consumption smoothing. Moreover when investors are net debtors, decreasing the interest rate too much would also lead to capital over-accumulation. Exiting the ZLB by reducing the effective real interest rate drives out monetary liquidity without providing alternative liquidity and solving the underlying asset scarcity problem.

A timidity trap   If the effective rate is not lowered completely to the shadow rate, this has an ambiguous effect on capital and output. Consider a slightly higher inflation θ. Besides a decline in the effective real interest rate, this also increases the cost of holding money. This negative effect dominates when money holdings are large, i.e., when leverage ϕ is small. The precise impact of θ is described in the following proposition.

Proposition 4 (Effect of steady-state inflation) *Define* $\theta_0(\phi) = (1/\phi - 1)^{\frac{1}{2}}\beta$, $\theta_T(\bar{l}, \phi) = \beta\alpha(1 - \phi)/(\alpha\phi + \bar{l})$ *and assume* $\bar{l}_{min}(\phi) < \bar{l} \leq \bar{l}_{max}(\phi)$ *as in Proposition 3. Then* $\beta < \theta_T(\bar{l}, \phi)$. *If* $\theta \geq \theta_T(\bar{l}, \phi)$, *then the steady state is cashless. If* $\beta < \theta \leq \theta_T(\bar{l}, \phi)$, *then the steady state is a liquidity trap and has the following properties:*

(i) *the real interest rate r is decreasing in θ;*

(ii) *if $\phi < 1/2$, the capital stock is U-shaped in θ, decreasing for $\beta < \theta < \theta_0(\phi)$ and increasing for $\theta_0(\phi) \leq \theta \leq \theta_T(\bar{l}, \phi)$; if $\phi \geq 1/2$, it is always increasing in θ;*

(iii) *still, an increase in θ from a value below $\theta_T(\bar{l}, \phi)$ to a value above $\theta_T(\bar{l}, \phi)$ necessarily increases the capital stock if $\bar{l} \leq 0$.*

Proof. See Appendix A. ∎

A similar analysis holds for small decreases in the nominal interest rate i.[20]

4.3 Fiscal Policy

In the baseline policy regime, the fiscal deficit τ^w adjusts to the policy mix $(\theta, \bar{l}^g)$. However, the fiscal deficit could also become the dominant policy parameter. In that case, either θ or $\bar{l}^g$ needs to adjust. However, we show here that the adjustment to a fiscal deficit cannot come from public debt in a liquidity trap. If a permanent fiscal deficit can be financed at all, it can only be through the inflation tax, so that the analysis of 4.2 applies. Aggressive fiscal policy can therefore bring the economy out of the liquidity trap and stimulate the economy (although it leads to an inefficient equilibrium). While this prediction is reminiscent of the standard effect of fiscal policy at the zero-lower-bound, the channel here is not an aggregate demand channel whereby fiscal policy creates inflation expectations that stimulate private consumption. Inflation increases to guarantees the solvency of the government, and higher inflation stimulates the investment capacities of investors.

Fiscal Deficit and Inflation Tax In the cashless steady state, government debt $\bar{l}^g$ can finance a fiscal deficit τ^w, as apparent through the government budget constraint (26). However, in the liquidity trap, $\bar{l}^g$ becomes irrelevant, and only a higher inflation θ can accommodate a higher fiscal deficit. To see this, aggregate the demand for bonds and money to get:[21]

$$\theta(m + \bar{l}y) = \theta s + (1 - \alpha)y \tag{27}$$

where $s = m^S + \bar{l}y$ is the aggregate demand for savings by investors as defined earlier. We can then rewrite the government budget constraint (26) in the liquidity trap as

$$(\theta - 1)[(1 - \alpha)/\theta + s/y - \bar{l}^w] = \frac{\tau^w}{y}, \tag{28}$$

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[20]A liquidity trap with a negative interest rate is a situation currently observed in several countries.
[21]Equation (27) follows from (16) taken in a liquidity trap steady state, together with the definition of s.

where s/y is given by (25). In the liquidity trap, the level of government debt $\bar{l}^g$ no longer appears in the government budget constraint. Instead $m + \bar{l}^g$ adjusts through m whenever $\bar{l}^g$ changes, because at the given liquidity trap interest rate the private sector is not willing to hold more government liabilities. Therefore, the composition of government liabilities changes without affecting its total amount, leaving the government budget constraint unaffected since in a liquidity trap, the composition of government liabilities does not matter.

This implies that with an increase in the fiscal deficit τ^w, only θ can adjust to maintain solvency by creating an inflation tax. In that context, a permanent increase in the fiscal deficit is necessarily inflationary. The fiscal deficit may therefore be effective in helping the economy getting out the liquidity trap, but only because it requires higher inflation-driven seigniorage. Government spending, by increasing the deficit, would have similar effects.

An Inflation Tax Laffer Curve There is, however, a limit to the fiscal income that can be generated through inflation. Indeed, a higher inflation also decreases the demand for government assets, which reduces seigniorage. We can show, by differentiating the LHS of (28), that there is a Laffer curve for inflation, where the maximum inflation tax is reached for $\theta = [(1 - \alpha) + \alpha(1 - \phi)\beta] / (\alpha\phi + \bar{l}^w)$. Beyond that point, a permanently higher fiscal deficit is unsustainable. In a liquidity trap, there is a limit to the use of fiscal policy, as there is a limit to both the issuance of public liabilities and to the inflation tax.

5 Extensions

Bubbles The existing literature has long shown that rational bubbles can obtain in environments with low enough real interest rates.[22] In our framework with scarce assets, bubbles can provide additional saving instruments to accommodate the demand for assets by S-investors. A bubble, when it emerges, provides enough liquidity to exit the ZLB. But, as we will show, it also constrains the real interest rate and prevents the natural equilibrium adjustment.

Consider an infinitely-lived asset in fixed unitary supply with no intrinsic value—a bubble. Denote z_t its relative price in terms of consumption goods. The real return of the bubble as

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[22]See Samuelson (1958), Tirole (1985), and more recently Martin and Ventura (2012).

of time t is z_{t+1}/z_t. For the bubble to be traded, this rate of return must be equal to the real interest rate: $z_{t+1}/z_t = r_{t+1}$. With r_{t+1} different from 1, the bubble would either asymptotically disappear or diverge to an infinite value. Then, a bubbly steady state necessarily has a zero *real* interest rate: $r = 1$. With positive long run inflation, $1 > 1/\theta$ so the bubble strictly dominates money as a saving instrument. Therefore, S-investors would hold the bubble and would not hold money.[23] In the case of autarkic investors, such a bubbly steady state is described by:

$$z = \alpha[(1 - \phi)\beta - \phi]y \tag{29}$$

$$k = \beta\alpha y - (1 - \beta)z \tag{30}$$

where (29) is the Euler equation of savers and (30) the aggregate budget constraint of investors. As can be seen from equations (19) and (21), the bubbly steady state is formally equivalent to a liquidity trap steady state with $m^S = z$ and $\theta = 1$. The bubble plays the same role as investor-held money in the liquidity trap, but offers a higher real return.

We show formally in the Online Appendix that a bubble can indeed help the economy exit the liquidity trap if $\theta > 1$. The bubble raises the nominal interest rate from $i = 1$ to $i = \theta$. S-investors then substitute the bubble for money in their portfolio. For a given money supply, this also reflates the economy as the price level increases to accommodate the lower money demand.

However, the bubbly steady state is qualitatively similar to a liquidity trap. As with money, holding the bubble takes out resources from investment and output is lower in the bubbly equilibrium than in the cashless steady state. In the intermediate case where $\phi_T \leq \phi < \phi_B$, where ϕ_B is a threshold value defined in the online Appendix, a bubble prevents the downward interest rate adjustment that would restore the cashless level of capital and output. In the case of low leverage $\phi < \phi_T$, bubbles increase the real interest rate, which may or may not increase capital and output compared to the liquidity trap. This is similar to the ambiguous effect of inflation described in Section 4.2.

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[23]With negative long-run inflation, bubbles would be dominated by money and could never arise in equilibrium.

Preference and Growth Shocks In the existing literature, the shock that brings the economy to the ZLB is often assumed to be an increase in the factor of time preference. This shock, by increasing the agents' propensity to save, has a negative effect on the interest rate. A reduction in the average growth rate of productivity has also been put forward as an explanation for the secular decrease in the interest rate and for hitting the ZLB. In fact, in an infinite-horizon model, the effect of a growth slowdown is isomorphic to an increase in the factor of time preference. We therefore restrict our analysis to the latter. We find that a permanent increase in β (alternatively, a permanent fall in steady-state growth), cannot generate a fall in the investment rate when the economy falls into a liquidity trap.

To study the effect of β on output, we make the simplifying assumption of autarkic investors: $\bar{l} = 0$. This is without loss of generality as the investors' net debt matters only in the cashless economy. We show in the online Appendix that an increase in β makes the long-run interest rate fall, and eventually hit the ZLB. In both the cashless and liquidity-trap steady states, an increase in β increases the investors' propensity to save, which increases the capital stock in the long run. As a result, whereas an increase in β can explain the emergence of a liquidity trap, it cannot explain the slowdown in investment. In the presence of trend growth, the same conclusions would hold in case of a growth slowdown. In particular, with lower trend growth, less investment is required to keep the capital stock on its trend. Therefore a given amount of saving leads to an upward shift in the capital intensity of production, and hence in the investment rate.

Financial Intermediation In the benchmark model, money is modeled as *outside money* directly supplied by the government. However, in practice, cash holdings usually take the form of deposits, which are a liability of banks, and could in principle be intermediated to capital investment. We show in the online Appendix that this is not the case. At the ZLB, banks are unable to channel deposits to credit-constrained I-investors for the same reason that savers are unable to do it in the benchmark model. Instead, banks increase their excess reserves at the central bank.

Inefficient saving technology The benchmark model assumes that bonds and money are the only available saving instruments. In the online Appendix, we extend the model by allowing

for an inefficient storage technology, with a rate of return $\sigma \in (\theta^{-1}, \beta^{-1})$ and concave installation costs. This technology starts being used by savers when the interest rate falls down to σ. Then, a moderate deleveraging shock reallocates savings to the storage technology, which crowds out "good" capital even in the cashless equilibrium. This reallocative effect is similar to the one studied by Buera and Nicolini (2016). With a large enough deleveraging shock, the economy falls into the liquidity trap, the use of inefficient storage is pinned down by the real rate of interest $1/\theta$, and higher money holdings crowd out capital as in the benchmark model. One difference with the benchmark model is that the shadow rate now has a strictly positive lower bound as ϕ goes to 0, since the storage technology prevents a complete collapse of intertemporal trade, arguably a more realistic feature.

Idiosyncratic Uncertainty   The benchmark model with deterministic transitions between saving and investing phases can be easily extended to stochastic transitions. We consider in the online Appendix a 2-state Markov process where an investor with no investment opportunity at time $t-1$ receives an investment opportunity at time t with probability $\omega \in (0, 1]$; while an investor with an investment opportunity at time $t-1$ receives no investment opportunity at time t. We then show that results from the benchmark model extend to the case of idiosyncratic uncertainty.

Partial Capital Depreciation   In the model, we assumed full capital depreciation. In the online Appendix, we allow the depreciation rate of capital to be lower than one, so that capital depreciates only partially from period to period. For consistency, we focus on the case where investors are net debtors $\bar{l} \leq 0$. All our results generalize provided some mild condition on $\bar{l}$, which is described in the online Appendix.

Nominal Government Bonds   We have assumed so far that government bonds were issued in real terms. In reality though, a large share of government bonds are nominal. In our deterministic setting, assuming that bonds are nominal instead of real is innocuous and all our results generalize to nominal bonds.

Nominal Rigidities and Transitional Dynamics   Since we have focused on the long term, we have assumed flexible prices. In order to discuss transitional dynamics in a meaningful way, we introduce nominal frictions with wage rigidities (see online Appendix). With sticky wages, a deleveraging shock large enough to move the economy to the ZLB creates a negative output gap in the short run, as in the existing New Keynesian literature. The intuition is best described by Equation (16), the market-clearing condition for money: $M_{t+1} = (1-\alpha)P_t y_t + M_{t+1}^S$. When the economy hits the ZLB, money demand by investors M^S increases. If the monetary authority does not react, adjustment has to come from a lower nominal output $P_t y_t$. If prices cannot adjust quickly, adjustment in the short run requires a drop in output. However, a sufficiently large monetary expansion could accommodate investors' money demand, which stabilizes output in the short run and the price level in the longer run.

The presence of nominal rigidities therefore allows for the analysis of short-run demand policies. But in the long run, the effects caused by the scarcity of assets prevail. Contrary to the New Keynesian literature, the economy stays at the ZLB with a lower capital stock and lower level of output, even after wages have adjusted and the output gap has closed.

6   Conclusions

The liquidity trap that followed the Global Financial Crisis has been more persistent than was expected. The liquidity trap has last even longer in Japan. In most countries, this has been accompanied by a slower-than-expected recovery and a surprising accumulation of money holdings. In this paper, we explored the long-term implications of a liquidity trap and found that a deleveraging shock may lead to a negative relationship between money and capital. We analyzed policies in a liquidity trap by examining their impact on the wedge between the effective real interest rate and the shadow rate.

While most of our analysis is conducted in a stylized benchmark model, the main mechanism is robust to many extensions. The extensions considered in the paper include bubbles, partial capital depreciation, idiosyncratic uncertainty, nominal bonds, or introducing an alternative saving technology or financial intermediaries. For analytical convenience, we consider a perma-nent deleveraging shock for investors, but the results would be similar with a very persistent

shock.

According to our results, long-term output declines in a liquidity trap only with investors' deleveraging. Other positive shocks to saving, like workers' deleveraging or an increase in the discount rate, may also lead to a liquidity trap, but they do not depress output in the long run. Therefore it is crucial to determine the factors that have led to a liquidity trap. Interestingly, Galí et al. (2012) suggest that financial shocks have played a key role in the slow recovery.

Overall, our approach is complementary to Keynesian analyses that stress the role of insufficient demand in a liquidity trap. While they describe a situation of negative output gap when the adjustment of prices is hampered by nominal rigidities, we show that low investment demand leads to lower potential output even after prices have fully adjusted. Our framework also enables to examine policies that are complementary to more standard demand management. In this context, we find that quantitative easing is ineffective at the ZLB and can deepen the liquidity trap. We also argue that it may be better to increase the shadow rate than decrease the effective real interest rate.

A   Proofs

We establish first the following Lemma:

Lemma 1 *The cashless and liquidity trap steady states are characterized as follows:*

(i) *In a cashless steady state,*

$$r^* = \frac{\alpha\phi + \bar{l}}{\beta\alpha(1-\phi)}, \qquad k^* = \left[\beta\alpha - \bar{l}(1/r^* - \beta)\right]^{\frac{1}{1-\alpha}}, \qquad m^{S*} = 0.$$

(ii) *In a liquidity-trap steady state,*

$$\hat{r} = 1/\theta, \qquad \hat{k} = \left(\frac{\beta^2 + \phi(\theta^2 - \beta^2)}{\theta/\alpha}\right)^{\frac{1}{1-\alpha}}, \qquad \hat{m}^S = \alpha\left[(1-\phi)\frac{\beta}{\theta} - \phi - \bar{l}/\alpha\right]\hat{k}^\alpha.$$

Proof. In a steady state, the money market equilibrium implies that $P_{t+1}/P_t = \theta$. As a result, $i = r\theta$.

In a steady state with $i^* > 1$, (14) and (15) are satisfied with $M^S = 0$. Equation (14) taken at the steady state gives r^*. Besides, (15) in the steady state gives:

$$k^*/y^* = \beta\alpha - \bar{l}(1/r^* - \beta)$$

which yields our result for k^*. This proves result (i).

In a steady state with $i = 1$, (14) and (15) are satisfied with $r = \hat{r} = \theta^{-1}$, which yields $\hat{k}/\hat{Y} = \frac{\beta^2 + \phi(\theta^2 - \beta^2)}{\theta\alpha^{-1}}$, from which we derive $\hat{k}$, and $\hat{m}^S = \alpha\left[(1-\phi)\frac{\beta}{\theta} - \phi\right]\hat{k}^\alpha - \bar{l}\hat{k}^\alpha$. This proves result (ii). ∎

A.1 Proof of Proposition 1

Consider a cashless steady state with $\bar{l} = 0$. According to Lemma 1, $r^* = \phi/[\beta(1-\phi)]$. We check that $0 < \beta r^* < 1$ as $\phi < \phi_{max}$ and that $i^* = \theta r^* \geq 1$ as $\phi \geq \phi_{max}$, which insures that the cashless steady state exists and is locally constrained. This proves result (i).

If $\phi < \phi_T$, then the steady state without money does not exist, as the implied nominal interest rate i^* would be below one. If there exists a steady state with $i = 1$, then it is a liquidity trap described by Lemma 1. According to Lemma 1, when $\bar{l} = 0$, $\hat{m}^S = \alpha\left[(1-\phi)\frac{\beta}{\theta} - \phi\right]\hat{k}^\alpha$, which is strictly positive when $\phi < \phi_T$. Besides, $\hat{r} = \theta$, which implies that $0 < \beta\hat{r} < 1$ under Assumption 1, and $\hat{r} > r^*$ for $\phi < \phi^T$. We also check that $\hat{k} = \left(\frac{\beta^2 + \phi(\theta^2 - \beta^2)}{\theta/\alpha}\right)^{\frac{1}{1-\alpha}} < k^* = (\beta\alpha)^{\frac{1}{1-\alpha}}$ for $\phi < \phi^T$. This proves result (ii). Results (iii) and (iv) derive naturally from Lemma 1.

A.2 Proof of Proposition 2

Consider a cashless steady state. Using Lemma 1, we check that $0 < \beta r^* < 1$ as $\phi_{min}(\bar{l}) < \phi < \phi_{max}(\bar{l})$ and that $i^* = \theta r^* > 1$ as $\phi > \phi_T(\bar{l})$, which insures that the cashless steady state exists and is locally constrained. This proves result (i).

If $\phi < \phi_T(\bar{l})$, then the steady state without money does not exist, as the implied nominal interest rate i^* would be below one. If there exists a steady state with $i = 1$, then it is a liquidity trap described by Lemma 1. According to Lemma 1, $\hat{m}^S = \alpha\left[(1-\phi)\frac{\beta}{\theta} - \phi - \bar{l}/\alpha\right]\hat{k}^\alpha$, which is strictly positive when $\phi < \phi_T(\bar{l})$. Besides, $\hat{r} = \theta$, which implies that $0 < \beta\hat{r} < 1$ under Assumption 1, and $\hat{r} > r^*$ for $\phi < \phi^T$. We also check that $\hat{k} = \left(\frac{\beta^2 + \phi(\theta^2 - \beta^2)}{\theta/\alpha}\right)^{\frac{1}{1-\alpha}} < k^* = k^* =$

$\left[\beta\alpha - \bar{l}(1/r^* - \beta)\right]^{\frac{1}{1-\alpha}}$ for $\phi < \phi^T$. This proves result (ii).

Regarding the properties of r, i and m^S, results (iii) and (iv) derive directly from Lemma 1. To derive the properties of k, we replace r^* in k^* to obtain

$$k^* = \left(\alpha\beta - \bar{l}\left[\frac{\alpha\beta(1-\phi)}{\alpha\phi + \bar{l}} - \beta\right]\right)^{1/(1-\alpha)} \tag{31}$$

We can see that k^* is increasing in ϕ for $\bar{l} > 0$, decreasing for $\bar{l} < 0$.

A.3 Proof of Proposition 3

Results (i) and (ii) derive directly from Lemma 1. Regarding the properties of r, i and m^S, results (iii) and (iv) derive directly from Lemma 1. To derive the properties of k, we use (31) and take the derivative of k with respect to $\bar{l}$. We find that k is decreasing in $\bar{l}$ whenever $P(\bar{l}) \geq 0$ with

$$P(\bar{l}) = \bar{l}^2 + 2\alpha\phi\bar{l} - \alpha^2\phi(1 - 2\phi)$$

This second-order polynomial admits two roots: $\bar{l}_{00} = -\alpha\phi - \alpha\sqrt{\phi}\sqrt{1-\phi}$ and $\bar{l}_0 = -\alpha\phi + \alpha\sqrt{\phi}\sqrt{1-\phi}$. As $\bar{l}_{00} < \bar{l}_{min}$, $\bar{l}_0$ is the only relevant solution. As a result, k is decreasing in $\bar{l}$ for $\bar{l}_{min} \leq \bar{l} \leq \bar{l}_0$ and increasing for $\bar{l}_0 \leq \bar{l} \leq \bar{l}_{max}$.

To show (iv), note that there exists cashless steady states with $\bar{l} < 0$ iif $\bar{l}_T(\phi) < 0$, which is the case when $\phi > \beta/(\beta + \theta)$. Besides, k is decreasing in $\bar{l}$ in the right neighborhood of $\bar{l}_T(\phi)$ iif $\bar{l}_0 > \bar{l}_T$, which is the case when $\phi > \beta^2/(\beta^2 + \theta^2)$. Since $\theta/\beta > 1$ by assumption, we have $\phi > \beta/(\beta + \theta)$ implies $\phi > \beta^2/(\beta^2 + \theta^2)$, hence the result.

A.4 Proof of Proposition 4

The proof derives from Lemma 1, with a threshold $\theta_T(\bar{l})$ defined such that $\phi = \phi_T(\bar{l})$ when $\theta = \theta_T(\bar{l})$. To derive result (ii), we take the derivative of k with respect to θ and show that it is negative for $\beta < \theta < (1/\phi - 1)^{\frac{1}{2}}\beta$ and positive for $(1/\phi - 1)^{\frac{1}{2}}\beta \geq \theta \geq \theta_T(\bar{l})$. To show (iii), it is enough to show that capital with $\theta = \beta$ is lower than capital with $\theta = \theta_T(\bar{l})$. With $\theta = \beta$, we have $k = (\alpha\beta)^{\frac{1}{1-\alpha}}$. With $\theta = \theta_T(\bar{l})$, the economy becomes cashless so we have $k = k^*$. Using the definition of k^* as given by Lemma 1, we know that $k^* \geq (\alpha\beta)^{\frac{1}{1-\alpha}}$ whenever $\bar{l} \leq 0$.

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