

**Alternative Approaches  
to Comparative  $n$ th-Degree Risk Aversion**

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Abstract: This paper extends the three main approaches to comparative risk aversion – the risk premium approach and the probability premium approach of Pratt (1964) and the comparative statics approach of Jindapon and Neilson (2007) – to study comparative  $n$ th-degree risk aversion. These extensions can accommodate trading off an  $n$ th-degree risk increase and an  $m$ th-degree risk increase for any  $m$  such that  $1 \leq m < n$ . It goes on to show that in the expected utility framework, all these general notions of comparative  $n$ th-degree risk aversion are equivalent, and can be characterized by the  $(n/m)$ th-degree Ross more risk aversion of Liu and Meyer (2013).

Key Words: Risk aversion; Comparative risk aversion; Risk premium; Probability premium; Downside risk aversion

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## 1. Introduction

For more than half a century now economists have used the Arrow-Pratt measure of risk aversion to compare how risk averse two individuals are. There are good reasons why. For one thing, the mathematical characterization is very simple, based on a ratio of the first two derivatives of the utility function. Second, and perhaps more importantly, the comparison based on the Arrow-Pratt measure was accompanied by some mathematically-equivalent behavioral conditions. Specifically, being everywhere higher on the Arrow-Pratt measure is equivalent to having a larger risk premium or having a larger probability premium for an actuarially-neutral risk (Pratt 1964). These behavioral, or choice-based, measures of risk aversion – i.e., the risk premium and the probability premium – have the advantage of being readily computed and compared in experiments investigating the factors that affect the strength of risk aversion.<sup>1</sup>

Recent experimental studies have demonstrated, in various contexts, a salient aversion to risk increases of 3<sup>rd</sup> and even higher degrees.<sup>2</sup> Moreover, 3<sup>rd</sup>-degree risk aversion (i.e., downside risk aversion or prudence), or even higher-degree risk aversion, has been shown to play critical roles in some important models of decision making under risk.<sup>3</sup> Along with these interests in higher-degree risk aversion, a question arises as to how to compare two individuals' relative strength of higher-degree risk aversion. Pratt's risk premium approach to comparative risk aversion has been generalized to deal with random initial wealth and comparative higher-degree risk aversion.<sup>4</sup> By comparison, his probability premium approach to comparative risk aversion, though extensively used in experiments investigating the strength of 2<sup>nd</sup>-degree risk aversion, has not played an important role in the study of comparative higher-degree risk aversion.

More recently, Jindapon and Neilson (2007) propose a new approach to comparative  $n$ th-degree risk aversion that is based on a comparative statics analysis. They show that an individual would always want to incur a larger monetary cost to reduce the  $n$ th-degree risk in wealth than another individual, if and only if the former is  $n$ th-degree Ross more risk averse than the latter.

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<sup>1</sup> For examples of choice-based risk aversion measures, see Holt and Laury (2002), Eckel and Grossman (2002), Andreoni and Sprenger (2011), Ebert and Wiesen (2014), Callen et al. (2014), and Grossman and Eckel (2015).

<sup>2</sup> For example, see Deck and Schlesinger (2010, 2014), Ebert and Wiesen (2011), Maier and Rieger (2011) and Noussair et al. (2014).

<sup>3</sup> One example is the self-protection decision. It has been shown that while a risk averse individual does not necessarily invest more in self-protection than a risk neutral individual, a downside risk averse (or a more downside risk averse) individual tends to invest less in self-protection than a downside risk neutral individual (or a less downside risk averse individual) (Chiu 2005, Eeckhoudt and Gollier 2005, Menegatti 2009, Denuit et al. 2016, Crainich et al. 2016 and Peter 2017). Another example is the precautionary saving/effort decision. It has been shown that as future income undergoes an  $n$ th-degree risk increase, the precautionary saving increases if and only if the utility function displays  $(n+1)$ th-degree risk aversion (Leland 1968, Sandmo 1970, Dreze and Modigliani 1972, Kimball 1990, Eeckhoudt and Schlesinger 2008, Eeckhoudt et al. 2012, Liu 2014, Wang et al. 2015, and Nocetti 2016).

<sup>4</sup> For example, see Ross (1981), Machina and Neilson (1987), Modica and Scarsini (2005), Jindapon and Neilson (2007), Crainich and Eeckhoudt (2008), Li (2009), and Denuit and Eeckhoudt (2010).

As Liu and Meyer (2013) argue, however, all these existing approaches to comparative  $n$ th-degree risk aversion essentially quantify  $n$ th-degree risk aversion by the willingness to trade a 1<sup>st</sup>-degree risk increase with an  $n$ th-degree risk increase. They propose a notion of “risk tradeoff” – the ratio of the reduction in expected utility caused by an  $n$ th-degree risk increase to that caused by an  $m$ th-degree risk increase – in order to quantify  $n$ th-degree risk aversion through the willingness to trade an  $m$ th-degree risk increase with an  $n$ th-degree risk increase for any  $m$  such that  $1 \leq m < n$ . They further show that an individual always has a larger tradeoff between an  $n$ th-degree risk increase and an  $m$ th-degree risk increase than another if and only if the former is  $(n/m)$ th-degree Ross more risk averse than the latter.

Nevertheless, Liu and Meyer’s risk tradeoff approach is not a direct generalization of the previously existing approaches to comparative risk aversion. Specifically, their notion of risk tradeoff cannot be interpreted as including either a risk premium or a probability premium as a special case. Further, although Liu and Meyer provide a comparative statics problem to accompany their analysis, much in the spirit of Jindapon and Neilson (2007), their comparative statics problem is not a direct generalization of that in Jindapon and Neilson (2007).

This paper extends the three main approaches to comparative risk aversion – the risk premium approach and the probability premium approach of Pratt (1964) and the comparative statics approach of Jindapon and Neilson (2007) – to study comparative  $n$ th-degree risk aversion, accommodating trading an  $m$ th-degree risk increase with an  $n$ th-degree risk increase for any  $m$  such that  $1 \leq m < n$ .

First, we propose a notion of the path-dependent  $m$ th-degree risk premium for an  $n$ th-degree risk increase, and interpret the existing risk premium concepts as the 1<sup>st</sup>-degree risk premium along some special paths. The relevant behavioral condition for one individual to be more  $n$ th-degree risk averse than another is that the former has a larger path-dependent  $m$ th-degree risk premium than the latter for every (random) initial wealth, every  $n$ th-degree risk increase, and every possible path of  $m$ th-degree increasing risk.

Second, we consider a situation where the individual compares random initial wealth to a binary compound lottery where the “good” state has less  $m$ th-degree risk than initial wealth and the “bad” state has higher  $n$ th-degree risk than initial wealth, with  $1 \leq m < n$ .<sup>5</sup> Generalizing Pratt’s probability premium, we look for the probability of the good state that makes the individual indifferent between initial wealth and the binary compound lottery. The relevant behavioral condition for one individual to be more  $n$ th-degree risk averse than another is that the former requires a higher probability on the good state than the latter for every initial wealth, every  $n$ th-degree risk increase, and every  $m$ th-degree risk decrease.

Third, we study a decision problem in which an individual faces an indexed path of random variables, and movements along the path involve precisely-defined reductions in  $n$ th-

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<sup>5</sup> The “good” or “bad” is from the perspective of an individual that is both  $m$ th-degree risk averse and  $n$ th-degree risk averse.

degree risk and increases in  $m$ th-degree risk. This formulation has as special cases both the comparative statics problem analyzed in Jindapon and Neilson (2007) and the portfolio choice problem analyzed in Pratt (1964), Ross (1981), and Machina and Neilson (1987). The relevant behavioral condition for one individual to be more  $n$ th-degree risk averse than another is that the former always chooses a random variable farther along the path than the latter. More simply, the  $n$ th-degree more risk averse individual chooses a random variable with less  $n$ th-degree risk but at the cost of more  $m$ th-degree risk.

All told, this paper demonstrates that when the expected utility framework is assumed, all these general behavioral notions of comparative  $n$ th-degree risk aversion are equivalent, and can be characterized by the  $(n/m)$ th-degree Ross more risk aversion of Liu and Meyer (2013).

The paper is organized as follows. Section 2 reviews notions of  $n$ th-degree increasing risk,  $n$ th-degree risk aversion, and  $(n/m)$ th-degree Ross more risk aversion. Section 3 presents three behavioral (i.e., choice-based) conditions comparing the  $n$ th-degree risk aversion of two individuals, as generalizations of the three main approaches to comparative risk aversion – the risk premium approach, the probability premium approach, and the comparative statics approach – respectively. The theorems in this section establish that, in the framework of expected utility, all these three behavioral conditions are characterized by the  $(n/m)$ th-degree Ross more risk averse condition. Section 4 shows how the theorems lend insight into 3rd-degree risk aversion, relating the notion of downside risk aversion to the notion of prudence. Section 5 offers some conclusions.

## 2. $n$ th-Degree Increases in Risk, $n$ th-Degree Risk Aversion, and $(n/m)$ th-Degree Ross More Risk Aversion.

Let  $F(x)$  and  $G(x)$  represent the cumulative distribution functions (CDFs) of two random variables whose supports are contained in a finite interval denoted  $[a, b]$  with no probability mass at point  $a$ . This implies that  $F(a) = G(a) = 0$  and  $F(b) = G(b) = 1$ . Letting  $F^{[1]}(x)$  denote  $F(x)$ , higher order cumulative functions are defined according to  $F^{[k]}(x) = \int_a^x F^{[k-1]}(y)dy$ ,  $k = 2, 3, \dots$ . Similar notation applies to  $G(x)$  and other CDFs.

For any integer  $n \geq 1$ , Ekern (1980) gives the following definition.

**Definition 1.**  $G(x)$  has more  $n$ th-degree risk (or is more  $n$ th-degree risky) than  $F(x)$  if

$$G^{[k]}(b) = F^{[k]}(b) \quad \text{for } k = 1, 2, \dots, n, \text{ and} \quad (1)$$

$$G^{[n]}(x) \geq F^{[n]}(x) \quad \text{for all } x \text{ in } [a, b] \text{ with “} \geq \text{” holding for some } x \text{ in } (a, b). \quad (2)$$

Condition (1) guarantees that the first  $n-1$  moments are held constant across the two distributions, and conditions (1) and (2) together imply that  $F(x)$  dominates  $G(x)$  in  $n$ th-degree

stochastic dominance. Thus, the  $n$ th-degree risk increase is a special case of  $n$ th-degree stochastic dominance in which the first  $n - 1$  moments are kept the same. This general definition of  $n$ th-degree risk increases has many well-known notions of stochastic changes as special cases. An increase in 1st-degree risk is a first-order stochastically dominated shift, which visually entails a leftward shift in probability mass. It implies (but is not equivalent to) a reduction in the mean. An increase in 2nd-degree risk holds the mean constant and spreads probability mass, which is the familiar mean-preserving spread of Rothschild and Stiglitz (1970). It implies (but is not equivalent to) an increase in the variance. Similarly, an increase in 3rd-degree risk holds the first two moments constant and shifts risk from high-wealth levels to low-wealth levels, which is the downside risk increase of Menezes et al. (1980). It implies (but is not equivalent to) a reduction in rightward skewness.

The fact that an  $n$ th-degree risk increase requires that the first  $n - 1$  moments remain constant places a restriction on the starting distribution  $F(x)$ . In particular, when  $F(x)$  is degenerate, placing all of its probability mass on a single outcome  $x_0$ , only 1st-degree and 2nd-degree risk increases are possible. A 1st-degree risk increase would entail a first-order stochastically dominated shift, as usual, and a 2nd-degree one would involve a mean-preserving spread. A 3rd-degree or higher risk increase would require that the variance of the new distribution be the same as that of the original distribution, and a degenerate distribution has zero variance. Consequently, 3rd-degree or higher risk increases are only well-defined when the starting distribution is nondegenerate.

Ekern (1980) also provides a definition of  $n$ th-degree risk aversion when the preferences have an expected utility representation. For any utility function  $u(x): [a, b] \rightarrow \mathbb{R}$ , assume that  $u \in C^\infty$ . Denote by  $u^{(k)}(x)$  the  $k$ th derivative of  $u(x)$ ,  $k = 1, 2, 3, \dots$ .

Definition 2. Decision maker  $u(x)$  is  $n$ th-degree risk averse if  $(-1)^{n+1} u^{(n)}(x) > 0$  for all  $x$  in  $[a, b]$ .

Note that  $u(x)$  is said to be *weakly*  $n$ th-degree risk averse when the strict inequality in Definition 2 is replaced with a weak one. 1st-degree risk aversion corresponds to an everywhere increasing utility function, and the usual 2nd-degree risk aversion corresponds to a concave utility function. If an individual exhibits all possible degrees of risk aversion his utility function will have derivatives that alternate in sign, beginning with a positive first derivative.

The relationship between the two concepts in Definitions 1 and 2 is given in Lemma 1 below that is proved by Ekern (1980).

Lemma 1.  $G(x)$  has more  $n$ th-degree risk than  $F(x)$  if and only if every  $n$ th-degree risk averse decision maker  $u(x)$  prefers  $F(x)$  to  $G(x)$ .

This result shows that  $n$ th-degree increases in risk are precisely the distribution changes that every  $n$ th-degree risk averse individual dislikes.

Another definition that is necessary for the analysis in this paper is  $(n/m)$ th-degree Ross more risk aversion, first described by Liu and Meyer (2013). Assume that  $m$  and  $n$  are two positive integers such that  $1 \leq m < n$ , and let the two utility functions  $u(x)$  and  $v(x)$  each be both  $n$ th-degree and  $m$ th-degree risk averse on  $[a, b]$ . The following definition of  $(n/m)$ th-degree Ross more risk aversion is from Liu and Meyer (2013).

**Definition 3.**  $u(x)$  is  $(n/m)$ th-degree Ross more risk averse than  $v(x)$  on  $[a, b]$  if

$$\frac{(-1)^{n+1}u^{(n)}(x)}{(-1)^{m+1}u^{(m)}(y)} \geq \frac{(-1)^{n+1}v^{(n)}(x)}{(-1)^{m+1}v^{(m)}(y)} \quad \text{for all } x, y \in [a, b], \quad (3)$$

or equivalently, if there exists  $\lambda > 0$ , such that  $\frac{u^{(n)}(x)}{v^{(n)}(x)} \geq \lambda \geq \frac{u^{(m)}(y)}{v^{(m)}(y)}$  for all  $x, y \in [a, b]$ .

Definition 3 includes many existing notions of one utility function being more risk averse than another as special cases. For  $n = 2, m = 1$  and  $y = x$ , condition (3) reduces to the familiar

Arrow-Pratt more risk averse condition:  $-\frac{u''(x)}{u'(x)} \geq -\frac{v''(x)}{v'(x)}$  for all  $x \in [a, b]$ . As Ross (1981)

points out, the behavioral conditions related to this characterization must have nonstochastic

initial wealth, and the stronger condition  $-\frac{u''(x)}{u'(y)} \geq -\frac{v''(x)}{v'(y)}$  for all  $x, y \in [a, b]$  – which is

referred to in the literature as Ross more risk aversion – allows for random initial wealth. For  $m = 1$ , Definition (3) reduces to Ross more risk aversion when  $n = 2$ , to Ross more downside risk aversion when  $n = 3$  (Modica and Scarsini 2005), and to Ross  $n$ th-degree more risk aversion for a general  $n \geq 2$  (Jindapon and Neilson 2007, Li 2009, and Denuit and Eeckhoudt 2010).

The following lemmas regarding the  $(n/m)$ th-degree Ross more risk averse condition will be used in proving the main results in the paper. Specifically, Lemma 2 is useful when using  $(n/m)$ th-degree Ross more risk aversion as a sufficient condition, and Lemma 3 is useful when showing  $(n/m)$ th-degree Ross more risk aversion as a necessary condition. A proof of Lemma 2 is given in Liu and Meyer (2013),<sup>6</sup> and a proof of Lemma 3 is provided in the appendix.

**Lemma 2.**  $u(x)$  is  $(n/m)$ th-degree Ross more risk averse than  $v(x)$  on  $[a, b]$  if and only if there exist  $\lambda > 0$  and  $\phi(x)$  with  $(-1)^{m+1}\phi^{(m)}(x) \leq 0$  and  $(-1)^{n+1}\phi^{(n)}(x) \geq 0$  for all  $x$  in  $[a, b]$  such that  $u(x) \equiv \lambda v(x) + \phi(x)$ .

<sup>6</sup> See the proof of their Theorem 1.

**Lemma 3.** If  $u(x)$  is NOT  $(n/m)$ th-degree Ross more risk averse than  $v(x)$  on  $[a, b]$ , then there exist  $\mu > 0$ ,  $[a_1, b_1] \subset (a, b)$  and  $[a_2, b_2] \subset (a, b)$  such that  $\phi(x) \equiv u(x) - \mu v(x)$  satisfies

$$\begin{aligned} (-1)^{n+1} \phi^{(n)}(x) &< 0 && \text{for all } x \in [a_1, b_1] \\ (-1)^{m+1} \phi^{(m)}(x) &> 0 && \text{for all } x \in [a_2, b_2] \end{aligned}$$

### 3. Alternative Approaches to Comparative $n$ th-Degree Risk Aversion

#### 3.1 The Risk Premium Approach

The best-known approach to comparative risk aversion involves the risk premium. In the original Arrow-Pratt analysis, the decision-maker has nonstochastic initial wealth  $w$  and faces an additive mean-zero risk  $\tilde{\varepsilon}$ . The risk premium  $\pi$  is the payment that satisfies the indifference condition  $w - \pi \sim w + \tilde{\varepsilon}$ . Ross (1981) extends the Arrow-Pratt analysis to random starting wealth levels, and defines the risk premium  $\pi$  according to

$$\tilde{w} - \pi \sim \tilde{y}, \quad (4)$$

where  $\tilde{w}$  is the random initial wealth and  $\tilde{y}$  a Rothschild-Stiglitz risk increase from  $\tilde{w}$ . In the expected utility framework, Ross shows that an individual always has a larger risk premium than another – for all  $\tilde{w}$  and  $\tilde{y}$  – if and only if the former is Ross more risk averse than the latter.<sup>7</sup>

Machina and Neilson (1987) extend Ross (1981) by defining a random risk premium. More precisely, suppose that  $\tilde{w}$  is the initial wealth,  $\tilde{y}$  is a Rothschild-Stiglitz risk increase from  $\tilde{w}$ , and  $\tilde{\eta}$  is a nonnegative random variable. The random risk premium  $\pi$  is a scalar satisfying the indifference condition

$$\tilde{w} - \pi \tilde{\eta} \sim \tilde{y}. \quad (5)$$

Machina and Neilson further show that, in the expected utility framework, an individual always has a larger random risk premium than another – for all  $\tilde{w}$ ,  $\tilde{y}$  and  $\tilde{\eta}$  – if and only if the former is Ross more risk averse than the latter.

Note that the left-hand side of (4) or (5) ( $\tilde{w} - \pi$  or  $\tilde{w} - \pi \tilde{\eta}$ ) is a 1<sup>st</sup>-degree risk increase from  $\tilde{w}$  when  $\pi > 0$ , and the right-hand side ( $\tilde{y}$ ) is a 2<sup>nd</sup>-degree risk increase from  $\tilde{w}$ . So the risk premium conditions (4) and (5) involve trading off a 1<sup>st</sup>-degree risk increase against a 2<sup>nd</sup>-degree one, along their respective “path” of 1<sup>st</sup>-degree risk increases. Take (5), for example.

The set  $\{\tilde{w} - \pi \tilde{\eta}\}_{\pi \in \mathbb{R}}$  constitutes a continuous, parameterized path indexed by the scalar  $\pi$ .<sup>8</sup>

Along this path, higher values of  $\pi$  correspond to increases in 1<sup>st</sup>-degree risk, and identifying  $\pi$

<sup>7</sup> This original notion of risk premium of Arrow-Pratt and Ross has been used to measure an individual’s aversion to higher-degree risk increases by Modica and Scarsini (2005), Crainich and Eeckhoudt (2008), Li (2009) and Denuit and Eeckhoudt (2010).

<sup>8</sup> Continuity is with respect to the topology of weak convergence.

in expression (5) is the same as finding the random variable on the path that is indifferent to  $\tilde{y}$ . A random variable further along the path involves more 1st-degree risk, and therefore a larger random risk premium, and consequently an individual who moves further along the path to reach indifference has a higher risk premium than one who does not move as far.

We can use this continuous path idea to formulate a general definition of the *path-dependent  $m$ th-degree risk premium* for an  $n$ th-degree risk increase, where  $1 \leq m < n$ , and uses it to measure an individual's  $n$ th-degree risk aversion in terms of an  $m$ th-degree risk increase. Let  $\tilde{w}$  be the random initial wealth and  $\tilde{y}$  be an  $n$ th-degree risk increase from  $\tilde{w}$ , and let

$\{\tilde{x}(\pi)\}_{\pi \in [A, B]}$  denote a continuous path of random variables, parameterized by  $\pi \in [A, B] \subset \mathbb{R}$ , such that  $\tilde{x}(0) = \tilde{w}$  and for every  $\pi' > \pi$  the random variable  $\tilde{x}(\pi')$  has more  $m$ th-degree risk than  $\tilde{x}(\pi)$  does, where  $1 \leq m < n$ . We refer to  $\{\tilde{x}(\pi)\}_{\pi \in [A, B]}$  as a *path of  $m$ th-degree increasing risk from  $\tilde{w}$* .

**Definition 4.** Suppose that  $\tilde{w}$  is the random initial wealth,  $\tilde{y}$  is an  $n$ th-degree risk increase from  $\tilde{w}$ , and  $\{\tilde{x}(\pi)\}_{\pi \in [A, B]}$  is a path of  $m$ th-degree increasing risk from  $\tilde{w}$ . The *path-dependent  $m$ th-degree risk premium* is the scalar  $\pi$  satisfying the indifference condition

$$\tilde{x}(\pi) \sim \tilde{y}. \quad (6)$$

Obviously,  $\{\tilde{w} - \pi\}_{\pi \in \mathbb{R}}$  and  $\{\tilde{w} - \pi\tilde{\eta}\}_{\pi \in \mathbb{R}}$  are examples of paths of 1<sup>st</sup>-degree increasing risk from  $\tilde{w}$ . The following examples are some paths of  $m$ th-degree increasing risk from  $\tilde{w}$  for  $m \geq 2$ . First,  $\{\tilde{x}(\pi) = \tilde{w} + \pi\tilde{\varepsilon}\}_{\pi \in [0, \infty)}$ , where  $\tilde{\varepsilon}$  is a mean-zero nondegenerate risk that is independent of  $\tilde{w}$ , is a path of 2<sup>nd</sup>-degree increasing risk from  $\tilde{w}$ . Second, suppose that  $\tilde{z}$  (with CDF  $H(x)$ ) has more  $m$ th-degree risk than  $\tilde{w}$  (with CDF  $F(x)$ ). Then  $\{\tilde{x}(\pi)\}_{\pi \in [0, 1]}$  is a path of  $m$ th-degree increasing risk from  $\tilde{w}$  if  $\tilde{x}(\pi)$  has a CDF of  $\pi H(x) + (1 - \pi)F(x)$ . In fact, assuming the expected utility framework and representing the preferences by utility function  $u(x)$ , the path-dependent  $m$ th-degree risk premium for an  $n$ th-degree risk increase from  $\tilde{w}$  to  $\tilde{y}$  along this path is given by  $\pi Eu(\tilde{z}) + (1 - \pi)Eu(\tilde{w}) = Eu(\tilde{y})$  or

$$\pi = \frac{Eu(\tilde{w}) - Eu(\tilde{y})}{Eu(\tilde{w}) - Eu(\tilde{z})}. \quad (7)$$

Note that the ratio in (7) is the “rate of substitution” between an  $n$ th-degree risk increase and an  $m$ th-degree risk increase defined in Liu and Meyer (2013). So, their rate of substitution is the path-dependent  $m$ th-degree risk premium for an  $n$ th-degree risk increase along a special path of  $m$ th-degree increasing risk from  $\tilde{w}$ ,  $\{\tilde{x}(\pi)\}_{\pi \in [0, 1]}$ , as discussed above.

It is straightforward to see that if the individual is both  $m$ th-degree and  $n$ th-degree risk averse, any path-dependent  $m$ th-degree risk premium for an  $n$ th-degree risk increase must be positive. Now consider two individuals,  $u$  and  $v$ , with different risk preferences. Take as given  $\tilde{w}$ ,  $\tilde{y}$ , and a path of  $m$ th-degree increasing risk from  $\tilde{w}$ ,  $\{\tilde{x}(\pi)\}_{\pi \in [A, B]}$ . If  $\pi_u$  and  $\pi_v$  satisfy  $\tilde{x}(\pi_u) \sim_u \tilde{y}$  and  $\tilde{x}(\pi_v) \sim_v \tilde{y}$ , respectively, and  $\pi_u > \pi_v$ , then this means that, compared to  $v$ , individual  $u$  must move further along the path of  $m$ th-degree increasing risk from  $\tilde{w}$  before offsetting the disutility caused by the  $n$ th-degree increase in risk entailed in  $\tilde{y}$ . More to the point, and much like the original Arrow-Pratt case, individual  $u$  is willing to accept a larger  $m$ th-degree risk increase to avoid an  $n$ th-degree risk increase than individual  $v$ .

If  $\pi_u \geq \pi_v$  for all  $\tilde{w}$ ,  $\tilde{y}$ , and paths of  $m$ th-degree increasing risk from  $\tilde{w}$ ,  $\{\tilde{x}(\pi)\}_{\pi \in [A, B]}$ , then  $u$  can be regarded as being more  $n$ th-degree risk averse than  $v$  when the willingness to pay for avoiding the  $n$ th-degree risk increase takes the form of an  $m$ th-degree risk increase. The following theorem provides a utility function-based characterization of the condition  $\pi_u \geq \pi_v$ , in the tradition of Pratt (1964), when the preferences of both  $u$  and  $v$  satisfy the axioms of expected utility, and are represented by utility functions  $u(x)$  and  $v(x)$ , respectively. The proof of the theorem is in the appendix.

**Theorem 1.** Suppose that two expected utility maximizers  $u(x)$  and  $v(x)$  are each both  $m$ th-degree risk averse and  $n$ th-degree risk averse everywhere. The path-dependent  $m$ th-degree risk premia satisfy  $\pi_u \geq \pi_v$  for every  $\tilde{w}$ , every  $\tilde{y}$  that is  $n$ th-degree riskier than  $\tilde{w}$  and every path of  $m$ th-degree increasing risk from  $\tilde{w}$ ,  $\{\tilde{x}(\pi)\}_{\pi \in [A, B]}$ , if and only if  $u(x)$  is  $(n/m)$ th-degree Ross more risk averse than  $v(x)$ .

The condition that both individuals are everywhere  $m$ th-degree risk averse plays the same role that increasing utility functions play in the standard Arrow-Pratt characterization of 2nd-degree comparative risk aversion. There the increasing utility functions imply that the individual dislikes increases in the risk premium, and here the  $m$ th-degree risk aversion implies that the individual dislikes movements farther along the path of  $m$ th-degree increasing risk from  $\tilde{w}$ .

### 3.2. The Probability Premium Approach

Along with the risk premium, Pratt (1964) also uses the probability premium as a measure of (global) risk aversion. Pratt defines the probability premium  $q$  according to the indifference condition

$$w \sim \begin{cases} w + \varepsilon & \text{with probability } \frac{1}{2} + q \\ w - \varepsilon & \text{with probability } \frac{1}{2} - q \end{cases} \quad (8)$$

where  $w$  is the nonrandom initial wealth and  $\varepsilon > 0$  is a constant. Pratt (1964) further shows that, in the expected utility framework, an individual  $u(x)$  always has a larger probability premium than another individual  $v(x)$  – for all  $w$  and  $\varepsilon$  – if and only if the former is Arrow-Pratt more risk averse than the latter.

Unlike the risk premium approach, the probability premium approach to comparative risk aversion has received little attention since Pratt (1964), perhaps because it was not used by Ross (1981) in the first generalization of the Arrow-Pratt conditions to random initial wealth.<sup>9</sup> We propose below a general formulation for using the probability premium to measure  $n$ th-degree risk aversion. Suppose that  $\tilde{w}$  is initial wealth,  $\tilde{y}$  is an  $n$ th-degree risk *increase* from  $\tilde{w}$ , and  $\tilde{z}$  is an  $m$ th-degree risk *decrease* from  $\tilde{w}$ . For an individual who is both  $n$ th-degree and  $m$ th-degree risk averse,  $\tilde{z} \succ \tilde{w} \succ \tilde{y}$ , where “ $\succ$ ” denotes the strict preference relationship. Consider a two-state compound lottery

$$\begin{cases} \tilde{z} & \text{with probability } p \\ \tilde{y} & \text{with probability } 1 - p \end{cases}.$$

As  $p$  increases continuously from 0 to 1, the above lottery goes from being dominated by  $\tilde{w}$  to being preferred to  $\tilde{w}$ . Assuming continuity of preferences, there is a  $p$  such that

$$\tilde{w} \sim \begin{cases} \tilde{z} & \text{with probability } p \\ \tilde{y} & \text{with probability } 1 - p \end{cases}. \quad (9)$$

Formally, the  $m$ th-degree probability premium for an  $n$ th-degree risk increase is defined below.

**Definition 5.** Suppose that  $\tilde{w}$  is the random initial wealth,  $\tilde{y}$  is an  $n$ th-degree risk *increase* from  $\tilde{w}$ , and  $\tilde{z}$  is an  $m$ th-degree risk *decrease* from  $\tilde{w}$ . The  $m$ th-degree probability premium for the  $n$ th-degree risk increase is the scalar  $p$  satisfying the indifference condition (9).

Note that for  $n = 2$  and  $m = 1$ , the  $m$ th-degree probability premium for the  $n$ th-degree risk increase includes Pratt’s probability premium as a special case. To see this, let  $\tilde{w} = w$ ,  $\tilde{z} = w + \varepsilon$  and  $\tilde{y}$  have two outcomes,  $w + \varepsilon$  and  $w - \varepsilon$ , with equal probability  $1/2$ . Then (9) becomes

$$w \sim \begin{cases} w + \varepsilon & \text{with probability } \frac{1}{2} + \frac{p}{2} \\ w - \varepsilon & \text{with probability } \frac{1}{2} - \frac{p}{2} \end{cases}$$

which is exactly the indifference condition (8) after relabeling  $\frac{p}{2}$  as  $q$ .

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<sup>9</sup> One exception is Jindapon (2010), who proposes a probability premium type measure of downside risk aversion based on the risk apportionment framework of Eeckhoudt and Schlesinger (2006). In addition, Eeckhoudt and Laeven (2015) give a graphical representation of the probability premium.

It is straightforward to see that if the individual is both  $m$ th-degree and  $n$ th-degree risk averse, then any  $m$ th-degree probability premium for an  $n$ th-degree risk increase lies in  $(0, 1)$ . Now consider two individuals,  $u$  and  $v$ , with different risk preferences. Given  $\tilde{w}$ ,  $\tilde{y}$  and  $\tilde{z}$ , if  $p_u$  and  $p_v$  satisfy (9) for  $\sim_u$  and  $\sim_v$ , respectively, and  $p_u > p_v$ , then this means that, compared to  $v$ , individual  $u$  requires a larger probability on the favorable state – in which an  $m$ th-degree risk decrease materializes – for the two-state compound lottery to be indifferent to the status quo.

If  $p_u \geq p_v$  for all  $\tilde{w}$ ,  $\tilde{y}$  and  $\tilde{z}$ , then  $u$  can be regarded as being more  $n$ th-degree risk averse than  $v$  when the necessary compensation to offset an  $n$ th-degree risk increase takes the form of an  $m$ th-degree risk *decrease*. The following theorem shows that in the framework of expected utility, the condition  $p_u \geq p_v$  is also characterized by  $(n/m)$ th-degree Ross more risk aversion. The proof of the theorem is in the appendix.

**Theorem 2.** Suppose that two expected utility maximizers  $u(x)$  and  $v(x)$  are each both  $m$ th-degree risk averse and  $n$ th-degree risk averse everywhere. The  $m$ th-degree probability premia satisfy  $p_u \geq p_v$  for every  $\tilde{w}$ ,  $\tilde{y}$  and  $\tilde{z}$  such that  $\tilde{y}$  is  $n$ th-degree more risky than  $\tilde{w}$  and  $\tilde{z}$  is  $m$ th-degree less risky than  $\tilde{w}$ , if and only if  $u(x)$  is  $(n/m)$ th-degree Ross more risk averse than  $v(x)$ .

Theorem 2 provides a straightforward way for understanding what  $(n/m)$ th-degree Ross more risk averse means. Individuals have initial random wealth given by  $\tilde{w}$ , and consider replacing it with a binary compound lottery that pays random variable  $\tilde{z}$  in the good state and random variable  $\tilde{y}$  in the bad state. What makes the bad state bad is that  $\tilde{y}$  is  $n$ th-degree riskier than  $\tilde{w}$ , and what makes the good state good is that  $\tilde{z}$  is  $m$ th-degree *less* risky than  $\tilde{w}$ . Choosing to move away from the status quo, then, involves trading off the  $n$ th-degree risk increase against the  $m$ th-degree risk reduction. The individual who is  $(n/m)$ th-degree Ross more risk averse requires a larger probability on the  $m$ th-degree risk reduction to keep him indifferent between the binary compound lottery and the status quo, which means for him the  $n$ th-degree risk increase weighs relatively more heavily in his decision than the  $m$ th-degree risk reduction does, compared to the other individual.

### 3.3. The Comparative Statics Approach

Jindapon and Neilson (2007) construct a decision problem in which an individual can reduce the  $n$ th-degree risk in the random wealth variable by incurring a monetary cost. Specifically, suppose that CDF  $G(x)$  has more  $n$ th-degree risk than CDF  $F(x)$ , and that by incurring a cost  $c(t)$ ,  $G(x)$  can be made into a less  $n$ th-degree risky CDF  $H(x, t) = tF(x) + (1-t)G(x)$ , where  $0 \leq t \leq 1$ ,  $c'(t) > 0$  and  $c''(t) > 0$ . The final wealth can be denoted as

$\tilde{w}(t) = \tilde{w}_1(t) - c(t)$ , where  $\tilde{w}_1(t)$  has a CDF of  $H(x, t)$ . An individual's problem is to choose  $t$  to maximize the expected utility. Jindapon and Neilson show that individual  $u(x)$  chooses a larger  $t$  than individual  $v(x)$  for all  $F(x)$  and  $G(x)$  such that  $G(x)$  has more  $n$ th-degree risk than  $F(x)$ , if and only if the former is  $(n/1)$ th-degree Ross more risk averse than the latter. They refer to their analysis as the comparative statics approach to comparative  $n$ th-degree risk aversion.<sup>10</sup>

Jindapon and Neilson's (2007) comparative statics approach is extended here to provide a choice-based behavioral characterization of the more general  $(n/m)$ th-degree Ross more risk aversion. Note that the final wealth in the Jindapon and Neilson setup,  $\tilde{w}(t) = \tilde{w}_1(t) - c(t)$ , undergoes a two-step change as  $t$  increases. Specifically, as  $t$  increases to  $t'$ , the change from  $\tilde{w}(t)$  to  $\tilde{w}(t')$  can be decomposed into the following two steps: From  $\tilde{w}(t) = \tilde{w}_1(t) - c(t)$  to  $\tilde{w}_1(t') - c(t)$  is an  $n$ th-degree risk decrease, which is an improvement for an individual who is  $n$ th-degree risk averse; from  $\tilde{w}_1(t') - c(t)$  to  $\tilde{w}(t') = \tilde{w}_1(t') - c(t')$  is a 1st-degree risk increase, which is a deterioration for an individual who prefers more to less. The definition below provides a general notion of changes in a random distribution that can be decomposed sequentially into an improvement and then a deterioration.

**Definition 6.**  $F(x)$  is *sequentially* less  $n$ th-degree risky and more  $m$ th-degree risky than  $G(x)$ , if there exists  $H(x)$  such that  $F(x)$  has less  $n$ th-degree risk than  $H(x)$ , and  $H(x)$  has more  $m$ th-degree risk than  $G(x)$ .

Now consider a parameterized wealth path represented by  $\tilde{w}(\alpha)$ , where as  $\alpha$  increases  $\tilde{w}(\alpha)$  becomes sequentially less  $n$ th-degree risky and more  $m$ th-degree risky. Moving down such a path, one reduces the  $n$ th-degree risk in wealth by increasing the  $m$ th-degree risk in wealth. The problem of an individual  $u(x)$  is

$$\max_{\alpha} Eu[\tilde{w}(\alpha)] \quad (10)$$

The solution to (10) is assumed to be unique and is denoted  $\alpha_u$ .

Intuitively, if  $u(x)$  is  $(n/m)$ th-degree Ross more risk averse than  $v(x)$ , then  $u(x)$  would choose to locate further down the path than  $v(x)$ . That is,  $\alpha_u \geq \alpha_v$ . Indeed, we have the following characterization theorem, which is proved in the appendix.

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<sup>10</sup>Watt and Vazquez (2013) provide an alternative comparative statics approach to comparative downside risk aversion.

Theorem 3.  $\alpha_u \geq \alpha_v$  for every wealth path  $\tilde{w}(\alpha)$  where, as  $\alpha$  increases,  $\tilde{w}(\alpha)$  becomes sequentially less  $n$ th-degree risky and more  $m$ th-degree risky, if and only if  $u(x)$  is  $(n/m)$ th-degree Ross more risk averse than  $v(x)$ .

There are four important situations that lead to wealth paths  $\tilde{w}(\alpha)$  where increases in  $\alpha$  make wealth sequentially less  $n$ th-degree risky and more  $m$ th-degree risky. The first relates to Jindapon and Neilson's (2007) comparative statics problem. Suppose that total wealth consists of two independent components, i.e.  $\tilde{w}(\alpha) = \tilde{w}_1(\alpha) + \tilde{w}_2(\alpha)$ , and as  $\alpha$  increases  $\tilde{w}_1(\alpha)$  becomes less  $n$ th-degree risky and  $\tilde{w}_2(\alpha)$  becomes more  $m$ th-degree risky. Note that the setup in Jindapon and Neilson (2007) is a special case of this situation for which  $m = 1$ . It can be immediately checked that as  $\alpha$  increases  $\tilde{w}(\alpha)$  becomes sequentially less  $n$ th-degree risky and more  $m$ th-degree risky. Then according to Theorem 3, an  $(n/m)$ th-degree Ross more risk averse individual would choose to have a less  $n$ th-degree risky first component and a more  $m$ th-degree risky second component.

The second situation incorporates binary compound lotteries, much like those in the probability premium setup. Suppose that nature rolls the dice to reveal two states with probability  $p$  (for State 1) and  $(1-p)$  (for State 2) respectively, that  $\tilde{w}_i(\alpha)$  is the random wealth obtained in state  $i$  ( $i = 1, 2$ ), and that as  $\alpha$  increases  $\tilde{w}_1(\alpha)$  becomes less  $n$ th-degree risky and  $\tilde{w}_2(\alpha)$  becomes more  $m$ th-degree risky. The ex ante total wealth can be depicted as:

$$\tilde{w}(\alpha) = \begin{cases} \tilde{w}_1(\alpha) & \text{with probability } p \\ \tilde{w}_2(\alpha) & \text{with probability } (1-p) \end{cases}$$

It can be readily checked that as  $\alpha$  increases  $\tilde{w}(\alpha)$  becomes sequentially less  $n$ th-degree risky and more  $m$ th-degree risky. Then according to Theorem 3, an  $(n/m)$ th-degree Ross more risk averse individual would choose to have a less  $n$ th-degree risky wealth in State 1 and a more  $m$ th-degree risky wealth in State 2. Thus, the intuition from the generalized comparative statics approach matches that from both the risk premium approach and the probability premium approach, where the  $(n/m)$ th-degree Ross more risk averse individual shows greater relative sensitivity to  $n$ th-degree than  $m$ th-degree risk.

The third situation arises when the wealth path is constructed through probability mixtures of two random variables. Suppose that  $F(x)$  is sequentially less  $n$ th-degree risky and more  $m$ th-degree risky than  $G(x)$ , and consider the convex combination of these two CDFs,

$J(x, \alpha) = \alpha F(x) + (1 - \alpha)G(x)$ . It can be shown that as  $\alpha$  increases  $J(x, \alpha)$  becomes sequentially less  $n$ th-degree risky and more  $m$ th-degree risky.<sup>11</sup>

Specifically, because  $F(x)$  is sequentially less  $n$ th-degree risky and more  $m$ th-degree risky than  $G(x)$ , there exists  $H(x)$  such that  $F(x)$  has less  $n$ th-degree risk than  $H(x)$ , and  $H(x)$  has more  $m$ th-degree risk than  $G(x)$ . To show that  $J(x, \alpha_2) = \alpha_2 F(x) + (1 - \alpha_2)G(x)$  is sequentially less  $n$ th-degree risky and more  $m$ th-degree risky than  $J(x, \alpha_1) = \alpha_1 F(x) + (1 - \alpha_1)G(x)$  for  $1 \geq \alpha_2 > \alpha_1 \geq 0$ , consider CDF  $K(x) = \alpha_1 F(x) + (\alpha_2 - \alpha_1)H(x) + (1 - \alpha_2)G(x)$ . It is easy to see that  $J(x, \alpha_2)$  is preferred to  $K(x)$  by every  $n$ th-degree risk averse individual, and therefore  $J(x, \alpha_2)$  has less  $n$ th-degree risk than  $K(x)$  (Lemma 1). Similarly, it can be seen that  $K(x)$  has more  $m$ th-degree risk than  $J(x, \alpha_1)$ . Then, by definition,  $J(x, \alpha_2) = \alpha_2 F(x) + (1 - \alpha_2)G(x)$  is sequentially less  $n$ th-degree risky and more  $m$ th-degree risky than  $J(x, \alpha_1) = \alpha_1 F(x) + (1 - \alpha_1)G(x)$ .

According to Theorem 3, in choosing among  $J(x, \alpha) = \alpha F(x) + (1 - \alpha)G(x)$  where  $\alpha \in [0, 1]$ , an  $(n/m)$ th-degree Ross more risk averse individual would choose to put a larger weight on  $F(x)$ .

Fourth, and finally, the original Arrow-Pratt portfolio choice condition is a very special case of Theorem 3. In the original problem, the individual has a fixed, nonstochastic amount to invest. Placing all the money in the riskless asset yields wealth  $x$  with probability one, and placing it all in the risky asset yields wealth  $\tilde{z}$ , which  $\tilde{z}$  is a random variable whose mean is strictly larger than  $x$ . The individual can place some fraction  $\alpha$  of his money in the riskless asset, in which case wealth is determined by the random variable  $\tilde{w}(\alpha) = \alpha x + (1 - \alpha)\tilde{z}$ . As  $\alpha$  increases  $\tilde{w}(\alpha)$  becomes sequentially less 2nd-degree risky and more 1st-degree risky, and by Theorem 3 the individual who is  $(2/1)$ nd-degree Ross more risk averse chooses a higher value of  $\alpha$ , that is, invests a larger fraction of his money in the riskless asset.

This version of the portfolio problem is a highly special case, though, because as noted in footnote 11, when  $n > 2$  the path created by the portfolio mixture is not one with sequentially less  $n$ th-degree risk and more  $m$ th-degree risk. If, however, there exists a set of different assets, indexed by  $\alpha$ , for which increases in  $\alpha$  mean that the asset has sequentially less  $n$ th-degree and more  $m$ th-degree risk, Theorem 3 would apply and the  $(n/m)$ th-degree Ross more risk averse investor chooses an asset with less  $n$ th-degree risk than his less risk averse counterpart.

#### 4. Downside risk aversion and prudence

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<sup>11</sup> In contrast, suppose that  $\tilde{x}$  is sequentially less  $n$ th-degree risky and more  $m$ th-degree risky than  $\tilde{z}$ . In general, it is *not* the case that as  $\alpha$  increases  $\tilde{w}(\alpha) = \alpha\tilde{x} + (1 - \alpha)\tilde{z}$  becomes sequentially less  $n$ th-degree risky and more  $m$ th-degree risky.

The three behavioral characterizations provide a means for understanding the difference between downside risk aversion and prudence. Kimball (1990) investigates prudence, measured by  $-u'''(x)/u''(x)$ , while Modica and Sarsini (2005) and Crainich and Eeckhoudt (2008) measure downside risk aversion using  $u'''(x)/u'(x)$ .<sup>12</sup> Clearly, both these ratios are natural extensions of the Arrow-Pratt measure to the 3<sup>rd</sup>-degree risk aversion.

The next two corollaries provide characterizations of prudence and downside risk aversion. Corollary 1 is an equivalence theorem related to comparative prudence, and Corollary 2 is its counterpart for comparative downside risk aversion.

**Corollary 1.** The following conditions are equivalent representations of agent  $u$  being more prudent than agent  $v$  over the interval  $[a, b]$ :

- (a)  $-u'''(x)/u''(y) \geq -v'''(x)/v''(y)$  for all  $x, y \in [a, b]$ .
- (b) Let  $\{\tilde{x}(\pi)\}_{\pi \in [A, B]}$  be a path of 2nd-degree increasing risk from  $\tilde{w}$  and  $\tilde{y}$  have more 3rd-degree risk than  $\tilde{w}$ . If  $\pi_u$  and  $\pi_v$  solve  $Eu(\tilde{x}(\pi)) = Eu(\tilde{y})$  and  $Ev(\tilde{x}(\pi)) = Ev(\tilde{y})$ , respectively, then  $\pi_u \geq \pi_v$ .
- (c) Let  $\tilde{z}$  have less 2nd-degree risk than  $\tilde{w}$  and  $\tilde{y}$  have more 3rd-degree risk than  $\tilde{w}$ . If  $p_u$  and  $p_v$  solve  $Eu(\tilde{w}) = pEu(\tilde{z}) + (1-p)Eu(\tilde{y})$  and  $Ev(\tilde{w}) = pEv(\tilde{z}) + (1-p)Ev(\tilde{y})$ , respectively, then  $p_u \geq p_v$ .
- (d) Let  $\tilde{w}(\alpha)$  be a parameterized wealth path where, as  $\alpha$  increases,  $\tilde{w}(\alpha)$  becomes sequentially less 3rd-degree risky and more 2nd-degree risky. If  $\alpha_u$  and  $\alpha_v$  maximize  $Eu[\tilde{w}(\alpha)]$  and  $Ev[\tilde{w}(\alpha)]$ , respectively, then  $\alpha_u \geq \alpha_v$ .

**Corollary 2.** The following conditions are equivalent representations of agent  $u$  being more downside risk averse than agent  $v$  over the interval  $[a, b]$ :

- (a)  $u'''(x)/u'(y) \geq v'''(x)/v'(y)$  for all  $x, y \in [a, b]$ .
- (b) Let  $\{\tilde{x}(\pi)\}_{\pi \in [A, B]}$  be a path of 1st-degree increasing risk from  $\tilde{w}$  and  $\tilde{y}$  have more 3rd-degree risk than  $\tilde{w}$ . If  $\pi_u$  and  $\pi_v$  solve  $Eu(\tilde{x}(\pi)) = Eu(\tilde{y})$  and  $Ev(\tilde{x}(\pi)) = Ev(\tilde{y})$ , respectively, then  $\pi_u \geq \pi_v$ .

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<sup>12</sup> Keenan and Snow (2002, 2017) and Liu and Meyer (2012) propose other measures of downside risk aversion that add a term of the Arrow-Pratt risk aversion to the ratio  $u'''(x)/u'(x)$ .

- (c) Let  $\tilde{z}$  have less 1st-degree risk than  $\tilde{w}$  and  $\tilde{y}$  have more 3rd-degree risk than  $\tilde{w}$ . If  $p_u$  and  $p_v$  solve  $Eu(\tilde{w}) = pEu(\tilde{z}) + (1-p)Eu(\tilde{y})$  and  $Ev(\tilde{w}) = pEv(\tilde{z}) + (1-p)Ev(\tilde{y})$ , respectively, then  $p_u \geq p_v$ .
- (d) Let  $\tilde{w}(\alpha)$  be a parameterized wealth path where, as  $\alpha$  increases,  $\tilde{w}(\alpha)$  becomes sequentially less 3rd-degree risky and more 1st-degree risky. If  $\alpha_u$  and  $\alpha_v$  maximize  $Eu[\tilde{w}(\alpha)]$  and  $Ev[\tilde{w}(\alpha)]$ , respectively, then  $\alpha_u \geq \alpha_v$ .

The two corollaries make clear the differences between downside risk aversion and prudence. Consider condition (b), for example. A more prudent individual is willing to accept a larger increase in the 2nd-degree risk to avoid an increase in the 3rd-degree risk than a less prudent individual is, while a more downside risk averse individual is willing to accept a larger increase in the 1st-degree risk to avoid an increase in the 3rd-degree risk than a less downside risk averse individual is. Using the less precise terminology from moments, a more prudent individual is willing to accept a larger increase in variance to avoid an increase in the leftward skewness, and a more downside risk averse individual is willing to accept a larger reduction in mean to avoid an increase in the leftward skewness.

Conditions (c) and (d) lead to basically the same interpretation. That is, compared with a less prudent individual, the more prudent individual weighs the 3rd-degree risk relatively more and the 2nd-degree risk relatively less; in contrast, compared with a less downside risk averse individual, the more downside risk averse individual weighs the 3rd-degree risk relatively more and the 1st-degree risk relatively less.

The same intuition covers other, higher-degree comparisons. Using the moment terminology, (4/1)th-degree Ross risk aversion concerns trading off changes in kurtosis against changes in mean, (4/2)th-degree Ross risk aversion concerns tradeoffs between kurtosis and variance, and (4/3)th-degree Ross risk aversion pertains to kurtosis and skewness.

## 5. Conclusion

More than half a century ago, Pratt (1964) uses two behavioral (or choice-based) conditions – which are based on the risk premium and the probability premium, respectively – to characterize the Arrow-Pratt more risk averse condition that is based on the famous Arrow-Pratt risk aversion measure,  $-u''(x)/u'(x)$ . These behavioral conditions regarding comparative risk aversion are important both because they have economic contents and because they can be readily implemented in experimental investigations into individual characteristics (e.g., gender, age, income, education, and religion affiliation) that affect the degree of risk aversion. These behavioral conditions do not depend on the expected utility framework to be meaningful and can be checked via experiments without explicit specifications of the utility function.

More recently, Liu and Meyer (2013) propose to use  $\frac{(-1)^{n+1}u^{(n)}(x)}{(-1)^{m+1}u^{(m)}(x)}$  as the  $(n/m)$ th-degree risk aversion measure for  $n$ th-degree risk aversion, and generalize the Arrow-Pratt more risk averse condition and the Ross more risk averse condition to the  $(n/m)$ th-degree Ross more risk aversion.

This paper generalizes the three main existing (behavioral or choice-based) approaches to comparative risk aversion – the risk premium approach and the probability premium approach due to Pratt (1964) and the comparative statics approach due to Jindapon and Neilson (2007) – for comparative  $n$ th-degree risk aversion that can accommodate trading off an  $n$ th-degree risk increase and an  $m$ th-degree risk increase for any  $m$  such that  $1 \leq m < n$ . It shows that when the expected utility framework is assumed, all these general notions of comparative  $n$ th-degree risk aversion are equivalent, and can be characterized by the  $(n/m)$ th-degree Ross more risk aversion.

In the future, economists and other social scientists may want to investigate the determining factors of the strength of 3<sup>rd</sup>- and higher-degree risk aversion, just as what they have extensively done for the 2<sup>nd</sup>-degree risk aversion. It is our hope that the results in this paper will deepen the understanding of, and help in creating alternative measures for, the intensity of  $n$ th-degree risk aversion.

## APPENDIX

### Proof of Lemma 3.

If  $u(x)$  is NOT  $(n/m)$ th-degree Ross more risk averse than  $v(x)$  on  $[a, b]$ , then there exist some  $y$  and  $z \in [a, b]$  such that

$$\frac{u^{(n)}(y)}{v^{(n)}(y)} < \frac{u^{(m)}(z)}{v^{(m)}(z)}.$$

Obviously, for such  $y$  and  $z$ , there exists  $\mu > 0$ , such that

$$\frac{u^{(n)}(y)}{v^{(n)}(y)} < \mu < \frac{u^{(m)}(z)}{v^{(m)}(z)},$$

which implies, due to continuity, that there exist  $[a_1, b_1] \subset (a, b)$  and  $[a_2, b_2] \subset (a, b)$  such that

$$\frac{u^{(n)}(y)}{v^{(n)}(y)} < \mu < \frac{u^{(m)}(z)}{v^{(m)}(z)}$$

for all  $y \in [a_1, b_1]$  and all  $z \in [a_2, b_2]$ .

Define  $\phi(x) \equiv u(x) - \mu v(x)$ . Differentiating yields

$$(-1)^{n+1} \phi^{(n)}(x) = (-1)^{n+1} u^{(n)}(x) - \mu (-1)^{n+1} v^{(n)}(x) < 0 \quad \text{for all } x \in [a_1, b_1]$$

$$(-1)^{m+1} \phi^{(m)}(x) = (-1)^{m+1} u^{(m)}(x) - \mu (-1)^{m+1} v^{(m)}(x) > 0 \quad \text{for all } x \in [a_2, b_2]$$

Q.E.D.

### Proof of Theorem 1.

The “if” part: Suppose that  $u(x)$  is  $(n/m)$ th-degree Ross more risk averse than  $v(x)$ . Then, according to Lemma 2, there exist  $\lambda > 0$  and  $\phi(x)$  with  $(-1)^{m+1} \phi^{(m)}(x) \leq 0$  and  $(-1)^{n+1} \phi^{(n)}(x) \geq 0$  for all  $x$  in  $[a, b]$  such that  $u(x) \equiv \lambda v(x) + \phi(x)$ . Note that  $\phi(x)$  is both weakly  $m$ th-degree risk tolerant – meaning that  $-\phi(x)$  is weakly  $m$ th-degree risk averse – and weakly  $n$ th-degree risk averse.

For every  $\tilde{w}$ , every  $\tilde{y}$  that is  $n$ th-degree riskier than  $\tilde{w}$  and every path of  $m$ th-degree increasing risk from  $\tilde{w}$ ,  $\{\tilde{x}(\pi)\}_{\pi \in [A, B]}$ ,  $\pi_u$  and  $\pi_v$  satisfy  $Eu(\tilde{x}(\pi_u)) = Eu(\tilde{y})$  and

$Ev(\tilde{x}(\pi_v)) = Ev(\tilde{y})$ , respectively, by definition. Further, we have

$$\begin{aligned} Eu(\tilde{x}(\pi_v)) &= \lambda Ev(\tilde{x}(\pi_v)) + E\phi(\tilde{x}(\pi_v)) \\ &= \lambda Ev(\tilde{y}) + E\phi(\tilde{x}(\pi_v)) \\ &\geq \lambda Ev(\tilde{y}) + E\phi(\tilde{y}) \\ &= Eu(\tilde{y}), \end{aligned}$$

where the inequality is from (i)  $\tilde{x}(\pi_u)$  has more  $m$ th-degree risk than  $\tilde{w}$  and  $\tilde{y}$  has more  $n$ th-degree risk than  $\tilde{w}$ , and (ii)  $\phi(x)$  is both weakly  $m$ th-degree risk tolerant and weakly  $n$ th-degree risk averse. Because  $Eu(\tilde{x}(\pi))$  is strictly decreasing in  $\pi$ , we have  $\pi_u \geq \pi_v$ .

The “only if” part: Suppose that the path-dependent  $m$ th-degree risk premia satisfy  $\pi_u \geq \pi_v$  for every  $\tilde{w}$ , every  $\tilde{y}$  that is  $n$ th-degree riskier than  $\tilde{w}$  and every path of  $m$ th-degree increasing risk from  $\tilde{w}$ ,  $\{\tilde{x}(\pi)\}_{\pi \in [A, B]}$ . Then, it must be the case that  $\pi_u \geq \pi_v$  for every  $\tilde{w}$ , every  $\tilde{y}$  that is  $n$ th-degree riskier than  $\tilde{w}$  and a special path of  $m$ th-degree increasing risk from  $\tilde{w}$  that is defined as follows.

Suppose that  $\tilde{z}$  (with CDF  $H(x)$ ) has more  $m$ th-degree risk than  $\tilde{w}$  (with CDF  $F(x)$ ). Then  $\{\tilde{x}(\pi)\}_{\pi \in [0, 1]}$  is a path of  $m$ th-degree increasing risk from  $\tilde{w}$  if  $\tilde{x}(\pi)$  has a CDF of  $\pi H(x) + (1 - \pi)F(x)$ . And for  $u(x)$ , the path-dependent  $m$ th-degree risk premium for an  $n$ th-degree risk increase from  $\tilde{w}$  to  $\tilde{y}$  along this path is given by  $\pi Eu(\tilde{z}) + (1 - \pi)Eu(\tilde{w}) = Eu(\tilde{y})$  or

$$\pi_u = \frac{Eu(\tilde{w}) - Eu(\tilde{y})}{Eu(\tilde{w}) - Eu(\tilde{z})}.$$

Similarly,

$$\pi_v = \frac{Ev(\tilde{w}) - Ev(\tilde{y})}{Ev(\tilde{w}) - Ev(\tilde{z})}.$$

So the given condition implies that  $\frac{Eu(\tilde{w}) - Eu(\tilde{y})}{Eu(\tilde{w}) - Eu(\tilde{z})} \geq \frac{Ev(\tilde{w}) - Ev(\tilde{y})}{Ev(\tilde{w}) - Ev(\tilde{z})}$  for every  $\tilde{w}$ , every

$\tilde{y}$  that is  $n$ th-degree riskier than  $\tilde{w}$  and every  $\tilde{z}$  that is  $m$ th-degree riskier than  $\tilde{w}$ . Note that the ratio on each side of the inequality is the “rate of substitution” between an  $n$ th-degree risk increase and an  $m$ th-degree risk increase defined in Liu and Meyer (2013). According to their Theorem 1, it must be the case that  $u(x)$  is  $(n/m)$ th-degree Ross more risk averse than  $v(x)$ .

Q.E.D.

### Proof of Theorem 2.

For any given  $\tilde{w}$ ,  $\tilde{y}$  and  $\tilde{z}$  such that  $\tilde{y}$  is  $n$ th-degree more risky than  $\tilde{w}$  and  $\tilde{z}$  is  $m$ th-degree less risky than  $\tilde{w}$ , define

$$U(p) \equiv Eu(\tilde{w}) - [pEu(\tilde{z}) + (1 - p)Eu(\tilde{y})].$$

Clearly,  $U'(p) = Eu(\tilde{y}) - Eu(\tilde{z}) < 0$  because  $u(x)$  is both  $m$ th-degree risk averse and  $n$ th-degree risk averse. For the same reason,  $U(0) > 0$  and  $U(1) < 0$ .

So  $\exists p_u \in (0, 1)$  such that  $U(p_u) = 0$ .  $V(p)$  for  $v(x)$  can be similarly defined, and  $\exists p_v \in (0, 1)$  such that  $V(p_v) = 0$ .

The “if” part: Suppose that  $u(x)$  is  $(n/m)$ th-degree Ross more risk averse than  $v(x)$ . Then, from Lemma 2, there exist  $\lambda > 0$  and  $\phi(x)$  with  $(-1)^{m+1}\phi^{(m)}(x) \leq 0$  and  $(-1)^{n+1}\phi^{(n)}(x) \geq 0$  for all  $x$  in  $[a, b]$  such that  $u(x) \equiv \lambda v(x) + \phi(x)$ . Note that  $\phi(x)$  is both weakly  $m$ th-degree risk tolerant – meaning that  $-\phi(x)$  is weakly  $m$ th-degree risk averse – and weakly  $n$ th-degree risk averse.

Evaluating  $U(p)$  at  $p_v$ , we have

$$\begin{aligned} U(p_v) &= Eu(\tilde{w}) - [p_v Eu(\tilde{z}) + (1 - p_v)Eu(\tilde{y})] \\ &= \lambda V(p_v) + E\phi(\tilde{w}) - [p_v E\phi(\tilde{z}) + (1 - p_v)E\phi(\tilde{y})] \\ &= E\phi(\tilde{w}) - [p_v E\phi(\tilde{z}) + (1 - p_v)E\phi(\tilde{y})] \\ &= p_v [E\phi(\tilde{w}) - E\phi(\tilde{z})] + (1 - p_v) [E\phi(\tilde{w}) - E\phi(\tilde{y})] \\ &\geq 0. \end{aligned}$$

The inequality above is from that (i)  $\tilde{z}$  has less  $m$ th-degree risk than  $\tilde{w}$  and  $\tilde{y}$  has more  $n$ th-degree risk than  $\tilde{w}$ , and (ii)  $\phi(x)$  is both weakly  $m$ th-degree risk tolerant and weakly  $n$ th-degree risk averse. Because  $U(p)$  is strictly decreasing in  $p$ , we have  $p_u \geq p_v$ .

The “only if” part: Suppose that  $p_u \geq p_v$  for all  $\tilde{w}$ ,  $\tilde{y}$  and  $\tilde{z}$  such that  $\tilde{y}$  is  $n$ th-degree more risky than  $\tilde{w}$  and  $\tilde{z}$  is  $m$ th-degree less risky than  $\tilde{w}$ . To prove that  $u(x)$  is  $(n/m)$ th-degree Ross more risk averse than  $v(x)$ , assume otherwise. Then, according to Lemma 3, there exist  $\mu > 0$ ,  $[a_1, b_1] \subset (a, b)$  and  $[a_2, b_2] \subset (a, b)$ , such that  $\phi(x) \equiv u(x) - \mu v(x)$  satisfies

$$\begin{aligned} (-1)^{n+1}\phi^{(n)}(x) &< 0 & \text{for all } x \in [a_1, b_1] \\ (-1)^{m+1}\phi^{(m)}(x) &> 0 & \text{for all } x \in [a_2, b_2] \end{aligned} \tag{A1}$$

Now denote the CDFs for  $\tilde{w}$ ,  $\tilde{y}$  and  $\tilde{z}$  as  $F(x)$ ,  $G(x)$  and  $H(x)$ , respectively, and choose  $F(x)$ ,  $G(x)$  and  $H(x)$  such that

$$\begin{cases} G^{[n]} - F^{[n]} > 0 & x \in (a_1, b_1) \\ G^{[n]} - F^{[n]} = 0 & x \notin (a_1, b_1) \end{cases} \quad \begin{cases} F^{[m]} - H^{[m]} > 0 & x \in (a_2, b_2) \\ F^{[m]} - H^{[m]} = 0 & x \notin (a_2, b_2) \end{cases}. \tag{A2}$$

Evaluating  $U(p)$  at  $p_v$ , we have

$$\begin{aligned}
U(p_v) &= \mu V(p_v) + E\phi(\tilde{w}) - [p_v E\phi(\tilde{z}) + (1-p_v)E\phi(\tilde{y})] \\
&= E\phi(\tilde{w}) - [p_v E\phi(\tilde{z}) + (1-p_v)E\phi(\tilde{y})] \\
&= p_v [E\phi(\tilde{w}) - E\phi(\tilde{z})] + (1-p_v)[E\phi(\tilde{w}) - E\phi(\tilde{y})] \\
&= p_v \int_a^b \phi(x) d[F(x) - H(x)] + (1-p_v) \int_a^b \phi(x) d[F(x) - G(x)] \\
&= p_v \int_a^b (-1)^{m+1} \phi^{(m)}(x) [H^{[m]}(x) - F^{[m]}(x)] dx + (1-p_v) \int_a^b (-1)^{n+1} \phi^{(n)}(x) [G^{[n]}(x) - F^{[n]}(x)] dx \\
&= p_v \int_{a_2}^{b_2} (-1)^{m+1} \phi^{(m)}(x) [H^{[m]}(x) - F^{[m]}(x)] dx + (1-p_v) \int_{a_1}^{b_1} (-1)^{n+1} \phi^{(n)}(x) [G^{[n]}(x) - F^{[n]}(x)] dx \\
&< 0.
\end{aligned}$$

The inequality in above is from (A1) and (A2). Because  $U(p)$  is strictly decreasing in  $p$ , we have  $p_u < p_v$ , a contradiction. Therefore,  $u(x)$  must be  $(n/m)$ th-degree Ross more risk averse than  $v(x)$ .

Q.E.D.

### Proof of Theorem 3.

The “if” part: Suppose that  $u(x)$  is  $(n/m)$ th-degree Ross more risk averse than  $v(x)$ . By Lemma 2, there exists  $\lambda > 0$  and  $\phi(x)$  such that  $u(x) \equiv \lambda v(x) + \phi(x)$ , where  $(-1)^{m+1} \phi^{(m)}(x) \leq 0$  and  $(-1)^{n+1} \phi^{(n)}(x) \geq 0$  for all  $x$ . Note that  $\phi(x)$  is both weakly  $m$ th-degree risk tolerant – meaning that  $-\phi(x)$  is weakly  $m$ th-degree risk averse – and weakly  $n$ th-degree risk averse.

We use proof by contradiction. To prove  $\alpha_u \geq \alpha_v$ , assume  $\alpha_u < \alpha_v$  instead. Note

$$Eu(\tilde{w}(\alpha_u)) - Eu(\tilde{w}(\alpha_v)) = \lambda [Ev(\tilde{w}(\alpha_u)) - Ev(\tilde{w}(\alpha_v))] + [E\phi(\tilde{w}(\alpha_u)) - E\phi(\tilde{w}(\alpha_v))].$$

The first bracket in the above expression is negative because the expected utility of  $v(x)$  is maximized at  $\alpha_v$ . Under the assumption  $\alpha_u < \alpha_v$ , the second bracket is nonpositive because that  $\phi(x)$  is both weakly  $m$ th-degree risk tolerant and weakly  $n$ th-degree risk averse, and that  $\tilde{w}(\alpha_v)$  is sequentially less  $n$ th-degree risky and more  $m$ th-degree risky than  $\tilde{w}(\alpha_u)$ . So

$Eu(\tilde{w}(\alpha_u)) - Eu(\tilde{w}(\alpha_v)) < 0$ , which contradicts that  $\alpha_u$  is the optimal choice for  $u(x)$ . Therefore, it must be the case that  $\alpha_u \geq \alpha_v$ .

The “only if” part: Suppose that  $\alpha_u \geq \alpha_v$  for every wealth path  $\tilde{w}(\alpha)$  where, as  $\alpha$  increases,  $\tilde{w}(\alpha)$  becomes sequentially less  $n$ th-degree risky and more  $m$ th-degree risky. To prove that  $u(x)$  is  $(n/m)$ th-degree Ross more risk averse than  $v(x)$ , assume otherwise. Then, according to Lemma 3, there exist  $\mu > 0$ ,  $[a_1, b_1] \subset (a, b)$  and  $[a_2, b_2] \subset (a, b)$  such that  $\phi(x) \equiv u(x) - \mu v(x)$  satisfies

$$\begin{aligned}
(-1)^{n+1} \phi^{(n)}(x) &< 0 && \text{for all } x \in [a_1, b_1] \\
(-1)^{m+1} \phi^{(m)}(x) &> 0 && \text{for all } x \in [a_2, b_2]
\end{aligned} \tag{A3}$$

Because  $\alpha_u \geq \alpha_v$ ,  $\tilde{w}(\alpha_u)$  is sequentially less  $n$ th-degree risky and more  $m$ th-degree risky than  $\tilde{w}(\alpha_v)$ . So there exists  $\tilde{z}$  such that  $\tilde{w}(\alpha_u)$  has less  $n$ th-degree risk than  $\tilde{z}$ , and  $\tilde{z}$  has more  $m$ th-degree risk than  $\tilde{w}(\alpha_v)$ .

Denote the CDFs for  $\tilde{w}(\alpha_u)$ ,  $\tilde{w}(\alpha_v)$  and  $\tilde{z}$  as  $F(x)$ ,  $G(x)$  and  $H(x)$ , respectively. Due to the arbitrariness of the wealth path  $\tilde{w}(\alpha)$ , we can choose  $F(x)$ ,  $G(x)$  and  $H(x)$  such that

$$\begin{cases} H^{[n]} - F^{[n]} > 0 & x \in (a_1, b_1) \\ H^{[n]} - F^{[n]} = 0 & x \notin (a_1, b_1) \end{cases} \quad \begin{cases} H^{[m]} - G^{[m]} > 0 & x \in (a_2, b_2) \\ H^{[m]} - G^{[m]} = 0 & x \notin (a_2, b_2) \end{cases}. \quad (\text{A4})$$

Then we have

$$\begin{aligned} & Eu(\tilde{w}(\alpha_u)) - Eu(\tilde{w}(\alpha_v)) \\ &= \mu [Ev(\tilde{w}(\alpha_u)) - Ev(\tilde{w}(\alpha_v))] + [E\phi(\tilde{w}(\alpha_u)) - E\phi(\tilde{w}(\alpha_v))] \\ &< E\phi(\tilde{w}(\alpha_u)) - E\phi(\tilde{w}(\alpha_v)) \\ &= [E\phi(\tilde{w}(\alpha_u)) - E\phi(\tilde{z})] + [E\phi(\tilde{z}) - E\phi(\tilde{w}(\alpha_v))] \\ &= \int_a^b \phi(x) d[F(x) - H(x)] + \int_a^b \phi(x) d[H(x) - G(x)] \\ &= \int_a^b (-1)^{n+1} \phi^{(n)}(x) [H^{[n]}(x) - F^{[n]}(x)] dx + \int_a^b (-1)^{m+1} \phi^{(m)}(x) [G^{[m]}(x) - H^{[m]}(x)] dx \\ &= \int_{a_1}^{b_1} (-1)^{n+1} \phi^{(n)}(x) [H^{[n]}(x) - F^{[n]}(x)] dx + \int_{a_2}^{b_2} (-1)^{m+1} \phi^{(m)}(x) [G^{[m]}(x) - H^{[m]}(x)] dx \\ &< 0. \end{aligned} \quad (\text{A5})$$

The first inequality above is from the fact that  $\alpha_v$  is the optimal choice for  $v(x)$ . The second inequality above is from (A3) and (A4). Note that (A5) implies that  $\alpha_u$  is *not* the optimal choice for  $u(x)$ , a contradiction. Therefore, it must be the case that  $u(x)$  is  $(n/m)$ th-degree Ross more risk averse than  $v(x)$ .

Q.E.D.

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