

Nash Equilibrium in Games with Quasi-Monotonic Best-Responses*

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Abstract

This paper develops a new existence result for pure-strategy Nash equilibrium. In succinct form, for a two-player game with scalar action sets, existence entails that one reaction curve be increasing and continuous and the other quasi-increasing (i.e, not have any downward jumps). The latter property amounts to strategic quasi-complementarities. The paper provides a number of ancillary results of independent interest, including sufficient conditions for a quasi-increasing argmax, and new sufficient conditions for uniqueness of fixed points. For maximal accessibility of the results, the main results are presented in a Euclidean setting. We argue that all these results have broad and elementary applicability by providing simple illustrations with commonly used models from applied microeconomic fields.

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1 Introduction

In the course of developing new game-theoretic models to describe economic behavior in various situations, the existence of Nash equilibrium often emerges as the first critical test to discriminate between alternative candidate models. In most economic settings, a long-standing preference for pure-strategy Nash equilibrium (henceforth, abbreviated PSNE) still constitutes the dominant norm. The primary requirement of existence applies to any type of investigation, independently of whether the analysis is meant to proceed along specific or general functional forms. In the latter case, existence of Nash equilibrium is virtually always obtained via the application of a fixed point theorem, following a long-standing practice going back to Nash (1951).

While Nash considered mixed-strategy equilibrium for finite games, Rosen (1965) extended his basic insight to the case of pure-strategy equilibrium and Euclidean action spaces. In this traditional approach, existence follows from Brouwer's (or Kakutani's) fixed point theorem, and is therefore predicated on the best response functions (or correspondences) being continuous on compact and convex action spaces. Stepping back to the payoff functions, the relevant properties are joint continuity in the strategies and quasi-concavity in own action.¹ For obvious reasons, the underlying method has come to be known as the topological approach.

In more recent times, a new approach to the existence of pure-strategy Nash equilibrium, which relies on the best response mapping (and thus the "reaction curves") being increasing functions (or selections) and the action spaces being complete lattices, was proposed by Topkis (1978, 1979). Based instead on Tarski's lesser known fixed point theorem for monotone functions (Tarski, 1955), this approach of an order-theoretic or algebraic nature has given rise to the recently much studied class of supermodular games. In addition to opening a new realm for addressing the fundamental issue of existence, this approach has also proven useful for the characterization of equilibrium properties, in particular with respect to comparative statics conclusions (Topkis, 1979, Vives, 1990, and Milgrom and Roberts, 1990).

The purpose of the present paper is to develop a new class of games that possess pure-strategy Nash equilibrium, which is not covered by either of the two aforementioned existence paradigms. In its most general form, the result pertains to two-player games with scalar action sets. This new

¹These requirements have since been partially relaxed in the recent literature dealing with discontinuous games.

result imposes different requirements on the two players' reaction curves. For one player, this curve must be both continuous and increasing, while for the other player all that is needed is that his reaction curve not possess any downward jump discontinuities. Fig. 1 below illustrates our basic fixed point result.

We follow a lesser known part of Tarski's (1955) classical paper, and call "quasi-increasing" functions that cannot have downward jumps.² The principal aim of this part of Tarski's paper is to prove an intersection point theorem between a quasi-increasing function and a quasi-decreasing function whenever these have the same domains and ranges (both complete chains) and the former starts above the latter and ends below it.³ An important special case of this Theorem obtains when one takes the quasi-decreasing function to be the identity function, in which case the result boils down to a fixed point theorem for quasi-increasing functions. Interestingly, variants of this fixed point theorem have been rediscovered in the economics literature and applied a number of times to establish existence of symmetric PSNE in symmetric oligopoly settings. As a result, the existence of a symmetric PSNE then follows at once from the existence of a fixed point for the common reaction curve to all the players.⁴

While the economic applications of this result have so far all shared the critical property that the underlying game is symmetric, the starting point for the present paper is the idea that the underlying logic may be used to establish existence of PSNE for asymmetric two-player games. To this end, it suffices to apply this fixed point theorem to the composition of the two reaction curves, one of which is a quasi-increasing function and the other an increasing and continuous function, upon making the key observation that the composition of two such functions is itself necessarily quasi-increasing. While this structure makes it clear that the underlying class of games overlaps with the two existing classes of games that are known to possess PSNEs, as described above, it is easy to see that it is not nested with either of them.

²Tarski's definition of quasi-increasing functions is given in purely lattice-theoretic terms for functions mapping chains into chains. The simpler version for real numbers, which is the focus of this paper, is formalized in Section 2.

³A quasi-decreasing function is defined by the dual property of not possessing any upward jump discontinuities. See Section 2 for a more detailed explanation and illustration of Tarski's intersection point theorem.

⁴See MacManus (1962, 1964), Roberts and Sonnenschein (1976), Reinganum (1982), Vives (1990), Milgrom and Roberts (1994), Amir (1996), Amir and Lambson (2000), and Hoernig (2003), among others.

It is instructive to provide some basic insight into the restrictiveness of the needed assumptions for this new existence result by comparing them to those underlying supermodular games as a benchmark. The comparison yields a mixed outcome. On the one hand, for one player, the present framework requires continuity in addition to upward monotonicity of the reaction curve. On the other hand, for the other player, the requirement of upward monotonicity has been relaxed to that of just quasi-monotonicity. Amending the standard terms to refer to these properties in the economics literature in a suitable way, one may designate such games as being characterized by continuous strategic complementarity for one player and limited strategic substitutability for the other player, or what we may call *strategic quasi-complementarity*. While the former property may be seen as combining the requirements from each of two known approaches to existence of PSNE so far, i.e., continuity and monotonicity, no such connection holds for the property of quasi-increasing reaction curves. Recall that the latter rules out only downward jump discontinuities.

Despite the partial link with the topological approach just alluded to, it is important to stress at the outset that the present approach is lattice-theoretic. One consequence of this fact is that the result admits an order dual for games where the reaction function of one player is continuous and decreasing and that of the other player is quasi-decreasing. While the two dual results are mathematically equivalent, they tend to apply to quite different classes of economic models (we shall have more to say on this point in the applications section below).

In delineating the proper scope for the new result at hand, it is important to point out two basic shortcomings. The first is that the players' strategy spaces must be a chain or a totally ordered set, a limitation that stems directly from the use of Tarski's intersection point theorem. Indeed, the latter result does not generalize to partially ordered sets. The second drawback is that this basic existence result does not extend at the same level of generality to games with more than two players. Multi-player extensions are possible, but only at the expense of some further assumptions. In one version, players 3 to n are added into the basic two-player setting in such a way that the new players satisfy type-symmetry with respect to one of the two basic players, so that the n -player game contains only two types of players. Another extension posits an aggregation property in each player's payoff and otherwise preserves the original two-player setting. These extensions to n -player games are beyond the scope of this paper.

While the existence conclusions described thus far constitute the main goal of the paper, the underlying analysis requires a number of ancillary results that are of independent interest. Some of these are crucially needed as building blocks to construct the basic machinery for the existence result, while others are useful supplementary results. The first of the building blocks is to derive some simple lemmas that capture the essential features of quasi-monotonic functions, provide a basic calculus for useful operations involving them, as well as sufficient conditions that isolate a useful subclass of quasi-monotonic functions (the so-called upper or lower Lipschitz functions). The second block consists in the elaboration of sufficient conditions on a parametric optimization problem to yield a quasi-monotonic argmax correspondence. To this end, we follow some existing work (in particular Granot and Veinott, 1985 and Curtat, 1996) and introduce parameter-dependent changes of variable that allow the desired conclusion to obtain in much the same way as the usual conclusion of monotonicity of argmax's.

In line with the theory of supermodular games, the basic existence results under consideration do not address uniqueness of PSNE in any way. Instead, the equilibrium set is shown to constitute a chain under great generality (this is akin to the result that the PSNE set for supermodular games is a complete lattice). Nonetheless, some key supplementary results provide novel sufficient conditions for the uniqueness of PSNE. While the underlying conditions are all related in one form or another to a contraction property in the reaction curves, they require this basic property to hold only in a local sense. As such, the results are more reminiscent of the uniqueness results in equilibrium theory that are based on degree theory (see e.g., Dierker, 1972). In addition to applying to some settings covered by the existence results given here, these uniqueness results could also apply more broadly to other classes of games. In particular, the uniqueness results apply to symmetric PSNE of symmetric games, for which we also state the basic underlying existence result that forms the general version of those that have appeared in specific oligopoly contexts (e.g., Roberts and Sonnenschein, 1976).

Last but certainly not least, as with any advance in abstract theory, one needs to address a crucial test: How broad is the scope of applicability of the novel results? A related subquestion is, how accessible is the overall tool kit developed to facilitate the use of the new results here? In order to make a compelling case that both questions can be answered along very positive lines, we provide several detailed examples of well known economic models for which new existence results

are obtained via the direct use of the examples of the paper. Furthermore, as we provide all the concomitant details of the various steps needed to apply the results for the different models, the reader can easily appreciate the practical value of the basic results of this paper. While a large variety of applications may be given, it suffices to develop the following selection of well established economic models: A hybrid duopoly model of price-quantity competition with differentiated products, a model of Bertrand competition with increasing returns for one firm, and a widely used model of non-cooperative provision of public goods. In addition, the Bertrand model is also used as a vehicle to illustrate some of the ancillary results of the paper.

The organization of the paper is as follows. We begin in Section 2 with a full exposition of the definitions of the new notions and a derivation of the basic results for real action sets. In Section 3, the new results are stated in the form of existence results for PSNE in games, along with the associated uniqueness results. Section 4 contains a detailed discussion of the economic applications of the new theory. Section 5 is a brief conclusion.

2 Quasi-monotone functions on \mathbb{R}

In the framework of real parameter and decision spaces, this section lays out all the fundamental notions and basic results that are needed as preliminaries for our new approach to the existence of pure-strategy Nash equilibrium (henceforth PSNE) for games with scalar real action spaces. The present theory is based on the properties of quasi-increasing and quasi-decreasing functions, introduced by Tarski (1955) for general totally ordered lattices (chains). We begin in Section 2.1 with the basic definitions and properties of quasi-monotone functions in one-dimensional Euclidean space along with some basic practical tests for this property. We provide useful sufficient conditions for quasi-increasingness in section 2.2, before moving to the analysis of parametric optimization problems that yield quasi-monotonic functions as optimal solutions in Section 2.3. Then Section 2.4 describes our fixed point results, which are used in section 3 for equilibrium existence in games.

2.1 Definition and basic properties

In the same article that contains his well known fixed point theorem for increasing maps on a complete lattice, Tarski (1955) also proved an intersection point theorem (his Theorem 3). Later, this subsection presents this much less known result for the special case of real-valued functions on a real domain. The main new concepts needed are the following.

Definition 1 *Let $X, Y \subset \mathbb{R}$. A function $f : X \rightarrow Y$ is quasi-increasing if for every $x \in X$,*

$$\limsup_{y \uparrow x} f(y) \leq f(x) \leq \liminf_{y \downarrow x} f(y). \quad (1)$$

f is quasi-decreasing if for every $x \in X$,

$$\liminf_{y \uparrow x} f(y) \geq f(x) \geq \limsup_{y \downarrow x} f(y). \quad (2)$$

A function is quasi-monotone if it is either quasi-increasing or quasi-decreasing.

Fig. 2 below illustrates the concepts of quasi-increasing and quasi-decreasing functions.

From the definition, the following facts follow easily:

Proposition 2 *Let $\lambda \geq 0$, $X \subset \mathbb{R}$ and $f, g : X \rightarrow \mathbb{R}$ be two functions. The following holds:*

1. *f is continuous if and only if it is both quasi-increasing and quasi-decreasing.*
2. *If f is increasing, then it is quasi-increasing. Analogously, if f is decreasing, then it is quasi-decreasing.⁵*
3. *If f is quasi-increasing and g is continuous and increasing, then $g \circ f$ and $f \circ g$ are also quasi-increasing.*
4. *If f is quasi-increasing and g is continuous and decreasing, then $g \circ f$ and $f \circ g$ are quasi-decreasing.*
5. *For the last two items, it is not sufficient that g be just increasing.*

⁵Throughout this paper, we call "increasing" any function that is weakly increasing (i.e., nondecreasing). We use "strictly increasing" for the strict version of this concepts. The same applies to decreasing functions.

6. If $f, g : X \rightarrow \mathbb{R}$ are quasi-increasing functions, then $\lambda f + g$ is quasi-increasing.

7. If f, g are quasi-decreasing functions, then $\lambda f + g$ is quasi-decreasing.

Proof. The proofs for (i) and (ii) are straightforward. Let us prove (iii). Since f is quasi-increasing, for every $x \in X$,

$$\limsup_{y \uparrow x} f(y) \leq f(x) \leq \liminf_{y \downarrow x} f(y).$$

Since g is continuous and increasing,

$$\limsup_{y \uparrow x} g(f(y)) = g\left(\limsup_{y \uparrow x} f(y)\right) \leq g(f(x)) \leq g\left(\liminf_{y \downarrow x} f(y)\right) = \liminf_{y \downarrow x} g(f(y)).$$

Therefore, $g \circ f$ is quasi-increasing. Similarly,

$$\limsup_{y \uparrow x} f(g(y)) \leq f\left(\limsup_{y \uparrow x} g(y)\right) = f(g(x)) = f\left(\liminf_{y \downarrow x} g(y)\right) \leq \liminf_{y \downarrow x} f(g(y)).$$

Thus, $f \circ g$ is quasi-increasing.

The proof of (iv) is analogous. To see (v), let $f, g : [0, 1] \rightarrow [0, 1]$ be defined by $f(x) = 1 - x$ and

$$g(x) = \begin{cases} \frac{x}{2} & \text{if } x < \frac{1}{2} \\ \frac{1}{2} & \text{if } x = \frac{1}{2} \\ \frac{1+x}{2} & \text{if } x > \frac{1}{2} \end{cases}$$

It is easy to see that f is continuous and therefore quasi-increasing, while g is increasing but not continuous. We have:

$$\limsup_{y \uparrow \frac{1}{2}} (g \circ f)(y) = \frac{3}{4} > \frac{1}{2} = (g \circ f)\left(\frac{1}{2}\right) > \frac{1}{4} = \liminf_{y \downarrow \frac{1}{2}} (g \circ f)(y),$$

and

$$\limsup_{y \uparrow \frac{1}{2}} (f \circ g)(y) = \frac{3}{4} > \frac{1}{2} = (f \circ g)\left(\frac{1}{2}\right) > \frac{1}{4} = \liminf_{y \downarrow \frac{1}{2}} (f \circ g)(y),$$

contradicting quasi-increasingness (1) both for $g \circ f$ and $f \circ g$.

To see (vi), let f and g be quasi-increasing functions. Using the fact that (a) $\liminf_{y \uparrow x} \lambda f(x) = \lambda \liminf_{y \uparrow x} f(x)$ and that a similar property holds for \limsup , and (b) the subadditivity of the \limsup operation and the superadditivity of the \liminf operation, we have:

$$\limsup_{y \uparrow x} (\lambda f(y) + g(y)) \leq \lambda \limsup_{y \uparrow x} f(y) + \limsup_{y \uparrow x} g(y) \leq \lambda f(x) + g(x),$$

and

$$\liminf_{y \downarrow x} (\lambda f(y) + g(y)) \geq \lambda \liminf_{y \downarrow x} f(y) + \liminf_{y \downarrow x} g(y) \geq \lambda f(x) + g(x).$$

This establishes (vi). The proof of (vii) is analogous. ■

Using this concept of quasi-increasing and quasi-decreasing functions, Tarski (1955) proved a theorem whose real-valued version reads as follows.

Theorem 3 (Tarski’s Intersection Point Theorem) *If $f : [a, b] \rightarrow \mathbb{R}$ is quasi-increasing, $g : [a, b] \rightarrow \mathbb{R}$ is quasi-decreasing, $f(a) \geq g(a)$ and $f(b) \leq g(b)$, then the set $\{x \in [a, b] : f(x) = g(x)\}$ is non-empty, and has as largest element $\vee\{x \in [a, b] : f(x) \geq g(x)\}$ and as smallest element $\wedge\{x \in [a, b] : f(x) \leq g(x)\}$.*

Theorem 3 is graphically illustrated in Fig. 3 below. The conditions $f(a) \geq g(a)$ and $f(b) \leq g(b)$ are indispensable. Fig. 2 above illustrates a case where these conditions fail and there is no intersection point.

Since Theorem 3 pertains to two functions with the same domains and the same ranges, it is more aptly called an intersection point theorem (between two curves). We shall refer to it as such, motivated also by the need to distinguish it from the well known fixed point theorem (Tarski, 1955).

Unaware of Tarski’s intersection point theorem, Milgrom and Roberts (1994) use (1) to define quasi-increasing functions, which they referred to as “continuous but for upward jumps.” Indeed, this terminology is quite revealing since one crucial implication of (1) is that jump discontinuities in quasi-increasing functions must be upward (likewise, jumps in quasi-decreasing functions must be downward).

Milgrom and Roberts (1994) proved a fixed-point result for quasi-increasing self maps on a compact interval, which can be obtained as a special case of Theorem 3 by taking f to be the mapping of interest and g to be the 45° line, in which case the extra conditions $f(a) \geq g(a)$ and $f(b) \leq g(b)$ are automatically satisfied (see Corollary 11). In fact, the latter result has a remarkable history in the economics literature in that special cases were independently discovered by McManus (1964) and Roberts and Sonnenschein (1976). These two studies used this fact as an intermediate result with the principal aim of establishing existence of a symmetric equilibrium for symmetric Cournot oligopoly. In their result, the key property of the underlying reaction curve (of any one

firm) in symmetric Cournot oligopoly with convex costs is that all its slopes are bounded below by -1 , which is a sufficient condition for a function to be quasi-increasing. For a generalization to the case of symmetric firms with non-convex costs, using Topkis's monotonicity result for the first time in this literature, see Amir and Lambson (2000) and Amir (1996).

We shall derive useful results for both quasi-increasing and quasi-decreasing functions. However, since properties related to quasi-decreasing functions can be obtained directly from analogous ones for quasi-increasing functions using standard duality arguments, we shall limit most of our discussion to quasi-increasing functions.

While the property of quasi-monotonicity might at first sight appear quite esoteric as far as its relevance to economic modeling is concerned, we shall derive functional and convenient sufficient conditions for quasi-increasingness that arise in quite natural ways in economics. The next subsection discusses some of these conditions.

2.2 On some subclasses of quasi-monotone functions

With the exception of the key implication of ruling out downward jumps, the general definition of quasi-increasing imposes hardly any useful structure on functions that would make them amenable to practical analysis from the perspective of economic applications. For instance, since every continuous function is both quasi-increasing and quasi-decreasing, quasi-monotonic functions may fail to possess left and right limits at points of their domains, or to possess any smoothness properties. In this section, we derive sufficient conditions for quasi-monotonicity that impart crucial structure to the associated subclass of functions, akin to that enjoyed by monotonic functions. Importantly, these sufficient conditions correspond precisely to properties that are naturally satisfied when quasi-monotonic functions arise as argmax's of parametric optimization problems whose objective functions satisfy some quasi-complementarity conditions to be identified below.

An important sub-class of quasi-increasing functions that arise naturally in economics is the class of lower-Lipschitz functions, defined as follows. A function $f : X \rightarrow \mathbb{R}$ is K -lower-Lipschitz if for some $K \in \mathbb{R}$, we have $f(x) - f(y) \geq K(x - y)$ for all $x, y \in X$ such that $x \geq y$.⁶ A function $f : X \rightarrow \mathbb{R}$ is K -upper-Lipschitz if $-f$ is K -lower-Lipschitz.

⁶Recall that f is K -Lipschitz if for some $K \geq 0$, we have $|f(x) - f(y)| \leq K|x - y|$ for all $x, y \in X$.

Lemma 4 *Let $X \subset \mathbb{R}$ and assume that a function $f : X \rightarrow \mathbb{R}$ is K -lower-Liipschitz (resp. K -upper-Liipschitz). Then*

- (a) *f is quasi-increasing (resp. quasi-decreasing), and*
- (b) *f is a function of bounded variation.*

Proof. (a) The property of lower-Liipschitz can be rewritten as $x \geq y \implies f(x) - Kx \geq f(y) - Ky$, that is, the function $\widehat{f}(x) = f(x) - Kx$ is increasing. Therefore, it is quasi-increasing by Proposition 2(ii). If we add to it the function $x \mapsto Kx$, which is continuous and therefore also quasi-increasing, the sum $x \mapsto f(x)$ is quasi-increasing by Proposition 2(vi).

(b) From part (a), we have $f(x) = \widehat{f}(x) + Kx$. Hence, when $K \geq 0$, f is increasing and when $K < 0$, f is the difference between two increasing functions. Thus in either case, f is a function of bounded variation.

The proof for the K -upper-Liipschitz case is analogous.⁷ ■

A useful consequence of this result is that lower-Liipschitz functions inherit all the useful properties of functions of bounded variation, such as the existence of a left and right limit at every point of their domain and differentiability almost everywhere.

The following lemma characterizes another subclass of quasi-monotone functions that will also prove useful for subsequent results.

Lemma 5 *Let $X, Y \subset \mathbb{R}$, $r : Y \rightarrow X$ and $\beta : X \times Y \rightarrow \mathbb{R}$ be two functions, and define $f : Y \rightarrow \mathbb{R}$ as $f(y) \equiv \beta(r(y), y)$.*

(a) *If r is increasing and β is increasing in its first argument and jointly continuous, then f is quasi-increasing.*

(b) *If in addition to the assumptions of (a), β is K -lower-Liipschitz in y for each x , then f is K -lower-Liipschitz.*

(c) *If in addition to the assumptions of (a), X, Y are compact intervals and β is continuously differentiable in y for each x , then f is K -lower-Liipschitz.*

Proof. (a) Let $\{y_n\}_{n \in \mathbb{N}}$ be such that $y_n \uparrow x$ and $f(y_n) \rightarrow \limsup_{y \uparrow x} f(y)$. Since $y_n \uparrow x$ and r

⁷Note that if $K \geq 0$, one can directly observe that f will be increasing and, therefore, trivially quasi-increasing. This lemma is useful, therefore, when $K < 0$.

is increasing, $r(y_1) \leq r(y_n) \leq r(x)$. Since r is increasing and $\{y_n\}_{n \in \mathbb{N}}$ is an increasing sequence, $\{r(y_n)\}_{n \in \mathbb{N}}$ converges, so that $r(y_n) \rightarrow \bar{r} \leq r(x)$. Since β is continuous, $f(y_n) = \beta(r(y_n), y_n) \rightarrow \beta(\bar{r}, x)$ and since β is increasing in its first coordinate, $\beta(\bar{r}, x) \leq \beta(r(x), x) = f(x)$, that is, $\limsup_{y \uparrow x} f(y) \leq f(x)$.

Taking $\{y_n\}_{n \in \mathbb{N}}$ to be such that $y_n \downarrow x$ and $f(y_n) \rightarrow \liminf_{y \downarrow x} f(y)$ and repeating the same arguments, we conclude that $\liminf_{y \downarrow x} f(y) \geq f(x)$. Thus, f is quasi-increasing.

(b) For any $y' > y$, since β is increasing in its first argument and r is increasing, we have

$$\frac{f(y') - f(y)}{y' - y} = \frac{\beta(r(y'), y') - \beta(r(y), y)}{y' - y} \geq \frac{\beta(r(y), y') - \beta(r(y), y)}{y' - y} \geq K.$$

Hence f is K -lower-Lipschitz.

(c) Since β is continuously differentiable in s for each x , and Y and X are compact, $\partial\beta/\partial y$ is uniformly bounded. Hence β is lower-Lipschitz in y for each x . The result follows from part (b). ■

One noteworthy aspect of the above result is that the function β is not required to be monotonic in its second argument y . In general, f as defined in this Lemma may well fail to be increasing.

We are now ready for an investigation of the emergence of the aforementioned subclasses of quasi-monotone functions as solutions to parametric optimization problems.

2.3 Sufficient conditions for quasi-increasing argmax

In what follows, we shall be primarily interested in quasi-monotone functions that arise as selections of players' reaction correspondences in a game. To that end, we must therefore investigate how such functions arise as solutions to parametric optimization problems. Thus we shall consider a selection $r : Y \rightarrow X$ of the correspondence $R : Y \rightrightarrows X$, with $X, Y \subset \mathbb{R}$ defined by:

$$r(y) \in R(y) \equiv \arg \max_{x \in X} M(x, y), \tag{3}$$

for some objective function $M : X \times Y \rightarrow \mathbb{R}$. We wish to provide conditions on M such that r is quasi-increasing in the parameter y .

It is useful first to recall that (at least one of the selections) r will be increasing if M satisfies a single-crossing condition with respect to $(x; y)$, i.e., if for any $(x'; y') \geq (x; y)$, we have⁸

$$M(x', y) \geq (>)M(x, y) \implies M(x', y') \geq (>)M(x, y'). \tag{4}$$

⁸As it is well known, this contradiction does not imply that all selections are increasing, but that at least the

For an argmax to be quasi-increasing in the parameter y , we need instead the notion of *shifted single-crossing*, which we introduce next via a judicious (non-separable) change of decision variable.

Let there be given a continuous function $\alpha : X \times Y \rightarrow Z$ that is strictly increasing in x for fixed y and increasing in y for fixed x . Here, Z is the range of α and it is given by $Z \equiv [\alpha(\underline{x}, \underline{y}), \alpha(\bar{x}, \bar{y})]$, where $\underline{x} = \inf X, \bar{x} = \sup X, \underline{y} = \inf Y$, and $\bar{y} = \sup Y$.

If one defines a new variable $z = \alpha(x, y)$, then since α is continuous and strictly increasing in x , there exists a (parametrized inverse) function $\beta : Z \times Y \rightarrow X$ such that $x = \beta(z, y)$. In other words, α has an inverse in its first argument, that is, there exists a function β satisfying $\alpha(\beta(z, y), y) = z$ and $\beta(\alpha(x, y), y) = x$. It is not difficult to see that β must be increasing in its first argument. Further, assume that β is continuous.

In view of the monotonicity properties of α , and of the fact that $X = [\underline{x}, \bar{x}]$ is independent of y , the set of feasible values of z for a fixed value of y is $Z(y) \equiv [\alpha(\underline{x}, y), \alpha(\bar{x}, y)]$, which is clearly ascending in y (since α is increasing in y). We have the following:

Definition 6 *Let $\alpha : X \times Y \rightarrow Z$ be continuous, strictly increasing in x and increasing in y , and $\beta(z, y)$ be the continuous inverse of α with respect to the first variable. A function $M : X \times Y \rightarrow \mathbb{R}$ satisfies a β -shifted single-crossing property with respect to $(x; y)$ if $\widetilde{M}(z, y) \equiv M(\beta(z, y), y)$ has the single-crossing property (4) with respect to $(z; y) \in Z \times Y$, that is, for any $(z'; y') \geq (z; y)$ we have*

$$\widetilde{M}(z', y) \geq (>) \widetilde{M}(z, y) \implies \widetilde{M}(z', y') \geq (>) \widetilde{M}(z, y'). \quad (5)$$

Moreover, M satisfies a β -shifted strict single crossing property with respect to $(x; y)$ if $\widetilde{M}(z, y) \equiv M(\beta(z, y), y)$ has the strict single-crossing property with respect to $(z; y) \in Z \times Y$, that is, for any $(z'; y') \geq (z; y)$, $(z'; y') \neq (z; y)$, we have

$$\widetilde{M}(z', y) \geq \widetilde{M}(z, y) \implies \widetilde{M}(z', y') > \widetilde{M}(z, y'). \quad (6)$$

maximal and minimal selections are. Throughout the paper, while we use the notions of increasing differences and single-crossing as sufficient conditions on an objective function to guarantee an increasing argmax, it is clear that we could as well use the interval dominance order (Quah and Strulovici, 2009) for the same purpose. (In fact, the latter is the most general condition of the three listed here when the action and parameter are real variables). In addition, since increasing differences is the only one that survives addition without any restrictions (see Quah and Strulovici, 2012 for more on this point), we shall make extensive use of this property in the applications section.

Naturally, a sufficient (but in general non-necessary) condition for the β -shifted single-crossing property is what one would naturally call β -shifted increasing differences, defined by $\widetilde{M}(z, y) \equiv M(\beta(z, y), y)$ having increasing differences in (z, y) , when β is continuous in (z, y) and strictly increasing in z . If both M and β are C^2 , this is equivalent to^{9,10}

$$\widetilde{M}_{12}(z, y) = [M_{11}(\beta(z, y), y)\beta_2(z, y) + M_{12}(\beta(z, y), y)]\beta_1(z, y) + M_1(\beta(z, y), y)\beta_{21}(z, y) \geq 0$$

Defining the set-valued functions $R : Y \rightrightarrows X$ and $Z^* : Y \rightrightarrows Z$ by

$$R(y) \equiv \arg \max_{x \in X} M(x, y) \tag{7}$$

and

$$Z^*(y) \equiv \arg \max_{z \in Z(y)} M(\beta(z, y), y), \tag{8}$$

we have a one-to-one mapping between selections r of R and selections z^* of Z^* , that is, given $z^*(\cdot) \in Z^*(\cdot)$, we have $r(y) = \beta(z^*(y), y) \in R(y)$ and, conversely, given $r(\cdot) \in R(\cdot)$, we have $z^*(y) = \alpha(r(y), y) \in Z^*(y)$.

Proposition 7 *Assume that $M : X \times Y \rightarrow \mathbb{R}$ satisfies a β -shifted single-crossing property with respect to $(x; y)$, with β jointly continuous and strictly increasing in its first argument. Assume that $R(y) \neq \emptyset$ for all $y \in Y$. Then, the following holds:*

- (a) *The maximal and minimal selections of R , \bar{r} and \underline{r} , are both quasi-increasing in y .*
- (b) *If, in addition, β is continuously differentiable, then \bar{r} and \underline{r} are both K -lower Lipschitz in y for some K .*
- (c) *If M satisfies the β -shifted strict single-crossing property, then all selections of R are quasi-increasing in y .*

Proof. (a) Since $M(\beta(z, y), y)$ has the single-crossing property with respect to $(z; y)$, and the feasible set $Z(y) \equiv [\alpha(\underline{x}, y), \alpha(\bar{x}, y)]$ is ascending in y (since α is increasing in y), we know from the Topkis-Milgrom-Shannon monotonicity theorem that the extremal selections of $S(y)$, \bar{s}^* and \underline{s}^* are increasing functions of y . Since β is increasing in its first coordinate and continuous by

⁹In this paper, we use subscripts in functions for partial derivatives.

¹⁰This equivalence is a well-known result about increasing differences.

assumption, the assumptions of Lemma 5(a) are satisfied by β and \bar{s}^* and \underline{s}^* and we conclude that $\bar{r}(y) = \beta(\bar{s}^*(y), y)$ and $\underline{r}(y) = \beta(\underline{s}^*(y), y)$ are both quasi-increasing in y .

(b) This follows directly from Lemma 5(c).

(c) It is well known that the strict single-crossing property implies that all selections are monotonic. Thus, the proof of (c) is similar to the proof of (a) and therefore omitted. ■

The following economic application illustrates the change of variable used above in the context of a familiar model. In particular, this provides some guidance as to how a suitable choice of the function α is arrived at. Several more relevant examples are given in the applications section.

Example 8 Consider a Bertrand duopoly with differentiated substitute products wherein firms 1 and 2 choose prices x, y (in some given price set $[0, \bar{p}]$) and face a demand system (D, \hat{D}) for their products, respectively. Assume linear cost functions with marginal costs c and \hat{c} . In what follows, we focus only on firm 1 (say). Its profit function is

$$F(x, y) = (x - c)D(x, y) \quad (9)$$

Its demand D is continuously differentiable and satisfies $D_1 < 0$ (the law of demand) and $D_2 > 0$ (products are substitutes in demand)

Our aim here is to show that Firm 1's reaction correspondence $f(y) = \arg \max_{x \in [0, \bar{p}]} (x - c)D(x, y)$ is quasi-decreasing in y , under the assumption that

$$D_2 D_1^2 - D[D_1 D_{12} - D_2 D_{11}] > 0 \text{ for all } (x, y). \quad (10)$$

To see this, let firm 1 respond by choosing its own output z instead of its price x , i.e. let $z = D(x, y)$. Since $D_1 < 0$, parametric inversion will yield a function h such that

$$x = h(z, y) \iff z = D(x, y).$$

As is easy to check, the partials of h and D are then related by

$$h_1 = \frac{1}{D_1}, h_2 = -\frac{D_2}{D_1}, \text{ and } h_{12} = \frac{1}{D_1^3}(D_2 D_{11} - D_1 D_{12}). \quad (11)$$

Given y , the best response problem of firm 1 may be equivalently viewed as

$$\max \left\{ \tilde{F}(z, y) \triangleq z[h(z, y) - c] : z \in [D(\bar{p}, y), D(0, y)] \right\}. \quad (12)$$

The first step is to derive conditions under which the argmax $z^*(y)$ is increasing in y . Since the feasible set $[D(\bar{p}, y), D(0, y)]$ is ascending in y (as $D_2 > 0$), by Topkis's Theorem, all the selections of $z^*(y)$ are increasing in y if \tilde{F} has strictly increasing differences in (z, y) . For this, it suffices that $\tilde{F}_{12}(z, y) = h_2(z, y) + zh_{12}(z, y) > 0$, for all (z, y) . Using (11), the latter is equivalent to

$$-\frac{D_2}{D_1} + D\frac{1}{D_1^3}(D_1D_{12} - D_2D_{11}) > 0,$$

which is the same as (10).

Since the argmax's of (9) and (12) are related by $z^*(y) = D(f(y), y)$ or $f(y) = h(z^*(y), y)$, and h is decreasing in its first argument, $f(y)$ is quasi-decreasing in y by Lemma 5.

The interpretation is that when firm 2 raises its price y , firm 1 may optimally react by increasing or decreasing its price x , but, in the latter case, never by so much that firm 1's output would end up increasing. In other words, while strategic complementarity in pricing decisions is allowed to any extent, a limited form of strategic substitutability can also be accommodated by condition (10). More precisely, condition (10) accommodates what we named strategic quasi-complementarity.

We shall return to this particular economic application below to illustrate other results from the present paper.

The following remark introduces a particular change of variable with a separable structure that will prove useful in some of the economic applications presented in section 4.

Remark 9 For many problems, a simple change of variable is as follows. Let $z = \alpha(x, y) = x + k(y)$ for some strictly increasing function k . Then for $Z^*(\cdot)$ to be increasing in y , it is sufficient that $\tilde{M}(z, y) \equiv M(z - k(y), y)$ has the single-crossing property with respect to $(z; y)$. A sufficient condition that is easy to check is that $\tilde{M}(z, y)$ has increasing differences with respect to (z, y) . When \tilde{M} and k are both twice continuously differentiable, this is equivalent to

$$\tilde{M}_{12}(z, y) = -M_{11}(z - k(y), y)k'(y) + M_{12}(z - k(y), y) \geq 0. \quad (13)$$

The idea of a change of variable to perform comparative statics of a non-monotonic sort has already appeared repeatedly in the literature on supermodular optimization and games. In a setting with scalar decision and parameter variables, Granot and Veinott (1985) define the notion of

doubly-increasing differences for an objective function as being the conjunction of the properties of increasing differences and of β -shifted increasing differences with an additively separable function (as in the previous remark). Curtat (1996) extends their result to multi-dimensional decision and parameter sets. Similar ideas have also appeared in the context of oligopoly applications with quasi-increasing reaction curves, e.g., in Amir (1996) and Amir and Lambson (2000), where the relevant change of variable is $z = x + y$.

The modern theory of monotone comparative statics is often qualified as being of a qualitative nature. Indeed, it aims to predict the directions of change of endogenous variables in response to changes in exogenous parameters, but usually not the associated magnitudes of these changes. In contrast, the conclusion that an argmax is quasi-decreasing in a parameter can be viewed as a comparative statics result of a non-monotonic and quantitative sort. As an illustration, consider the Bertrand duopoly example above. The derived conclusion may be re-stated as $f'(y) \leq -D_2(f(y), y)/D_1(f(y), y)$, for almost all y (w.r.t. Lebesgue measure),¹¹ which provides a lower bound on the rate of decrease of $f(y)$ as y changes. If one adds the reasonable further assumption on demand that $D_2(f(y), y)/|D_1(f(y), y)| < 1$ for all y , then one can conclude the firm 1 never lowers its price by as much as the increase in its rival's price, a conclusion of a clearly quantitative nature.

Observing that a similar (dual) argument can handle the derivation of upper bounds on the rate of the change of argmax's, as will be illustrated in the last section below, this method can easily be used to provide sufficient conditions on the players' reaction curves in a game to constitute contraction mappings, thus ensuring uniqueness of PSNE. For instance, Amir (1996) uses such arguments to establish uniqueness of Cournot equilibrium.

2.4 Our fixed point results

This subsection states the simplest form of our basic fixed point result in the Euclidean case, and discusses its direct connection to Tarski's intersection point Theorem. To fix ideas, the functions may be thought of as selections from players' best response correspondences in a strategic game. In the next section, we provide sufficient conditions directly on the payoff functions of a game that yield the following properties on players' best responses.

¹¹This is justified since f is a function of bounded variation (Lemma 4).

Theorem 10 (Fixed Point Theorem) *Let $f : [a, b] \rightarrow [c, d]$ be quasi-increasing, $g : [c, d] \rightarrow [a, b]$ be continuous and increasing, and define $h(x, y) = (g(y), f(x))$. Then there exists $(\bar{x}, \bar{y}) \in [a, b] \times [c, d]$ such that $h(\bar{x}, \bar{y}) = (\bar{x}, \bar{y})$.*

Proof. By Proposition 2(iii), the composition $f \circ g : X \rightarrow X$ is quasi-increasing. Let $\iota : X \rightarrow X$ be the identity. Since ι is continuous, Proposition 2(i) implies it is quasi-decreasing. If $\inf X = \underline{x}$ and $\sup X = \bar{x}$, we have $f \circ g(\underline{x}) \geq \underline{x} = \iota(\underline{x})$ and $f \circ g(\bar{x}) \leq \bar{x} = \iota(\bar{x})$. Therefore, the assumptions of Tarski's Intersection Point Theorem 3 are satisfied for $f \circ g$ and ι and we conclude that the set $\bar{X}_1 \equiv \{x \in X : f(g(x)) = x\}$ is nonempty. If $\bar{x} \in \{x \in [a, b] : g \circ f(x) = x\}$ and we define $\bar{y} = f(\bar{x})$, then (\bar{x}, \bar{y}) is a fixed point of h . ■

From this proof, it is clear that the key idea here is to translate Tarski's result from an intersection point result to a fixed point theorem for an important subclass of bivariate maps, namely those that are formed by the conjunction of two one-dimensional functions, as is the best response mapping for a two-player game. Put differently, the idea is to translate an intersection point result between two functions with the same domains and the same ranges to an intersection point result between two functions with interchanged domains and ranges, in line with the usual graphical depiction of intersecting reaction curves in economics as a simple way of representing PSNE.

A well-known interesting and immediate corollary is the following.

Corollary 11 (a) *Let $f : [a, b] \rightarrow [a, b]$ be quasi-increasing. Then f has a fixed-point.*

(b) *Let $f : [a, b] \rightarrow [a, b]$ be such that $\frac{f(x') - f(x)}{x' - x} \geq -k$, for some $k \geq 0$ and any $x', x \in [a, b], x \neq x'$. Then f has a fixed-point.*

Proof. (a) Simply apply Theorem 10 to f and $g(x) = x$ (the identity).

(b) The given slope condition means that f is k -lower-Lipschitz, and hence quasi-increasing (Lemma 4). Then use part (a). ■

As noted earlier, the result in part (b) with $k = 1$ was proved and used by MacManus (1964) and Roberts and Sonnenschein (1976) to establish existence of symmetric Cournot equilibrium in symmetric Cournot oligopoly with convex costs.¹² The latter property alone ensures that each firm's reaction curve has all its slopes above -1 (though it may be discontinuous), so that each

¹²The proof of this result by Mac Manus is not fully rigorous.

firm always reacts to rivals' output in a way that increases total output. Existence then follows from this property alone, even though the game is neither of strategic substitutes nor of strategic complements. Amir and Lambson (2000) extends this result to oligopolies with some level of non-convex costs. Milgrom and Roberts (1994) prove part (a) independently and use it to conduct comparative statics of equilibrium points.

The following theorem is the order-dual of Theorem 10, that is, it says that if f is quasi-decreasing and g is continuous and decreasing, then $h(x, y) = (g(y), f(x))$ has a fixed point. The economic models to which it might apply can be substantially different from those associated with Theorem 10, as will be confirmed by some of the applications later on.

Theorem 12 *Let $f : [a, b] \rightarrow [c, d]$ be quasi-decreasing, $g : [c, d] \rightarrow [a, b]$ be continuous and decreasing, and define $h(x, y) = (g(y), f(x))$. Then there exists $(\bar{x}, \bar{y}) \in [a, b] \times [c, d]$ such that $h(\bar{x}, \bar{y}) = (\bar{x}, \bar{y})$.*

The fixed point theorems presented in this section will be translated into equilibrium existence results for games in section 3. The next subsection deals with the issue of uniqueness of fixed points of quasi-monotone functions and will be useful to establish uniqueness of PSNE.

2.5 Uniqueness of fixed points

In some applications, beyond the issue of existence, the uniqueness of fixed points is often a highly desirable property. In fact, most studies in applied microeconomics postulate game-theoretic models with a unique PSNE, independently of whether specific functional forms are adopted or not. The standard methods used to establish uniqueness of fixed points or PSNEs typically rely on dominant diagonal conditions on payoff functions or, equivalently, on contraction arguments for best response mappings (Rosen, 1965; Milgrom and Roberts, 1990). Both of these conditions are generally postulated to hold in a global sense. In this section, we present two results that establish the uniqueness of fixed points in the present setting without requiring global contraction arguments. Our first result of this form allows the function to be quasi-decreasing (instead of continuous, or often even smooth) and satisfy a local contraction property along the diagonal, that is, only at

candidate fixed points.¹³

Proposition 13 *Let $f : [a, b] \rightarrow [a, b]$ be quasi-decreasing and satisfy the following property: for every $x \in [a, b]$ such that $x = f(x)$, we have*

$$\limsup_{y \rightarrow x} \frac{f(y) - f(x)}{y - x} < 1. \quad (14)$$

Then f has at most one fixed point.

Proof. Suppose that $\bar{x}, \bar{y} \in [a, b]$ are both fixed points of f , with (say) $\bar{x} < \bar{y}$. Because of (14), there are neighborhoods U_1 of \bar{x} and U_2 of \bar{y} such that $y \in U_1 \cap ([a, b] \setminus \{\bar{x}\}) \implies \frac{f(y) - \bar{x}}{y - \bar{x}} < 1$ and $y \in U_2 \cap ([a, b] \setminus \{\bar{y}\}) \implies \frac{f(y) - \bar{y}}{y - \bar{y}} < 1$. Then we can pick $y_1 \in U_1 \cap (\bar{x}, \bar{y})$ and $y_2 \in U_2 \cap (\bar{x}, \bar{y})$, $y_1 < y_2$ such that $\frac{f(y_1) - \bar{x}}{y_1 - \bar{x}} < 1$ and $\frac{f(y_2) - \bar{y}}{y_2 - \bar{y}} < 1$. Hence $f(y_1) < y_1$ and $f(y_2) > y_2$.

Define the function $g(y) = f(y) - y$ on $[y_1, y_2]$. Since f is quasi-decreasing and $-y$ is continuous, hence quasi-decreasing, g is quasi-decreasing by Proposition 2(vii). Moreover, $g(y_1) < 0 < g(y_2)$. We can apply Tarski's intersection point theorem (Theorem 3) to the function g thus defined and the constant function $c(y) = 0, \forall y$, which is continuous and thus quasi-increasing. The supremum of $\{y \in [y_1, y_2] : 0 \geq g(y)\}$ is contained in, and is the supremum of $\{y \in [y_1, y_2] : 0 = g(y)\}$. Denote by $\hat{y} \in [y_1, y_2]$ this supremum. It is clear that $\hat{y} < y_2$, since $g(y_2) > 0$ and \hat{y} is the highest fixed point of f in $[y_1, y_2]$. Using again property (14) for \hat{y} , we can find $\hat{y} \in (\hat{y}, y_2)$ such that $f(\hat{y}) < \hat{y}$. Defining as before $\hat{g} : [\hat{y}, y_2] \rightarrow \mathbb{R}$ by $\hat{g}(y) = f(y) - y$, we see that it is a quasi-decreasing function satisfying the assumptions of Theorem 3. Therefore, there exists $\tilde{y} \in (\hat{y}, y_2)$, such that $0 = \hat{g}(\tilde{y}) = f(\tilde{y}) - \tilde{y}$, that is, $\tilde{y} \in [y_1, y_2]$ is a fixed point of f and $\tilde{y} > \hat{y} > \hat{y}$, which contradicts the fact that \hat{y} is the highest fixed point of f in $[y_1, y_2]$. This contradiction establishes the property. ■

Remark 14 *While this Proposition might a priori appear to be directly suitable for use as a uniqueness argument only for symmetric PSNE of symmetric games, one can also use it for two-player asymmetric games by applying it to the composition of the two players' reaction functions, which maps (say) player 1's action space to itself. Indeed, it is well known that the set of fixed points of such a composition coincides with the set of PSNEs of the game (Vives, 1990). In this form, all that is needed for a unique PSNE is that the mentioned composition is 1-upper Lipschitz in a*

¹³To underscore the novelty of this result, an application to Bertrand competition is presented in Section 4.

neighborhood of any fixed point (as captured by (14)), and not necessarily a global contraction (i.e., a globally 1-upper and 1-lower Lipschitz function).

Relying on first order conditions under smoothness assumptions, another convenient test for the uniqueness of a fixed point that is non-global in character can be given in the form of the following sufficient condition.

Proposition 15 *Let $X = [a, b] \subset \mathbb{R}$ and $n \in \mathbb{N}$. Assume that $M : X^n \rightarrow \mathbb{R}$ is differentiable in its first coordinate and satisfies the following:*

$$x, x' \in X, x' > x \text{ and } M_1(x, x, \dots, x) \leq 0 \text{ imply } M_1(x', x', \dots, x') < 0. \quad (15)$$

Then a function $r : X \rightarrow X$ satisfying $r(x) \in \arg \max_{y \in X} M(y, x, \dots, x)$ has at most one fixed point.

Proof. The proof rules out multiple interior fixed points, and then multiple corner fixed points. Since $r(x) \in \arg \max_{y \in X} M(y, x, \dots, x)$ and M is differentiable in its first variable, we must have $M_1(r(x), x, \dots, x) = 0$ if x is an interior point of X . Assume that x and x' are interior fixed points of r , with $x' > x$. Then, $M_1(x, x, \dots, x) = 0$ and $M_1(x', x', \dots, x') = 0$, but this contradicts (15). Assume now that the endpoint a is a fixed point of r . In this case, we must have $M_1(a, \dots, a) \leq 0$. By (15), $M_1(x', \dots, x') < 0$ for all $x' > a$, which shows that there is no other fixed point of r above a . Similarly, if the endpoint b is a fixed point of r , then $M_1(b, \dots, b) \geq 0$ and (15) implies that $M_1(x, \dots, x) > 0$ for all $x < b$. Therefore, there are no other fixed point of r below b . This concludes the proof. ■

The above proposition will be used to establish uniqueness of symmetric equilibria in symmetric games later on. A slight adaptation of the results above can yield analogous uniqueness results for asymmetric games.

Interestingly, (15) may be seen as a strict dual single-crossing condition for the partial $M_1(x, \dots, x)$, viewed as a function of one variable. As such, it becomes transparent that a sufficient condition for (15) is that $M_1(a, a, \dots, a)$ is strictly decreasing in a , which (if M is twice continuously differentiable) is in turn implied by

$$M_{11}(a, \dots, a) + \sum_{j \neq 1} M_{1j}(a, a, \dots, a) < 0.$$

The latter condition is of the dominant diagonal type; it says that any row sum of the Hessian matrix of M is negative. Nevertheless, this condition is in general significantly less restrictive than the typical related conditions in the literature, in that it is not required to hold globally, but rather only along the diagonal of the domain X^n .

3 Pure-strategy Nash equilibrium in games

This section contains our results about the existence and uniqueness of pure-strategy Nash equilibrium (henceforth PSNE) in games. Its main objective is to translate our fixed points results into conclusions about the existence of PSNE. In the process, the relevant sufficient conditions shall be placed on the primitives of the game.

We begin by describing results for two players games in subsection 3.1. Then n -player symmetric games are the object of subsection 3.2.

3.1 Two-player games

Consider a two-player strategic game with action spaces X and Y and payoff functions $F, G : X \times Y \rightarrow \mathbb{R}$. The following result translates the assumptions of Theorem 10 onto sufficient conditions on the primitives of the game.

Theorem 16 *Assume that X and Y are compact intervals in \mathbb{R} , F and G are upper semi-continuous in own action, and that:*

(a) *F satisfies a β -shifted single-crossing property with respect to $(x; y)$ for some β that is continuous and increasing, and*

(b) *G is strictly quasi-concave in y for each fixed x , and satisfies the single-crossing property with respect to $(y; x)$.*

Then the set E of PSNE is non-empty.

Proof. Due to the upper semi-continuity assumption, the best-reply correspondences have non-empty values. By Proposition 7, the maximal and minimal selections of the best-reply correspondence for the first player, $\bar{x}(\cdot)$ and $\underline{x}(\cdot)$, are quasi-increasing.

From the assumption that G is strictly quasi-concave and upper semi-continuous in y , we know that the best-reply correspondence is a single-valued continuous function denoted \bar{y} . From the single-crossing property, $\bar{y}(\cdot)$ is increasing in x . By Theorem 10, there exists (x^*, y^*) such that $\bar{x}(y^*) = x^*$ and $\bar{y}(x^*) = y^*$, that is, (x^*, y^*) is a PSNE of the game and E is non-empty. ■

It is instructive to contrast this result with its counterpart from the theory of supermodular games for the present setting. While the latter relies on Tarski's well known fixed point theorem for increasing maps, the present result is based on a reinterpretation of Tarski's intersection point theorem as a fixed point theorem for bivariate maps that arise as best-response maps of two-player games. In terms of scope, the two approaches are not nested. On the one hand, for player 1, the present result imposes less structure since his reaction curve is only quasi-decreasing and not necessarily increasing. On the other hand, for player 2, the present result requires continuity of the reaction curve in addition to monotonicity while the latter property is all that is needed for supermodular games. Put differently, the present result relaxes strategic complementarity to strategic quasi-complementarity (i.e., allows a limited form of strategic substitutability) for one player, but imposes continuity as an extra condition on the reaction curve of the other player.

Since the result relies on a mix of continuity and generalized monotonicity conditions, it may be regarded as a synthesis of the two existing methodologies for establishing existence of PSNE in general games: the classical (topological) approach via Brouwer's or Kakutani's fixed point theorem (Nash, 1951 and Rosen, 1965) and the algebraic (supermodularity) approach via Tarski's fixed point theorem (Topkis, 1979).

Theorem 16 admits an order-dual, which is as follows.

Theorem 17 *Assume that X and Y are compact intervals in \mathbb{R} , and F and G are upper semi-continuous in own action. Let α be a continuous function on $X \times Y$ that is strictly decreasing in x and decreasing in y , and $\beta(\cdot, y)$ be the inverse of α with respect to the first variable. If*

- (a) $\hat{F}(z, y) \equiv F(\beta(z, y), y)$ satisfies the dual single-crossing property with respect to $(x; y)$, and
- (b) G is strictly quasi-concave in y for each fixed x , and satisfies the dual single-crossing property with respect to $(y; x)$.

Then the set of PSNE of this game is non-empty.

Proof. If the order on (say) player 2's action set is reversed, the assumptions of this theorem turn into those of Theorem 16. Hence, the conclusion follows from the latter result. ■

While equivalent to Theorem 16 from a mathematical point of view, in terms of economics applications, this order-dual will apply to models that are quite distinct from those of Theorem 16. In fact, this version will be directly invoked in some of the applications later on.

One can extend Theorem 17 to also deliver uniqueness of PSNE by using the novel insight from Proposition 13.

Proposition 18 *In addition to the assumptions of Theorem 17, assume that the composition of the two reaction curves $f \circ g$ satisfies (14) at all of its fixed points. Then there exists a unique PSNE.*

Proof. Since the composition $f \circ g$ is quasi-decreasing, the uniqueness conclusion follows directly from Proposition 13, when one takes into account that the set of PSNE coincides with the set of fixed points of the composition $f \circ g$. ■

At the level of generality at which they are stated, Theorem 16 and its order-dual (Theorem 17) are a priori valid only for two-player games. Nevertheless, these results can be extended to n -player games upon the inclusion of some additional assumptions on the primitives. This can be done in a number of different ways. The first and more natural extension is to work with symmetric players. This is considered in subsection 3.2 below.

3.2 N-player symmetric games

Consider a n -player game, where all players have the same action space X , assumed to be a compact interval in \mathbb{R} , and the same payoff function $F : X \times X^{n-1} \rightarrow \mathbb{R}$, where the first entry is the player's own action. With the usual abuse of notation, we can write a joint action vector $x \in X^n$ as (x_i, x_{-i}) for any $i \in N$.

We now state a basic existence result for symmetric games, special versions of which have surfaced in the economics literature a number of times in specific contexts (e.g., Amir and Lambson, 2000 and Milgrom and Roberts, 1994). This result provides conditions on the payoff function that lead to the (common) reaction correspondence satisfying Corollary 11(a).

Theorem 19 *Assume that*

(a) *F is upper semi-continuous in own action (or first entry);*

(b) *for each $a \in X$, $\widehat{F}(x, a) \equiv F(x, a, \dots, a)$ satisfies a β -shifted single-crossing property with respect to $(x; a)$ for some $\beta : X \times X \rightarrow X$ that is continuous and increasing.*

Then the set of symmetric PSNE is non-empty.

Proof. The assumption that F is upper semi-continuous in its first entry guarantees that the best-reply correspondence is non-empty. By Proposition 7, the maximal best-reply restricted to symmetric actions $\bar{x} : X \rightarrow X$ is quasi-increasing. By Corollary 11, the set of fixed points of \bar{x} is a non-empty chain. Since a fixed point of \bar{x} is an equilibrium, the set of equilibria is non-empty. ■

For the next uniqueness result, existence of PSNE may be guaranteed either by the continuity of the (common) reaction curve or by the fact that it is quasi-increasing.

Theorem 20 *Assume that F is C^2 and*

(a) *for each $a \in X$, $\widehat{F}(x, a) \equiv F(x, a, \dots, a)$ is either strictly quasi-concave in x or satisfies a β -shifted single-crossing property with respect to $(x; a)$ for some $\beta : X \times X \rightarrow X$ that is continuous and increasing.*

(b) *for each $a \in X$, \widehat{F} satisfies $\widehat{F}_{11}(a, a) + \widehat{F}_{12}(a, a) < 0$.*

Then, there exists a unique symmetric PSNE of this game.

Proof. The existence of a symmetric PSNE follows from either of the two assumptions in (a), the quasi-concavity in x or the β -shifted single-crossing property, see Theorem 19. Since F is C^2 , we know from (b) that for each \bar{a} , $\widehat{F}_{11}(x, \bar{a}) + \widehat{F}_{12}(x, \bar{a}) < 0$ for all (x, a) in some neighborhood of (\bar{a}, \bar{a}) . Hence, with the change of variable $z = x + a$, $\widehat{F}(z - a, a)$ has strongly increasing differences in (z, a) , i.e., $\partial \widehat{F} / \partial a$ is strictly increasing in z ; see Amir (1996b) or [Topkis, 1998, p. 79]. Thus, all the selections of $z^*(a) \equiv \widehat{F}(z - a, a)$ are strictly increasing in a . In other words, $x^*(a)$ has all its slopes strictly less than 1 in a neighborhood of \bar{a} . The uniqueness of symmetric PSNE then follows from Proposition 13. ■

Again, the main novelty in the underlying argument is that the assumption in part (b) generates a local contraction property for the reaction curve along the diagonal, which is not required to hold in a global sense. Due to the latter point, multiple asymmetric PSNEs are not ruled out.

Before considering an extension to a class of n -player games, we provide a new uniqueness result for symmetric pure-strategy Nash equilibrium for symmetric normal-form games, which is of independent interest for many potential economic applications.

Theorem 21 *Assume that:*

- (a) F is differentiable in its first variable (own action) and
- (b) For any $x', x \in X$ with $x' > x$, $F_1(x, x, \dots, x) \leq 0 \implies F_1(x', x', \dots, x') < 0$.

Then, there exists at most one symmetric equilibrium of this game.

Proof. This follows directly from Proposition 15. ■

Observe here that the given assumptions do not preclude the existence of other PSNEs as long as they are asymmetric.

A sufficient condition for the assumption in Theorem 21(b) is that $\partial_1 F_i(x, \dots, x)$ is strictly decreasing in x (see comments after Proposition 15).

We now provide an illustration of Proposition 13 in a familiar setting.

Example 22 *Consider the symmetric version of the Bertrand model of Example 8 with demand function $D(x, y)$. We shall show that there can be at most one symmetric Bertrand equilibrium provided one assumes that demand satisfies (10) and*

$$D_2(x, x) < |D_1(x, x)| \text{ for all } x. \quad (16)$$

Recall that in Example 8, we showed that $f(y)$ is quasi-decreasing in y , due to Assumption (10), or equivalently that $z^(y) = D(f(y), y)$ is increasing in y . Consider any candidate fixed point $y_0 = f(y_0)$. Since $D(f(y), y)$ is increasing in y , it is differentiable a.e. in y (with respect to Lebesgue measure). Therefore, we can find a sequence $y_n \rightarrow y_0$ such that $D(f(y), y)$ is differentiable at y_n for each n , so that one has*

$$D_1(f(y_n), y_n) f'(y_n) + D_2(f(y_n), y_n) \geq 0 \text{ or } f'(y_n) \leq -\frac{D_2(f(y_n), y_n)}{D_1(f(y_n), y_n)}.$$

Taking limsup on both sides yields (in view of (16))

$$\limsup_{n \rightarrow \infty} f'(y_n) \leq -\frac{D_2(y_0, y_0)}{D_1(y_0, y_0)} < 1.$$

As this holds at all candidate symmetric PSNE, there is a unique one by Proposition 13.

Some key aspects of the uniqueness result here are worth stressing:

- (a) The argument extends to any number of firms along the same lines (using a two-player example here is just for simplicity of notation).
- (b) Continuity of reaction curves is not needed, quasi-decreasingness being sufficient.
- (c) The reaction curve is required to be contractive only locally, around fixed points; it need not be a global contraction. For the latter property, one would need to strengthen Assumption (A3) to hold at all pairs (x, y) . This would be a much more restrictive assumption, which is not satisfied by many known (non-linear) demand systems.
- (d) As made clear here, as a uniqueness result, Proposition 13 is quite convenient for direct use.

4 Some Selected Applications

This section presents a selection of well known models in applied microeconomics for which the results of the present paper apply quite naturally and in a straightforward manner to yield novel results. Despite the fact that some of these models have extensive literature dealing specifically with the existence of PSNEs, the results proposed below constitute either significant generalizations of their counterparts in the literature or new versions that are not nested with existing ones. Since the results presented below are actually new to the separate literatures dealing with each model, we present the results in the form of formal propositions with concise proofs.

The first two applications below serve to illustrate how our results can be used to tackle the issue of PSNE existence for classes of games that cannot be handled via the existing approaches, thus giving some economic insight as to how the key asymmetry between the two players' reaction curves can arise naturally. The third application illustrates how the present approach can lead to alternative sufficient conditions for the existence of PSNE for a well known model, relative to the existing approach.

4.1 Price-quantity duopoly

In their pioneering paper on linear duopoly, Singh and Vives (1984) introduced a hybrid notion of duopoly wherein one firm is a price setter and the other firm a quantity setter. Singh and

Vives (1984) analyzed only the case of linear demands and derived closed-form PSNEs. In their specification, the reaction curves are both linear, downward-sloping for the price setter and upward-sloping for the quantity setter. Owing to this two-way monotonicity, no lattice-theoretic argument can establish existence of PSNE in more general cases. Instead, one would therefore a priori resort back to a Brouwer-type fixed point argument to obtain PSNEs (and thus impose continuity of the reaction curves). Nevertheless, in this subsection, we invoke our main existence result to provide the very first general PSNE existence result for this hybrid duopoly.

Consider an asymmetric duopoly with differentiated substitute products and linear costs c_1 and c_2 . Denote the direct and inverse demand functions by $D^i(p_1, p_2)$ and $P^i(q_1, q_2)$, $i = 1, 2$, where (p_1, p_2) and (q_1, q_2) are the prices and the outputs chosen by the two firms. In order to introduce the new hybrid duopoly game, we need to also consider the related standard price (Bertrand) and quantity (Cournot) games.

For the price (Bertrand) game, firm i 's payoff function is

$$\Pi_B^i(p_1, p_2) = (p_i - c_i)D^i(p_1, p_2).$$

For the quantity (Cournot) game, firm i 's payoff function is

$$\Pi_C^i(q_1, q_2) = q_i[P^i(q_1, q_2) - c_i].$$

To define the price-quantity duopoly, assume that D^i and P^i are smooth (for convenience only), and that $D_i^i < 0$ and $P_i^i < 0$ (i.e., the Law of Demand).¹⁴ To avoid technicalities, we also assume that the firms' outputs and prices lie in compact sets.¹⁵ For the products to be substitutes, we need to postulate that

$$\frac{\partial D^i(p_1, p_2)}{\partial p_j} > 0 \text{ and } \frac{\partial P^i(q_1, q_2)}{\partial q_j} < 0.$$

By taking parametrized inverse functions, there are functions H_1 and H_2 such that

$$p_1 = P^1(q_1, q_2) \iff q_1 = H^1(p_1, q_2)$$

¹⁴In this section, subscripts denote partial differentiation with respect to the indicated variable(s). Throughout this section, we assume that the primitive functions of each model are twice continuously differentiable. This is only for convenience in establishing the relevant complementarities via Topkis's cross partial test.

¹⁵This may be justified on economic grounds too by postulating for instance that firms face capacity constraints and regulatory price ceilings.

and

$$q_2 = D^2(p_1, p_2) \iff p_2 = H^2(p_1, q_2)$$

The payoffs in the price-quantity duopoly (with firm 1 as the price setter) are then

$$F(p_1, q_2) = (p_1 - c_1)H^1(p_1, q_2)$$

and

$$G(p_1, q_2) = q_2[H^2(p_1, q_2) - c_2]$$

We can now state our main result for this section. A discussion and economic interpretation of the assumptions used here follows the proof below.

Proposition 23 *Assume that the original demand system satisfies the following conditions*

$$P_2^1(q_1, q_2) + q_1 P_{12}^1(q_1, q_2) < 0, \text{ for all } q_1, q_2 \geq 0 \quad (17)$$

$$D_1^2(p_1, p_2) + (p_2 - c_2)D_{21}^2(p_1, p_2) > 0, \text{ for all } p_1, p_2 \geq 0 \quad (18)$$

and $\frac{1}{D^2(p_1, p_2)}$ is strictly convex in p_2 , or

$$D^2(p_1, p_2)D_{22}^2(p_1, p_2) - 2[D_2^2(p_1, p_2)]^2 < 0 \quad (19)$$

Then the price-quantity duopoly possesses a PSNE.

Proof. Denote the reaction correspondences in the price-quantity duopoly by

$$r_1(q_2) = \arg \max_{p_1} F(p_1, q_2) \text{ and } r_2(p_1) = \arg \max_{q_2} G(p_1, q_2).$$

Consider the change of variable

$$z_1 = H^1(p_1, q_2) \iff p_1 = h^1(z_1, q_2)$$

then firm 1's payoff function can be rewritten as

$$\tilde{F}(z_1, q_2) = z_1[h^1(z_1, q_2) - c_1]$$

For firm 1, we aim to show that $z_1^* = \arg \max_{z_1} \tilde{F}(z_1, q_2)$, or equivalently $H^1(r_1(q_2), q_2)$, is decreasing in q_2 , so that $r_1(q_2)$ is quasi-increasing. To this end, a sufficient condition is that $\tilde{F}(z_1, q_2)$ has decreasing differences in (z_1, q_2) , or

$$h_2^1(z_1, q_2) + z_1 h_{12}^1(z_1, q_2) < 0 \quad (20)$$

Using the simple formulas that relate the partial derivatives of a function and its parametric inverse (11) a first time, it is easy to see that (20) is equivalent to

$$-\frac{H_2^1(p_1, q_2)}{H_1^1(p_1, q_2)} + \frac{H^1(p_1, q_2)}{[H_1^1(p_1, q_2)]^3} [H_2^1(p_1, q_2)H_1^1(p_1, q_2) - H_{12}^1(p_1, q_2)H_1^1(p_1, q_2)] < 0$$

which via (11) is in turn equivalent to (17) upon simplification.

As for firm 2, to show continuity of $r_2(p_1)$, (19) implies that $\Pi_B^2(p_1, p_2)$ is strictly quasi-concave in p_2 (Caplin and Nalebuff, 1991). Since $G(p_1, q_2)$ is obtained from $\Pi_B^2(p_1, p_2)$ via a strictly monotonic transformation of firm 2's action, this implies that $G(p_1, q_2)$ is strictly quasi-concave in q_2 . Therefore, $r_2(p_1)$ is a continuous function.

With the change of decision variable $z_2 = H^2(p_1, q_2)$ for firm 2, a similar two-step procedure as for firm 1 above show that $r_2(p_1)$ is increasing if (18) holds.

The existence of PSNE follows from our basic existence result (Theorem 16). ■

For a good understanding of the result, we now interpret the specific role played by each of the assumptions in familiar contexts, and then relate this Proposition to known results on the existence of PSNEs in the two standard oligopolies. Condition (17) is known to make the payoff of firm 1 in the standard Cournot game submodular, and thus firm 1's reaction curve decreasing. Condition (18) is known to make the payoff of firm 1 in the standard Bertrand game supermodular, and thus firm 1's reaction curve increasing. It follows that the three assumptions are quite natural since Cournot and Bertrand duopolies are typically games of strategic substitutes and complements respectively. Finally, (19) guarantees the quasi-concavity of firm 2's payoff in own price in the Bertrand game.

Significantly, it turns out that these familiar structural conditions on the two standard duopolies also constitute minimal sufficient conditions to make the price-quantity duopoly enjoy strategic quasi-complementarities and thus possess PSNEs according to the results of the present paper. It follows that the basic structure imposed by the approach to PSNE existence in the present paper

is as natural for the hybrid duopoly as the familiar strategic complementarity and substitutability are to the standard Bertrand and Cournot duopolies respectively.

4.2 Bertrand Competition with increasing returns for one firm

Consider a Bertrand oligopoly with differentiated substitute products wherein firms 1 and 2 choose prices x and y (in a given price set $[0, \bar{p}]$) and face a demand system (D^1, D^2) for their products, such that $D_i^i < 0$ (the Law of Demand) and $D_j^i > 0, i, j = 1, 2$, and $i \neq j$ (goods are substitutes). With $C_1(\cdot)$ and $C_2(y) = c_2y$ denoting the firms' cost functions, the profit function of firms 1 and 2 are then

$$F(x, y) = xD^1(x, y) - C_1[D^1(x, y)]$$

and

$$G(x, y) = (y - c_2)D^2(x, y).$$

Using basic insights from the theory of supermodular games, Vives (1990) and Milgrom and Shannon (1994) derive sufficient conditions that imply existence of PSNE via the strategic complementarity of the game. Likewise, for the classical approach using Brouwer's fixed point theorem, sufficient conditions for continuous reaction curves are easily written down (e.g., Caplin and Nalebuff, 1991 or Vives, 1999). Though based on different arguments, both approaches require a convex cost function (or decreasing returns to scale) for both firms. In fact, with (at least one) general concave cost function, it is easy to see that the above approaches do not extend, and thus, not surprisingly, no existence result is known so far. In this subsection, we provide the first such result, which imposes no restriction on one firm's cost function, so in particular it may well be concave (locally or globally).

Assumptions

(A1) (i) $D^2(x, y)$ is log-submodular (i.e., $D^2D_{12}^2 - D_1^2D_2^2 \leq 0$)

(ii) $\frac{1}{D^2(x, y)}$ is strictly convex in x for each y , i.e., $D^2(x, y)D_{11}^2(x, y) - 2[D_1^2(x, y)]^2 < 0$ for all (x, y) .

(A2) $D_2^1(D_1^1)^2 - D^1 [D_1^1D_{12}^1 - D_2^1D_{11}^1] > 0$ for all (x, y) .

Assumption (A1) is well known to yield a reaction curve that is downward-sloping (part i) and continuous (part ii), as seen in the proof. (A1) (i) is quite restrictive, as it requires D_{12}^2 to be strongly negative.¹⁶ However, given the paucity of existence results under increasing returns in oligopoly in general, one would not expect a high level of generality.

The new assumption here is (A2). It clearly imposes a very mild restriction on the demand function. Indeed, the third term is always negative, and the first term has a high tendency for a negative sign. In addition, it is sufficient (but not necessary) for (A2) to hold to have $D_2^1 D_{11}^1 - D_1^1 D_{12}^1 < 0$. The latter is equivalent to D_1^1/D_2^1 being decreasing in x , which is a very general condition.

Importantly, the following existence result imposes no assumptions at all on the cost function of firm 1.

Proposition 24 *Under Assumptions (A1)-(A3), the Bertrand game has a PSNE.*

Proof. For firm 2, by Assumption (A1)(ii), $G(x, y)$ is strictly quasi-concave in y for each x , and thus the reaction correspondence $f(y)$ of firm 2 is a continuous function (Caplin and Nalebuff, 1991).

In addition, $\log G(x, y) = \log(y - c_2) + \log D^2(x, y)$ so (A1(i)) implies that $\log G(x, y)$ is super-modular, so the reaction curve $r_2(x)$ is decreasing (Milgrom and Shannon, 1994).

For firm 1, the fact that (A2) implies that the reaction curve $r_1(y)$ is quasi-decreasing was already proved in Example 8.

Existence of PSNE then follows from Theorem 17. ■

A noteworthy point is that this existence result imposes no restrictions at all on the cost function of firm 1 (other than continuity). Therefore, a key implication of this result is that a quasi-decreasing reaction curve is a natural property for a Bertrand firm when its cost function has increasing returns to scale (even strong ones), either in a local or a global sense. In contrast, in such cases, in general, the other two common properties, upward-monotonicity and continuity, can easily fail to hold.

¹⁶One example of a demand function satisfying this assumptions is

4.3 Provision of public goods

We consider the standard model of noncooperative provision of public goods (e.g., Bergstrom, Blume and Varian, 1986, or BBV), and modify it in that agents contribute an input to the production of the public good output. Let agent $i \in \{1, 2\}$, with utility function $U^i(\cdot, \cdot)$ and wealth w_i , contribute $x_i \in [0, w_i]$ as input to produce the public good (whole price is normalized at 1), and spend the rest of his wealth ($w_i - x_i$) on consumption of a composite good with price p . Denoting by $f(\cdot)$ the production function, with $f'(\cdot) > 0$, the payoffs of agents 1 and 2 are then

$$F(x_1, x_2) = U^1[w_1 - px_1, f(x_1 + x_2)]$$

and

$$G(x_1, x_2) = U^2[w_2 - px_2, f(x_1 + x_2)].$$

In what follows, we aim to make minimal assumptions to obtain existence of PSNE via Theorem ???. In addition, to make the problem interesting for the approach at hand, we refrain from making any concavity assumptions on the utility and production functions, so that the existing approach based on continuous reaction curves (as in BBV, 1986) does not apply. The need for our approach becomes critical if production has increasing returns to scale (either globally or even locally).

(A1) U^1 satisfies

(i) $U_2^1 f'' - pU_{21}^1 f' + U_{22}^1 (f')^2 > 0$

(ii) $U^1[w_1 - px_1, f(x_1 + x_2)]$ is strictly quasi-concave in x_1 .

(A2) U^2 satisfies $U_{12}^2 f' - pU_{11}^2 > 0$, $i = 1, 2$.

We now discuss the meaning and plausibility of these assumptions. The assumption here that departs most from the usual setting of BBV is (A1)(i). It is seen by inspection that it is easier for (A1)(i) to hold when the utility function is linear in the public good (i.e., $U_{22}^1 = 0$), and submodular ($U_{12}^1 < 0$) or additively separable (i.e., $U_{12}^1 = 0$), and the production function is convex or has increasing returns (in the relevant range). Clearly, these conditions need not hold simultaneously for (A1)(i) to be valid.

While Assumption (A1)(ii) puts a bound on the extent of increasing returns for f , and thus goes against (A1)(i), the two can be seen to be mutually compatible. As an example, let $U^1(x_1, x_2) = x_2 \log x_1$ and $f(z) = z^2$.

As to Assumption (A2), it amounts to assuming that the private good (good 1) is a normal good for agent 2 (in the sense of standard consumer theory). To see this, note that along with the first order condition $-pU_1^2 = U_2^2 f'$, (A2) amounts to $U_1^2 U_{12}^2 - U_2^2 U_{11}^2 > 0$. Hence, (A2) is very general.

Proposition 25 *Under Assumption (A0)-(A1), the game possesses a PSNE.*

Proof. For agent 1, by Assumption (A1)(i) and Topkis's Theorem, the reaction curve $f(x_2)$ is upward-sloping since

$$\begin{aligned} \frac{\partial F(x_1, x_2)}{\partial x_1 \partial x_2} &= U_2^1 f'' - pU_{21}^1 f' + U_{22}^1 (f')^2 \\ &> 0 \text{ by (A1)} \end{aligned} \quad (21)$$

Due to Assumption (A1)(ii), $f(x_2)$ is also a continuous function.

For agent 2, consider the change of variable $z = x_1 + x_2$, and rewrite his payoff as

$$\tilde{G}(x_1, z) = U^2[w_2 - p(z - x_1), f(z)]$$

Then

$$\begin{aligned} \frac{\partial \tilde{G}(z, x_2)}{\partial z \partial x_2} &= -p^2 U_{11}^2 + pU_{12}^2 f' \\ &> 0 \text{ by (A2)} \end{aligned}$$

Hence, by Topkis's Theorem, given that the feasible set for z , i.e., $[x_1, \infty)$, is ascending in x_1 , $z^*(x_1) = \arg \max_{z \geq x_1} \tilde{G}(x_1, z)$ is increasing in x_1 . Since $g(x_1) = z^*(x_1) - x_1$, it follows that $g(\cdot)$ has all slopes ≥ -1 , i.e., $\frac{g(x_1') - g(x_1)}{x_1' - x_1} \geq -1$, which implies that $g(x_1)$ is quasi-increasing.

Existence of PSNE then follows from Theorem 16. ■

The fact that the only assumption that is required for agent is (A2), i.e., normality, irrespective of the nature of the returns to scale in production, is quite remarkable. It follows that quasi-increasingness is a natural and robust property for agents' reaction functions in tis standard public good provision setting.

The following consequence is noteworthy for the original BBV model, which is obtained from the above formulation by letting $f(x) = x$ (the identity function).

Corollary 26 Consider the BBV model but only under the assumptions that

(i) the public good (good 2) is inferior for agent 1 (i.e., that $-pU_{21}^1 + U_{22}^1 > 0$) and

(ii) U^1 is strictly jointly quasi-concave.

(iii) the private good (good 1) is a normal good for agent 2 (i.e., A2 holds).

Then the game possesses a PSNE.

Proof. From the previous proof, F has increasing differences since (see (21))

$$\frac{\partial F(x_1, x_2)}{\partial x_1 \partial x_2} = -pU_{21}^1 + U_{22}^1 > 0$$

From Assumption (ii), f is also a continuous function.

The proof that g is quasi-increasing (but not necessarily continuous) follows directly from the proof of the previous result.

Existence of PSNE then follows from Theorem 16. ■

In contrast, BBV's existence result assumed that (i) both the private and the public goods are normal for both players, and (ii) both utility functions are strictly jointly quasi-concave. The present result reflects a significant generalization for both types of assumptions (we stress that, here, U^2 need not be quasi-concave).

4.4 Final remarks

The applications to hybrid duopoly and to standard duopoly with increasing returns provide nice illustrations of the scope for strategic pseudo-complementarities in various simple games in industrial organization, for which strategic complementarity would not be as appropriate. The duality between pseudo-complementarities in hybrid duopoly on the one hand and strategic complementarity in Bertrand duopoly and substitutability in Cournot duopoly on the other is a novel insight in oligopoly theory in general. Likewise, the fact that a Bertrand firm that produces with increasing returns to scale will always have a quasi-decreasing reaction curve is noteworthy in itself. Similar insights can be obtained by natural modifications in well known models, such as introducing scale economies in production.

As illustrated in the public goods example, the results of the present paper also apply in various models for which they require alternative sufficient conditions on the primitives that may or may

not be nested with known conditions that make the standard supermodular approach go through. For the BBV model, our conditions are strictly more general than existing ones. However, one can easily apply our results to the standard Cournot or Bertrand models with differentiated products and obtain sufficient conditions on demand and costs that are not nested with existing ones (this is available from the authors upon request). Finally, one can also apply our results to some of the models proposed by Monaco and Sabarwal (2016) as games with strategic complementarity for one player and strategic substitutability for the other player.

5 Conclusion

By building on an intersection point theorem due to Tarski (1955), the main result of this paper demonstrates that a pure-strategy Nash equilibrium exists in two-player games when one reaction curve is continuous and increasing and the other has no downward jumps (though it may well have upward jumps). We elaborate in some detail on functions with the latter property, called quasi-increasing in Tarski (1955), by deriving a number of results on natural operations involving such functions. In particular, these results include sufficient conditions on an objective function for quasi-increasing functions to arise as argmax's of parametric optimization problems.

Some novel uniqueness results are also proved, which rely on a local (instead of the commonly used global) contraction property, and require quasi-increasingness instead of continuity of the best response maps.

The special case of symmetric n -player games is also covered, thus unifying some existing results dealing mostly with Cournot oligopoly.

In an important part of the paper, we argue that the new results here have a promising scope of application for a wide variety of economic models, including a hybrid duopoly model (of price and quantity competition), a Bertrand model with increasing returns for one firm, and a public good provision game. We illustrate in elementary ways all the various steps needed to actually apply some of the results of this paper for each of these models, tacitly establishing that strategic quasi-complementarity (or a quasi-increasing reaction curve) forms a convenient relaxation of strategic complementarity, and arises naturally in well-known economic models.

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