Fractional Group Identification[*]

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Abstract

We study group identification problems, the objective of which is to classify agents into groups based on individual opinions. Our point of departure from the literature is to allow membership to be fractional, to qualify the extent of belonging. Examining implications of independence of irrelevant opinions, we identify and characterize four nested families of rules. The four families include the weighted-average rules, which are obtained by taking a weighted average of all entries of a problem, and the fractional consent rules, which adapt the consent rules from the binary model to our multinary setup, balancing two principles in group identification, namely liberalism and social consent. Existing characterizations of the one-vote rules, the consent rules, and the liberal rule follow from ours.

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1 Introduction

1.1 Motivation

Ethno-racial identity is a multi-faceted issue and it is not rare to find individuals seeing themselves as belonging to several groups, whether it be due to racial or cultural reasons. Over 9 million Americans report more than one race as their identity and from 2000 to 2010, their population grew three times as fast as the monoracial population.[1] The 2011 United Kingdom Census reveals that the population with mixed heritage approximately doubled over a decade[2] and around 7 percent of children grow up in multi-ethnic households.[3] In Brazil, as much as 43.4 percent of the 190-million population self-identify with multiple racial backgrounds.[4] These increasing numbers of individuals of mixed race direct our attention to a question: How much does each racial group contribute to one's identity?

The group identification literature, initiated by a seminal work of Kasher and Rubinstein (1997), takes a social choice approach to the problem of classifying individuals into (racial) groups. A typical group identification problem consists of individual opinions on who belong to which group and the objective is to aggregate them into a single social decision. One may propose some properties that a reasonable (decision) rule should satisfy and characterize a rule or a family of rules by means of such properties. Starting from the binary case where the question is essentially how to partition the agents into members and non-members of a given group (Kasher and Rubinstein, 1997; Samet and Schmeidler, 2003; Sung and Dimitrov, 2005; Dimitrov et al., 2007; Houy, 2007; and Çengelci and Sanver, 2010), the literature has developed to allow the group under study to vary (Miller, 2008; Cho and Ju, 2015) and identify three or more groups simultaneously (Cho and Ju, 2017).

A limitation with the existing studies is that they only deal with deterministic

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[1]United States Census Bureau, *The Two or More Races Populations: 2010*, published 2012.

[2]United Kingdom Office for National Statistics, *2011 Census: Key Statistics and Quick Statistics for Local Authorities in the United Kingdom*, published 2013.

[3]United Kingdom Office for National Statistics, *2011 Census Analysis: What Does the 2011 Census Tell Us about Inter-ethnic Relationships?*, published 2014.

[4]Brazilian Institute of Geography and Statistics, *Demographic Census 2010: General Characteristics of the Population, Religion, and Persons with Disabilities*, published 2010.

membership. If an agent belongs to a group, his identity is fully determined by that group. Yet an individual's identity may not be so clear-cut; it may spread across several groups, each constituting a differing amount of identity. To take account of this possibility, we study fractional membership in a multinary identification model. In our model, it is possible, for instance, that $\frac{2}{3}$ of an agent's identity comes from group a and $\frac{1}{3}$ from group b.

The shortage and drawbacks of decision rules in the multinary literature also call for attention to fractional membership. A good inventory of rules are available for binary identification problems. Examples include the consent rules (Samet and Schmeidler, 2003), the agreement rules, and the nomination rules (Miller, 2008). In particular, the consent rules afford the flexibility of accommodating various degrees of liberalism and democracy. In contrast, the multinary literature has only the family of one-vote rules, most members of which induce a very skewed distribution of decision power across agents. Further, the multinary model does not admit rules that have strengths similar to those of the consent rules. This is due in part to the deterministic nature of membership, prohibiting multiple agents from being decisive on the same issue. One way of overcoming the limitation is to permit fractional membership and hence fractional decisiveness.

These observations motivate us to seek new possibilities that arise when fractional membership is introduced to the identification model, both multinary and binary. We identify four families of rules and justify them with axiomatic characterizations.

1.2 The Model

Specifically, we extend the multinary model of Cho and Ju (2017), where there are three or more groups to be identified. Each agent has an opinion, stating to which one of the groups each agent (including himself) belongs. An (identification) problem is simply a profile of such opinions and is represented by an $n \times n$ matrix, where entry (i, j) is agent i's opinion about agent j (n is the number of agents). A (social) decision specifies the agents' fractional membership to all groups; that is, to each agent corresponds a profile of non-negative fractions adding up to one, representing the extent to which the

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agent belongs to the groups.[5] Our main axiom for rules is independence of irrelevant opinions, which requires that identification of a given group should rely only on the opinions about that group. The axiom is implicit in the binary model of Kasher and Rubinstein (1997) and Samet and Schmeidler (2003) and the variable group model of Miller (2008). Cho and Ju (2017) explicitly formulate the axiom in a context where multiple groups are identified at the same time. The axiom has a natural analog in our fractional model: if the opinions on group a, say, remain the same, so should the fractional decisions on group a (while the fractional decisions on the other groups may vary).

1.3   Main Results

With independence of irrelevant opinions imposed, a new family of rules emerge that value individual opinions potentially differently to reach a social decision. A typical rule in this family is associated with a profile of weights, each corresponding to an agent. Given a problem, the rule determines agent i's membership by taking a weighted average of all entries in the problem, using i's weights the rule is equipped with. We call these rules the weighted-average rules. We show that the weighted-average rules are the only rules satisfying independence of irrelevant opinions and a full-range condition called deterministic full range (Theorem 1). Deterministic full range requires that for each agent i and each group a, the range of a rule should include a decision where i is a full (i.e., with fraction 1) member of group a; and the axiom is weaker than unanimity. The one-vote rules (Miller, 2008), which determine each agent's identity using one fixed entry of a problem, are special cases of the weighted-average rules whose distribu-

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[5]Note the asymmetric formulation of opinions and decisions: opinions are deterministic whereas decisions are fractional. One may model the fractional membership problem in such a way that opinions and decisions have the same algebraic structure, similar to and extending Rubinstein and Fishburn (1986). However, we introduce fractional membership only to decisions because we are interested in new possibilities that become available with a minimal departure from the existing models and because eliciting opinions in the fractional form from agents may be informationally demanding in some contexts (especially with a large number of agents). Some other strands of literature keep similar asymmetries between input (opinions) and output (decisions). In probabilistic public choice (Gibbard, 1977), agents submit their preferences over sure alternatives while a lottery over those alternatives are chosen as an outcome. In probabilistic assignment (Bogomolnaia and Moulin, 2001) and school choice (Abdulkadiroğlu and Sönmez, 2003), agents' preferences are defined over objects but they are assigned lotteries over the objects.
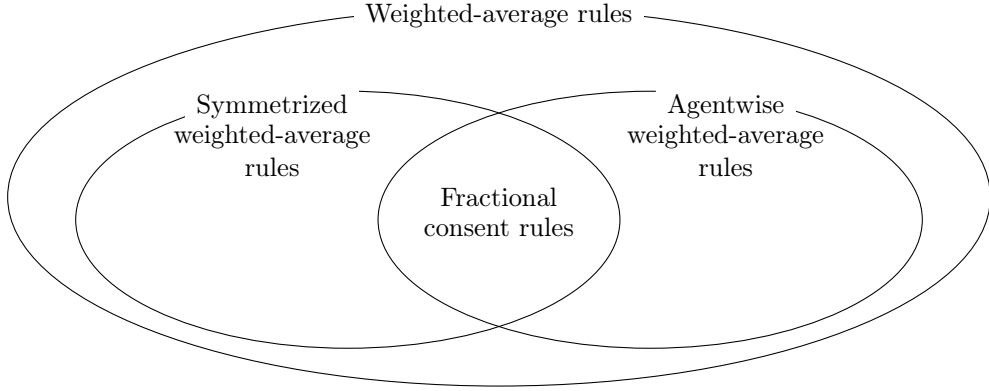
Figure 1: **Characterizations of four families of rules.** The weighted-average rules are characterized by independence of irrelevant opinions and deterministic full range. Further imposing symmetry leads to a subfamily of symmetrized weighted-average rules, and weak agentwise change to a subfamily of agentwise weighted-average rules. At the intersection of the two subfamilies lie the fractional consent rules. The latter rules are also the ones generated by the fractional approval functions.

tion of weights is degenerate. Thus, our characterization of the weighted-average rules generalizes that of the one-vote rules by Cho and Ju (2017).

Within the family of weighted-average rules, we invoke further axioms and narrow attention to two subfamilies. First, we consider symmetry, a fairness property requiring that agents' names should have no impact on membership. Symmetry isolates the family of symmetrized weighted-average rules (Theorem 2). This family is a convex set with five extreme points, which can be used as building blocks to construct a rule that suits our purpose at hand. Two rules stand out among the five. One is the liberal rule (Kasher and Rubinstein, 1997), which grants each agent the full right to choose his membership. Also known as self-identification, the liberal rule is the most common way of collecting information on ethnicity and race (for example, most censuses use it). The other rule is what we call the almost-column rule, according to which agent i's membership is the simple average of the entries in column i except entry (i,i), that is, all opinions about i except his self-opinion. The liberal and almost-column rules also have a connection to the fractional consent rules, which we discuss below. In the family of symmetrized weighted-average rules, the liberal rule is the only deterministic one and the main characterization of Cho and Ju (2017) follows as a corollary. Figure 1

summarizes our characterizations.

A drawback shared by some weighted-average rules motivates another subfamily. The definition of weighted-average rules allows agent i's membership to hinge on agent j's opinion about agent h. This is an issue when we pursue identification on an individual basis and it is foreshadowed by the axioms that give rise to the weighted-average rules, namely independence of irrelevant opinions and deterministic full range: none of the axioms has a component that enables a rule to distinguish columns from rows in identification problems. Therefore, we invoke another axiom, weak agentwise change, which requires that there exist at least one case where *opinions about i* are decisive enough to influence his membership in some deterministic fashion. Weak agentwise change is implied by some axioms studied in the literature, such as agentwise unanimity by Kasher and Rubinstein (1997) ("consensus" as the authors call it) and a different type of an independence axiom by Samet and Schmeidler (2003). Weak agentwise change yields the family of agentwise weighted-average rules (Theorem 3). For an agentwise weighted-average rule, the weights determining agent i's membership are placed only on those entries of a problem that directly concern him, that is, those in column i.

We also extend an important family of rules from the binary identification literature to our setting. To distinguish notions, let us call an aggregation function that operates on binary problems an approval function. In the binary setup, Samet and Schmeidler (2003) introduce the consent approval functions, for which the consent quotas that parametrize the family serve to balance liberalism and social consent. A consent approval function socially approves an agent's self-opinion if a consent quota is met and otherwise put him in the group opposite to his choice. Adapting such a decision scheme to a multinary environment is not straightforward because an exogenous, asymmetric tie-breaking rule should come into play whenever an agent's view of himself is not accompanied by sufficient support from others.

To circumvent this difficulty, note that a multinary identification problem contains several pieces of binary information: it tells us, for each group, who belong to that group. Therefore, a multinary problem can equivalently be represented as a list of binary problems. Then we may apply approval functions to each of the binary problems

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separately and combine the resulting binary decisions into a single multinary decision. In this "decomposition" process, one may attempt to work with the consent approval functions by Samet and Schmeidler (2003) but all of them except the liberal one turn out to fail.

At the heart of the failure is the deterministic nature of the consent approval functions, which motivates a fractional extension. In the binary setting, we propose the fractional consent approval functions and characterize them by the same set of axioms as Samet and Schmeidler (2003): monotonicity, agentwise identification ("independence" as the authors call it), and symmetry (Theorem 4). When applied in the decomposition process, the fractional consent approval functions generate a family of rules parametrized by a number between zero and one (Theorem 5). The parameter captures the degree of compromise between liberalism and social consent in the multinary setup and therefore we call such rules the fractional consent rules. Denoting by w the parameter associated with a fractional consent rule, each agent i determines fraction w of his membership and the remaining fraction $1 - w$ rests in the hands of the other agents, each responsible for fraction $\frac{1-w}{n-1}$ of i's membership. The fractional consent rules are a special case of both the symmetrized weighted-average rules and the agentwise weighted-average rules. Thus, they are the only rules satisfying independence of irrelevant opinions, deterministic full range, symmetry, and weak agentwise change (Theorem 6).

Our results indicate that the small departure of permitting decisions to be fractional unveils a new family of rules that afford the flexibility of basing individual identity on several opinions. This strength cannot be achieved by deterministic rules because of their very deterministic nature: two or more entries of a problem cannot be decisive at the same time. The weighted-average rules, on the other hand, can compromise on their decisiveness by assigning weights to them and the distribution of weights represents the emphasis we place on one's self-opinion and others' opinions about him.

The connection between Cho and Ju (2017) and our results parallels that between Gibbard-Satterthwaite Impossibility Theorem (Gibbard, 1973; Satterthwaite, 1975) and Gibbard Random Dictatorship Theorem (Gibbard, 1977) in social choice theory. In a voting environment where agents submit ordinal preferences over alternatives, the

first theorem says, each strategy-proof voting scheme is dictatorial (assuming that the scheme includes at least three outcomes). When a voting scheme is allowed to be probabilistic, the second theorem shows, strategy-proofness (together with ex post efficiency) implies random dictatorship: a fixed distribution over agents determines the dictator at random, who then chooses his most preferred alternative. In sum, the passage from the deterministic to the probabilistic setup adds randomizations over the voting schemes that are already available in the deterministic setup. The weighted-average rules generalize the one-vote rules in much the same way as random dictatorship does dictatorship. The analogy, however, weakens when symmetry is invoked. The axiom is compatible with independence of irrelevant opinions in the deterministic identification model whereas a similar axiom, anonymity, is not compatible with strategy-proofness in the deterministic voting model. Further, the fractional identification model admits symmetric rules—e.g., the almost-column rule—that cannot be expressed as a convex combination of symmetric deterministic rules. The strategy-proof decision schemes in the deterministic voting model, namely dictatorial rules, constitute extreme points of the set of all strategy-proof voting schemes in the probabilistic voting model.

1.4   Related Literature

Group identification begins with an axiomatic analysis of Kasher and Rubinstein (1997). In a binary setup where agents are to be identified as members or non-members of a group under question, they characterize the liberal rule and derive an impossibility result. Samet and Schmeidler (2003) propose and characterize the consent rules that depending on the choice of parameters, can embed varying degrees of liberalism and social consent in decisions. Other papers studying the binary model include Sung and Dimitrov (2005), Dimitrov et al. (2007), Houy (2007), and Çengelci and Sanver (2010). On a more general domain where opinions can be "neutral", Ju (2010, 2013) explores decisiveness of an agent or a group of agents and characterizes self-dependency, a hallmark of liberalism, by a weaker set of axioms.

Most closely related to this paper is Cho and Ju (2017), who consider the multinary problem of identifying three or more groups simultaneously. Noting an implicit assumption in the binary model, they introduce independence of irrelevant opinions,

an adaptation of Arrow's (1951) independence axiom in preference aggregation theory to the group identification setting. They show that the one-vote rules uniquely satisfy the axiom (together with non-degeneracy) and that only the liberal rule obtains once symmetry is additionally imposed. Clearly, the weighted-average rules in this paper are a fractional counterpart of the one-vote rules and our characterizations extend Cho and Ju (2017).

On the other hand, Miller (2008) considers a variable group model with the focus on consistency (or "separability" as he calls it), a relational property that binds decisions across groups; e.g., the decision on group "a and b" should be the conjunction of the decisions on group "a" and on group "b". The one-vote rules first appear in this context and Miller (2008) shows that they are the only rules satisfying consistency (together with non-degeneracy). While different axioms are imposed in different setups, Miller (2008) and Cho and Ju (2017) both characterize the one-vote rules. To clarify the distinction, Cho and Ju (2015) introduce an extended setup that subsumes the two papers and derive stronger results.

The rest of the paper proceeds as follows. In Section 2, we set up the model and introduce various rules and axioms. In Section 3, we investigate the connection between independence of irrelevant opinions and a stronger, simplifying property called decomposability. Characterizations of the weighted-average rules and two subfamilies thereof are in Section 4 and those of the fractional consent rules are in Section 5.

2   The Model

A finite set of agents seek to determine their membership to several groups, with membership being allowed to be fractional. Let $N \equiv \{1, \cdots, n\}$ be the set of agents and $G \equiv \{k_1, \cdots, k_m\}$ the set of groups ($n \geq 2$ and $m \geq 3$). Agents are denoted by i and j, and groups by k and ℓ. Each agent $i \in N$ has an **opinion** $P_i \equiv (P_{ij})_{j \in N} \in G^N$, where for each $j \in N$, $P_{ij} = k \in G$ means that i believes j to be a member of group k. A (multinary) **problem** is an opinion profile $P \equiv (P_i)_{i \in N}$. We often treat individual opinions as row vectors ($1 \times n$) and problems as matrices ($n \times n$), with P_{ij} being entry (i, j) of P. Let $\mathcal{P} \equiv G^{N \times N}$ be the set of all problems. For each $P \in \mathcal{P}$ and each

$i \in N$, let $\boldsymbol{P^i} \equiv (P_{ji})_{j \in N}$ be column i of P, which is the opinions about i; and let $\boldsymbol{P^{-i}} \equiv (P^j)_{j \in N \setminus \{i\}}$ be all the columns of P but column i.

Let $\boldsymbol{\Delta(G)}$ be the set of distributions over G. A (social) **decision** is a profile $\boldsymbol{x} \equiv (x_i)_{i \in N}$, where for each $i \in N$, $\boldsymbol{x_i} \equiv (x_{ik})_{k \in G} \in \Delta(G)$ is agent i's fractional membership to all groups. Let $\boldsymbol{X} \equiv (\Delta(G))^N$ be the set of all decisions. Given decision $x \in X$, for each $i \in N$ and each $k \in G$, we interpret x_{ik} as an index measuring the extent to which agent i belongs to group k (in particular, we do *not* interpret x_{ik} as the probability of agent i belonging to group k). An agent may affirm affiliation to multiple racial groups and our formulation with fractions attempts to capture the composition of his identity.

A (social decision) **rule** is a mapping $f : \mathcal{P} \to X$, associating with each problem a decision. Decision $x \in X$ is **deterministic** if for each $i \in N$, x_i is a degenerate distribution (i.e., for some $k \in G$, $x_{ik} = 1$). A rule is **deterministic** if it always produces a deterministic decision. Decisions and rules are **fractional** if they are not deterministic.

In our model, decisions are allowed to be fractional while opinions are not. One may consider an alternative setup where fractional opinions are aggregated into a fractional decision, which adapts algebraic aggregation theory of Rubinstein and Fishburn (1986) to group identification. We do not pursue the alternative for two reasons. First, we seek to explore advantages of decision rules that emerge with only a minimal departure from the literature. As our analysis shows, fractional decisions suffice to unveil new possibilities. Second is the clear simplicity our model has. With only deterministic opinions permitted, the task of formulating an opinion is simple: an agent need only to choose one group for each agent. On the other hand, a fractional opinion consists of n distributions over the groups, which may be too demanding an informational requirement in some contexts, especially when a large number of agents are involved. There are models in other literature where the input and outcome for a collective decision problem have asymmetric structures. Gibbard (1977) studies a probabilistic public choice model where agents have preferences over sure alternatives and a lottery over those alternatives are to be chosen. The literature on probabilistic assignment (Bogomolnaia and Moulin, 2001) and school choice (Abdulkadiroğlu and Sönmez, 2003)

applies Gibbard's (1977) approach to the allocation of indivisible commodities.

Below we use the following notation. For each $k \in G$, $\boldsymbol{k_{n \times n}}$ is a problem in $\mathcal{P}$ consisting of k's only; a row vector $\boldsymbol{k_{1 \times n}}$ and a column vector $\boldsymbol{k_{n \times 1}}$ are similarly defined. For all $x_i, y_i \in \Delta(G)$, let $||\boldsymbol{x_i} - \boldsymbol{y_i}|| = \max_{k \in G} |x_{ik} - y_{ik}|$. Note that $||x_i - y_i|| = 1$ if and only if for some $k \in G$, one of x_{ik} and y_{ik} is 1 and the other is 0. In particular, if there is $k \in G$ with $x_{ik}, y_{ik} > 0$, then $||x_i - y_i|| < 1$.

2.1 Rules

In the deterministic group identification models (Samet and Schmeidler, 2003; Miller, 2008; Cho and Ju, 2017), both binary and multinary, the following rules have been studied. The **liberal rule**, denoted $\boldsymbol{L}$, classifies each agent into the group of his own choice; i.e., for each $P \in \mathcal{P}$, each $i \in N$, and each $k \in G$, $P_{ii} = k$ implies $L_{ik}(P) = 1$. A rule f is a **one-vote rule** if each agent's membership is determined by a single fixed entry for all problems; i.e., for each $i \in N$, there is $(j, h) \in N^2$ such that for each $P \in \mathcal{P}$ and each $k \in G$, $P_{jh} = k$ implies $f_{ik}(P) = 1$—we call (j, h) the **decisive entry** for i. The decisive entry for i is not required to be in column i, in which case, the membership decision for i is based on an opinion about some other agent. An **agentwise one-vote rule** is a one-vote rule such that for each $i \in N$, the decisive entry for agent i is in column i.

We can easily extend the one-vote rules to the fractional setup by introducing "fractional decisiveness". A rule f is a **weighted-average rule** if for each $i \in N$, there exists $\alpha_i \in \Delta(N^2)$ such that for each $P \in \mathcal{P}$ and each $k \in G$, $f_{ik}(P) = \sum_{(j,h) \in N^2 : P_{jh} = k} \alpha_i(j, h)$ (where $\alpha_i(j, h)$ is the weight assigned to entry (j, h) by α_i). We call $(\alpha_i)_{i \in N}$ the **weights associated with f**. By definition, a weighted-average rule is a convex combination[6] of the one-vote rules.

An interesting subfamily of the weighted-average rules consists of those rules that do not discriminate agents on the basis of their names (the latter property is called symmetry and is defined below). These rules are most succinctly described as a convex combination of several extreme points of the subfamily, which we define now. The

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[6]Convex combinations of rules are defined in the usual way. Given rules f and g and $\lambda \in [0, 1]$, the rule $\lambda f + (1 - \lambda)g$ is defined as for each $P \in \mathcal{P}$, each $i \in N$, and each $k \in G$, $(\lambda f + (1 - \lambda)g)_{ik}(P) = \lambda f_{ik}(P) + (1 - \lambda)g_{ik}(P)$.

almost-column rule is the rule f such that for each $P \in \mathcal{P}$ and each $i \in N$, $f_i(P)$ is the simple average of $(P_{ji})_{j \in N \setminus \{i\}}$ (i.e., for each $k \in G$, $f_{ik}(P) = \frac{1}{n-1} |\{j \in N \setminus \{i\} : P_{ji} = k\}|$). The **almost-row rule** is the rule f such that for each $P \in \mathcal{P}$ and each $i \in N$, $f_i(P)$ is the simple average of $(P_{ij})_{j \in N \setminus \{i\}}$. The **almost-diagonal rule** is the rule f such that for each $P \in \mathcal{P}$ and each $i \in N$, $f_i(P)$ is the simple average of $(P_{jj})_{j \in N \setminus \{i\}}$. The **almost-off-diagonal rule** is the rule f such that for each $P \in \mathcal{P}$ and each $i \in N$, $f_i(P)$ is the simple average of $(P_{jh})_{j,h \in N \setminus \{i\}, j \neq h}$ (this rule is well-defined only when $n \geq 3$). A rule is a **symmetrized weighted-average rule** if it is a convex combination of the liberal, almost-column, almost-row, almost-diagonal, and almost-off-diagonal rules.[7] The family of symmetrized weighted-average rules has only one deterministic rule: the liberal rule.

As is the case with the one-vote rules, a weighted-average rule is allowed to put positive weights on the opinions not about agent i (i.e., entries outside column i) when determining i's membership, which is not desirable when we seek a decision on an individual basis. A rule f is an **agentwise weighted-average rule** if it is an weighted-average rule with the associated weights $(\alpha_i)_{i \in N}$, say, such that for each $i \in N$, α_i puts positive weights only on the opinions about i (i.e., $\sum_{j \in N} \alpha_i(j, i) = 1$).

In the binary setup, Samet and Schmeidler (2003) introduce the family of consent rules. The family is parametrized by two numbers, each of which serves as a quota to be satisfied in order for an agent's self-opinion to be socially approved (different quotas apply depending on whether the agent sees himself as belonging to a given group). While the consent rules do not easily extend to the deterministic multinary setup (see Section 5 for details), the fractional multinary setup admits a generalization. Say that a rule is a **fractional consent rule** if there is $w \in [0, 1]$ such that for each $P \in \mathcal{P}$, each $i \in N$, and each $k \in G$, $f_{ik}(P) = w\mathbf{1}_{\{P_{ii} = k\}} + \frac{1-w}{n-1} |\{j \in N \setminus \{i\} : P_{ji} = k\}|$, where $\mathbf{1}_{\{P_{ii} = k\}}$ is an indicator function. Denote by $\boldsymbol{f^w}$ the fractional consent rule associated with w. Under a fractional consent rule, each i's membership is a weighted average of his self-opinion and the others' opinions about him; the others' opinions about him are

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[7] To give an explicit formula, a rule f is a symmetrized weighted-average rule if it is a weighted-average rule with the associated weights $(\alpha_i)_{i \in N}$, say, and there exist $s, t, u, v, w \in [0, 1]$ with $s + (n-1)t + (n-1)u + (n-1)w + (n^2 - 3n + 2)w = 1$ satisfying the following: for each $i \in N$, (i) $\alpha_i(i, i) = s$; (ii) for each $j \in N \setminus \{i\}$, $\alpha_i(j, j) = t$; (iii) for each $j \in N \setminus \{i\}$, $\alpha_i(j, i) = u$; (iv) for each $j \in N \setminus \{i\}$, $\alpha_i(i, j) = v$; and (v) for all $j, h \in N \setminus \{i\}$ with $j \neq h$, $\alpha_i(j, h) = w$.

all valued equally (but not necessarily as equally as i's self-opinion); and the weights $(w, 1-w)$ applied when taking a weighted average is the same across all agents. Clearly, the fractional consent rules constitute the intersection of the symmetrized weighted-average rules and the agentwise weighted-average rules.

2.2 Axioms

Does the identity of an Asian depend on whether other agents view him as a White or as a Black? If agent i changes his opinion about agent j from White to Black, should the change be considered additional support for j's membership to the group of Asians? In the context of ethnic classification, there is no clear order that defines the relationship among ethnic groups.[8] Therefore, each ethnicity should be treated as an independent entity and when identifying a group, changes in opinions about other groups should be dismissed as irrelevant. In fact, the latter inter-group independence is implicit in the binary identification literature such as Kasher and Rubinstein (1997) and Samet and Schmeidler (2003)[9]: for identification of a group, opinions only about that group are solicited, thus ruling out the possibility from the outset that opinions about other groups may affect membership to the group. In the deterministic identification model, Cho and Ju (2017) formulate the independence property as an axiom and call it independence of irrelevant opinions, noting its resemblance to Arrow's (1951) independence axiom in preference aggregation theory. In our fractional setting, the independence axiom requires that an agent's *fractional* membership to a group should be invariant with respect to changes in the opinions about the other groups. A formal expression of this idea is as follows.

Independence of Irrelevant Opinions. Let $P, P' \in \mathcal{P}$ and $k \in G$. Suppose that for all $i, j \in N$, $P_{ij} = k$ if and only if $P'_{ij} = k$. Then for each $i \in N$, $f_{ik}(P) = f_{ik}(P')$.

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[8]On surface, one could argue, based on skin color, that Asian is between White and Black, to justify a (linear) order "White–Asian–Black". Yet the claim is immediately challenged by genetic distances between human populations. The genetic variation across White and Black is lower than that across White and Asian, suggesting that White is closer to Black than to Asian; see, e.g., Tishkoff and Kidd (2004), Kidd et al. (2004), and Ayub et al. (2003). In fact, the genetic distance map across ethnic groups does not fit into any order.

[9]The following papers also assume the same inter-group independence in the binary model: Sung and Dimitrov (2005), Dimitrov et al. (2007), Houy (2007), Çengelci and Sanver (2010).

Concerning fairness, we require that all agents be treated symmetrically regardless of their names. This idea is expressed by a permutation over the agents, which represents a change in their names. Let $\pi : N \to N$ be a permutation. For each $P \in \mathcal{P}$, let $\boldsymbol{P_\pi} \equiv (P_{\pi(i),\pi(j)})_{i,j \in N}$ and $\boldsymbol{f_\pi(P)} \equiv \big(f_{\pi(i)}(P)\big)_{i \in N}$ (for each $x \in X$, x_π is defined similarly). The following property, due to Samet and Schmeidler (2003), says that name changes should not affect the membership decision for the agents.

Symmetry. For each $P \in \mathcal{P}$ and each permutation $\pi : N \to N$, $f(P_\pi) = f_\pi(P)$.

In the deterministic case, independence of irrelevant opinions has strong implications when combined with the following axiom, which requires that no agent's membership should be fixed.

Non-degeneracy. For each $i \in N$, there are $P, P' \in \mathcal{P}$ such that $f_i(P) \neq f_i(P')$.

Non-degeneracy, however, is too weak in the fractional setup to pin down a family of rule we are interested in. Thus, we consider a slightly stronger requirement: when restricted to deterministic decisions, a rule should have a full range for each agent.

Deterministic Full Range. For each $i \in N$ and each $k \in G$, there is $P \in \mathcal{P}$ such that $f_{ik}(P) = 1$.

Deterministic full range says nothing about what decision a rule should assign to particular problems. Yet in the presence of independence of irrelevant opinions, the axiom indeed prescribes reasonable decisions to those problems that consist of a single group. The following property is stronger than deterministic full range.

Unanimity. For each $i \in N$ and each $k \in G$, $f_{ik}(k_{n \times n}) = 1$.

While independence of irrelevant opinions leads to quite a small class of rules, it has no bite on how opinions about agent i should be valued when agent j's membership is determined. In fact, none of the axioms introduced so far compels a rule to distinguish columns from rows, which is why a rule satisfying independence of irrelevant opinions may put a positive weight on the information that does not directly concern agent j. Now we introduce two axioms that allow us to remedy such problems. In a binary setup, Kasher and Rubinstein (1997) consider an agent-by-agent version of the above unanimity property, requiring that if all agents have the same opinion about an agent,

then the social decision should respect that opinion. This property can be adapted to the multinary setup as follows.

Agentwise Unanimity. For each $P \in \mathcal{P}$, each $i \in N$, and each $k \in G$, if $P^i = k_{n \times 1}$, then $f_{ik}(P) = 1$.[10]

To weaken agentwise unanimity, consider problems $P, P' \in \mathcal{P}$. Suppose that for some agent i, P and P' differ only in the opinions about i (i.e., $P^i \neq P'^i$ and $P^{-i} = P'^{-i}$). If the membership decision for i is never affected as the problem changes from P to P', one could argue that opinions about i play no role in determining his identity, which is not desirable. Thus, we require that there exist a case where opinions about i are decisive enough to influence his membership in some deterministic fashion.

Weak Agentwise Change. For each $i \in N$, there are $P, P' \in \mathcal{P}$ such that $P^i \neq P'^i$, $P^{-i} = P'^{-i}$, and $\|f_i(P) - f_i(P')\| = 1$.

Even when combined with unanimity, weak agentwise change is weaker than agentwise unanimity. Samet and Schmeidler (2003) propose a property saying that to determine an agent's membership, a rule should focus only on the opinions about him—they call the property "independence". Clearly, the independence axiom has direct bearing on decisions across agents and it is stronger than weak agentwise change in the presence of unanimity.

3   Decomposability

A simple way of solving a multinary problem is to transform it into a list of "binary" problems (one for each group), obtain a "binary" decision for each binary problem, and then combine all binary decisions into a multinary decision. This approach, called *decomposition*, turns out to be very close to the requirement of independence of irrelevant opinions in the deterministic setup (Cho and Ju, 2017). In this section, we re-examine that relationship in the fractional case.

Given $P \in \mathcal{P}$, for each $k \in G$, the **binary problem concerning group k derived from P**, denoted $\boldsymbol{B^{P,k}} \in \{0,1\}^{N \times N}$, is defined as for all $i, j \in N$, $B_{ij}^{P,k} = 1$ if $P_{ij} = k$

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[10]Kasher and Rubinstein (1997) call this property consensus.

and $B_{ij}^{P,k} = 0$ otherwise. Let $\mathcal{B} \equiv \{0,1\}^{N \times N}$ be the set of all binary problems. Our definition of opinion requires that each agent be a member of one and only one group. Thus, each multinary problem P can alternatively be represented by m binary problems $(B^{P,k})_{k \in G}$. For each $B \in \mathcal{B}$, let $|\boldsymbol{B}| \equiv \sum_{i,j \in N} B_{ij}$ be the number of ones in problem B. A **binary decision** is a profile $b \equiv (b_i)_{i \in N} \in [0,1]^N$, where for each $i \in N$, b_i is agent i's fractional membership to the group under question. A list of m binary decisions $(b^k)_{k \in G}$ translates to a proper multinary decision if and only if for each $i \in N$, $\sum_{k \in G} b_i^k = 1$.

Since a multinary problem can be expressed as a collection of binary problems, one may ask if a rule can also be expressed as a function operating on binary problems. Formally, an **approval function** is a mapping $\varphi : \mathcal{B} \to [0,1]^N$, associating with each binary problem $B \in \mathcal{B}$ a binary decision $\varphi(B) \in [0,1]^N$. A natural counterpart of the weighted-average rules for approval functions are those with fixed weights on the entries of a binary problem: an approval function φ is a **weighted-average approval function** if for each $i \in N$, there exists $\alpha_i \in \Delta(N^2)$ such that for each $B \in \mathcal{B}$, $\varphi_i(B) = \sum_{(j,h) \in N^2 : B_{jh}=1} \alpha_i(j,h)$.

The following property requires that a rule be expressed in the form of an approval function.

Decomposability. There is an approval function $\varphi : \mathcal{B} \to [0,1]^N$ such that for each $P \in \mathcal{P}$, each $i \in N$, and each $k \in G$, $f_{ik}(P) = \varphi_i(B^{P,k})$.

In this case, we say that $\boldsymbol{f}$ **is represented by** $\boldsymbol{\varphi}$ and $\boldsymbol{\varphi}$ **generates** $\boldsymbol{f}$. Clearly, a weighted-average rule is decomposable, represented by a weighted-average approval function. Further, decomposability is stronger than independence of irrelevant opinions. An approval function can serve to represent a decomposable rule only if it satisfies a number of properties. Exploring the latter properties is instructive since it simplifies our investigation of independence of irrelevant opinions and decomposability to that of properties of approval functions, which are more analytically tractable. To define such properties, an approval function φ is **m-unit-additive** if for all m binary problems $B^1, \ldots, B^m \in \mathcal{B}$, $\sum_{k \in G} B^k = 1_{n \times n}$ implies $\sum_{k \in G} \varphi(B^k) = 1_{1 \times n}$. It is **unanimous** if $\varphi(0_{n \times n}) = 0_{1 \times n}$ and $\varphi(1_{n \times n}) = 1_{1 \times n}$. Given $B \in \mathcal{B}$, the **dual binary problem of** $\boldsymbol{B}$ is denoted $\overline{\boldsymbol{B}} \equiv 1_{n \times n} - B$ (i.e., for all $i, j \in N$, $(\overline{B})_{ij} = 1 - B_{ij}$); similarly, given $b \in [0,1]^N$, the **dual binary decision of** $\boldsymbol{b}$ is denoted $\overline{\boldsymbol{b}} \equiv 1_{1 \times n} - b$. The **dual of** $\boldsymbol{\varphi}$,

denoted by $\boldsymbol{\varphi^d}$, is the approval function such that for each $B \in \mathcal{B}$, $\varphi^d(B) = \overline{\varphi(\overline{B})}$. We say that φ is **self-dual** if $\varphi = \varphi^d$. Finally, φ is **monotonic** if for all $B, B' \in \mathcal{B}$ such that $B \leq B'$, $\varphi(B) \leq \varphi(B')$.

Now we show that the ability of an approval function to represent a decomposable rule is equivalent to m-unit-additivity.

Proposition 1. *An approval function represents a decomposable rule if and only if it is m-unit-additive.*

Proof. First, we prove the "only if" part. Suppose that an approval function φ represents a decomposable rule f. Let $B^1, \ldots, B^m \in \mathcal{B}$ be such that $\sum_{k \in G} B^k = 1_{n \times n}$. Then there exists $P \in \mathcal{P}$ such that for each $k \in G$, $B^k = B^{P,k}$. Since φ represents f, for each $i \in N$ and each $k \in G$, $f_{ik}(P) = \varphi_i(B^{P,k})$. Thus, by the definition of fractional decisions, $\sum_{k \in G} \varphi_i(B^k) = \sum_{k \in G} \varphi_i(B^{P,k}) = \sum_{k \in G} f_{ik}(P) = 1$ and φ is m-unit-additive.

Next, to prove the "if" part, let φ be an m-unit-additive approval function. Define a rule f using φ as follows: for each $P \in \mathcal{P}$, each $i \in N$, and each $k \in G$, $f_{ik}(P) = \varphi_i(B^{P,k})$. Then f is well-defined by m-unit-additivity of φ. Thus, f is decomposable and is represented by φ. $\qquad\square$

In the deterministic setup, decomposability is equivalent to the combination of independence of irrelevant opinions and non-degeneracy. The latter equivalence, however, fails in our fractional model. The following example makes this point.

Example 1 (*A rule satisfying independence of irrelevant opinions and non-degeneracy, but not decomposability*)**.** We consider a variant of the one-vote rules. Fix $a \in G$ and for each $i \in N$, $(j_i, h_i) \in N^2$. A rule f^* is such that each agent $i \in N$ belongs to group a with at least fraction $\frac{1}{2}$ and the remaining fraction is determined solely by entry (j_i, h_i). That is, for each $P \in \mathcal{P}$ and each $i \in N$, (i) if $P_{j_i h_i} = a$, then $f_{ia}^*(P) = 1$; and (ii) if $P_{j_i h_i} = b \in G \backslash \{a\}$, then $f_{ia}^*(P) = \frac{1}{2} = f_{ib}^*(P)$. Clearly, f^* is independent of irrelevant opinions and non-degenerate. However, f^* is not decomposable. To see this, suppose, by contradiction, that f^* is decomposable and is represented by an approval function φ. Consider agent $1 \in N$. Since $f_{1a}^*(a_{n \times n}) = 1$, decomposability implies that $\varphi_1(1_{n \times n}) = 1$. Now let $b \in G \backslash \{a\}$. Since $f_{1b}^*(b_{n \times n}) = \frac{1}{2}$, decomposability again implies $\varphi_1(1_{n \times n}) = \frac{1}{2}$, a contradiction. $\qquad\triangle$

For our purpose, it is enough to identify an axiom that when combined with independence of irrelevant opinions, implies decomposability. It turns out that deterministic full range suffices. To prove this, we first establish that under the assumption of independence of irrelevant opinions, deterministic full range is equivalent to unanimity.

Lemma 1. *In the presence of independence of irrelevant opinions, deterministic full range is equivalent to unanimity.*

Proof. We only show that under the assumption of independence of irrelevant opinions, deterministic full range implies unanimity (the converse is clear). Let f be a rule satisfying independence of irrelevant opinions and deterministic full range. For each $k \in G$, define an approval function $\varphi^k : \mathcal{B} \to [0,1]^N$ as follows: for each $B \in \mathcal{B}$ and each $i \in N$, $\varphi_i^k(B) = f_{ik}(P)$, where $P \in \mathcal{P}$ is such that $B^{P,k} = B$. The m approval functions $(\varphi^k)_{k \in G}$ are well-defined because f satisfies independence of irrelevant opinions.

Step 1: *For each $i \in N$ and each $k \in G$, $\varphi_i^k(0_{n \times n}) = 0$.*

Let $i \in N$ and $k \in G$. By deterministic full range, there exists $P \in \mathcal{P}$ such that $f_{ik}(P) = 1$. Let $\ell, h \in G \setminus \{k\}$ be distinct. Let $P' \in \mathcal{P}$ be such that for all $j, j' \in N$, (i) $P'_{jj'} = k$ if and only if $P_{jj'} = k$; and (ii) $P'_{jj'} = h$ if and only if $P_{jj'} \neq k$. By independence of irrelevant opinions, $f_{ik}(P') = f_{ik}(P) = 1$, so that $f_{i\ell}(P') = 0$. Thus, $\varphi_i^\ell(0_{n \times n}) = \varphi_i^\ell(B^{P',\ell}) = f_{i\ell}(P') = 0$. Since our choice of k is arbitrary, the claim follows.

Step 2: *f is unanimous.*

Suppose, by contradiction, that there exist $i \in N$ and $k \in G$ such that $f_{ik}(k_{n \times n}) < 1$. Then, there is $\ell \in G \setminus \{k\}$ such that $f_{i\ell}(k_{n \times n}) > 0$. That is, $\varphi_i^\ell(0_{n \times n}) = \varphi_i^\ell(B^{k_{n \times n},\ell}) = f_{i\ell}(k_{n \times n}) > 0$, contradicting Step 1. $\qquad \square$

Now we show that decomposability follows from independence of irrelevant opinions and deterministic full range.

Proposition 2. *Independence of irrelevant opinions and deterministic full range together imply decomposability.*

Proof. Let f be a rule satisfying independence of irrelevant opinions and deterministic full range. Then f is represented by a profile of m approval functions $(\varphi^k)_{k \in G}$ (as in the proof of Lemma 1). By Lemma 1, f is unanimous. Now, we proceed in two steps.

Step 1: *For each $P \in \mathcal{P}$, each $i \in N$, and each $k \in G$, if k is not one of the entries of P, then $f_{ik}(P) = 0$.*

Let $P \in \mathcal{P}$, $i \in N$, and $k \in G$. Assume that k is not one of entries of P. Let $\ell \in G \backslash \{k\}$ and consider $P' \equiv \ell_{n \times n}$. By Lemma 1, $f_{i\ell}(P') = 1$, so that $f_{ik}(P') = 0$. Now applying independence of irrelevant opinions to P and P', $f_{ik}(P) = f_{ik}(P') = 0$.

Step 2: $\varphi^1 = \varphi^2 = \cdots = \varphi^m$.

Suppose, by contradiction, that there are $k, \ell \in G$ such that $\varphi^k \neq \varphi^\ell$. Then there are $B \in \mathcal{B}$ and $i \in N$ such that $\varphi_i^k(B) \neq \varphi_i^\ell(B)$. Let $h \in G \setminus \{k, \ell\}$. Let $P \in \mathcal{P}$ be such that for all $j, j' \in N$, (i) $P_{jj'} = h$ if and only if $B_{jj'} = 0$; and (ii) $P_{jj'} = k$ if and only if $B_{jj'} = 1$. Similarly, let $P' \in \mathcal{P}$ be such that (i) $P'_{jj'} = h$ if and only if $B_{jj'} = 0$; and $P'_{jj'} = \ell$ if and only if $B_{jj'} = 1$. By construction, $B^{P,k} = B^{P',\ell} = B$. Also, since $B^{P,h} = B^{P',h}$, independence of irrelevant opinions implies $f_{ih}(P) = f_{ih}(P')$. Since for each $a \in G \setminus \{k, h\}$, a is not one of the entries of P, Step 1 implies $f_{ia}(P) = 0$. Thus, $\sum_{k' \in G} f_{ik'}(P) = f_{ik}(P) + f_{ih}(P) = 1$. That is, $f_{ih}(P) = 1 - f_{ik}(P) = 1 - \varphi_i^k(B^{P,k}) = 1 - \varphi_i^k(B)$. Similarly, for each $a \in G \setminus \{\ell, h\}$, $f_{ia}(P') = 0$, so that $f_{ih}(P') = 1 - \varphi_i^\ell(B)$. However, $\varphi_i^k(B) \neq \varphi_i^\ell(B)$ implies $f_{ih}(P) \neq f_{ih}(P')$, a contradiction. $\square$

In the fractional setup, decomposability alone cannot guarantee some basic properties of approval functions, which stands in contrast with what is known in the deterministic setup. For instance, an approval function representing a decomposable rule may violate all of unanimity, monotonicity, and self-duality. The following example illustrates this point.

Example 2. Let $N = \{1, 2\}$ and $G = \{a, b, c\}$. Let $t \in [0, \frac{1}{2}]$ and define an approval function φ as follows: for each $B \in \mathcal{B}$ and each $i \in N$, $\varphi_i(B) = t$ if $B = 0_{2 \times 2}$; $\frac{t+1}{4}$ if $|B| = 1$; $\frac{1-t}{2}$ if $|B| = 2$; $\frac{3-5t}{4}$ if $|B| = 3$; and $1 - 2t$ if $B = 1_{2 \times 2}$. It is simple to show that φ is m-unit-additive (for each $t \in [0, \frac{1}{2}]$). Define a rule f by means of φ as follows: for each $P \in \mathcal{P}$, each $i \in N$, and each $k \in G$, $f_{ik}(P) = \varphi_i(B^{P,k})$. By construction, f is represented by φ and therefore, it is decomposable. Nevertheless, φ is not unanimous (neither is f) unless $t = 0$. Also, φ is not monotonic unless $t \in [0, \frac{1}{3}]$. Finally, φ is not self-dual unless $t = 0$. $\triangle$

The approval function defined in Example 2 can work to define a decomposable

rule while failing several reasonable properties. Our next result indicates that we may escape those failures by additionally imposing deterministic full range.

Proposition 3. *Let f be a rule satisfying decomposability and deterministic full range. Then an approval function φ that represents f is unanimous, self-dual, and monotonic.*

Proof. Suppose that f satisfies decomposability and deterministic full range and that f is represented by φ. By Lemma 1, f is unanimous. By Proposition 1, φ is m-unit-additive.

To show that φ is unanimous, let $i \in N$. For each $k \in G$, $1 = f_{ik}(k_{n\times n}) = \varphi_i(B^{k_{n\times n},k}) = \varphi_i(1_{n\times n})$. Let $B^1 = 1_{n\times n}$ and for each $k \in G \setminus \{1\}$, $B^k = 0_{n\times n}$. Then, $1 = \sum_{k\in G} \varphi_i(B^k) = \varphi_i(1_{n\times n}) + (m-1)\varphi_i(0_{n\times n}) = 1 + (m-1)\varphi_i(0_{n\times n})$. Thus, $\varphi_i(0_{n\times n}) = 0$.

To show that φ is self-dual, let $B^1 = B$, $B^2 = \overline{B}$ and for each $k \in G \setminus \{1,2\}$, $B^k = 0_{n\times n}$. Since $\sum_{k\in G} B^k = 1_{n\times n}$, m-unit-additivity and unanimity imply $1_{1\times n} = \sum_{k\in G} \varphi(B^k) = \varphi(B) + \varphi(\overline{B})$. Thus, $\varphi(B) = \overline{\varphi(\overline{B})}$.

To show that φ is monotonic, suppose, by contradiction, that there are $i \in N$ and $B, B' \in \mathcal{B}$ with $B \leq B'$ such that $\varphi_i(B) > \varphi_i(B')$. Let $B^1 = B$ and $B^2 = \overline{B'}$. Let $B^3, \ldots, B^m \in \mathcal{B}$ be such that $\sum_{k\in G} B^k = 1_{n\times n}$ (such $B^3, \ldots, B^m$ exist since $B^1 + B^2 \leq B + \overline{B} = 1_{n\times n}$). By m-unit-additivity and self-duality,

$$
\begin{aligned}
1 &= \sum_{k\in G} \varphi_i(B^k) \\
&= \varphi_i(B) + \varphi_i(\overline{B'}) + \sum_{k\in G\setminus\{1,2\}} \varphi_i(B^k) \\
&> \varphi_i(B') + \varphi_i(\overline{B'}) + \sum_{k\in G\setminus\{1,2\}} \varphi_i(B^k) \\
&= 1 + \sum_{k\in G\setminus\{1,2\}} \varphi_i(B^k),
\end{aligned}
$$

a contradiction. $\square$

4   Weighted-average Rules

With preliminary observations on independence of irrelevant opinions at hand, we are now ready to explore its consequences in detail. In the deterministic case, indepen-

dence of irrelevant opinions and non-degeneracy characterize the one-vote rules (Cho and Ju, 2017). An exact counterpart of the latter characterization in the fractional setup—namely that the two axioms characterize the weighted-average rules—does not hold (see Example 1). However, the weighted-average rules are the only rules satisfying independence of irrelevant opinions and the stronger axiom of deterministic full range.

Theorem 1. *A rule satisfies independence of irrelevant opinions and deterministic full range if and only if it is a weighted-average rule.*

Proof. In the proof, we use the following notation. A binary problem $B \in \mathcal{B}$ is a **unit binary problem** if $|B| = 1$ (i.e., there is only one unity in B). For each $(j,h) \in N^2$, let $\boldsymbol{U^{jh}} \in \mathcal{B}$ be the unit binary problem such that $U_{jh}^{jh} = 1$.

We omit the simple proof of the "if" part. To prove the "only if" part, let f be a rule satisfying independence of irrelevant opinions and deterministic full range. By Proposition 2, f is decomposable and is represented by an approval function φ. Recall that decomposability of f is equivalent to m-unit-additivity of φ. Moreover, since f is unanimous and decomposable, φ is unanimous, self-dual, and monotonic. Now it suffices to show that φ is a weighted-average approval function.

Step 1: *For each $i \in N$ and each $B \in \mathcal{B} \backslash \{0_{n \times n}\}$, $\varphi_i(B) = \sum_{(j,h) \in N^2 : B_{jh} = 1} \varphi_i(U^{jh})$.*

We prove the claim by induction. Clearly, $(*)$ the claim is true for each $B \in \mathcal{B}$ with $|B| = 1$. Let $\ell \in \mathbb{N}$ be such that $\ell < n^2$. Suppose that $(**)$ for each $i \in N$ and each $B \in \mathcal{B}$ with $|B| \leq \ell$, $\varphi_i(B) = \sum_{(j,h) \in N^2 : B_{jh} = 1} \varphi_i(U^{jh})$. Let $i \in N$ and let $B \in \mathcal{B}$ be such that $|B| = \ell + 1$. Define m binary problems $(B^k)_{k \in G}$ as follows: (i) $|B^1| = 1$, $|B^2| = \ell$, and $B^1 + B^2 = B$; (ii) $B^3 = \overline{B}$; and (iii) for each $k \in G \backslash \{1, 2, 3\}$, $B^k = 0_{n \times n}$. Since $\sum_{k \in G} B^k = 1_{n \times n}$, m-unit-additivity implies $\sum_{k \in G} \varphi_i(B^k) = 1$. By unanimity, for each $k \in G \backslash \{1, 2, 3\}$, $\varphi_i(B^k) = 0$. Now self-duality and the induction hypothesis $(**)$ imply $\varphi_i(B) = \overline{\varphi_i(\overline{B})} = 1 - \varphi_i(\overline{B}) = \varphi_i(B^1) + \varphi_i(B^2) = \sum_{(j,h) \in N^2 : B_{jh} = 1} \varphi_i(U^{jh})$. Finally, the claim follows from $(*)$ and the induction argument.

Step 2: *φ is a weighted-average approval function.*

Let $i \in N$. For each $(j,h) \in N^2$, let $\alpha_i(j,h) \equiv \varphi_i(U^{jh})$. By unanimity and Step 1, $\sum_{(j,h) \in N^2} \alpha_i(j,h) = \sum_{(j,h) \in N^2} \varphi_i(U^{jh}) = \varphi_i(1_{n \times n}) = 1$. Thus, $\alpha_i \in \Delta(N^2)$. Now by construction and Step 1, φ is a weighted-average approval function, with the associated weights $(\alpha_i)_{i \in N}$. $\square$

Remark 1. The axioms in the theorem are logically independent. The variants of the one-vote rules in Example 1 satisfy independence of irrelevant opinions but not deterministic full range. It is simple to construct rules satisfying the latter axiom but not the former. △

The characterization of the one-vote rules in the deterministic setup follows as a simple corollary if we restrict Theorem 1 to the class of deterministic rules. To prove, suppose that f is a deterministic rule satisfying independence of irrelevant opinions and non-degeneracy. Then f satisfies unanimity (and hence deterministic full range). By Theorem 1, f is a weighted-average rule. Now the deterministic nature of f implies that it is a one-vote rule.

Corollary 1 (Cho and Ju, 2017). *A deterministic rule satisfies independence of irrelevant opinions and non-degeneracy if and only if it is a one-vote rule.*

Next, we narrow down to two subfamilies of the weighted-average rules by imposing two more axioms, one at a time, in addition to the axioms in Theorem 1. First, we consider a fairness axiom, symmetry. In the deterministic case, the axiom singles out the liberal rule in the family of one-vote rules. With fractional membership permitted, however, a few other rules emerge and any convex combination of those rules satisfies symmetry as well as other axioms. The following result shows that the symmetrized weighted-average rules obtain once symmetry is additionally imposed.

Theorem 2. *Assume that there are at least three agents ($n \geq 3$). A rule satisfies independence of irrelevant opinions, deterministic full range, and symmetry if and only if it is a symmetrized weighted-average rule.*[11]

Proof. We only prove the "only if" part. Let f be a rule satisfying the three axioms. By Theorem 1, f is a weighted-average rule, with the associated weights $(\alpha_i)_{i \in N} \in (\Delta(N^2))^N$. Throughout the proof, let $a, b \in G$ be distinct groups.

Step 1: *For each $i \in N$ and all $j, h, j', h' \in N \backslash \{i\}$ with $j \neq h$ and $j' \neq h'$, $\alpha_i(j, h) = \alpha_i(j', h')$ (this step applies only when $n \geq 3$).*

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[11] In light of Theorem 1 and logical independence of the axioms in that theorem, it is simple to verify that the axioms in Theorem 2 are independent.

Assume that $n \geq 4$. We first prove the claim when $j = j'$. Let $i \in N$. Let $j, h, h' \in N \setminus \{i\}$ be all distinct. Let $P \in \mathcal{P}$ be such that $P_{jh} = a$ and all the other entries of P are b. Then $f_{ia}(P) = \alpha_i(j, h)$. Let $\pi : N \to N$ be a transposition that swaps h and h' only. Then $f_{ia}(P_\pi) = \alpha_i(j, h')$. Using $\pi(i) = i$ and symmetry, $f_i(P) = f_{\pi(i)}(P) = f_i(P_\pi)$, so that $\alpha_i(j, h) = f_{ia}(P) = f_{ia}(P_\pi) = \alpha_i(j, h')$.

Next, we prove the claim when $h = h'$. Let $i \in N$. Let $j, j', h \in N \setminus \{i\}$ be all distinct. Let $P \in \mathcal{P}$ be such that $P_{jh} = a$ and all the other entries of P are b. Then $f_{ia}(P) = \alpha_i(j, h)$. Let $\pi : N \to N$ be a transposition that swaps j and j' only. Then $f_{ia}(P_\pi) = \alpha_i(j', h)$. Using $\pi(i) = i$ and symmetry, $f_i(P) = f_{\pi(i)}(P) = f_i(P_\pi)$, so that $\alpha_i(j, h) = f_{ia}(P) = f_{ia}(P_\pi) = \alpha_i(j', h)$.

Combining the previous two arguments proves the claim for $n \geq 4$. The claim for $n = 3$ follows from a simpler argument. Write $N = \{1, 2, 3\}$. Let $i \in N$, say, $i = 1$. Then we need only to show that $\alpha_1(2, 3) = \alpha_1(3, 2)$. Let $P \in \mathcal{P}$ be such that $P_{23} = a$ and all the other entries of P are b. Let $\pi : N \to N$ be a transposition that swaps 2 and 3 only. Then $f_{1a}(P) = \alpha_1(2, 3)$ and $f_{1a}(P_\pi) = \alpha_1(3, 2)$. Using $\pi(1) = 1$ and symmetry, $f_1(P) = f_{\pi(1)}(P) = f_1(P_\pi)$, so that $\alpha_1(2, 3) = f_{1a}(P) = f_{1a}(P_\pi) = \alpha_1(3, 2)$.

Step 2: *For each $i \in N$ and all $j, h \in N \setminus \{i\}$, (i) $\alpha_i(j, j) = \alpha_i(h, h)$; (ii) $\alpha_i(j, i) = \alpha_i(h, i)$; and (iii) $\alpha_i(i, j) = \alpha_i(i, h)$.*

This can be proved by an argument similar to that in Step 1.

Step 3: *Concluding.*

By symmetry, $\alpha_1(1, 1) = \alpha_2(2, 2) = \cdots = \alpha_n(n, n)$. Let $s \equiv \alpha_1(1, 1)$. By Steps 1 and 2, for each $i \in N$, we may let $t_i \equiv \alpha_i(j, j)$ for some $j \in N \setminus \{i\}$; $u_i \equiv \alpha_i(j, i)$ for some $j \in N \setminus \{i\}$; $v_i \equiv \alpha_i(i, j)$ for some $j \in N \setminus \{i\}$; and $w_i \equiv \alpha_i(j, h)$ for some $j, h \in N \setminus \{i\}$ with $j \neq h$. Again by symmetry, (t_i, u_i, v_i, w_i) does not depend on i and we can write $t \equiv t_1 = \cdots = t_n$, $u \equiv u_1 = \cdots = u_n$, $v \equiv v_1 = \cdots = v_n$, and $w \equiv w_1 = \cdots = w_n$. Since $\alpha_i \in \Delta(N^2)$, $s + (n-1)t + (n-1)u + (n-1)v + (n^2 - 3n + 2)w = 1$. Now it is clear that f is obtained by taking a convex combination of the liberal, almost-diagonal, almost-column, almost-row, and almost-off-diagonal rules, with weights s, $(n-1)t$, $(n-1)u$, $(n-1)v$, and $(n^2 - 3n + 2)w$, respectively. $\qquad\square$

Remark 2. When there are only two agents $(n = 2)$, the almost-diagonal, almost-column, and almost-row rules are all deterministic and the almost-off-diagonal rule

does not exist. Thus, for $n = 2$, the rules satisfying the three axioms in the theorem are of the following form: there exist weights $s, t, u, v \in [0, 1]$ with $s + t + u + v = 1$ such that for each $P \in \mathcal{P}$ and all $i, j \in N$ with $i \neq j$, $f_i(P)$ is the weighted average of P_{ii}, P_{ij}, P_{ji}, and P_{jj}, with the associated weights s, t, u, and v, respectively. This characterization for the two-agent case follows from the proof of the above theorem (the only difference is that Step 1 is no longer needed). $\triangle$

The characterization of the liberal rule in the deterministic setup obtains as a corollary to Theorem 2. Except the liberal rule, the other four extreme points of the set of symmetrized weighted-average rules are all fractional. Thus, a deterministic rule satisfying independence of irrelevant opinions, non-degeneracy, and symmetry, should put zero weight on the four extreme points, implying that it is the liberal rule.

Corollary 2 (Cho and Ju, 2017). *Assume that there are at least three agents ($n \geq 3$). A deterministic rule satisfies independence of irrelevant opinions, non-degeneracy, and symmetry if and only if it is the liberal rule.*

Next, we turn our attention to another subfamily of the weighted-average rules. First, observe that a weighted-average rule can base the membership decision for agent i on opinions about other agents. This is because neither independence of irrelevant opinions nor deterministic full range contains a component that governs how membership decisions should be related across agents. With the aid of weak agentwise change, we can ensure that positive weights are placed only on opinions about i.

Theorem 3. *A rule satisfies independence of irrelevant opinions, deterministic full range, and weak agentwise change if and only if it is an agentwise weighted-average rule.*

Proof. The simple proof of the "if" part is omitted. To prove the "only if" part, let f be a rule satisfying the three axioms. By Theorem 1, f is a weighted-average rule, with, say, the associated weights $(\alpha_i)_{i \in N} \in (\Delta(N^2))^N$. Let $i \in N$. It suffices to show that for each $(j, h) \in N^2$ with $h \neq i$, $\alpha_i(j, h) = 0$. Suppose, by contradiction, that for some $(j, h) \in N^2$ with $h \neq i$, $\alpha_i(j, h) > 0$. Let $P, P' \in \mathcal{P}$ be such that $P^i \neq P'^i$ and $P^{-i} = P'^{-i}$. Letting $k \equiv P_{jh} = P'_{jh} \in G$, $f_{ik}(P) \geq \alpha_i(j, h) > 0$

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and $f_{ik}(P') \geq \alpha_i(j, h) > 0$. Thus, $||f_i(P) - f_i(P')|| < 1$, violating weak agentwise change. $\qquad \square$

Remark 3. To verify the independence of the axioms in the theorem, Theorem 1 provides rules satisfying all but weak agentwise change. The variant of the one-vote rules in Example 1 such that for each $i \in N$, the decisive entry (j_i, h_i) for i is in column i (i.e., $h_i = i$), satisfies all but deterministic full range. Finally, for a rule satisfying all but independence of irrelevant opinions, fix $a \in G$ and consider the following: for each $P \in \mathcal{P}$ and each $i \in N$, (i) if there is $k \in G$ with $P^i = k_{n \times 1}$, then $f_{ik}(P) = 1$; (ii) otherwise, $f_{ia}(P) = 1$. $\qquad \triangle$

Since agentwise unanimity implies both unanimity and weak agentwise change, we obtain the following corollary.

Corollary 3. *A rule satisfies independence of irrelevant opinions and agentwise unanimity if and only if it is an agentwise weighted-average rule.*

5 Fractional Consent Rules

The focus of this section is on extending the consent rules in the binary setup (Samet and Schmeidler, 2003) to our model and characterizing them. Recall that we refer to rules in the binary setup as approval functions. The spirit underlying a consent approval function is to approve an agent's self-opinion if and only if the number of agents agreeing with him exceeds a given quota. In the binary setup, it is obvious what the decision should be when the quota is not met: if i views himself as a member (and a non-member), then the decision will be a non-member (and a member, respectively). A similar spirit can be implemented in the multinary setup, for instance, by specifying a default membership against each group (Cho and Ju, 2017). That is, define a mapping $\delta : G \rightarrow G$ and whenever agent i's self-opinion is $k \in G$ and he does not gain sufficient support from other agents, put i in group $\delta(k)$. Such rules, however, fail to inherit many strengths from their binary origin. Groups are not treated symmetrically and introduction of the mapping δ is not natural, compared to how a consent approval function deals with the case of insufficient social consent. Further, they violate independence of irrelevant opinions: membership to group $\delta(k)$ depends on opinions about group k.

One way of circumventing these issues is to use consent approval functions to generate rules via the reverse decomposition process in Section 3. Yet most consent approval functions, except the liberal one, cannot serve this purpose. The key reason is that the consent approval functions are deterministic. Therefore, we first generalize them to a fractional setup and then investigate the rules they generate.

An approval function φ is a **fractional consent approval function** if there exist non-decreasing functions $s : \{1, \cdots, n\} \to [0,1]$ and $t : \{1, \cdots, n\} \to [0,1]$ satisfying the following: (i) for each $B \in \mathcal{B}$ and each $i \in N$, (i.a) when $B_{ii} = 1$, $\varphi_i(B) = s\left(|\{j \in N : B_{ji} = 1\}|\right)$, and (i.b) when $B_{ii} = 0$, $\varphi_i(B) = 1 - t\left(|\{j \in N : B_{ji} = 0\}|\right)$; and (ii) for each $h \in \{1, \cdots, n\}$, $s(h) + t(n - h + 1) \geq 1$. Denote by $\boldsymbol{\varphi^{st}}$ the fractional consent approval function associated with functions s and t. Condition (ii) and the non-decreasing property of s and t ensure that φ^{st} is monotonic. If s and t assume only the values of 0 and 1, φ^{st} is deterministic and it reduces to a consent approval function as defined by Samet and Schmeidler (2003). Further, in such case, there exist $s^*, t^* \in \{1, \cdots, n+1\}$ such that for each $h \in \{1, \cdots, n\}$, (i) $h < s^*$ implies $s(h) = 0$ and $h \geq s^*$ implies $s(h) = 1$; and (ii) $h < t^*$ implies $t(h) = 0$ and $h \geq t^*$ implies $t(h) = 1$. The numbers s^* and t^* are the consent quotas that parametrize the family of deterministic consent approval functions in Samet and Schmeidler (2003).

The fractional consent approval functions can be characterized by several axioms. In addition to the properties of approval functions introduced in Section 3, we consider the following. An approval function φ satisfies **agentwise identification** if for each $i \in N$ and all $B, B' \in \mathcal{B}$ with $B^i = B'^i$, $\varphi_i(B) = \varphi_i(B')$. Also, φ satisfies **symmetry** if for each $B \in \mathcal{B}$ and each permutation $\pi : N \to N$, $\varphi(B_\pi) = \varphi_\pi(B)$ (where B_π and $\varphi_\pi(B)$ are similarly defined as P_π and $f_\pi(P)$ in Section 2.2). Finally, φ satisfies **deterministic full range** if for each $i \in N$, there exist $B, B' \in \mathcal{B}$ such that $\varphi_i(B) = 0$ and $\varphi_i(B') = 1$. When restricted to deterministic approval functions, deterministic full range is indeed equivalent to non-degeneracy. Note that a fractional consent approval function φ^{st} satisfies deterministic full range if and only if $s(n) = t(n) = 1$. Deterministic full range is weaker than unanimity; it is also weaker than self-duality when restricted to the fractional consent approval functions (the latter implication does not hold in general).

The following result characterizes the fractional consent approval functions by mono-

tonicity, agentwise identification, and symmetry.

Theorem 4. *An approval function satisfies monotonicity, agentwise identification, and symmetry if and only if it is a fractional consent approval function.*[12]

Proof. We only prove the "only if" part. Let φ be an approval function satisfying the three axioms in the theorem. By agentwise identification and symmetry, there is a function $\theta : \{0,1\} \times \{0,1,\cdots,n-1\} \to [0,1]$ such that for each $B \in \mathcal{B}$ and each $i \in N$, $\varphi_i(B) = \theta\left(B_{ii}, |\{j \in N\backslash\{i\} : B_{ji} = 1\}|\right)$. By monotonicity, $\theta(1,\cdot)$ and $\theta(0,\cdot)$ are both non-decreasing. Define functions s and t from $\{1,\cdots,n\}$ to $[0,1]$ as for each $h \in \{1,\cdots,n\}$, $s(h) = \theta(1, h-1)$ and $t(h) = 1 - \theta(0, n-h)$. Clearly, s and t are non-decreasing and $\varphi = \varphi^{st}$.

It remains to show that for each $h \in \{1,\cdots,n\}$, $s(h) + t(n-h+1) \geq 1$. Let $h \in \{1,\cdots,n\}$. Fix agent $i \in N$. Let $B \in \mathcal{B}$ be such that $B_{ii} = 0$ and B^i has $h-1$ ones and $n-h+1$ zeros. Let $B' \in \mathcal{B}$ be the same as B except that $B'_{ii} = 1$. Then $\varphi_i(B) = 1 - t(n-h+1)$ and $\varphi_i(B') = s(h)$. Since monotonicity implies $\varphi_i(B) \leq \varphi_i(B')$, it follows that $s(h) + t(n-h+1) \geq 1$. $\square$

The characterization of the deterministic consent approval functions (Samet and Schmeidler, 2003) follows from Theorem 4: a deterministic approval function satisfies monotonicity, agentwise identification, and symmetry if and only if it is a deterministic consent approval function.

Then what rules do the fractional consent approval functions generate when it is used to represent a rule? Our next result shows that they are indeed the fractional consent rules as long as deterministic full range is imposed. Recall that the family of fractional consent rules is parametrized by $w \in [0,1]$: agent i's self-opinion is assigned weight w and each other agent's opinion about i is assigned weight $\frac{1-w}{n-1}$. When $w = 1$, we have the liberal rule, which puts full weight on his self-opinion. On the other hand, when $w = 0$, we have the almost-column rule, which puts full weight on the others' opinions about i, or "social consent" on i's membership, and the weight is evenly distributed to the $n-1$ opinions. Therefore, a choice of w between 1 and 0 corresponds to a

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[12]The logical independence of the axioms in this theorem is clear from Samet and Schmeidler (2003). One may adapt their examples (of deterministic approval functions) to get fractional approval functions.

compromise between two principles in determining individual identity: liberalism and social consent. This is the fractional analog of the principle that underpins the consent rules in the binary setup. The consent quotas of a given consent rule determines when an agent's self-opinion is socially approved, similar to the role of w.

Theorem 5. *Suppose that a rule is represented by a fractional consent approval function satisfying deterministic full range. Then the rule is a fractional consent rule.*

Proof. Suppose that a rule f is represented by a fractional consent approval function φ^{st} satisfying deterministic full range. Since f satisfies decomposability and deterministic full range, Proposition 3 implies that φ^{st} is unanimous and self-dual. Thus, $s(n) = t(n) = 1$ and for each $h \in \{1, \cdots, n\}$, $s(h) = t(h)$. Also, recall that φ^{st} is m-unit-additive.

Let $h \in \{1, \cdots, n-1\}$. Fix agent $1 \in N$. Let $B^1, \cdots, B^m \in \mathcal{B}$ be such that

    (i) for each $j \leq h$, $B^1_{j1} = 1$, and for each $j \geq h+1$, $B^1_{j1} = 0$;

    (ii) for each $j \leq h+1$, $B^2_{j1} = 0$, and for each $j \geq h+2$, $B^2_{j1} = 1$;

    (iii) $B^3_{h+1,1} = 1$ and for each $j \neq h+1$, $B^3_{j1} = 0$;

    (iv) $B^4 = \cdots = B^m = 0_{n \times n}$; and

    (v) $B^1 + \cdots + B^m = 1_{n \times n}$.[13]

By m-unit-additivity of φ^{st}, $1 = \varphi_i(B^1) + \cdots + \varphi_i(B^m) = s(h) + (1 - t(h+1)) + (1 - t(n-1)) + 0 + \cdots + 0$. Since self-duality implies $t(h+1) = s(h+1)$ and $t(n-1) = s(n-1)$, simplifying the latter equation yields $s(h+1) - s(h) = 1 - s(n-1)$. Thus, $s(2) - s(1) = s(3) - s(2) = \cdots = s(n) - s(n-1)$. Let $w \equiv s(1) = t(1) \in [0,1]$. Then for each $h \in \{1, \cdots, n-1\}$, $s(h+1) - s(h) = t(h+1) - t(h) = \frac{1-w}{n-1}$. Now it is clear that f is the fractional consent rule f^w associated with w. $\qquad \square$

Another characterization obtains when we note that each fractional consent rule is a convex combination of the liberal and almost-column rules. As shown by Theorem 2, the latter two rules are extreme points of the set of rules satisfying independence of irrelevant opinions, deterministic full range, and symmetry. The other three extreme points—the almost-row, almost-diagonal, almost-off-diagonal rules—determine i's membership using opinions that do not directly concern him, in violation of weak agentwise

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[13]Such $B^1, \cdots, B^m$ can be constructed because G has at least three groups ($m \geq 3$). When $m = 3$, condition (iv) would not be needed.

change. Thus, additionally imposing weak agentwise change, we have the following characterization.

Theorem 6. *Assume that there are at least three agents ($n \geq 3$). A rule satisfies independence of irrelevant opinions, deterministic full range, symmetry, and weak agentwise change if and only if it is a fractional consent rule.*

A corollary to the above theorem and Corollary 3 is that the fractional consent rules are the only rules satisfying independence of irrelevant opinions, agentwise unanimity, and symmetry.

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