Optimal Two-stage Auctions with Costly Information Acquisition*

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Abstract

We study optimal two-stage mechanisms in an auction environment where bidders are endowed with original estimates ("types") about their private values and can further learn their true values of the object for sale by incurring an entry cost. We first derive an integral form of the envelope formula as required by incentive compatible two-stage mechanisms, based on which we demonstrate that the optimality of the generalized Myerson allocation rule is robust to our setting with costly information acquisition. Optimal entry is thus to admit the set of bidders that maximizes expected *virtual surplus* adjusted by both the second-stage signal and entry cost. We show that our optimal entry and allocation rules are both IR and IC implementable. Our analytical framework is general enough to encompass many existing models in the literature on auctions with costly entry.

Keywords: Two-stage auctions, entry, information acquisition, sequential screening, handicap auctions, optimal mechanisms.
JEL Classification: D44, D80, D82.

1 Introduction

In high-valued asset sales, buyers often need to go through a due diligence process before developing final bids. Due diligence is usually a process to update or acquire information about the value of the asset for sale or to prepare for the bidding process (e.g., to establish qualifications to bid). This process is costly and is usually modeled as entry as it is closely monitored by the auctioneer. For a sale of an asset worth billions of dollars, the entry cost can run from tens of thousands to millions of dollars.[1]

Given the substantial entry cost, it is unrealistic to assume that whoever is interested would necessarily go through the costly entry process. The success of a sale thus very much relies on whether the

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[1] A more detailed description of a typical due diligence process is provided in Section 4.

most qualified bidders would commit to the due diligence process and participate in the final sale. Mainly motivated by the need for entry screening, variants of two-stage selling mechanisms have emerged in the real world. A leading example of the two-stage auction procedure is known as indicative bidding, which is commonly used in sales of complicated business assets with very high values. It works as follows: the auctioneer actively markets the assets to a large group of potentially interested buyers. The potential buyers are then asked to submit non-binding bids, based on which a final set of bidders is shortlisted to advance to the second stage. The auctioneer then communicates only with these final bidders, providing them with extensive access to information about the assets,[2] and finally runs the auction (typically using binding sealed bids). The use of this two-stage auction procedure is quite widespread. For example, in response to the restructuring of the electric power industry in the U.S. – which was designed to separate power generation from transmission and distribution – billions of dollars of electrical generating assets were divested through this two-stage auction procedure over the last two decades.[3] This two-stage auction procedure is also commonly used in privatization, takeover, and merger and acquisition contests.[4] Finally, it is commonly used in the institutional real estate market, which has an annual sales volume in the order of $60 to $100 billion.[5]

Ye (2007) was the first study of indicative bidding based on the assumption of costly information acquisition.[6] Ye's analysis suggests that the current design of indicative bidding cannot reliably select the most qualified bidders for the final sale, as there does not exist a symmetric, strictly increasing equilibrium bid function in the indicative bidding stage. In a more recent paper, by restricting indicative bids to a finite discrete domain, Quint and Hendricks (2013) show that a symmetric equilibrium exists in weakly-monotone strategies. But again, the highest-value bidders are not always selected, as bidder types "pool" over a finite number of bids. Without safely selecting the most qualified bidders for the final sale, the mechanism is less likely to be optimal in maximizing expected revenue. What the optimal mechanism is in this two-stage auction environment remains an open question in the literature, and this paper seeks to provide an answer.

We model the two-stage auction environment as follows. Before entry, each potential bidder is endowed with a private signal, α_i, which can be regarded as her pre-entry "type." After entry (by incurring a common entry cost, c), each bidder i fully observes her (private) value v_i, which is positively correlated with her pre-entry type. Given costly entry, it is not feasible for all potential bidders to be included in the final sale. As such, a general mechanism must consist of an entry-right allocation stage to shortlist bidders into the sale. Mainly for the tractability of our analysis, we assume that shortlisting occurs simultaneously in a single round. So in effect, we restrict our search of optimal mechanisms to the class

[2]Data rooms, which are described in Section 4, are typically set up to facilitate bidders' due diligence process.

[3]A list of industry examples using this two-stage auction design can be found in Ye (2007).

[4]Leading examples include the privatization of the Italian Oil and Energy Corporation (ENI), the acquisition of Ireland's largest cable television provider Cablelink Limited, and the takeover contest for South Korea's second largest conglomerate Daewoo Motors.

[5]See Foley (2003) for a detailed account.

[6]Boone and Goeree (2009) provide an analysis of pre-qualifying auctions, which are similar to indicative bidding.

of two-stage mechanisms, with the first stage allocating entry rights and the second stage allocating the asset. The focus on two-stage mechanisms should be regarded as a constraint, which is fully discussed in Section 4.[7]

Despite the potential complication due to both sequential screening and costly information acquisition, we are able to completely characterize the optimal revenue-maximizing two-stage mechanisms. Our analysis benefits greatly from recent developments in the literature of sequential screening (e.g., Courty and Li, 2000; Esö and Szentes, 2007; Pavan, Segal, and Toikka, 2014; and Bergemann and Wambach, 2015).[8] In particular, our analysis follows Esö and Szentes closely, and our technical contribution is to extend their analysis to dynamic auctions with costly information acquisition. We first derive an integral form of the envelope formula as a necessary condition for incentive compatibility for our two-stage mechanisms, which extends the validity of the envelope theorem to dynamic auctions with costly information acquisition. Based on this derived envelope formula, we are able to show that the optimal allocation rule of the asset in our second stage is the same as that identified by Esö and Szentes, which requires that, among the shortlisted bidders, the asset be allocated to the bidder with the highest *virtual value* adjusted by the second-stage signal. Our analysis thus suggests that the optimality of the generalized Myerson optimal allocation rule (adjusted by second-round signals) is robust to the dynamic auction setting with costly entry. The first-stage entry right allocation mechanism is new to the original Esö-Szentes framework, and we show that the optimal entry rule is to admit the set of bidders that gives rise to the maximum expected *virtual surplus* (adjusted by both the second-stage signal and entry cost). Alternatively, given the regularity assumption and that buyers are *ex ante* symmetric in our model, the optimal entry rule is to admit the bidders in descending order of their pre-entry "types", the highest type first, the second highest type second, etc., provided that their marginal contribution to the expected virtual surplus is positive. Therefore, the optimal number of shortlisted bidders typically depends on the reported type profile from the potential bidders, which is endogenously determined. We then show that specific payment rules can be constructed in each stage to implement both optimal entry and allocation rules truthfully.

For an important setting where one's value is linear in her first signal, Esö and Szentes show that their optimal mechanism can be implemented over two rounds via a so-called handicap auction: in the first round (before observing the second-stage signal), each buyer selects a "price premium" by paying a fee according to a pre-announced schedule. In the second round (after observing the second-stage signals), buyers compete in a second-price or English auction, where the winner obtains the object at a price equal to the second-highest bid plus the price premium selected from the first round. In our setting with entry, the implementation is presumably more complicated, as optimal entry needs to be implemented prior to the final auction. Indeed, now the implementation requires that an (optimal) entry rule be augmented

[7] In Appendix B, we consider the model with two potential bidders; we are able to fully characterize optimal mechanisms allowing for sequential shortlisting.

[8] Early work on dynamic contracting with a single agent are due to Baron and Besanko (1984) and Riordan and Sappington (1987).

to the handicap auction. So in our case the optimal mechanism is implemented via a two-stage auction, with the first stage being an auction for entry rights (as well as the price premia) and the second stage being a second-price or English auction for the asset.

Other than the connection with sequential screening and dynamic auctions mentioned above, our paper is related to the literature on information acquisition in auctions (see, for example, Persico, 2000; Compte and Jehiel, 2001; and Rezende, 2013). These papers focus on bidders' incentives to acquire information in different auction formats. Our paper differs from theirs in that we follow the normative approach to identify optimal mechanisms with information acquisition.

To the extent that information acquisition is modeled as entry, our paper is closely related to the growing literature on auctions with costly entry.[9] This literature can be summarized into three branches. In the first branch, bidders are assumed to possess no private information before entry and they learn their private values or signals only after entry (see, for example, McAfee and McMillan, 1987; Engelbrecht-Wiggans, 1993; Tan, 1992; Levin and Smith, 1994; and Ye, 2004). In the second branch, it is assumed that bidders are endowed with private information about their values but have to incur entry costs to participate in an auction (see, for example, Samuelson, 1985; Stegeman, 1996; Campbell, 1998; Menezes and Monteiro, 2000; Tan and Yilankaya, 2006; Cao and Tian, 2009; and Lu, 2009). Finally, in the third branch, bidders are endowed with some private information before entry, and are able to acquire additional private information after entry (Ye, 2007; Quint and Hendricks, 2013). The framework in this current paper nests all the models mentioned above as special cases. Our paper thus characterizes optimal mechanisms for a very general framework in the literature on auctions with costly entry.

Our research is also related to a small literature on auctions of entry rights. In a pioneering work, Fullerton and McAfee (1999) introduce auctions for entry rights to shortlist contestants for a final tournament. Ye (2007) extends their approach to the setting of two-stage auctions described above. Our current approach differs from theirs in the way the set of finalists is determined: while in their approach the number of finalists to be selected is fixed and pre-announced, in our entry right allocation mechanism the selection of shortlisted bidders is contingent on the reported bid profile, making the number of finalists endogenously determined. For this reason the entry right allocation mechanism examined in this research is more general.[10]

In another relevant paper, Lu and Ye (2013) explore optimal two-stage mechanisms in an environment where bidders are characterized by heterogenous and private information acquisition costs before entry. In that setting the pre-entry "type" is the entry cost, which is neither correlated to nor part of the value of the asset for sale. As such, there is no benefit to make the second-stage mechanism contingent on the reports of the pre-entry types, resulting in a much simpler characterization of optimal mechanisms. The setting in this current paper is different, as the pre-entry "type" is correlated to the value of the asset, hence there are potential gains to make the second-stage mechanism contingent on first-stage reports.

[9]See Bergemann and Välimäki (2006) for a thoughtful survey of this literature.

[10]In fact, it resembles multi-unit auctions with endogenously determined supply (see, e.g., McAdams, 2007).

Indeed, in our current setting, the optimal allocation and payment rules in the second stage do depend on the first-stage reports. Therefore the characterization of optimal mechanisms is more demanding in this work, and the implementation of the optimal mechanism is also more sophisticated.

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 characterizes the optimal mechanism and its auction implementation. Section 4 discusses main assumptions/restrictions in our analysis and the robustness of our results. Section 5 concludes.

2 The Model

The information structure in our model is closest to that in Esö and Szentes (2007). The main differences are that in Esö and Szentes, the additional information is controlled by the seller, and they focus on the seller's incentive to *disclose* (without observing) additional signals to the buyers. In our setting, however, it is costly for the bidders to *acquire* additional information, and we focus on the bidders' incentive for information acquisition (entry). In addition, all buyers are included in the final sale in Esö and Szentes, but due to costly entry in our setting, generally not all buyers will be willing to participate in the final auction. As such, we will additionally consider entry mechanisms – which is the major difference from the analysis in Esö and Szentes.

Formally, a single indivisible asset is offered for sale to N potentially interested buyers. The seller and bidders are assumed to be risk neutral. The seller's own valuation for the asset is normalized to 0. Buyer i's true valuation for the asset is v_i. However, initially she only observes a noisy signal of it, α_i, which is her private information and can be interpreted as her original "type". After incurring a common information acquisition cost (or entry cost) of $c(>0)$, bidder i fully observes her ex post value, v_i. The pairs (α_i, v_i) are assumed to be independent across i.[11]

Ex ante, α_i follows distribution $F(\cdot)$ with its associated density $f(\cdot)$ on support $[\underline{\alpha},\overline{\alpha}]$.[12] We assume that f is positive on the interval $[\underline{\alpha},\overline{\alpha}]$ and satisfies the monotone hazard rate condition; that is, $f/(1-F)$ is weakly increasing. Given α_i, the ex post value v_i follows distribution $H_{\alpha_i} \equiv H(\cdot|\alpha_i)$ with its density $h_{\alpha_i} \equiv h(\cdot|\alpha_i)$ over support $[\underline{v},\overline{v}] \subset \mathbb{R}$.[13] The values N and c and distributions F and H_{α_i} are all common knowledge.

Following the signal orthogonalization technique introduced by Esö and Szentes (2007),[14] there exist functions u and s_i, such that $u(\alpha_i, s_i) \equiv v_i$, where u is strictly increasing in both arguments, and s_i is

[11] As in Esö and Szentes (2007) and Pavan, Segal, and Toikka (2014), this assumption rules out the possibility of full rent extraction (Crémer and McLean, 1988).

[12] Esö and Szentes allow α_i's to be drawn from different distributions. Our procedure can be extended to accommodate asymmetric distributions for α_i's. For ease of characterizing our optimal entry right allocation rule, we assume that α_i's are drawn from a common distribution, so that bidders are *ex ante* symmetric. Note that with different realizations of α_i's, bidder heterogeneity before entry is still captured in our model.

[13] Following the dynamic mechanism design literature, we assume that the support of v_i is independent of the first-stage signal α_i.

[14] The use of this technique has become standard in the literature (see, e.g., Pavan, Segal, and Toikka, 2014, and Bergemann and Wambach, 2015).

independent of α_i. In particular, s_i can be constructed as follows:

$$s_i = H(v_i|\alpha_i),$$

which is the percentile of the value realization to bidder i.[15] Thus given type α_i and signal s_i, the valuation can be computed as

$$v_i = H^{-1}_{\alpha_i}(s_i) \equiv u(\alpha_i, s_i).$$

We will denote the c.d.f. of s_i by G_i.[16]

We maintain the following assumptions that are adopted in Esö and Szentes (2007):

Assumption 1. $(\partial H_\alpha(v)/\partial\alpha)/h_\alpha(v)$ is increasing in v.

Assumption 2. $(\partial H_\alpha(v)/\partial\alpha)/h_\alpha(v)$ is increasing in α.

Esö and Szentes show that Assumption 1 is equivalent to $u_{12} \leq 0$ and Assumption 2 is equivalent to $u_{11}/u_1 \leq u_{12}/u_2$. Assumption 1 thus states that the marginal impact of the new information on buyer i's value is decreasing in her type α_i. Assumption 2 implies that an increase in α_i, holding $u(\alpha_i, s_i)$ constant, weakly decreases the marginal value of α_i. Assumptions 1 and 2 can thus be interpreted as a kind of substitutability in buyer i's posterior valuation between α_i and s_i.

Example 1. (Ye, 2007): Each potential bidder is endowed with a private value component α_i before entry; after entry, each buyer learns another private value component s_i, where s_i is independent of α_i. The ex post value $u(\alpha_i, s_i) = \alpha_i + s_i$. By the linearity of $u(\alpha_i, s_i)$, Assumptions 1 and 2 hold.

Example 2. (Adapted from Esö and Szentes, 2007): v_i is drawn from a normal distribution with mean μ and precision (inverse variance) τ_0. The pre-entry type, α_i, is normally distributed with mean v_i and precision τ_v. After entry, the buyer can observe her true value, v_i. It is clear that v_i and α_i are strictly affiliated. The distribution of α_i, which is normal, satisfies the hazard rate condition. The cdf of v_i conditional on α_i, H_{α_i}, is normal with mean $(\tau_0\mu + \tau_v\alpha_i)/(\tau_0 + \tau_v)$ and precision $\tau_0 + \tau_v$. Define $s_i = H_{\alpha_i}(v_i)$ and let $u(\alpha_i, s_i) = H^{-1}_{\alpha_i}(s_i) \equiv v_i$. Obviously u is strictly increasing in s_i. It can be verified that $u_1(\alpha_i, s_i) = \tau_v/(\tau_0 + \tau_v)$, which is a constant. Therefore, u is linear and strictly increasing in α_i. Hence Assumptions 1 and 2 hold.

Since information acquisition is modeled as entry in our setting, we consider a mechanism design framework in which the seller exercises entry control. In addition, we restrict our analysis to two-stage mechanisms: the first stage is the entry right allocation mechanism, and the second stage is the private

[15] It is easily seen that s_i is uniformly distributed over $[0,1]$, and is hence statistically independent of the initial information α_i.

[16] G_i could be assumed to be uniform on $[0,1]$. More generally, all s_i's satisfying $u(\alpha_i, s_i) \equiv v_i$ are positive monotonic transformation of each other (Lemma 1 in Esö and Szentes).

good provision mechanism. Note that in this mechanism design framework, the second-stage mechanism can be made contingent on the first-stage reports.

We restrict to direct mechanisms where agents report their types truthfully at each stage on the equilibrium path. We assume that all shortlisted bidders are disclosed and the first-stage reported profile α is revealed to all admitted bidders so that the first-stage entry allocation and payments are immediately verifiable.[17] This revelation policy turns out to be "optimal", in the sense that no other revelation policy (e.g., not revealing or partially revealing α) can generate a higher expected revenue to the seller. For this reason, our restriction to fully revealing α is without loss of generality in our search for optimal mechanisms. A detailed discussion is relegated to Section 4. As in Pavan, Segal, and Toikka (2014), the revelation policy concerned in this paper is about the first-stage information and outcome, including the agents' first-stage reports, their payments, and the agents being shortlisted. In our paper, the principal has no control over the ways in which the second-stage new information is revealed to bidders. A shortlisted bidder will be fully informed about her true value v_i after incurring the entry cost. As such, we are not concerned about the discriminatory information disclosure issue studied in Li and Shi (2013).

As in Esö and Szentes, we can focus on equivalent direct mechanisms that require bidders to report s_i's, rather than v_i's. Note that reporting (α'_i, v'_i) is equivalent to reporting $(\alpha'_i, s'_i = H_{\alpha'_i}(v'_i))$.

Let $\mathbf{N} = \{1, 2, ..., N\}$ denote the set of all the potential buyers and $2^{\mathbf{N}}$ denote the collection of all the subsets (subgroups) of $\mathbf{N}$, including the empty set, ϕ. The first-stage mechanism is characterized by the shortlisting rule $A^g(\alpha)$ and payment rule $x_i(\alpha), i = 1, 2, ..., N$. Given the reported profile α, the shortlisting rule, $A^g : [\underline{\alpha}, \overline{\alpha}]^N \to [0,1]$, assigns a probability to each subgroup $g \in 2^{\mathbf{N}}$, where $\sum_{g \in 2^{\mathbf{N}}} A^g(\alpha) = 1$. The payment rule $x_i : [\underline{\alpha}, \overline{\alpha}]^N \to \mathbb{R}$, specifies bidder i's first-stage payment given the reported profile α.

Given the first-stage reported profile α, and that group g is shortlisted, the second-stage mechanism is characterized by $p_i^g(\alpha, \mathbf{s}^g)$, the probability that the asset is allocated to buyer $i \in g$, and $t_i^g(\alpha, \mathbf{s}^g)$, the payment to the seller made by buyer $i \in g, \forall g \in 2^{\mathbf{N}}$.

3 The Analysis

We start with the second stage. Suppose group g is shortlisted, and the profile $\tilde{\alpha}$ reported in the first stage is revealed as public information to the shortlisted bidders.

First, suppose α is truthfully reported at the first stage and group g is shortlisted. Assume that they follow the recommendation and incur the information acquisition cost c to discover $\mathbf{s}^g$.[18]

Given the announced α and s_i, define the interim winning probability and expected payment rule as $P_i^g(\alpha, s_i) = E_{\mathbf{s}_{-i}^g} p_i^g(\alpha, \mathbf{s}^g)$ and $T_i^g(\alpha, s_i) = E_{\mathbf{s}_{-i}^g} t_i^g(\alpha, \mathbf{s}^g)$, where $\mathbf{s}_{-i}^g = \mathbf{s}^g \setminus \{s_i\}$, $\forall i \in g$ and $\forall g \in 2^{\mathbf{N}}$. Then bidder

[17] In Esö and Szentes, there is no such need for interim verification, as their allocation and payment rules are executed at the end of the mechanism.

[18] As will be shown, the equilibrium expected profit from going forward is positive for a buyer upon entry, so in equilibrium, a bidder does have an incentive to follow the recommendation to acquire (costly) information and participate in the final auction once admitted (as dropping out only results in zero profit).

i's second-stage interim expected payoff when she observes s_i but reports $\hat{s}_i$ is as follows:

$$\tilde{\pi}_i^g(\alpha;s_i,\hat{s}_i) = E_{\mathbf{s}_{-i}^g}[u(\alpha_i,s_i)p_i^g(\alpha,\hat{s}_i,\mathbf{s}_{-i}^g) - t_i^g(\alpha,\hat{s}_i,\mathbf{s}_{-i}^g)] = u(\alpha_i,s_i)P_i^g(\alpha,\hat{s}_i) - T_i^g(\alpha,\hat{s}_i).$$

The second-stage incentive compatibility (IC) conditions require that

$$\tilde{\pi}_i^g(\alpha;s_i,\hat{s}_i) \leq \tilde{\pi}_i^g(\alpha;s_i,s_i), \forall g,\alpha,s_i,\hat{s}_i. \tag{1}$$

First, the following lemma is standard in the traditional screening literature:

Lemma 1. Suppose α is truthfully revealed from the first stage and $P_i^g(\alpha,s_i), \forall i \in g$, is continuous and weakly increasing in s_i where g denotes the group being shortlisted, then the second-stage incentive compatibility condition (1) holds if and only if

$$\tilde{\pi}_i^g(\alpha;s_i,s_i) = \tilde{\pi}_i^g(\alpha;\hat{s}_i,\hat{s}_i) + \int_{\hat{s}_i}^{s_i} u_2(\alpha_i,\tau)P_i^g(\alpha,\tau)d\tau, \forall s_i > \hat{s}_i, \forall i \in g. \tag{2}$$

(2) is an integral form of the envelope formula. Next, we consider the case when $\hat{\alpha}_i$ instead of α_i is reported by bidder i while others report their types truthfully. As demonstrated in Esö and Szentes (2007), whenever a bidder had misreported her type in the first stage, she would "correct" her lie in the second stage. Formally in our setting, suppose α_{-i} is truthfully revealed from the first stage and the second-stage mechanism is incentive-compatible given a truthfully revealed α. Then buyer i of type α_i who reported $\hat{\alpha}_i$ in the first round will report $\hat{s}_i = \sigma_i(\alpha_i,\hat{\alpha}_i,s_i)$ if she observes s_i in the second stage such that[19]

$$u(\alpha_i,s_i) = u(\hat{\alpha}_i,\sigma_i(\alpha_i,\hat{\alpha}_i,s_i)). \tag{3}$$

Reporting $\hat{s}_i$ after a lie $\hat{\alpha}_i$ is equivalent to revealing v_i truthfully regardless of the first-stage report. The optimality of this strategy has been established in general for the Markov environments by Pavan, Segal, and Toikka (2014). Our two-stage setting resembles the Markov environment defined in Pavan, Segal, and Toikka since the agents' payoffs only depend on their second-stage true types (v_i's) and the allocation outcome, but not on their first-stage true types. For this reason, an agent's reporting incentive in the second stage depends only on her current type and her first-stage report, but not on her first-stage true type.

Note that $\hat{s}_i$ does not depend on α_{-i}, g, or $\mathbf{s}_{-i}^g$. Define

$$\begin{aligned}
\tilde{\pi}_i^g(\alpha,\hat{\alpha}_i;s_i,\hat{s}_i) &= E_{\mathbf{s}_{-i}^g}[u(\alpha_i,s_i)p_i^g(\alpha_{-i},\hat{\alpha}_i,\hat{s}_i,\mathbf{s}_{-i}^g) - t_i^g(\alpha_{-i},\hat{\alpha}_i,\hat{s}_i,\mathbf{s}_{-i}^g)] \\
&= u(\alpha_i,s_i)P_i^g(\alpha_{-i},\hat{\alpha}_i,\hat{s}_i) - T_i^g(\alpha_{-i},\hat{\alpha}_i,\hat{s}_i);
\end{aligned}$$

[19]The existence of $\sigma_i(\cdot,\cdot,\cdot)$ relies on the assumption that the support of v_i does not depend on the first-stage signal α_i.

$$\tilde{\pi}_i^g(\alpha_i,\hat{\alpha}_i;\alpha_{-i}) \quad = \quad E_{s_i}\tilde{\pi}_i^g(\alpha,\hat{\alpha}_i;s_i,\hat{s}_i=\sigma_i(\alpha_i,\hat{\alpha}_i,s_i)).$$

$\tilde{\pi}_i^g(\alpha_i,\hat{\alpha}_i;\alpha_{-i})$ is the expected second-stage payoff for the type-α_i bidder if she reported $\hat{\alpha}_i$ in the first stage (and everyone else reported truthfully) given her opponents' types being α_{-i}. Parallel to Lemma 5 in Esö and Szentes, we can show the following lemma:

Lemma 2. Suppose α_{-i} is truthfully revealed from the first stage and the second-stage mechanism is incentive-compatible given a truthfully revealed α. If buyer i of type α_i who reported $\hat{\alpha}_i$ in the first stage is shortlisted in group g_i, her expected payoff from the second stage is given by

$$\tilde{\pi}_i^{g_i}(\alpha_i,\hat{\alpha}_i;\alpha_{-i})=\tilde{\pi}_i^{g_i}(\hat{\alpha}_i,\hat{\alpha}_i;\alpha_{-i})+\int\int_{\hat{\alpha}_i}^{\alpha_i}u_1(y,s_i)P_i^{g_i}(\hat{\alpha}_i,\alpha_{-i},\sigma_i(y,\hat{\alpha}_i,s_i))dydG_i(s_i). \tag{4}$$

Throughout, g_i will be used to denote the group including bidder i. (4) should again be regarded as an integral form of the envelope formula: the winning probability ($P_i^{g_i}$) is now obtained when evaluating at $\hat{s}_i=\sigma_i(y,\hat{\alpha}_i,s_i)$ (which is optimal given the first-round "lie" $\hat{\alpha}_i$). We are now ready to consider the first-stage IC mechanism.

Let $\pi_i(\alpha_i,\hat{\alpha}_i)$ be the expected payoff (net of the entry cost) for a type-α_i bidder who reports $\hat{\alpha}_i$ in the first stage. By (3), we have

$$\begin{aligned}\pi_i(\alpha_i,\hat{\alpha}_i) &= E_{\alpha_{-i}}\left\{\sum_{g_i}A^{g_i}(\hat{\alpha}_i,\alpha_{-i})[\tilde{\pi}_i^{g_i}(\alpha_i,\hat{\alpha}_i;\alpha_{-i})-c]-x_i(\hat{\alpha}_i,\alpha_{-i})\right\} \qquad (5)\\ &= E_{\alpha_{-i}}\left\{\sum_{g_i}A^{g_i}(\hat{\alpha}_i,\alpha_{-i})\left[E_{s_i}\left(u(\alpha_i,s_i)P_i^{g_i}(\alpha_{-i},\hat{\alpha}_i,\hat{s}_i)-T_i^{g_i}(\alpha_{-i},\hat{\alpha}_i,\hat{s}_i)\right)-c\right]\right\}-x_i(\hat{\alpha}_i),\end{aligned}$$

where $\hat{s}_i=\sigma_i(\alpha_i,\hat{\alpha}_i,s_i)$ and $x_i(\hat{\alpha}_i)=E_{\alpha_{-i}}x_i(\hat{\alpha}_i,\alpha_{-i})$.

The following lemma characterizes the bidder's expected payoff in an IC two-stage mechanism with costly entry.

Lemma 3. If the two-stage mechanism is incentive compatible and $E_{\alpha_{-i}}A^{g_i}(\alpha_i,\alpha_{-i})P_i^{g_i}(\alpha_i,\alpha_{-i},s_i)$ is continuous in α_i then buyer i's expected payoff (as a function of her pre-entry type) can be expressed as

$$\pi_i(\alpha_i,\alpha_i)=\pi_i(\underline{\alpha},\underline{\alpha})+\int_{\underline{\alpha}}^{\alpha_i}\int u_1(y,s_i)\cdot\sum_{g_i}\left[E_{\alpha_{-i}}A^{g_i}(y,\alpha_{-i})P_i^{g_i}(y,\alpha_{-i},s_i)\right]dG_i(s_i)dy. \tag{6}$$

Proof. See Appendix A. □

Note that $\sum_{g_i}\left[E_{\alpha_{-i}}A^{g_i}(y,\alpha_{-i})P_i^{g_i}(y,\alpha_{-i},s_i)\right]$ is buyer i's equilibrium probability of eventually winning the asset with signals (y,s_i) in our setting. Thus (6) is also an integral form of the envelope formula.

Under a set of regularity conditions, which basically require that each agent's expected utility be a sufficiently well behaved function of her private information, Pavan, Segal, and Toikka (2014) show that the envelope formula continues to hold in the dynamic mechanism design setting. Lemma 3 can be regarded as an extension of their result to a dynamic mechanism design setting with costly information acquisition.

3.1 The Optimal Two-stage Mechanisms

We are now ready to derive the seller's expected payoff from an IC two-stage mechanism. By Lemma 3, we have

$$\begin{aligned}
& E\pi_i(\alpha_i,\alpha_i) \\
= \;& \pi_i(\underline{\alpha},\underline{\alpha}) + \int_{\underline{\alpha}}^{\overline{\alpha}}\int_{\underline{\alpha}}^{\alpha_i}\int u_1(y,s_i)\cdot\sum_{g_i}\left[E_{\alpha_{-i}}A^{g_i}(y,\alpha_{-i})P_i^{g_i}(y,\alpha_{-i},s_i)\right]dG_i(s_i)dydF(\alpha_i) \\
= \;& \pi_i(\underline{\alpha},\underline{\alpha}) + \int_{\underline{\alpha}}^{\overline{\alpha}}\frac{1-F(\alpha_i)}{f(\alpha_i)}\int u_1(\alpha_i,s_i)\cdot\sum_{g_i}\left[E_{\alpha_{-i}}A^{g_i}(\alpha_i,\alpha_{-i})P_i^{g_i}(\alpha_i,\alpha_{-i},s_i)\right]dG_i(s_i)dF(\alpha_i) \\
= \;& \pi_i(\underline{\alpha},\underline{\alpha}) + E_\alpha\left\{\sum_{g_i}A^{g_i}(\alpha_i,\alpha_{-i})\left[\int\frac{1-F(\alpha_i)}{f(\alpha_i)}u_1(\alpha_i,s_i)P_i^{g_i}(\alpha_i,\alpha_{-i},s_i)dG_i(s_i)\right]\right\}.
\end{aligned}$$

The second equality above is due to Fubini's Theorem. Thus

$$\sum_{i=1}^N E\pi_i(\alpha_i,\alpha_i) = \sum_{i=1}^N \pi_i(\underline{\alpha},\underline{\alpha}) + E_\alpha\left\{\sum_g A^g(\alpha)E_{\mathbf{s}}\left[\sum_{i\in g}p_i^g(\alpha,\mathbf{s}^g)\frac{1-F(\alpha_i)}{f(\alpha_i)}u_1(\alpha_i,s_i)\right]\right\}.$$

The total expected surplus from the two-stage mechanism is

$$TS = E_\alpha\sum_g\left\{A^g(\alpha)E_{\mathbf{s}}\left[\sum_{i\in g}p_i^g(\alpha,\mathbf{s}^g)u(\alpha_i,s_i) - |g|c\right]\right\}.$$

The seller's expected revenue is thus given by

$$\begin{aligned}
ER \;&=\; TS - \sum_{i=1}^N E\pi_i(\alpha_i,\alpha_i) \\
&=\; E_\alpha\sum_g\left\{A^g(\alpha)E_{\mathbf{s}}\left[\sum_{i\in g}p_i^g(\alpha,\mathbf{s}^g)\left(u(\alpha_i,s_i) - \frac{1-F(\alpha_i)}{f(\alpha_i)}u_1(\alpha_i,s_i)\right) - |g|c\right]\right\} - \sum_{i=1}^N \pi_i(\underline{\alpha},\underline{\alpha}), \qquad (7)
\end{aligned}$$

where $A^g(\alpha)$ is the shortlisting rule and $p_i^g(\alpha,\mathbf{s}^g)$ is the second-stage allocation rule. To maximize ER subject to IC and IR (individual rationality), the seller sets $\pi_i(\underline{\alpha},\underline{\alpha}) = 0$ for all $i = 1,2,\ldots,N$; i.e., no rent should be given to the buyer with the lowest possible (pre-entry) type.

Define the *virtual value adjusted by the second-stage signal* as follows:

$$w(\alpha_i, s_i) = u(\alpha_i, s_i) - \frac{1 - F(\alpha_i)}{f(\alpha_i)} u_1(\alpha_i, s_i). \tag{8}$$

From the expression of the expected revenue, we can derive the optimal allocation rules in both stages as follows, provided that some suitable monotonicity conditions hold. At the second stage, given the revealed α and the shortlisted group g, $\forall \mathbf{s}^g$,[20]

$$p_i^{*g}(\alpha, \mathbf{s}^g) = \begin{cases} 1 & \text{if } i = \arg\max_{j \in g}\{w(\alpha_j, s_j)\} \text{ and } w(\alpha_i, s_i) \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad \forall g, \forall i \in g. \tag{9}$$

So as also identified by Esö and Szentes, the asset should be awarded to the bidder with the highest non-negative virtual value adjusted by the second-stage signal, which is a generalization of the optimal allocation rule in Myerson (1981). Our analysis thus shows that the generalized Myerson allocation rule is robust to settings with costly entry, which affects the final allocation only through its effect on the entry right allocation rule.

Define the expected *virtual surplus* (the virtual value less the entry cost) as follows:

$$w^{*g}(\alpha) = E_{\mathbf{s}}\left[\sum_{i \in g} p_i^{*g}(\alpha, \mathbf{s}^g) w(\alpha_i, s_i) - |g| c\right].$$

Then at the first stage, contingent on the revealed α, the optimal shortlisting rule is as follows:[21]

$$A^{*g}(\alpha) = \begin{cases} 1 & \text{if } g = \arg\max_{\tilde{g}}\{w^{*\tilde{g}}(\alpha)\} \text{ and } w^{*g}(\alpha) \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad \forall g. \tag{10}$$

The optimal shortlisting rule admits the set of bidders that gives rise to the maximal expected virtual surplus. Alternatively, the optimal shortlisting rule admits the bidders in descending order of their marginal contribution to the expected virtual surplus – the bidder with the highest contribution first, the bidder with the second-highest contribution second, etc. – provided that their marginal contribution is positive. Let $g^*(\alpha)$ denote the set of bidders admitted under the optimal shortlisting rule.

Similarly to Esö and Szentes, following Assumptions 1 and 2, we can establish the following properties of the optimal second-stage allocation rule:[22]

Corollary 1. (i) $p_i^{*g_i}(\alpha, \mathbf{s}^{g_i})$ increases in both α_i and s_i, $\forall i \in g_i$, $\forall g_i$, α_{-i}, and $\mathbf{s}_{-i}^{g_i}$, which implies that $P_i^{*g_i}(\alpha_i, \alpha_{-i}, s_i)$ increases in both α_i and s_i, $\forall g_i$, α_{-i}; (ii) If $\alpha_i > \hat{\alpha}_i$, $s_i < \hat{s}_i$ and $u(\alpha_i, s_i) = u(\hat{\alpha}_i, \hat{s}_i)$, then $p_i^{*g_i}(\alpha_i, \alpha_{-i}, s_i, \mathbf{s}_{-i}^{g_i}) \geq p_i^{*g_i}(\hat{\alpha}_i, \alpha_{-i}, \hat{s}_i, \mathbf{s}_{-i}^{g_i})$, which implies $P_i^{*g_i}(\alpha_i, \alpha_{-i}, s_i) \geq P_i^{*g_i}(\hat{\alpha}_i, \alpha_{-i}, \hat{s}_i), \forall g_i$, α_{-i}.

[20] Ties occur with probability zero and are hence ignored.
[21] Again ties occur with probability zero and are hence ignored.
[22] Assumption 2 is used to show property (ii).

Property (ii) above suggests that whenever $\alpha_i > \hat{\alpha}_i$, $s_i < \hat{s}_i$ and $u(\alpha_i, s_i) = u(\hat{\alpha}_i, \hat{s}_i)$, the optimal allocation rule favors the "truth-telling" pair (α_i, s_i).

Given α_i, let $s(\alpha_i)$ be defined such that $w(\alpha_i, s(\alpha_i)) = 0$. To identify properties of the optimal shortlisting rule, we define a truncated random variable as follows:

$$w_i^+(\alpha_i, s_i) = \begin{cases} w(\alpha_i, s_i) & \text{if } w(\alpha_i, s_i) \geq 0 \text{ or equivalently } s_i \geq s(\alpha_i) \\ 0 & \text{otherwise} \end{cases} \forall i.$$

Note that conditional on α, w_i^+'s are independent across $i \in g$.

Let $\Delta\tilde{S}^g(\alpha_i; \alpha_{-i})$ denote buyer i's marginal contribution to the expected virtual surplus, $i \in g$, then

$$\Delta\tilde{S}^g(\alpha_i; \alpha_{-i}) = \tilde{S}(\alpha^g) - \tilde{S}(\alpha_{-i}^g), i \in g, \forall \alpha^g,$$

where $\alpha_{-i}^g = \alpha^g \setminus \{\alpha_i\}$ and

$$\tilde{S}(\alpha^g) = E_{\mathbf{s}^g} \max_{i \in g}\{w_i^+(\alpha_i, s_i)\}, \forall g, \forall \alpha^g.$$

The following two properties are obvious:

(1) $\Delta\tilde{S}^g(\alpha_i; \alpha_{-i})$ increases with α_i, and decreases with $\alpha_j, \forall j \neq i, \forall i \in g, \forall g$.

(2) $\Delta\tilde{S}^g(\alpha_i; \alpha_{-i}) \geq \Delta\tilde{S}^{g'}(\alpha_i; \alpha_{-i})$, $\forall \alpha_{-i}, \forall i \in g, \forall g \subset g'$.

The revenue-optimal shortlisting rule can be alternatively described as follows. For given α, we can rank all α_is from the highest to the lowest. The seller admits bidders one by one in descending order of α_i's as long as the bidder's marginal contribution to the expected virtual surplus is nonnegative, i.e.

$$\Delta\tilde{S}^g(\alpha_i; \alpha_{-i}) - c = \tilde{S}(\alpha^g) - \tilde{S}(\alpha_{-i}^g) - c \geq 0,$$

where g denotes the group of bidders with the highest $|g|$ types before entry.

Two properties follow immediately from the optimal shortlisting rule A^{*g}:

Corollary 2. (i) Given α_{-i}, if bidder i with α_i is shortlisted, then she would also be shortlisted with a higher type $\tilde{\alpha}_i (> \alpha_i)$; (ii) Given α_{-i}, bidder i will be shortlisted as long as α_i is higher than a threshold $\hat{\alpha}_i(\alpha_{-i})$. As α_i increases, the shortlisted group weakly shrinks. As α_i increases from $\hat{\alpha}_i(\alpha_{-i})$, the bidders in $g^*(\alpha) \setminus \{i\}$ would be excluded one by one (with the lowest type originally shortlisted being excluded first).

We are now ready to show that the optimal final good allocation and entry right allocation rules (9) and (10) are truthfully implementable by some well constructed payment rules in both stages.

Theorem 1. Under Assumptions 1 and 2, the optimal final good allocation and entry right allocation rules (9) and (10) are IR and IC implementable.

Proof. $u(\alpha_i, s_i)$ increases with s_i and by Assumption 1, $u_1(\alpha_i, s_i)$ (weakly) decreases with s_i. This implies that $w(\alpha_i, s_i)$ increases with s_i. By the final good allocation rule (9), the winning probability $P_i^{*g}(\alpha, s_i)$ is weakly increasing in s_i. By Lemma 1, the second-stage mechanism is incentive compatible (given α and g). Thus, given the truthfully revealed α and shortlisted group g, a second-stage payment rule, say, $t_i^{*g}(\alpha, \mathbf{s}^g), \forall i \in g, \forall g$, can be constructed to truthfully implement the second-stage allocation rule $p_i^{*g}(\alpha, \mathbf{s}^g), \forall i \in g, \forall g$ while maintaining the second-stage IR constraints (to participate in the second-stage mechanism), i.e. $\tilde{\pi}_i^g(\alpha, \alpha_i; s_i, s_i) \geq 0$ on equilibrium path. This resembles the Myerson (1981) setting with asymmetric bidders.

We use $\tilde{\pi}_i^{*g_i}(\alpha_i, \hat{\alpha}_i; \alpha_{-i})$ to denote the second-stage expected payoff to buyer i of type α_i if she announces $\hat{\alpha}_i$ and is shortlisted in group g_i, given that everyone else announces α_{-i} truthfully at the first stage. $\tilde{\pi}_i^{*g_i}(\alpha_i, \hat{\alpha}_i; \alpha_{-i})$ is well defined given Lemma 2. Therefore, when buyer i of type α_i announces $\hat{\alpha}_i$ while others reveal α_{-i} truthfully, her first-stage expected payoff can be written as follows:

$$\pi_i^*(\alpha_i, \hat{\alpha}_i) = E_{\alpha_{-i}} \left\{ \sum_{g_i} A^{*g_i}(\hat{\alpha}_i, \alpha_{-i})[\tilde{\pi}_i^{*g_i}(\alpha_i, \hat{\alpha}_i; \alpha_{-i}) - c] - x_i^*(\hat{\alpha}_i, \alpha_{-i}) \right\},$$

where x_i^* is the first-stage payment rule.

Next, we will show that the optimal shortlisting rule (10) is truthfully implementable by a properly chosen first-stage payment rule x_i^*, together with the second-stage payment rules t_i^{*g} chosen above.

Note that by (5), we have

$$\pi_i^*(\alpha_i, \alpha_i) = E_{\alpha_{-i}} \left\{ \sum_{g_i} A^{*g_i}(\alpha_i, \alpha_{-i})[\tilde{\pi}_i^{*g_i}(\alpha_i, \alpha_i; \alpha_{-i}) - c] - x_i^*(\alpha_i, \alpha_{-i}) \right\}. \tag{11}$$

Construct the first-stage payment rule as follows:

$$\begin{aligned} x_i^*(\alpha) \quad &= \quad \sum_{g_i} A^{*g_i}(\alpha_i, \alpha_{-i})[\tilde{\pi}_i^{*g_i}(\alpha_i, \alpha_i; \alpha_{-i}) - c] \\ &\quad - \int_{\underline{\alpha}}^{\alpha_i} \int u_1(y, s_i) \cdot \sum_{g_i} \left[E_{\alpha_{-i}} A^{*g_i}(y, \alpha_{-i}) P_i^{*g_i}(y, \alpha_{-i}, s_i) \right] dG_i(s_i) dy \end{aligned} \tag{12}$$

Substituting (12) into (11), we can verify that

$$\pi_i^*(\alpha_i, \alpha_i) = \int_{\underline{\alpha}}^{\alpha_i} \int u_1(y, s_i) \cdot \sum_{g_i} \left[E_{\alpha_{-i}} A^{*g_i}(y, \alpha_{-i}) P_i^{*g_i}(y, \alpha_{-i}, s_i) \right] dG_i(s_i) dy,$$

which is precisely equation (6) with $\pi_i^*(\underline{\alpha}, \underline{\alpha}) = 0$ (the optimality requirement). Note that $\pi_i^*(\alpha_i, \alpha_i) \geq 0$, so IR is satisfied in the first stage.

Suppose that all buyers except i report their types α_{-i} truthfully. Consider buyer i with α_i contemplating to misreport $\hat{\alpha}_i < \alpha_i$. The deviation payoff is

$$\Delta = \pi_i^*(\alpha_i, \hat{\alpha}_i) - \pi_i^*(\alpha_i, \alpha_i) = [\pi_i^*(\alpha_i, \hat{\alpha}_i) - \pi_i^*(\hat{\alpha}_i, \hat{\alpha}_i)] + [\pi_i^*(\hat{\alpha}_i, \hat{\alpha}_i) - \pi_i^*(\alpha_i, \alpha_i)].$$

Since (6) is satisfied by the construction of $x_i^*(\alpha)$, we have

$$\pi_i^*(\hat{\alpha}_i,\hat{\alpha}_i)-\pi_i^*(\alpha_i,\alpha_i) = -\int_{\hat{\alpha}_i}^{\alpha_i}\int u_1(y,s_i)\cdot\sum_{g_i}\left[E_{\alpha_{-i}}A^{*g_i}(y,\alpha_{-i})P_i^{*g_i}(y,\alpha_{-i},s_i)\right]dG_i(s_i)dy.$$

Recall the definition of $\pi_i^*(\alpha_i,\hat{\alpha}_i)$ above, we have from Lemma 2 that

$$\pi_i^*(\alpha_i,\hat{\alpha}_i)-\pi_i^*(\hat{\alpha}_i,\hat{\alpha}_i) = \int_{\hat{\alpha}_i}^{\alpha_i}\int u_1(y,s_i)\cdot\sum_{g_i}\left[E_{\alpha_{-i}}A^{*g_i}(\hat{\alpha}_i,\alpha_{-i})P_i^{*g_i}(\hat{\alpha}_i,\alpha_{-i},\sigma_i(y,\hat{\alpha}_i,s_i))\right]dG_i(s_i)dy.$$

Therefore, we have

$$\begin{aligned}\Delta \;=\; & \int_{\hat{\alpha}_i}^{\alpha_i}E_{\alpha_{-i}}\sum_{g_i}A^{*g_i}(y,\alpha_{-i})\int u_1(y,s_i)[P_i^{*g_i}(\hat{\alpha}_i,\alpha_{-i},\sigma_i(y,\hat{\alpha}_i,s_i))-P_i^{*g_i}(y,\alpha_{-i},s_i)]dG_i(s_i)dy\\ & +\int_{\hat{\alpha}_i}^{\alpha_i}E_{\alpha_{-i}}\sum_{g_i}[A^{*g_i}(\hat{\alpha}_i,\alpha_{-i})-A^{*g_i}(y,\alpha_{-i})]\int u_1(y,s_i)P_i^{*g_i}(\hat{\alpha}_i,\alpha_{-i},\sigma_i(y,\hat{\alpha}_i,s_i))dG_i(s_i)dy. \qquad (13)\end{aligned}$$

From Corollary 1 (ii), we have $P_i^{*g_i}(\hat{\alpha}_i,\alpha_{-i},\sigma_i(y,\hat{\alpha}_i,s_i))-P_i^{*g_i}(y,\alpha_{-i},s_i)\le 0$, which implies that the first term in Δ is negative.

We now consider the second term in Δ when $y>\hat{\alpha}_i$. By Corollary 2, the optimal shortlisting rule implies that given α_{-i}, when buyer i is admitted with a higher α_i, she must be admitted to a group with a weakly smaller size. If y and $\hat{\alpha}_i$ are admitted in the same group, then $A^{*g_i}(\hat{\alpha}_i,\alpha_{-i})=A^{*g_i}(y,\alpha_{-i})$ and this term in Δ is zero.

We now turn to the case where $g^*(\hat{\alpha}_i,\alpha_{-i})\supset g^*(y,\alpha_{-i})\supset\{i\}$. Note that $A^{*g_i}(\cdot,\alpha_{-i})$ is 1 for the shortlisted group, and 0 for all other groups. Therefore,

$$\begin{aligned}& \sum_{g_i}[A^{*g_i}(\hat{\alpha}_i,\alpha_{-i})-A^{*g_i}(y,\alpha_{-i})]u_1(y,s_i)P_i^{*g_i}(\hat{\alpha}_i,\alpha_{-i},\sigma_i(y,\hat{\alpha}_i,s_i))\\ = \;& u_1(y,s_i)[P_i^{*g^*(\hat{\alpha}_i,\alpha_{-i})}(\hat{\alpha}_i,\alpha_{-i},\sigma_i(y,\hat{\alpha}_i,s_i))-P_i^{*g^*(y,\alpha_{-i})}(\hat{\alpha}_i,\alpha_{-i},\sigma_i(y,\hat{\alpha}_i,s_i))]\\ \le \;& 0,\end{aligned}$$

which implies that the second term in Δ is negative. Since $g^*(\hat{\alpha}_i,\alpha_{-i})\supset g^*(y,\alpha_{-i})\supset\{i\}$, we must have $P_i^{*g^*(\hat{\alpha}_i,\alpha_{-i})}(\hat{\alpha}_i,\alpha_{-i},\sigma_i(y,\hat{\alpha}_i,s_i))\le P_i^{*g^*(y,\alpha_{-i})}(\hat{\alpha}_i,\alpha_{-i},\sigma_i(y,\hat{\alpha}_i,s_i))$, i.e. entrant i wins with a smaller probability if a strictly bigger group is shortlisted.

A similar argument can be used to rule out deviation to $\hat{\alpha}_i>\alpha_i$. □

It is worth noting that Assumptions 1 and 2 are sufficient but not necessary for the optimal entry rule to be truthfully implementable: the necessary and sufficient condition is that Δ defined in (13) is non-positive, which is also the *integral monotonicity* condition characterized by Pavan, Segal, and Toikka (2014).

When $u(\alpha_i,s_i)$ is linear in α_i, i.e., when $u(\alpha_i,s_i)=u_1\alpha_i+r(s_i)$ for some constant u_1 and function r, we will demonstrate that the optimal mechanism can be implemented via a two-stage auction, with the first stage being an auction for both entry rights and price premia and the second stage being a second-price or English auction for the final good. This two-stage auction can be regarded as a handicap auction introduced in Esö and Szentes, augmented by an additional auction at the entry stage.[23]

More specifically, our two-stage auction works as follows. The first stage is an all-pay auction, where bidders need to pay what they bid, regardless of being awarded entry rights or not. Suppose buyer i, knowing her type α_i, bids an amount b_i, $i = 1, 2, ..., N$. After all the first-stage bids are collected, underlying types will be recovered based on a recovery function, x^{*-1}, such that buyer i's perceived type $\widehat{\alpha}_i$ is $x^{*-1}(b_i)$, $i = 1, 2, ..., N$. Given the recovered type profile $\{\widehat{\alpha}_i\}_{i=1}^N$, the entry rights are implemented according to the optimal entry rule (10), and a "price premium" is determined for each shortlisted buyer according to the following premium schedule: $p(\widehat{\alpha}_i) = u_1(1 - F(\widehat{\alpha}_i))/f(\widehat{\alpha}_i)$. Both the recovery function x^{*-1} and the premium determination rule p are made public at the outset of the game, which remain common knowledge throughout the auction process. Upon being admitted, each entrant bidder will incur the information acquisition cost and participate in the second-round bidding. The second stage is a traditional second-price or English auction with a zero reserve price, but the winner is required to pay her premium over the price.[24] This mechanism is referred to as the handicap auction in Esö and Szentes, since the buyers compete under unequal conditions: a bidder with a smaller premium has an advantage. In our setting, the handicap auction is modified so that the optimal entry rule is also implemented after the first-round bidding. In Esö and Szentes, buyers pay fees regardless of winning the final good or not; in our setting, buyers pay b_i's regardless of being admitted to the final sale or not. For this reason, the first-stage auction is a variant of the all-pay auction.

The implementation in our setting is established by showing that a properly selected $x^*(\cdot)$ constitutes a symmetric (strictly) increasing equilibrium bid function in the (reduced) all-pay auction game, with the second stage being replaced by its associated equilibrium payoffs. A major step in the proof is to establish that $x^*(\alpha_i)$ is strictly increasing for $\alpha_i \in [\alpha^*, \overline{\alpha}]$, where $\alpha^* \in (\underline{\alpha}, \overline{\alpha})$ is the minimum type that could possibly be allocated with an entry right in equilibrium. The proof is tedious, which is available upon request.

3.2 Applications

Our optimal mechanism analysis is general enough to encompass many existing models in the literature on auctions with costly entry. Below we demonstrate how we can apply our general optimal mechanism to special models previously studied.

1. Bidders do not have pre-entry types and only learn about their values after entry (e.g., McAfee and McMillan, 1987; Tan, 1992; and Levin and Smith, 1994). In this case, $u(\alpha_i, s_i) = s_i$. Hence $w(\alpha_i, s_i) = s_i$, which implies that the optimal auction is *ex post* efficient, and the optimal entry is to select a set of bidders that results in the maximal expected social surplus. Since bidders are

[23]The assumption that u_1 is constant is satisfied in both Examples 1 and 2.

[24]Should there be only one entrant, the price premium for this sole entrant becomes the effective reserve price.

identical before entry, optimal entry is entirely characterized by n^*, the optimal number of bidders to be selected. The implementation is somewhat simple: the second round is a standard auction (first-price, second-price, or English auction – no price premium is involved). The first round (entry stage) is to select exactly n^* bidders, and whomever selected is required to pay an upfront entry fee e^*, which is set so that no rent is left for the entrants *ex ante*.

2. Bidders know their values before entry, and entry is merely a bid preparation process (without value updating) (e.g. Samuelson,1985; Stegeman, 1996; Campbell, 1998; Menezes and Monteiro, 2000; Tan and Yilankaya, 2006; Cao and Tian, 2009; and Lu, 2009). In this setting, $u(\alpha_i, s_i) = \alpha_i$, and hence $w(\alpha_i, s_i) = \alpha_i - (1 - F(\alpha_i))/f(\alpha_i)$. It is easily verified that according to Theorem 1, the optimal allocation rules can be described as follows: the bidder with the highest "type" (α_i) is admitted as the sole entrant to win the item, as long as her contribution to the virtual surplus $w(\alpha_i, s_i) - c$ is positive. The optimal mechanism can be implemented as follows: each buyer pays what she bids in the first stage (regardless of being admitted or not), and the only entrant wins the item at a price equal to her price premium determined from her first-round bid.

3. Each bidder is endowed with pre-entry type α_i, and learns an additional private value component s_i (e.g., Ye, 2007; Quint and Hendricks, 2013). The total value is given by $u(\alpha_i, s_i) = \alpha_i + s_i$.[25] Hence $w(\alpha_i, s_i) = \alpha_i + s_i - (1 - F(\alpha_i))/f(\alpha_i)$. The optimal second-stage allocation rule thus requires that the asset be allocated to the entrant bidder with the highest virtual value $w(\alpha_i, s_i)$ provided that it is nonnegative. The optimal entry rule requires that bidders be admitted in descending order of their pre-entry types, as long as their contribution to the expected virtual surplus is nonnegative.

   To further illustrate the optimal entry rule, we assume that α_i is distributed uniformly over $[0,1]$ and s_i follows a Bernoulli distribution, taking value 1 ("High") with probability q and 0 ("Low") with probability $1-q$. Then $w(\alpha_i, s_i) = 2\alpha_i + s_i - 1$. If only one buyer (the one with the highest type $\alpha_{(1)}$) is admitted, the expected virtual surplus is given by $w_1 = E\left(2\alpha_{(1)} + s_1 - 1\right) - c = 2\alpha_{(1)} + q - 1 - c$. So the optimal number of entrants $n^* \geq 1$ if $2\alpha_{(1)} + q - 1 - c \geq 0$. For ease of computation we assume that $\alpha_{(1)} \geq \alpha_{(2)} \geq .5$ (so that the virtual values from the top two bidders are guaranteed to be non-negative). If two top buyers are admitted, the expected virtual surplus is given by

$$
\begin{aligned}
w_2 &= E\left[\max\left\{2\alpha_{(1)} + s_1 - 1, 2\alpha_{(2)} + s_2 - 1\right\}\right] - 2c \\
&= E\left[\max\left\{2\alpha_{(1)} + s_1, 2\alpha_{(2)} + s_2\right\}\right] - 1 - 2c \\
&= \Pr(s_1 = 1)\cdot\left(2\alpha_{(1)} + 1\right) + \Pr(s_1 = s_2 = 0)\cdot 2\alpha_{(1)} + \Pr(s_1 = 0, s_2 = 1)\cdot\left(2\alpha_{(2)} + 1\right) - 1 - 2c \\
&= q\cdot\left(2\alpha_{(1)} + 1\right) + (1-q)^2\cdot 2\alpha_{(1)} + (1-q)q\cdot\left(2\alpha_{(2)} + 1\right) - 1 - 2c
\end{aligned}
$$

   So the optimal number of entrants $n^* \geq 2$ if the incremental expected virtual surplus $\Delta w = w_2 - w_1 =$

[25] Note that the common support assumption stated in footnote 13 is violated for this linear valuation model. Eso and Szente (???) also use such an example but explain how it can be reconciled with the common support assumption.

$q(1-q)\left[1-2\left(\alpha_{(1)}-\alpha_{(2)}\right)\right]-c \geqslant 0$. Continuing this procedure of calculation,[26] it can be verified that $n^* \geq n$ if $q(1-q)^{n-1}\left[1-2\left(\alpha_{(1)}-\alpha_{(n)}\right)\right]-c \geqslant 0$,[27] or

$$\alpha_{(1)}-\alpha_{(n)} \leq \frac{1}{2}\left[1-\frac{c}{q(1-q)^{n-1}}\right]. \tag{14}$$

This condition is intuitive: the admission of the n-th highest buyer is more likely to be justified if (1) the probability that she will turn out to be the winner in the second round is sufficiently high; (2) the entry cost is sufficiently low; or (3) her type is sufficiently close to the highest type. It is thus clear that the optimal number of entrants, n^*, is determined by the following conditions:

$$\alpha_{(1)}-\alpha_{(n^*)} \leq \frac{1}{2}\left[1-\frac{c}{q(1-q)^{n^*-1}}\right],\ \alpha_{(1)}-\alpha_{(n^*+1)} > \frac{1}{2}\left[1-\frac{c}{q(1-q)^{n^*}}\right].$$

4 Discussion

4.1 Revelation Policy

In our preceding analysis, we have focused on the revelation policy so that the first-stage reports are fully revealed to the shortlisted bidders. Due to this particular revelation policy, one concern is that there might be some loss of generality in identifying optimal mechanisms. To address this concern, we next identify an upper bound for the expected revenue that can be achieved by examining a relaxed setting by dropping the IC and IR constraints for the shortlisted bidders in the second stage so that all shortlisted bidders must incur entry costs to learn their second-stage signals as in our original setup, and regardless of their second-stage signals, they must participate in the second-stage selling mechanism and report truthfully their second stage signals. As a result, regardless of the disclosure policy of the first-stage reports, the highest possible expected revenue achievable in this relaxed setting should impose an upper bound for the expected revenue that can be obtained in our original setup, where the bidders' second-stage IC and IR must both be satisfied. A useful observation is that in the relaxed setting, bidders can only misreport their first-stage signals, and the shortlisted buyers' beliefs on buyers' first-stage type profiles have no impact on their second-stage decisions (as shortlisted bidders must enter and truthfully report their second-stage signals). This observation implies that the revelation policy of the first-stage reports is not relevant to the mechanism design in the relaxed setting. Consequently, the highest expected revenue attainable in this relaxed setting does not depend on the prevailing disclosure policy of the first-stage signals. We next proceed to identify this bound.

In the relaxed setting, the mechanisms are specified exactly the same as in Section 2. All potential bidders report their types α_i, giving rise to a reported type profile α. The mechanism specifies the first-stage shortlisting rule $A^g(\alpha)$ and payment rule $x_i(\alpha_i, \alpha_{-i})$. Every shortlisted bidder j incurs cost c to

[26] We continue to consider the case $\alpha_{(1)} > \alpha_{(2)} \geq \ldots \geq \alpha_{(n)} \geq .5$ so that the virtual value from these buyers will be positive.

[27] The addition of the nth highest buyer only contributes to the expected virtual surplus when she turns out to be the only one having a good "shot" in the second stage (i.e., $s_n = 1$, while $s_1 = \ldots = s_{n-1} = 0$).

discover her second-stage signal s_j. The second-stage selling mechanism specifies the winning probability $p_i^g(\alpha, \mathbf{s}^g)$ and payment rule $t_i^g(\alpha, \mathbf{s}^g)$, $\forall i \in g$, $\forall g \in 2^N$.

Recall that $P_i^g(\alpha, s_i) = E_{\mathbf{s}_{-i}^g} p_i^g(\alpha, \mathbf{s}^g)$ and $T_i^g(\alpha, s_i) = E_{\mathbf{s}_{-i}^g} t_i^g(\alpha, \mathbf{s}^g)$. For shortlisted bidder $i \in g_i$ with type α_i, her interim expected payoff when she reports $\hat{\alpha}_i$ and others report truthfully is given by

$$\pi_i(\alpha_i, \hat{\alpha}_i) = E_{\alpha_{-i}}\{\sum_{g_i} A^{g_i}(\hat{\alpha}_i, \alpha_{-i})[E_{s_i}((u(\alpha_i, s_i)P_i^{g_i}(\hat{\alpha}_i, \alpha_{-i}, s_i) - T_i^{g_i}(\hat{\alpha}_i, \alpha_{-i}, s_i)) - c] - x_i(\hat{\alpha}_i, \alpha_{-i})\}. \tag{15}$$

Applying the envelope theorem, the IC condition $\pi_i(\alpha_i, \alpha_i) \geq \pi_i(\alpha_i, \hat{\alpha}_i)$ leads to the following necessary condition:

$$\frac{d\pi_i(\alpha_i, \alpha_i)}{d\alpha_i} = \frac{\partial \pi_i(\alpha_i, \hat{\alpha}_i)}{\partial \alpha_i}|_{\hat{\alpha}_i = \alpha_i} = E_{\alpha_{-i}}\{\sum_{g_i} A^{g_i}(\alpha_i, \alpha_{-i}) E_{s_i}[u_1(\alpha_i, s_i) P_i^{g_i}(\alpha_i, \alpha_{-i}, s_i)]\}.$$

Therefore, we have

$$\begin{aligned}
\pi_i(\alpha_i, \alpha_i) &= \pi_i(\underline{\alpha}, \underline{\alpha}) + E_{\alpha_{-i}} \int_{\underline{\alpha}}^{\alpha_i} \sum_{g_i} A^{g_i}(y, \alpha_{-i}) E_{s_i}[u_1(y, s_i) P_i^{g_i}(y, \alpha_{-i}, s_i)] dy \\
&= \pi_i(\underline{\alpha}, \underline{\alpha}) + E_{\alpha_{-i}} \int_{\underline{\alpha}}^{\alpha_i} \int u_1(y, s_i) \cdot \sum_{g_i} A^{g_i}(y, \alpha_{-i}) P_i^{g_i}(y, \alpha_{-i}, s_i) dG_i(s_i) dy.
\end{aligned}$$

Note that the above expression is exactly the same as (6), which implies that the seller's expected revenue must be the same as in (7); in other words, the upper bound of the expected revenue in the relaxed setting is achieved in our original setting. In this sense, there is no loss of generality to derive optimal mechanisms by only considering mechanisms that fully reveal the buyers' first-stage reports to all admitted bidders.

4.2 Modeling Information Acquisition as Entry

Another important aspect in our analysis is that we model information acquisition as entry. An implication is that information acquisition is mandatory, in the sense that a bidder is not allowed to bid without going through the "due diligence" process. This assumption is due to the specific institutional setup we are trying to model. For example, "data rooms" are usually provided by the selling party to disclose a large amount of confidential data to bidders during the due diligence process. A typical data room is a continually monitored space that the bidders and their advisers will visit in order to inspect and report on the various documents and data made available. Often only one bidder at a time will be allowed to enter a data room. Teams involved in large due diligence processes will typically remain available throughout the process. Such teams often consist of a number of experts in different fields, hence the overall cost of keeping such groups on call near to the data room is often extremely high.[28] In a typical electrical

[28] See Vallen and Bullinger (1999) for a detailed description of the due diligence process in a typical electric power plant sale in the US.

generating asset sale as studied by Ye (2007), before submitting a final bid, each bidder (more precisely, bidding team) usually needs to go through the due diligence process to meet with senior management and personnel, study equipment conditions and operating history, evaluate supply contracts and employment agreements, etc. This process is strictly controlled and closely monitored by the auctioneer (typically an investment banker serving as the financial advisor for the selling party). Given the complexity and high-stakes nature of the sale, it is very unlikely that a seller would be comfortable accepting a bid from someone who did not go through such an important information acquisition process. As such, we believe that it is appropriate to model information acquisition as entry for such an environment. From both theoretical and practical points of view, it would be interesting to identify optimal mechanisms in environments where bidders are allowed to bid without having to go through information acquisition (and information acquisition may not be observable or contractible). Such an analysis would be more involved, however, as we will need to worry that the informed and uninformed buyers may mimic each other.

4.3 Sequential Shortlisting

Finally, we restrict our search for optimal mechanisms to the class of two-stage mechanisms (with a single shortlisting stage). A consequence is that if some bidders are excluded from entry after the first stage, the seller cannot go back to these bidders after the second-stage bidding. For a more general characterization of optimal mechanisms, we should allow for sequential shortlisting so that the mechanism may potentially consist of multiple stages or rounds, rather than only two. For example, the seller may select a single bidder or a subset of bidders to go through due diligence and submit final bids, and if the seller is not satisfied with any offer, he can go back to the unselected first-round bidders and invite another bidder or another subset of bidders to go through due diligence and submit final bids. This process can then repeat itself, until the seller finds a satisfactory offer. Such mechanisms can be much more complicated. First of all, the seller will need to determine the order of bidders to invite for conducting due diligence (i.e., who should be invited first and who second, etc.). Given that bidders are heterogenous before entry, it is desirable to make the optimal "ordering" or "sequencing" of entry contingent on their pre-entry types. In Appendix B, we analyze such a general mechanism with two potential bidders. Restricting to two potential bidders allows us to fully characterize optimal mechanisms with sequential shortlisting. Our main results are as follows. First, the object is allocated to the shortlisted bidder with the highest virtual value $w(\alpha_i, s_i)$, provided that it is positive. Second, without discounting, there is no need to shortlist both bidders at the same stage. Third, as long as one bidder should be shortlisted, the bidder with the higher first-stage signal should be shortlisted first. Fourth, the other bidder should be shortlisted in the second round if and only if her expected contribution to the virtual surplus is positive (conditional on all the available information, in particular, the second-stage signal revealed by the first shortlisted bidder). While a full analysis with an arbitrary number of potential bidders would be too tedious and hence not attempted,[29] we believe that our results based on the two-bidder case should be robust. For the general case with any arbitrary number of potential bidders, we conjecture that the optimal final good allocation rule should be the same as characterized in our main analysis (with single-round shortlisting); with sequential shortlisting, however, the optimal shortlisting rule should be modified, so that at each round, at most one bidder is shortlisted, and a new bidder is shortlisted at a given round if and only if, conditional on all the revealed information up to this round, her expected contribution to the virtual surplus is positive. Since single-round shortlisting can be trivially replicated by sequential shortlisting, the optimal mechanism characterized in the main text must be revenue-dominated by the optimal mechanism allowing for sequential shortlisting. This is true, however, when there is no time discounting. When time discounting is taken into account, an obvious drawback of running a multi-stage mechanism is the potential for delay, which would be too costly and therefore favors a more time-efficient two-stage mechanism. We believe that this consideration, along with the practical difficulty in administering multiple rounds of the due diligence process,[30] leads to the "norm" of the two-stage auction format widely used in the real world. This is also the main justification for why we restrict our analysis to the class of two-stage auctions.

5 Concluding Remarks

Our paper contributes to the literature on two fronts. First, it characterizes optimal two-stage mechanisms for an environment of two-stage auctions, which are commonly employed in sales of complicated and high-valued business assets, procurements, privatization, takeover, and merger and acquisition contests. Our analysis is general enough to nest many existing studies in the literature of auctions with costly entry. Second, our paper contributes to the literature on sequential screening by introducing costly entry into a dynamic auction framework. Entry provides a natural setting for sequential information acquisition; on the other hand, entry also makes the optimal mechanism design more challenging, as now it must balance information acquisition at the entry stage and information elicitation in the final good allocation stage, which are interdependent.

Implementation of the optimal mechanism characterized in this paper may face some practical obstacles. First, the industry may not be comfortable with the idea of paying entry fees whether or not they win the object eventually, and this is the major reason, we believe, that contributes to the common use of nonbinding indicative bidding. Second, the optimal mechanism is so complicated that the industry bidders might face great difficulties in developing bidding strategies for both rounds (although such a concern is

[29] In particular, with a general number of agents, establishing the incentive compatibility of sequential shortlisting and selling rules is much more involved. This is especially the case when we consider the incentive compatibility at the first stage. For example, if agent i over-reports her type α_i, then she has a better chance to be shortlisted. At the same time this changes the chances of other agents to be shortlisted. For different type profiles, this impact would be different and there would be too many possibilities to analyze. As a result, establishing the incentive compatibility in a similar way as for the two-agent case will be much more challenging.

[30] Just Imagine, for example, the hassle of arranging multiple meetings with senior management.

alleviated to some extent if professional or sophisticated experts are hired to help). For these reasons the nature of our analysis is primarily normative, offering a "market design" approach to guide a potential refinement of an extremely important transaction procedure widely used in the industry. Despite this limitation, our analysis does conform to the "norm" of business in at least two aspects. First, a defining feature of our optimal mechanism is the shortlisting rule, which is also central in the two-stage auction practices. Second, we demonstrate that the optimal number shortlisted is endogenously determined (by the first-stage bids), which is also consistent with the fact that in real sales, the number of finalists is often not pre-determined.[31]

Our analysis offers a theoretical benchmark for evaluating various two-stage auctions currently used in the real world. The information structure modeled in this research has recently received attention not only from theorists but also from econometricians and empiricists. For example, Marmer, Shneyerov, and Xu (2013) and Gentry and Li (2014) have successfully proposed nonparametric specification tests on a so-called affiliated-signal (AS) model with entry, and Roberts and Sweeting (2013) estimate a parametric variant of the AS model using data on California timber auctions. The affiliated-signal models can be regarded as a special case in the framework studied in our paper, and the optimal mechanism characterized in this paper may potentially serve as a calibration benchmark for counter-factual simulations for related empirical works to come.

[31]For example, in the ongoing sale of PGW (Philadelphia Gas Works), a recent application of two-stage auctions, a "smaller number" of firms were invited to submit final bids after the first round – although this number was neither pre-announced nor disclosed (CBS Phily, November 19, 2013, "Sell-off of Philadelphia's Natural Gas Utility Goes To Binding Bidding," by Mike Dunn).

6 Appendix A: Proofs

Proof of Lemma 3: Let g_i denote any subset that includes i. By (5) and Lemma 2, we have

$$
\begin{aligned}
& \pi_i(\alpha_i,\hat{\alpha}_i) \\
= \; & \pi_i(\hat{\alpha}_i,\hat{\alpha}_i)+E_{\alpha_{-i}}\left\{\sum_{g_i} A^{g_i}(\hat{\alpha}_i,\alpha_{-i})[\tilde{\pi}_i^{g_i}(\alpha_i,\hat{\alpha}_i;\alpha_{-i})-\tilde{\pi}_i^{g_i}(\hat{\alpha}_i,\hat{\alpha}_i;\alpha_{-i})]\right\} \\
= \; & \pi_i(\hat{\alpha}_i,\hat{\alpha}_i)+E_{\alpha_{-i}}\left\{\sum_{g_i} A^{g_i}(\hat{\alpha}_i,\alpha_{-i})\int\int_{\hat{\alpha}_i}^{\alpha_i} u_1(y,s_i)P_i^{g_i}(\hat{\alpha}_i,\alpha_{-i},\sigma_i(y,\hat{\alpha}_i,s_i))dydG_i(s_i)\right\}.
\end{aligned}
$$

Thus for $\hat{\alpha}_i<\alpha_i$, $\pi_i(\alpha_i,\hat{\alpha}_i)\le\pi_i(\alpha_i,\alpha_i)$ implies that

$$
\pi_i(\alpha_i,\alpha_i)-\pi_i(\hat{\alpha}_i,\hat{\alpha}_i)\ge E_{\alpha_{-i}}\left\{\sum_{g_i} A^{g_i}(\hat{\alpha}_i,\alpha_{-i})\int\int_{\hat{\alpha}_i}^{\alpha_i} u_1(y,s_i)P_i^{g_i}(\hat{\alpha}_i,\alpha_{-i},\sigma_i(y,\hat{\alpha}_i,s_i))dydG_i(s_i)\right\}.
$$

Similarly,

$$
\begin{aligned}
& \pi_i(\hat{\alpha}_i,\alpha_i) \\
= \; & \pi_i(\alpha_i,\alpha_i)+E_{\alpha_{-i}}\left\{\sum_{g_i} A^{g_i}(\alpha_i,\alpha_{-i})[\tilde{\pi}_i^{g_i}(\hat{\alpha}_i,\alpha_i;\alpha_{-i})-\tilde{\pi}_i^{g_i}(\alpha_i,\alpha_i;\alpha_{-i})]\right\} \\
= \; & \pi_i(\alpha_i,\alpha_i)-E_{\alpha_{-i}}\left\{\sum_{g_i} A^{g_i}(\alpha_i,\alpha_{-i})\int\int_{\hat{\alpha}_i}^{\alpha_i} u_1(y,s_i)P_i^{g_i}(\alpha_i,\alpha_{-i},\sigma_i(y,\alpha_i,s_i))dydG_i(s_i)\right\}.
\end{aligned}
$$

Thus for $\hat{\alpha}_i<\alpha_i$, $\pi_i(\hat{\alpha}_i,\alpha_i)\le\pi_i(\hat{\alpha}_i,\hat{\alpha}_i)$ implies that

$$
\pi_i(\alpha_i,\alpha_i)-\pi_i(\hat{\alpha}_i,\hat{\alpha}_i)\le E_{\alpha_{-i}}\left\{\sum_{g_i} A^{g_i}(\alpha_i,\alpha_{-i})\int\int_{\hat{\alpha}_i}^{\alpha_i} u_1(y,s_i)P_i^{g_i}(\alpha_i,\alpha_{-i},\sigma_i(y,\alpha_i,s_i))dydG_i(s_i)\right\}.
$$

So

$$
\begin{aligned}
& E_{\alpha_{-i}}\left\{\sum_{g_i} A^{g_i}(\hat{\alpha}_i,\alpha_{-i})\int \frac{\int_{\hat{\alpha}_i}^{\alpha_i} u_1(y,s_i)P_i^{g_i}(\hat{\alpha}_i,\alpha_{-i},\sigma_i(y,\hat{\alpha}_i,s_i))dy}{\alpha_i-\hat{\alpha}_i}dG_i(s_i)\right\} \\
\le \; & \frac{\pi_i(\alpha_i,\alpha_i)-\pi_i(\hat{\alpha}_i,\hat{\alpha}_i)}{\alpha_i-\hat{\alpha}_i}\le E_{\alpha_{-i}}\left\{\sum_{g_i} A^{g_i}(\alpha_i,\alpha_{-i})\int \frac{\int_{\hat{\alpha}_i}^{\alpha_i} u_1(y,s_i)P_i^{g_i}(\alpha_i,\alpha_{-i},\sigma_i(y,\alpha_i,s_i))dy}{\alpha_i-\hat{\alpha}_i}dG_i(s_i)\right\}.
\end{aligned}
$$

By Fubini's Theorem, we have

$$
\begin{aligned}
& E_{\alpha_{-i}}\left\{\sum_{g_i} A^{g_i}(\hat{\alpha}_i,\alpha_{-i})\int \frac{\int_{\hat{\alpha}_i}^{\alpha_i} u_1(y,s_i)P_i^{g_i}(\hat{\alpha}_i,\alpha_{-i},\sigma_i(y,\hat{\alpha}_i,s_i))dy}{\alpha_i-\hat{\alpha}_i}dG_i(s_i)\right\} \\
= \; & \sum_{g_i}\int \frac{\int_{\hat{\alpha}_i}^{\alpha_i} u_1(y,s_i)E_{\alpha_{-i}}\left[A^{g_i}(\hat{\alpha}_i,\alpha_{-i})P_i^{g_i}(\hat{\alpha}_i,\alpha_{-i},\sigma_i(y,\hat{\alpha}_i,s_i))\right]dy}{\alpha_i-\hat{\alpha}_i}dG_i(s_i).
\end{aligned}
$$

Since $A^{g_i}, P_i^{g_i} \leq 1$, and u is concave in α_i, we have

$$\frac{\int_{\hat{\alpha}_i}^{\alpha_i} u_1(y,s_i) E_{\alpha_{-i}}\left[A^{g_i}(\hat{\alpha}_i,\alpha_{-i}) P_i^{g_i}(\hat{\alpha}_i,\alpha_{-i},\sigma_i(y,\hat{\alpha}_i,s_i))\right] dy}{\alpha_i - \hat{\alpha}_i} \leq \frac{\int_{\hat{\alpha}_i}^{\alpha_i} u_1(y,s_i) dy}{\alpha_i - \hat{\alpha}_i} \leq u_1(\hat{\alpha}_i, s_i).$$

By assumption $u_1(\hat{\alpha}_i, s_i)$ has a finite expectation with respect to s_i. Hence, by the Lebesgue convergence theorem,

$$\begin{aligned}
& \lim_{\hat{\alpha}_i \to \alpha_i} E_{\alpha_{-i}}\left\{\sum_{g_i} A^{g_i}(\hat{\alpha}_i,\alpha_{-i}) \int \frac{\int_{\hat{\alpha}_i}^{\alpha_i} u_1(y,s_i) P_i^{g_i}(\hat{\alpha}_i,\alpha_{-i},\sigma_i(y,\hat{\alpha}_i,s_i)) dy}{\alpha_i - \hat{\alpha}_i} dG_i(s_i)\right\} \\
= & \sum_{g_i} \int \lim_{\hat{\alpha}_i \to \alpha_i} \frac{\int_{\hat{\alpha}_i}^{\alpha_i} u_1(y,s_i) E_{\alpha_{-i}}\left[A^{g_i}(\hat{\alpha}_i,\alpha_{-i}) P_i^{g_i}(\hat{\alpha}_i,\alpha_{-i},\sigma_i(y,\hat{\alpha}_i,s_i))\right] dy}{\alpha_i - \hat{\alpha}_i} dG_i(s_i) \\
= & \sum_{g_i} \int \lim_{\hat{\alpha}_i \to \alpha_i} \left\{u_1(\hat{\alpha}_i,s_i) E_{\alpha_{-i}}\left[A^{g_i}(\hat{\alpha}_i,\alpha_{-i}) P_i^{g_i}(\hat{\alpha}_i,\alpha_{-i},s_i)\right]\right\} dG_i(s_i) \\
= & \sum_{g_i} \int \left\{u_1(\alpha_i,s_i) E_{\alpha_{-i}}\left[A^{g_i}(\alpha_i,\alpha_{-i}) P_i^{g_i}(\alpha_i,\alpha_{-i},s_i)\right]\right\} dG_i(s_i) \\
= & E_{\alpha_{-i}}\left\{\sum_{g_i} A^{g_i}(\alpha_i,\alpha_{-i}) \int u_1(\alpha_i,s_i) P_i^{g_i}(\alpha_i,\alpha_{-i},s_i) dG_i(s_i)\right\}.
\end{aligned}$$

The third equality above is due to the assumption that $E_{\alpha_{-i}}\left[A^{g_i}(\hat{\alpha}_i,\alpha_{-i}) P_i^{g_i}(\hat{\alpha}_i,\alpha_{-i},s_i)\right]$ is continuous in $\hat{\alpha}_i$ (which is guaranteed as long as both A^{g_i} and $P_i^{g_i}$ are continuous a.e. in $[\underline{\alpha},\overline{\alpha}]^{|g_i|}$).

Analogously, we can show that

$$\begin{aligned}
& \lim_{\hat{\alpha}_i \to \alpha_i} E_{\alpha_{-i}}\left\{\sum_{g_i} A^{g_i}(\alpha_i,\alpha_{-i}) \int \frac{\int_{\hat{\alpha}_i}^{\alpha_i} u_1(y,s_i) P_i^{g_i}(\alpha_i,\alpha_{-i},\sigma_i(y,\alpha_i,s_i)) dy}{\alpha_i - \hat{\alpha}_i} dG_i(s_i)\right\} \\
= & E_{\alpha_{-i}}\left\{\sum_{g_i} A^{g_i}(\alpha_i,\alpha_{-i}) \int u_1(\alpha_i,s_i) P_i^{g_i}(\alpha_i,\alpha_{-i},s_i) dG_i(s_i)\right\}.
\end{aligned}$$

Thus the left derivative of $\pi_i(\alpha_i,\alpha_i)$ is given by

$$\frac{d\pi_i^-(\alpha_i,\alpha_i)}{d\alpha_i} = E_{\alpha_{-i}}\left\{\sum_{g_i} A^{g_i}(\alpha_i,\alpha_{-i}) \int u_1(\alpha_i,s_i) P_i^{g_i}(\alpha_i,\alpha_{-i},s_i) dG_i(s_i)]\right\}.$$

Working with the case $\hat{\alpha}_i > \alpha_i$, we can obtain the right derivative of $\pi_i(\alpha_i,\alpha_i)$, which is given by

$$\frac{d\pi_i^+(\alpha_i,\alpha_i)}{d\alpha_i} = E_{\alpha_{-i}}\left\{\sum_{g_i} A^{g_i}(\alpha_i,\alpha_{-i}) \int u_1(\alpha_i,s_i) P_i^{g_i}(\alpha_i,\alpha_{-i},s_i) dG_i(s_i)\right\}.$$

Therefore, we conclude that $\pi_i(\alpha_i) = \pi_i(\alpha_i,\alpha_i)$ is differentiable everywhere, and

$$\pi_i'(\alpha_i) = E_{\alpha_{-i}}\left\{\sum_{g_i} A^{g_i}(\alpha_i,\alpha_{-i}) \int u_1(\alpha_i,s_i) P_i^{g_i}(\alpha_i,\alpha_{-i},s_i) dG_i(s_i)\right\}$$

$$= \int u_1(\alpha_i, s_i) \cdot \sum_{g_i} \left[E_{\alpha_{-i}} A^{g_i}(\alpha_i, \alpha_{-i}) P_i^{g_i}(\alpha_i, \alpha_{-i}, s_i) \right] dG_i(s_i)$$

Since $\pi_i'(\alpha_i)$ is bounded over $[\underline{\alpha}, \overline{\alpha}]$, π_i satisfies a Lipschitz condition and hence it can be recovered from its derivative, which gives rise to (6).

7 Appendix B: An Analysis Allowing Sequential Shortlisting

We consider two potential bidders who are endowed with types (α_1, α_2). If bidder i incurs information acquisition cost c, her *ex post* value v_i will be randomly drawn from a distribution indexed by α_i. Again we follow the orthogonalization procedure introduced by Esö and Szentes (2007) to write $v_i = u(\alpha_i, s_i)$.

Unlike the single-round shortlisting modeled in the main text, we now consider a general mechanism allowing for sequential shortlisting.

Stage I: the auctioneer shortlists a subset g of bidders, where $g \in \{\{1\},\{2\},\{1,2\}\}$ with probability $A_g^I(\alpha_1, \alpha_2)$, and transfers are defined by $x_i^I(\alpha_1, \alpha_2), i = 1,2$. Note that $A_g^I(\alpha_1, \alpha_2) \geq 0$ and $\sum_g A_g^I(\alpha_1, \alpha_2) \leq 1$.

Stage II: If bidder i is shortlisted to discover her value v_i by incurring the cost c, she announces the percentile of her value s_i. Contingent on $(\alpha_1, \alpha_2, s_i)$, bidder $-i$ is also shortlisted with probability $A_{-i}^{II}(\alpha_1, \alpha_2, s_i)$, and the transfer is denoted as $x_k^{II}(\alpha_1, \alpha_2, s_i), k = 1,2$. If bidder $-i$ is shortlisted, then she incurs the cost c to discover s_{-i}, and is required to announce her s_{-i}.

Stage III: If no one is shortlisted thus far, the game ends. if $\{i\}$ is shortlisted thus far, the winning probability is defined as $p_i^{III}(\alpha_1, \alpha_2, s_i)$, and the transfer is defined by $x_i^{III}(\alpha_1, \alpha_2, s_i)$.

If bidder 1 is first shortlisted and bidder 2 is subsequently shortlisted, then the winning probabilities are defined as $p_{i,(1,2)}^{III}(\alpha_1, \alpha_2, s_1, s_2)$, $i = 1,2$, and the transfers are given by $x_{i,(1,2)}^{III}(\alpha_1, \alpha_2, s_1, s_2)$, $i = 1,2$, where $(1,2)$ indicates the order of bidders being shortlisted. $p_{i,(2,1)}^{III}(\alpha_1, \alpha_2, s_1, s_2)$ and $x_{i,(2,1)}^{III}(\alpha_1, \alpha_2, s_1, s_2)$, $i = 1,2$ are similarly defined.

If both bidders are shortlisted at the first stage, then the winning probabilities are defined as $p_{i,\{1,2\}}^{III}(\alpha_1, \alpha_2, s_1, s_2), i = 1,2$, and the transfers are given by $x_{i,\{1,2\}}^{III}(\alpha_1, \alpha_2, s_1, s_2), i = 1,2$, where $\{1,2\}$ denotes the set of bidders being shortlisted.

We first establish an upper bound on expected revenue by assuming that bidders' second-round signals, the s_i's, are observed by the auctioneer, thus the mechanism can be made contingent on the s_i's directly. As a result, the IC conditions after the first stage can be ignored. Consider bidder i with type α_i, if she reports α_i' while the other bidder is reporting truthfully, her expected profit is given by

$$\begin{aligned}
&\pi_i(\alpha_i, \hat{\alpha}_i) \\
= \; & E_{\alpha_j, s_i, s_j}\{-x_i(\hat{\alpha}_i, \alpha_j) + A_{\{i,j\}}^I(\hat{\alpha}_i, \alpha_j)[u(\alpha_i, s_i) p_{i,\{i,j\}}^{III}(\hat{\alpha}_i, \alpha_j, s_i, s_j) - x_{i,\{i,j\}}^{III}(\hat{\alpha}_i, \alpha_j, s_i, s_j) - c] \\
& + A_{\{i\}}^I(\hat{\alpha}_i, \alpha_j)[A_j^{II}(\hat{\alpha}_i, \alpha_j, s_i)(u(\alpha_i, s_i) p_{i,(i,j)}^{III}(\hat{\alpha}_i, \alpha_j, s_i, s_j) - x_{i,(i,j)}^{III}(\hat{\alpha}_i, \alpha_j, s_i, s_j)) \\
& \qquad + (1 - A_j^{II}(\hat{\alpha}_i, \alpha_j, s_i))(u(\alpha_i, s_i) p_i^{III}(\hat{\alpha}_i, \alpha_j, s_i) - x_i^{III}(\hat{\alpha}_i, \alpha_j, s_i)) - x_i^{II}(\hat{\alpha}_i, \alpha_j, s_i) - c]
\end{aligned}$$

$$+A^I_{\{j\}}(\hat{\alpha}_i,\alpha_j)[A^{II}_i(\hat{\alpha}_i,\alpha_j,s_j)(u(\alpha_i,s_i)p^{III}_{i,(j,i)}(\hat{\alpha}_i,\alpha_j,s_i,s_j)-x^{III}_{i,(j,i)}(\hat{\alpha}_i,\alpha_j,s_i,s_j)-c)-x^{II}_i(\hat{\alpha}_i,\alpha_j,s_j)]\}.$$

Applying the envelope theorem, the IC condition $\pi_i(\alpha_i,\alpha_i)\geq\pi_i(\alpha_i,\hat{\alpha}_i)$ leads to the following necessary condition:

$$\begin{aligned}
&\frac{d\pi_i(\alpha_i,\alpha_i)}{d\alpha_i}\\
=\ &\frac{\partial\pi_i(\alpha_i,\hat{\alpha}_i)}{\partial\alpha_i}|_{\hat{\alpha}_i=\alpha_i}\\
=\ &E_{\alpha_j,s_i,s_j}\{A^I_{\{i,j\}}(\alpha_i,\alpha_j)u_1(\alpha_i,s_i)p^{III}_{i,\{i,j\}}(\alpha_i,\alpha_j,s_i,s_j)+A^I_{\{i\}}(\alpha_i,\alpha_j)[A^{II}_j(\alpha_i,\alpha_j,s_i)u_1(\alpha_i,s_i)p^{III}_{i,(i,j)}(\alpha_i,\alpha_j,s_i,s_j)\\
&+(1-A^{II}_j(\alpha_i,\alpha_j,s_i))u_1(\alpha_i,s_i)p^{III}_i(\alpha_i,\alpha_j,s_i)]+A^I_{\{j\}}(\alpha_i,\alpha_j)A^{II}_i(\alpha_i,\alpha_j,s_j)u_1(\alpha_i,s_i)p^{III}_{i,(j,i)}(\alpha_i,\alpha_j,s_i,s_j)\}
\end{aligned}\tag{16}$$

Therefore, we have

$$\pi_i(\alpha_i,\alpha_i)=\pi_i(\underline{\alpha},\underline{\alpha})+\int_{\underline{\alpha}}^{\alpha_i}\frac{d\pi_i(y,y)}{dy}dy.$$

$$\begin{aligned}
\sum_{i=1}^{2}E\pi_i(\alpha_i,\alpha_i) &= \sum_{i=1}^{2}\pi_i(\underline{\alpha},\underline{\alpha})+\sum_{i=1}^{2}\int_{\underline{\alpha}}^{\overline{\alpha}}\int_{\underline{\alpha}}^{\alpha_i}\frac{d\pi_i(y,y)}{dy}dy\,dF(\alpha_i)\\
&= \sum_{i=1}^{2}\pi_i(\underline{\alpha},\underline{\alpha})+\sum_{i=1}^{2}E_{\alpha_i}\left[\frac{1-F(\alpha_i)}{f(\alpha_i)}\frac{d\pi_i(\alpha_i,\alpha_i)}{d\alpha_i}\right]
\end{aligned}$$

The expected total surplus from the mechanism is

$$\begin{aligned}
TS\ =\ &E_{\alpha,\mathbf{s}}\{A^I_{\{i,j\}}(\alpha_i,\alpha_j)[u(\alpha_i,s_i)p^{III}_{i,\{i,j\}}(\alpha_i,\alpha_j,s_i,s_j)+u(\alpha_j,s_j)p^{III}_{j,\{i,j\}}(\alpha_i,\alpha_j,s_i,s_j)-2c]\\
&+A^I_{\{i\}}(\alpha_i,\alpha_j)A^{II}_j(\alpha_i,\alpha_j,s_i)[u(\alpha_i,s_i)p^{III}_{i,(i,j)}(\alpha_i,\alpha_j,s_i,s_j)+u(\alpha_j,s_j)p^{III}_{j,(i,j)}(\alpha_i,\alpha_j,s_i,s_j)-2c]\\
&+A^I_{\{i\}}(\alpha_i,\alpha_j)(1-A^{II}_j(\alpha_i,\alpha_j,s_i))[u(\alpha_i,s_i)p^{III}_i(\alpha_i,\alpha_j,s_i)-c]\\
&+A^I_{\{j\}}(\alpha_i,\alpha_j)A^{II}_i(\alpha_i,\alpha_j,s_j)[u(\alpha_j,s_j)p^{III}_{j,(j,i)}(\alpha_i,\alpha_j,s_i,s_j)+u(\alpha_i,s_i)p^{III}_{i,(j,i)}(\alpha_i,\alpha_j,s_i,s_j)-2c]\\
&+A^I_{\{j\}}(\alpha_i,\alpha_j)(1-A^{II}_i(\alpha_i,\alpha_j,s_j))[u(\alpha_j,s_j)p^{III}_j(\alpha_i,\alpha_j,s_j)-c]\}.
\end{aligned}$$

Recall that

$$w(\alpha_i,s_i)=u(\alpha_i,s_i)-\frac{1-F(\alpha_i)}{f(\alpha_i)}u_1(\alpha_i,s_i).\tag{17}$$

$$\begin{aligned}
ER\ =\ &TS-\sum_{i=1}^{2}E\pi_i(\alpha_i,\alpha_i)\\
=\ &E_{\alpha,\mathbf{s}}\{A^I_{\{i,j\}}(\alpha_i,\alpha_j)[w(\alpha_i,s_i)p^{III}_{i,\{i,j\}}(\alpha_i,\alpha_j,s_i,s_j)+w(\alpha_j,s_j)p^{III}_{j,\{i,j\}}(\alpha_i,\alpha_j,s_i,s_j)-2c]\\
&+A^I_{\{i\}}(\alpha_i,\alpha_j)A^{II}_j(\alpha_i,\alpha_j,s_i)[w(\alpha_i,s_i)p^{III}_{i,(i,j)}(\alpha_i,\alpha_j,s_i,s_j)+w(\alpha_j,s_j)p^{III}_{j,(i,j)}(\alpha_i,\alpha_j,s_i,s_j)-2c]\\
&+A^I_{\{i\}}(\alpha_i,\alpha_j)(1-A^{II}_j(\alpha_i,\alpha_j,s_i))[w(\alpha_i,s_i)p^{III}_i(\alpha_i,\alpha_j,s_i)-c]\\
&+A^I_{\{j\}}(\alpha_i,\alpha_j)A^{II}_i(\alpha_i,\alpha_j,s_j)[w(\alpha_j,s_j)p^{III}_{j,(j,i)}(\alpha_i,\alpha_j,s_i,s_j)+w(\alpha_i,s_i)p^{III}_{i,(j,i)}(\alpha_i,\alpha_j,s_i,s_j)-2c]
\end{aligned}$$

$$+A^{I}_{\{j\}}(\alpha_i,\alpha_j)(1-A^{II}_{i}(\alpha_i,\alpha_j,s_j))[w(\alpha_j,s_j)p^{III}_{j}(\alpha_i,\alpha_j,s_j)-c]\}-\sum_{i=1}^{2}\pi_i(\underline{\alpha},\underline{\alpha}). \tag{18}$$

Based on (18), we can first pin down the optimal asset allocation rule when both bidders are shortlisted (simultaneously or sequentially). Terms

$$w(\alpha_i,s_i)p^{III}_{i,\{i,j\}}(\alpha_i,\alpha_j,s_i,s_j)+w(\alpha_j,s_j)p^{III}_{j,\{i,j\}}(\alpha_i,\alpha_j,s_i,s_j)-2c \tag{19}$$

and

$$w(\alpha_i,s_i)p^{III}_{i,(i,j)}(\alpha_i,\alpha_j,s_i,s_j)+w(\alpha_j,s_j)p^{III}_{j,(i,j)}(\alpha_i,\alpha_j,s_i,s_j)-2c \tag{20}$$

are maximized if we define

$$\begin{aligned} & p^{III*}_{i,\{i,j\}}(\alpha_i,\alpha_j,s_i,s_j)=p^{III*}_{i,(i,j)}(\alpha_i,\alpha_j,s_i,s_j)=p^{III*}_{i,(j,i)}(\alpha_i,\alpha_j,s_i,s_j) \\ = & \begin{cases} 1 & \text{if } i=\arg\max_{j\in\{1,2\}}\{w(\alpha_j,s_j)\} \text{ and } w(\alpha_i,s_i)\geq 0 \\ 0 & \text{otherwise} \end{cases} \quad \forall i\in\{1,2\}. \end{aligned}$$

Second, we pin down the optimal asset allocation rule when only one bidder is shortlisted. The term

$$w(\alpha_i,s_i)p^{III}_{i}(\alpha_i,\alpha_j,s_i)-c$$

is maximized if we define

$$p^{III*}_{i}(\alpha_i,\alpha_j,s_i)=\begin{cases} 1 & \text{if } w(\alpha_i,s_i)\geq 0 \\ 0 & \text{otherwise} \end{cases} \quad \forall i\in\{1,2\}. \tag{21}$$

Define

$$w^{+}_{i}(\alpha_i,s_i)=\begin{cases} w(\alpha_i,s_i) & \text{if } w(\alpha_i,s_i)\geq 0 \text{ or equivalently } s_i\geq s(\alpha_i) \\ 0 & \text{otherwise} \end{cases} \quad \forall i.$$

Proposition 1. The object is allocated to the shortlisted bidder with the highest positive virtual value $w^{+}(\alpha_i,s_i)$.

Now we are ready to pin down the optimal shortlisting rule for the relaxed problem. Recall that given α_i, $s(\alpha_i)$ is defined such that $w(\alpha_i,s(\alpha_i))=0$.

Lemma 4. There is no loss of generality in choosing $A^{I*}_{\{i,j\}}(\alpha_i,\alpha_j)=0$. In other words, there is no need to shortlist both bidders at stage I.

Proof. Shortlisting two bidders in a single round can be duplicated by shortlisting bidder i first, and then admitting the other regardless of the signal s_i possessed by the first bidder admitted. □

Note terms (19) and (20) can both be written as

$$\max\{w^+(\alpha_i,s_i),w^+(\alpha_j,s_j)\}-2c.$$

Lemma 5. Given that a bidder i is shortlisted in stage I and announces s_i, then the other bidder j is admitted in stage II if and only if

$$E_{s_j}\max\{w^+(\alpha_j,s_j)-w^+(\alpha_i,s_i),0\}-c>0.$$

Proof. Suppose bidder i is shortlisted in stage I. We have $A^I_{\{i\}}(\alpha_i,\alpha_j)=1$ and $A^I_{\{j\}}(\alpha_i,\alpha_j)=0$. Upon knowing s_i, to maximize expected revenue, the auctioneer should set $A^{II*}_j(\alpha_i,\alpha_j,s_i)=1$ if $E_{s_j}\max\{w^+(\alpha_j,s_j)-w^+(\alpha_i,s_i),0\}-c>0$ and $A^{II*}_j(\alpha_i,\alpha_j,s_i)=0$ otherwise. □

Lemma 6. For $\alpha_i>\alpha_j$, given that one bidder must be shortlisted in stage I, then bidder i must be shortlisted.

Proof. Suppose bidder i is shortlisted in stage I. We have $A^I_{\{i\}}(\alpha_i,\alpha_j)=1$ and $A^I_{\{j\}}(\alpha_i,\alpha_j)=0$. Upon knowing s_i, according to Lemma 5, we should set $A^{II*}_j(\alpha_i,\alpha_j,s_i)=1$ if $E_{s_j}\max\{w^+(\alpha_j,s_j)-w^+(\alpha_i,s_i),0\}-c>0$ and $A^{II*}_j(\alpha_i,\alpha_j,s_i)=0$ otherwise.

Suppose bidder j is shortlisted in stage I. We have $A^I_{\{j\}}(\alpha_i,\alpha_j)=1$ and $A^I_{\{i\}}(\alpha_i,\alpha_j)=0$. Upon knowing s_j, according to Lemma 5, we should set $A^{II*}_i(\alpha_i,\alpha_j,s_j)=1$ if $E_{s_i}\max\{w^+(\alpha_i,s_i)-w^+(\alpha_j,s_j),0\}-c>0$ and $A^{II*}_i(\alpha_i,\alpha_j,s_j)=0$ otherwise.

We next show that, given $\alpha_i>\alpha_j$, the expected revenue from shortlisting bidder i first is higher than that from shortlisting j first, i.e.,

$$\begin{aligned}
& E_{s_i,s_j}\{A^{II*}_j(\alpha_i,\alpha_j,s_i)[w(\alpha_i,s_i)p^{III*}_{i,(i,j)}(\alpha_i,\alpha_j,s_i,s_j)+w(\alpha_j,s_j)p^{III*}_{j,(i,j)}(\alpha_i,\alpha_j,s_i,s_j)-2c] \\
& +(1-A^{II*}_j(\alpha_i,\alpha_j,s_i))[w(\alpha_i,s_i)p^{III*}_i(\alpha_i,\alpha_j,s_i)-c]\} \\
\geq\ & E_{s_i,s_j}\{A^{II*}_i(\alpha_i,\alpha_j,s_j)[w(\alpha_j,s_j)p^{III*}_{j,(j,i)}(\alpha_i,\alpha_j,s_i,s_j)+w(\alpha_i,s_i)p^{III*}_{i,(j,i)}(\alpha_i,\alpha_j,s_i,s_j)-2c] \\
& +(1-A^{II*}_i(\alpha_i,\alpha_j,s_j))[w(\alpha_j,s_j)p^{III*}_j(\alpha_i,\alpha_j,s_j)-c]\}.
\end{aligned} \tag{22}$$

Define $\hat{s}_i(c;\alpha_i,\alpha_j)\in[\underline{s},\bar{s}]$ such that

$$E_{s_j}\max\{w^+(\alpha_j,s_j)-w^+(\alpha_i,s_i=\hat{s}_i),0\}-c=0. \tag{23}$$

So $\hat{s}_i(c;\alpha_i,\alpha_j)$ defines a cutoff for the realization of s_i: bidder j is shortlisted if and only if $s_i<\hat{s}_i(c;\alpha_i,\alpha_j)$.

If

$$E_{s_j}\max\{w^+(\alpha_j,s_j)-w^+(\alpha_i,s_i=\underline{s}),0\}-c<0,$$

we define $\hat{s}_i(c;\alpha_i,\alpha_j) = \underline{s}$, i.e., bidder j should never be shortlisted regardless of the value of s_i.

If

$$E_{s_j} \max\{w^+(\alpha_j,s_j) - w^+(\alpha_i,s_i = \bar{s}),0\} - c > 0,$$

we define $\hat{s}_i(c;\alpha_i,\alpha_j) = \bar{s}$, i.e., bidder j should always be shortlisted regardless of the value of s_i.

$\hat{s}_j(c;\alpha_i,\alpha_j)$ is analogously defined.

Given $\alpha_i > \alpha_j$, we can show that $\hat{s}_j(c;\alpha_i,\alpha_j) \geq \hat{s}_i(c;\alpha_i,\alpha_j)$. Without loss of generality, we focus on the case where both $\hat{s}_j(c;\alpha_i,\alpha_j)$ and $\hat{s}_i(c;\alpha_i,\alpha_j) \in (\underline{s},\bar{s})$.

$\forall$ $s_i \leq \hat{s}_i(c;\alpha_i,\alpha_j)$, define $\hat{s}_j(s_i;\alpha_i,\alpha_j)$ such that $w^+(\alpha_j,s_j = \hat{s}(s_i;\alpha_i,\alpha_j)) - w^+(\alpha_i,s_i) = 0$. So $\hat{s}_j(s_i;\alpha_i,\alpha_j)$ is the cutoff of s_j so that bidder j's contribution to the (positive) virtual value is positive iff $s_j \geq \hat{s}_j(s_i;\alpha_i,\alpha_j)$.

We are now ready to show (22).

First note that both LHS and RHS of (22) are decreasing in c. When $c = 0$, we have $LHS = RHS$. LHS equals $E_{s_i}[w(\alpha_i,s_i)p_i^{III*}(\alpha_i,\alpha_j,s_i)] - c$ = $E_{s_i}[w^+(\alpha_i,s_i)] - c$ when $c > c_i$, where c_i is defined by $\hat{s}_i(c_i;\alpha_i,\alpha_j) = \underline{s}$ (so bidder j would not be shortlisted if $c > c_i$). We define c_j analogously. Then $c_i < c_j$ as

$$c_i = E_{s_j} \max\{w^+(\alpha_j,s_j) - w^+(\alpha_i,\underline{s}),0\} < E_{s_i} \max\{w^+(\alpha_i,s_i) - w^+(\alpha_j,\underline{s}),0\} = c_j.$$

Clearly, LHS becomes $E_{s_i}[w^+(\alpha_i,s_i)] - c$ when $c \geq c_i$; and RHS becomes $E_{s_j}[w^+(\alpha_j,s_j)] - c$ when $c \geq c_j$.

To establish the desired inequality (22), it is sufficient to show that LHS decreases more slowly than RHS as c increases.

Note the LHS can be alternatively written as

$$LHS = [E_{s_i} w^+(\alpha_i,s_i) - c] + \int_{\underline{s}}^{\hat{s}_i(c;\alpha_i,\alpha_j)} [\int_{\hat{s}_j(s_i;\alpha_i,\alpha_j)}^{\bar{s}} (w^+(\alpha_j,s_j) - w^+(\alpha_i,s_i))G'(s_j)ds_j - c]G'(s_i)ds_i.$$

Similarly,

$$RHS = [E_{s_j} w^+(\alpha_j,s_j) - c] + \int_{\underline{s}}^{\hat{s}_j(c;\alpha_i,\alpha_j)} [\int_{\hat{s}_i(s_j;\alpha_i,\alpha_j)}^{\bar{s}} (w^+(\alpha_i,s_i) - w^+(\alpha_j,s_j))G'(s_i)ds_i - c]G'(s_j)ds_j.$$

By (23), we have

$$\int_{\hat{s}_j(\hat{s}_i(c;\alpha_i,\alpha_j);\alpha_i,\alpha_j)}^{\bar{s}} (w^+(\alpha_j,s_j) - w^+(\alpha_i,\hat{s}_i(c;\alpha_i,\alpha_j)))G'(s_j)ds_j - c = 0$$

Therefore,

$$\begin{aligned} & \frac{\partial LHS}{\partial c} \\ = \quad & -1 - \int_{\underline{s}}^{\hat{s}_i(c;\alpha_i,\alpha_j)} G'(s_i)ds_i \end{aligned}$$

$$
\begin{aligned}
&+[\int_{\hat{s}_j(\hat{s}_i(c;\alpha_i,\alpha_j);\alpha_i,\alpha_j)}^{\bar{s}}(w^+(\alpha_j,s_j)-w^+(\alpha_i,\hat{s}_i(c;\alpha_i,\alpha_j)))G'(s_j)ds_j-c]G'(\hat{s}_i(c;\alpha_i,\alpha_j))\frac{\partial\hat{s}_i(c;\alpha_i,\alpha_j)}{\partial c}\\
= \quad & -1-\int_{\underline{s}}^{\hat{s}_i(c;\alpha_i,\alpha_j)}G'(s_i)ds_i.
\end{aligned}
$$

Similarly,

$$
\frac{\partial RHS}{\partial c}=-1-\int_{\underline{s}}^{\hat{s}_j(c;\alpha_i,\alpha_j)}G'(s_j)ds_j.
$$

Thus

$$
\frac{\partial RHS}{\partial c}\leq\frac{\partial LHS}{\partial c},
$$

as $\hat{s}_i(c;\alpha_i,\alpha_j)\leq\hat{s}_j(c;\alpha_i,\alpha_j)$. (22) thus holds. □

Given Lemma 6, we have the following proposition:

Proposition 2. If $\alpha_i>\alpha_j$, then bidder j is not shortlisted in stage I, and bidder i is shortlisted in stage I iff $E_{s_i}[w^+(\alpha_i,s_i)]-c>0$. Bidder j is shortlisted in stage II iff bidder i is shortlisted in stage I and $s_i<\hat{s}_i(c;\alpha_i,\alpha_j)$ as defined in (23). If bidder i is not shortlisted in stage I, then no buyer is shortlisted in both stages.

We next show that the shortlisting and allocation rules derived above are incentive compatible. We continue to assume, without loss of generality, that at most one bidder is shortlisted in stage I.

The key observation is still that if bidder i deviates by announcing a first-stage signal $\alpha'_i\neq\alpha_i$, then when asked to announce the second-stage signal, she would correct the lie by announcing $s'_i=\sigma_i(\alpha_i,\alpha'_i,s_i)$ which together with α'_i reveals the true value $u_i(\alpha_i,s_i)$, i.e.

$$
u_i(\alpha_i,s_i)=u_i(\alpha'_i,s'_i).
$$

This is true either in the second or third stage when bidder i is asked to announce her type. The reason is that type (α'_i,s'_i) would reveal s'_i truthfully if she reveals α'_i truthfully. The argument is essentially the same as illustrated by Esö and Szentes (2007).

Given that (α_i,α_j) is truthfully announced at stage I, we first construct payment rules such that (s_i,s_j) is announced in later stages. In the third stage, regardless of the number of bidders shortlisted, a payment rule can be constructed as a second-price auction with bidder-specific reserves contingent on their own first-stage signals to implement the optimal allocation rule in the third stage in weakly dominant strategies.

For the second-stage payment rule $x_i^{II}(\alpha_i,\alpha_j,s_i)$, it can be constructed following the envelope condition. If announcing s'_i, the bidder i's expected payoff at stage II conditional on s_i,α_i,α_j is given by

$$
\begin{aligned}
&\pi_i^{II}(s'_i|s_i;\alpha_i,\alpha_j)\\
= \quad & E_{s_j}\{[A_j^{II}(\alpha_i,\alpha_j,s'_i)(u(\alpha_i,s_i)p_{i,(i,j)}^{III}(\alpha_i,\alpha_j,s'_i,s_j)-x_{i,(i,j)}^{III}(\alpha_i,\alpha_j,s'_i,s_j))
\end{aligned}
$$

$$+(1-A_j^{II}(\alpha_i,\alpha_j,s_i'))(u(\alpha_i,s_i)p_i^{III}(\alpha_i,\alpha_j,s_i')-x_i^{III}(\alpha_i,\alpha_j,s_i'))-x_i^{II}(\alpha_i,\alpha_j,s_i')-c]\}.$$

The mechanism will be IC as long as the single crossing and the monotonicity conditions hold. The monotonicity condition is about $A_j^{II}(\alpha_i,\alpha_j,s_i')p_{i,(i,j)}^{III}(\alpha_i,\alpha_j,s_i',s_j)+(1-A_j^{II}(\alpha_i,\alpha_j,s_i'))p_i^{III}(\alpha_i,\alpha_j,s_i')]$. Let

$$\begin{aligned}\zeta &= E_{s_j}[A_j^{II}(\alpha_i,\alpha_j,s_i')p_{i,(i,j)}^{III}(\alpha_i,\alpha_j,s_i',s_j)+(1-A_j^{II}(\alpha_i,\alpha_j,s_i'))p_i^{III}(\alpha_i,\alpha_j,s_i')]\\ &= E_{s_j}p_{i,(i,j)}^{III}(\alpha_i,\alpha_j,s_i',s_j)+[1-A_j^{II}(\alpha_i,\alpha_j,s_i')][p_i^{III}(\alpha_i,\alpha_j,s_i')-E_{s_j}p_{i,(i,j)}^{III}(\alpha_i,\alpha_j,s_i',s_j)].\end{aligned}$$

Define s_i^c such that $A_j^{II}(\alpha_i,\alpha_j,s_i')=1$ iff $s_i'\le s_i^c$. Thus $\zeta=E_{s_j}p_{i,(i,j)}^{III}(\alpha_i,\alpha_j,s_i',s_j)$ when $s_i'\le s_i^c$ and $\zeta=p_i^{III}(\alpha_i,\alpha_j,s_i')$ when $s_i'>s_i^c$. Clearly, both parts increase in s_i'. Moreover, $p_i^{III}(\alpha_i,\alpha_j,s_i')\ge E_{s_j}p_{i,(i,j)}^{III}(\alpha_i,\alpha_j,s_i',s_j)$, $\forall s_i'$. Thus ζ increases in s_i'.

To check the single crossing condition, first we have

$$\begin{aligned}&\frac{\partial\pi_i^{II}(s_i'|s_i;\alpha_i,\alpha_j)}{\partial s_i}\\ =\;& E_{s_j}\{\frac{\partial u(\alpha_i,s_i)}{\partial s_i}[A_j^{II}(\alpha_i,\alpha_j,s_i')p_{i,(i,j)}^{III}(\alpha_i,\alpha_j,s_i',s_j)+(1-A_j^{II}(\alpha_i,\alpha_j,s_i'))p_i^{III}(\alpha_i,\alpha_j,s_i')]\}\\ =\;& \frac{\partial u(\alpha_i,s_i)}{\partial s_i}\zeta.\end{aligned}$$

We thus have $\partial\pi_i^{II}(s_i'|s_i;\alpha_i,\alpha_j)/\partial s_i\partial s_i'>0$, as ζ increases in s_i' and $\partial u(\alpha_i,s_i)/\partial s_i>0$.

Recall that, if bidder i deviates by announcing a first-stage signal $\alpha_i'\neq\alpha_i$, then when asked to announce the second-stage signal, she would correct the lie by announcing $s_i'=\sigma_i(\alpha_i,\alpha_i',s_i)$ which together with α_i' reveals the true value $u_i(\alpha_i,s_i)$, i.e., $u(\alpha_i,s_i)=u(\alpha_i',s_i')$. Let $\sigma_i=\sigma_i(\alpha_i,\alpha_i',s_i)$.

$$\begin{aligned}&\pi_i(\alpha_i',\alpha_i)\\ =\;& E_{\alpha_j,s_i,s_j}\{-x_i(\alpha_i',\alpha_j)+A_{\{i\}}^I(\alpha_i',\alpha_j)[A_j^{II}(\alpha_i',\alpha_j,\sigma_i)(u(\alpha_i,s_i)p_{i,(i,j)}^{III}(\alpha_i',\alpha_j,\sigma_i,s_j)-x_{i,(i,j)}^{III}(\alpha_i',\alpha_j,\sigma_i,s_j))\\ &+(1-A_j^{II}(\alpha_i',\alpha_j,\sigma_i))(u(\alpha_i,s_i)p_i^{III}(\alpha_i',\alpha_j,\sigma_i)-x_i^{III}(\alpha_i',\alpha_j,\sigma_i))-x_i^{II}(\alpha_i',\alpha_j,\sigma_i)-c]\\ &+A_{\{j\}}^I(\alpha_i',\alpha_j)[A_i^{II}(\alpha_i',\alpha_j,s_j)(u(\alpha_i,s_i)p_{i,(j,i)}^{III}(\alpha_i',\alpha_j,\sigma_i,s_j)-x_{i,(j,i)}^{III}(\alpha_i',\alpha_j,\sigma_i,s_j)-c)-x_i^{II}(\alpha_i',\alpha_j,s_j)]\}.\end{aligned}$$

Define $\Delta=\pi_i(\alpha_i',\alpha_i)-\pi_i(\alpha_i,\alpha_i)=[\pi_i(\alpha_i',\alpha_i)-\pi_i(\alpha_i',\alpha_i')]+[\pi_i(\alpha_i',\alpha_i')-\pi_i(\alpha_i,\alpha_i)]$. We consider the two terms one by one.

Use π_i^{II} and π_i^{III} to denote player i's payoff at stages II and III respectively. Similar to Lemma 1 in the main text, we have

$$\begin{aligned}&\pi_i^{II}(s_i|s_i;\alpha_i,\alpha_j)\\ =\;& \pi_i^{II}(s_i'|s_i';\alpha_i,\alpha_j)+\int_{s_i'}^{s_i}u_2(\alpha_i,t)\{A_j^{II}(\alpha_i,\alpha_j,t)E_{s_j}p_{i,(i,j)}^{III}(\alpha_i,\alpha_j,t,s_j)+(1-A_j^{II}(\alpha_i,\alpha_j,t))p_i^{III}(\alpha_i,\alpha_j,t)\}dt.\end{aligned}$$

Define

$$\pi_i^{III}(s_i|s_i;s_j,\alpha_i,\alpha_j) = u(\alpha_i,s_i)p_{i,(j,i)}^{III}(\alpha_i,\alpha_j,s_i,s_j) - x_{i,(j,i)}^{III}(\alpha_i,\alpha_j,s_i,s_j) - c.$$

We have

$$\pi_i^{III}(s_i|s_i;s_j,\alpha_i,\alpha_j) = \pi_i^{III}(s_i'|s_i';s_j,\alpha_i,\alpha_j) + \int_{s_i'}^{s_i} u_2(\alpha_i,t)p_{i,(j,i)}^{III}(\alpha_i,\alpha_j,t,s_j)dt.$$

Let $\sigma_i = \sigma_i(\alpha_i,\alpha_i',s_i)$. Since $u(\alpha_i,s_i) = u(\alpha_i',\sigma_i(\alpha_i,\alpha_i',s_i))$, we have

$$\begin{aligned}
& \pi_i(\alpha_i',\alpha_i) \\
= \; & E_{\alpha_j,s_i,s_j}\{-x_i(\alpha_i',\alpha_j) + A_{\{i\}}^I(\alpha_i',\alpha_j)[A_j^{II}(\alpha_i',\alpha_j,\sigma_i)(u(\alpha_i',\sigma_i)p_{i,(i,j)}^{III}(\alpha_i',\alpha_j,\sigma_i,s_j) - x_{i,(i,j)}^{III}(\alpha_i',\alpha_j,\sigma_i,s_j)) \\
& +(1-A_j^{II}(\alpha_i',\alpha_j,\sigma_i))(u(\alpha_i',\sigma_i)p_i^{III}(\alpha_i',\alpha_j,\sigma_i) - x_i^{III}(\alpha_i',\alpha_j,\sigma_i)) - x_i^{II}(\alpha_i',\alpha_j,\sigma_i) - c] \\
& +A_{\{j\}}^I(\alpha_i',\alpha_j)[A_i^{II}(\alpha_i',\alpha_j,s_j)(u(\alpha_i',\sigma_i)p_{i,(j,i)}^{III}(\alpha_i',\alpha_j,\sigma_i,s_j) - x_{i,(j,i)}^{III}(\alpha_i',\alpha_j,\sigma_i,s_j) - c) - x_i^{II}(\alpha_i',\alpha_j,s_j)]\} \\
= \; & E_{\alpha_j}[-x_i(\alpha_i',\alpha_j)] + E_{\alpha_j,s_i}\{A_{\{i\}}^I(\alpha_i',\alpha_j)\pi_i^{II}(\sigma_i(\alpha_i,\alpha_i',s_i)|\sigma_i(\alpha_i,\alpha_i',s_i);\alpha_i',\alpha_j)\} \\
& +E_{\alpha_j,s_i,s_j}\{A_{\{j\}}^I(\alpha_i',\alpha_j)[A_i^{II}(\alpha_i',\alpha_j,s_j)\pi_i^{III}(\sigma_i(\alpha_i,\alpha_i',s_i)|\sigma_i(\alpha_i,\alpha_i',s_i);s_j,\alpha_i',\alpha_j) - x_i^{II}(\alpha_i',\alpha_j,s_j)]\}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
& \pi_i(\alpha_i',\alpha_i') \\
= \; & E_{\alpha_j}[-x_i(\alpha_i',\alpha_j)] + E_{\alpha_j,s_i}\{A_{\{i\}}^I(\alpha_i',\alpha_j)\pi_i^{II}(\sigma_i(\alpha_i',\alpha_i',s_i)|\sigma_i(\alpha_i',\alpha_i',s_i);\alpha_i',\alpha_j)\} \\
& +E_{\alpha_j,s_i,s_j}\{A_{\{j\}}^I(\alpha_i',\alpha_j)[A_i^{II}(\alpha_i',\alpha_j,s_j)\pi_i^{III}(\sigma_i(\alpha_i',\alpha_i',s_i)|\sigma_i(\alpha_i',\alpha_i',s_i);s_j,\alpha_i',\alpha_j) - x_i^{II}(\alpha_i',\alpha_j,s_j)]\} \\
= \; & E_{\alpha_j}[-x_i(\alpha_i',\alpha_j)] + E_{\alpha_j,s_i}\{A_{\{i\}}^I(\alpha_i',\alpha_j)\pi_i^{II}(s_i|s_i;\alpha_i',\alpha_j)\} \\
& +E_{\alpha_j,s_i,s_j}\{A_{\{j\}}^I(\alpha_i',\alpha_j)[A_i^{II}(\alpha_i',\alpha_j,s_j)\pi_i^{III}(s_i|s_i;s_j,\alpha_i',\alpha_j) - x_i^{II}(\alpha_i',\alpha_j,s_j)]\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \pi_i(\alpha_i',\alpha_i) \\
= \; & \pi_i(\alpha_i',\alpha_i') + E_{\alpha_j,s_i}\{A_{\{i\}}^I(\alpha_i',\alpha_j)[\pi_i^{II}(\sigma_i(\alpha_i,\alpha_i',s_i)|\sigma_i(\alpha_i,\alpha_i',s_i);\alpha_i',\alpha_j) - \pi_i^{II}(s_i|s_i;\alpha_i',\alpha_j)]\} \\
& +E_{\alpha_j,s_i,s_j}\{A_{\{j\}}^I(\alpha_i',\alpha_j)A_i^{II}(\alpha_i',\alpha_j,s_j)[\pi_i^{III}(\sigma_i(\alpha_i,\alpha_i',s_i)|\sigma_i(\alpha_i,\alpha_i',s_i);s_j,\alpha_i',\alpha_j) - \pi_i^{III}(s_i|s_i;s_j,\alpha_i',\alpha_j)]\} \\
= \; & \pi_i(\alpha_i',\alpha_i') + E_{\alpha_j,s_i}\{A_{\{i\}}^I(\alpha_i',\alpha_j)\int_{s_i}^{\sigma_i(\alpha_i,\alpha_i',s_i)} u_2(\alpha_i',t)[A_j^{II}(\alpha_i',\alpha_j,t)E_{s_j}p_{i,(i,j)}^{III}(\alpha_i',\alpha_j,t,s_j) \\
& +(1-A_j^{II}(\alpha_i',\alpha_j,t))p_i^{III}(\alpha_i',\alpha_j,t)]dt\} + E_{\alpha_j,s_i,s_j}A_{\{j\}}^I(\alpha_i',\alpha_j)A_i^{II}(\alpha_i',\alpha_j,s_j)\int_{s_i}^{\sigma_i(\alpha_i,\alpha_i',s_i)} u_2(\alpha_i',t)p_{i,(j,i)}^{III}(\alpha_i',\alpha_j,t,s_j)dt.
\end{aligned}$$

Let $t = \sigma_i(y,\alpha_i',s_i)$. We have

$$\begin{aligned}
& \pi_i(\alpha_i',\alpha_i) \\
= \; & \pi_i(\alpha_i',\alpha_i') + E_{\alpha_j,s_i}\{A_{\{i\}}^I(\alpha_i',\alpha_j)\int_{\alpha_i'}^{\alpha_i} u_2(\alpha_i',\sigma_i(y,\alpha_i',s_i))[A_j^{II}(\alpha_i',\alpha_j,\sigma_i(y,\alpha_i',s_i))E_{s_j}p_{i,(i,j)}^{III}(\alpha_i',\alpha_j,\sigma_i(y,\alpha_i',s_i),s_j)
\end{aligned}$$

$$+(1-A_j^{II}(\alpha_i',\alpha_j,\sigma_i(y,\alpha_i',s_i)))p_i^{III}(\alpha_i',\alpha_j,\sigma_i(y,\alpha_i',s_i))]\sigma_{i1}(y,\alpha_i',s_i)dy\} \tag{24}$$
$$+E_{\alpha_j,s_i,s_j}A^I_{\{j\}}(\alpha_i',\alpha_j)A_i^{II}(\alpha_i',\alpha_j,s_j)\int_{\alpha_i'}^{\alpha_i} u_2(\alpha_i',\sigma_i(y,\alpha_i',s_i))p^{III}_{i,(j,i)}(\alpha_i',\alpha_j,\sigma_i(y,\alpha_i',s_i),s_j)\sigma_{i1}(y,\alpha_i',s_i)dy.$$

Note that $u(\alpha_i,s_i)=u(\alpha_i',\sigma_i(\alpha_i,\alpha_i',s_i))$ implies

$$u_1(\alpha_i,s_i)=u_2(\alpha_i',\sigma_i(\alpha_i,\alpha_i',s_i))\sigma_{i1}(\alpha_i,\alpha_i',s_i).$$

Recall that $A^I_{\{i,j\}}(\alpha_i,\alpha_j)=0$ for the optimal shortlisting rule identified above. Applying the Myersonian procedure as in the proof of Lemma 3, incentive compatibility leads to

$$\begin{aligned}
&\frac{d\pi_i(\alpha_i,\alpha_i)}{d\alpha_i}\\
= \;& E_{\alpha_j,s_i}\{A^I_{\{i\}}(\alpha_i,\alpha_j)u_1(\alpha_i,s_i)[A_j^{II}(\alpha_i,\alpha_j,s_i)E_{s_j}p^{III}_{i,(i,j)}(\alpha_i,\alpha_j,s_i,s_j)+(1-A_j^{II}(\alpha_i,\alpha_j,s_i))p_i^{III}(\alpha_i,\alpha_j,s_i)]\}\\
&+E_{\alpha_j,s_i,s_j}\{A^I_{\{j\}}(\alpha_i,\alpha_j)A_i^{II}(\alpha_i,\alpha_j,s_j)u_1(\alpha_i,s_i)p^{III}_{i,(j,i)}(\alpha_i,\alpha_j,s_i,s_j)\}>0.
\end{aligned} \tag{25}$$

We are now ready to show $\Delta=\pi_i(\alpha_i',\alpha_i)-\pi_i(\alpha_i,\alpha_i)=[\pi_i(\alpha_i',\alpha_i)-\pi_i(\alpha_i',\alpha_i')]-[\pi_i(\alpha_i,\alpha_i)-\pi_i(\alpha_i',\alpha_i')]\le 0$. Without loss of generality, we assume $\alpha_i'<\alpha_i$.

By (25), we have

$$\begin{aligned}
&\pi_i(\alpha_i,\alpha_i)-\pi_i(\alpha_i',\alpha_i')\\
=\;&\int_{\alpha_i'}^{\alpha_i}\int_0^1 u_1(y,s_i)\{E_{\alpha_j}\{A^I_{\{i\}}(y,\alpha_j)[A_j^{II}(y,\alpha_j,s_i)E_{s_j}p^{III}_{i,(i,j)}(y,\alpha_j,s_i,s_j)+(1-A_j^{II}(y,\alpha_j,s_i))p_i^{III}(y,\alpha_j,s_i)]\}\\
&+E_{\alpha_j,s_j}\{A^I_{\{j\}}(y,\alpha_j)A_i^{II}(y,\alpha_j,s_j)p^{III}_{i,(j,i)}(y,\alpha_j,s_i,s_j)\}\}dG(s_i)dy.
\end{aligned}$$

By (24), we have

$$\begin{aligned}
&\pi_i(\alpha_i',\alpha_i)-\pi_i(\alpha_i',\alpha_i')\\
=\;&\int_{\alpha_i'}^{\alpha_i}\int_0^1 u_1(y,s_i)\{E_{\alpha_j}\{A^I_{\{i\}}(\alpha_i',\alpha_j)[A_j^{II}(\alpha_i',\alpha_j,\sigma_i(y,\alpha_i',s_i))E_{s_j}p^{III}_{i,(i,j)}(\alpha_i',\alpha_j,\sigma_i(y,\alpha_i',s_i),s_j)\\
&\qquad+(1-A_j^{II}(\alpha_i',\alpha_j,\sigma_i(y,\alpha_i',s_i)))p_i^{III}(\alpha_i',\alpha_j,\sigma_i(y,\alpha_i',s_i))]\}\\
&+E_{\alpha_j,s_j}\{A^I_{\{j\}}(\alpha_i',\alpha_j)A_i^{II}(\alpha_i',\alpha_j,s_j)p^{III}_{i,(j,i)}(\alpha_i',\alpha_j,\sigma_i(y,\alpha_i',s_i),s_j)\}\}dG(s_i)dy.
\end{aligned}$$

Note $p^{III}_{i,(i,j)}=p^{III}_{i,(j,i)}$ for the proposed mechanism. To show $\Delta\le 0$, it suffices to show that $\forall y\in[\alpha_i',\alpha_i],s_i,\alpha_j,s_j$,

$$\begin{aligned}
&[A^I_{\{i\}}(y,\alpha_j)A_j^{II}(y,\alpha_j,s_i)+A^I_{\{j\}}(y,\alpha_j)A_i^{II}(y,\alpha_j,s_j)]p^{III}_{i,(i,j)}(y,\alpha_j,s_i,s_j)+A^I_{\{i\}}(y,\alpha_j)(1-A_j^{II}(y,\alpha_j,s_i))p_i^{III}(y,\alpha_j,s_i)\\
\ge\;&[A^I_{\{i\}}(\alpha_i',\alpha_j)A_j^{II}(\alpha_i',\alpha_j,\sigma_i(y,\alpha_i',s_i))+A^I_{\{j\}}(\alpha_i',\alpha_j)A_i^{II}(\alpha_i',\alpha_j,s_j)]p^{III}_{i,(i,j)}(\alpha_i',\alpha_j,\sigma_i(y,\alpha_i',s_i),s_j)\\
&\qquad+A^I_{\{i\}}(\alpha_i',\alpha_j)(1-A_j^{II}(\alpha_i',\alpha_j,\sigma_i(y,\alpha_i',s_i)))p_i^{III}(\alpha_i',\alpha_j,\sigma_i(y,\alpha_i',s_i)).
\end{aligned} \tag{26}$$

Note that

$$
\begin{aligned}
p_i^{III}(y,\alpha_j,s_i) &\geq p_i^{III}(\alpha_i',\alpha_j,\sigma_i(y,\alpha_i',s_i)),\\
p_{i,(i,j)}^{III}(y,\alpha_j,s_i,s_j) &\geq p_{i,(i,j)}^{III}(\alpha_i',\alpha_j,\sigma_i(y,\alpha_i',s_i),s_j),\\
p_i^{III}(y,\alpha_j,s_i) &\geq p_{i,(i,j)}^{III}(y,\alpha_j,s_i,s_j),\\
p_i^{III}(\alpha_i',\alpha_j,\sigma_i(y,\alpha_i',s_i)) &\geq p_{i,(i,j)}^{III}(\alpha_i',\alpha_j,\sigma_i(y,\alpha_i',s_i),s_j).
\end{aligned}
$$

The first two inequalities above are due to the fact that the virtual value for bidder i is lower if she deviates in such a way ($\alpha_i' < \alpha_i$) in the first stage, and corrects the lie later (the same arguments as in Corollary 1 in Esö and Szentes and in our main analysis with simultaneous shortlisting). The last two inequalities above are due to the fact that bidder i wins with a lower chance if bidder j is shortlisted.

The LHS of (26) is bidder i's winning probability if she announces truthfully her first-stage signal y and second-stage signal s_i. The RHS of (26) is bidder i's winning probability if she under-reports by announcing α_i' at the first stage and then correcting the lie at the second stage by announcing $\sigma_i(y,\alpha_i',s_i)$.

Note that we have

$$
A_{\{i\}}^I(\alpha_i',\alpha_j)(1-A_j^{II}(\alpha_i',\alpha_j,\sigma_i(y,\alpha_i',s_i))) \leq A_{\{i\}}^I(y,\alpha_j)(1-A_j^{II}(y,\alpha_j,s_i)) \tag{27}
$$

since $A_{\{i\}}^I(\alpha_i',\alpha_j) \leq A_{\{i\}}^I(y,\alpha_j)$ and $A_j^{II}(\alpha_i',\alpha_j,\sigma_i(y,\alpha_i',s_i))) \geq A_j^{II}(y,\alpha_j,s_i)$.

We are now ready to prove the desired inequality $\Delta \leq 0$. First note that the following result is sufficient but not necessary for the desired inequality:

$$
\begin{aligned}
& [A_{\{i\}}^I(y,\alpha_j)A_j^{II}(y,\alpha_j,s_i)+A_{\{j\}}^I(y,\alpha_j)A_i^{II}(y,\alpha_j,s_j)] \\
\geq\ & [A_{\{i\}}^I(\alpha_i',\alpha_j)A_j^{II}(\alpha_i',\alpha_j,\sigma_i(y,\alpha_i',s_i))+A_{\{j\}}^I(\alpha_i',\alpha_j)A_i^{II}(\alpha_i',\alpha_j,s_j)].
\end{aligned} \tag{28}
$$

We will consider different cases in order:

Case 1 $y < \alpha_j$. Then we have $\alpha_i' < y < \alpha_j$, which means $A_{\{i\}}^I(y,\alpha_j) = A_{\{i\}}^I(\alpha_i',\alpha_j) = 0$. We must have $A_{\{j\}}^I(y,\alpha_j) = A_{\{j\}}^I(\alpha_i',\alpha_j)$ and $A_i^{II}(y,\alpha_j,s_j) \geq A_i^{II}(\alpha_i',\alpha_j,s_j)$. This means that (28) holds. By (27), we have the desired inequality $\Delta \leq 0$.

Case 2 $\alpha_i' < \alpha_j < y$. We have $A_{\{i\}}^I(\alpha_i',\alpha_j) = 0$.

If $A_{\{i\}}^I(y,\alpha_j) = 0$, then $A_{\{j\}}^I(y,\alpha_j) = A_{\{j\}}^I(\alpha_i',\alpha_j) = 0$. We have (28), hence $\Delta \leq 0$.

If $A_{\{i\}}^I(y,\alpha_j) = 1$, then $A_{\{j\}}^I(y,\alpha_j) = 0$. If $A_{\{j\}}^I(\alpha_i',\alpha_j) = 0$, we have (28), done.

If $A_{\{j\}}^I(\alpha_i',\alpha_j) = 1$, then we need to compare $A_j^{II}(y,\alpha_j,s_i)$ and $A_i^{II}(\alpha_i',\alpha_j,s_j)$: $A_i^{II}(\alpha_i',\alpha_j,s_j) = 0$, or $A_i^{II}(\alpha_i',\alpha_j,s_j) = 1$ and $A_j^{II}(y,\alpha_j,s_i) = 1$: we have (28), done; $A_i^{II}(\alpha_i',\alpha_j,s_j) = 1$ but $A_j^{II}(y,\alpha_j,s_i) = 0$: since $A_{\{i\}}^I(y,\alpha_j) = 1$, we have LHS of (26) $= p_i^{III}(y,\alpha_j,s_i)$; and RHS of (26) $= p_{i,(i,j)}^{III}(\alpha_i',\alpha_j,\sigma_i(y,\alpha_i',s_i),s_j)$. Also done since $p_i^{III}(y,\alpha_j,s_i) \geq p_{i,(i,j)}^{III}(\alpha_i',\alpha_j,\sigma_i(y,\alpha_i',s_i),s_j)$.

Case 3 $\alpha_j < \alpha'_i < y$. This means that $A^I_{\{j\}}(y,\alpha_j) = A^I_{\{j\}}(\alpha'_i,\alpha_j) = 0$. Thus RHS of (28) $= A^I_{\{i\}}(\alpha'_i,\alpha_j)A^{II}_j(\alpha'_i,\alpha_j,\sigma_i(y,\alpha'_i,s_i))$ and LHS of (28) $= A^I_{\{i\}}(y,\alpha_j)A^{II}_j(y,\alpha_j,s_i)$.

If $A^I_{\{i\}}(\alpha'_i,\alpha_j) = 0$, we have $A^I_{\{i\}}(y,\alpha_j)A^{II}_j(y,\alpha_j,s_i) \geq A^I_{\{i\}}(\alpha'_i,\alpha_j)A^{II}_j(\alpha'_i,\alpha_j,\sigma_i(y,\alpha'_i,s_i))$. We then have (28), done.

If $A^I_{\{i\}}(\alpha'_i,\alpha_j) = 1$, we have $A^I_{\{i\}}(y,\alpha_j) = 1$. Note that $A^{II}_j(y,\alpha_j,s_i) \leq A^{II}_j(\alpha'_i,\alpha_j,\sigma_i(y,\alpha'_i,s_i))$ since the virtual value of bidder i is lower when she deviates, as shown by Esö and Szentes (2007). In this case, we have LHS of (26) $= A^{II}_j(y,\alpha_j,s_i)p^{III}_{i,(i,j)}(y,\alpha_j,s_i,s_j) + (1 - A^{II}_j(y,\alpha_j,s_i))p^{III}_i(y,\alpha_j,s_i)$, and RHS of (26) $= A^{II}_j(\alpha'_i,\alpha_j,\sigma_i(y,\alpha'_i,s_i))p^{III}_{i,(i,j)}(\alpha'_i,\alpha_j,\sigma_i(y,\alpha'_i,s_i),s_j) + (1 - A^{II}_j(\alpha'_i,\alpha_j,\sigma_i(y,\alpha'_i,s_i)))p^{III}_i(\alpha'_i,\alpha_j,\sigma_i(y,\alpha'_i,s_i))$. Note that

$$
\begin{aligned}
& A^{II}_j(\alpha'_i,\alpha_j,\sigma_i(y,\alpha'_i,s_i))p^{III}_{i,(i,j)}(\alpha'_i,\alpha_j,\sigma_i(y,\alpha'_i,s_i),s_j) + (1 - A^{II}_j(\alpha'_i,\alpha_j,\sigma_i(y,\alpha'_i,s_i)))p^{III}_i(\alpha'_i,\alpha_j,\sigma_i(y,\alpha'_i,s_i)) \\
\leq\ & A^{II}_j(\alpha'_i,\alpha_j,\sigma_i(y,\alpha'_i,s_i))p^{III}_{i,(i,j)}(y,\alpha_j,s_i,s_j) + (1 - A^{II}_j(\alpha'_i,\alpha_j,\sigma_i(y,\alpha'_i,s_i)))p^{III}_i(y,\alpha_j,s_i) \\
\leq\ & A^{II}_j(y,\alpha_j,s_i)p^{III}_{i,(i,j)}(y,\alpha_j,s_i,s_j) + (1 - A^{II}_j(y,\alpha_j,s_i))p^{III}_i(y,\alpha_j,s_i).
\end{aligned}
$$

Thus (26) holds. Hence we also have $\Delta \leq 0$.

Proposition 3. The shortlisting rule and allocation rule characterized above are incentive compatible.

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