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Ordinal Potentials in Smooth Games

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Abstract. While smooth exact potential games are easily characterized in terms of the cross-derivatives of players' payoff functions, an analogous differentiable characterization of ordinal or generalized ordinal potential games has been elusive for a long time. In this paper, it is shown that the existence of a generalized ordinal potential in a smooth game with multidimensional strategy spaces is crucially linked to the *semipositivity* (Fiedler and Pták, 1966) of a modified Jacobian matrix on the set of interior strategy profiles at which at least two first-order conditions hold. Our findings imply, in particular, that any generalized ordinal potential game must exhibit pairwise strategic complements or substitutes at any interior Cournot-Nash equilibrium. Moreover, provided that there are more than two players, the cross-derivatives at any interior equilibrium must satisfy a rather stringent equality constraint. The two conditions, which may be conveniently condensed into a local variant of the differentiable condition for weighted potential games, are made explicit for sum-aggregative games, symmetric games, and two-person zero-sum games. For the purpose of illustration, the results are applied to classic games, including probabilistic all-pay contests with heterogeneous valuations, models of mixed oligopoly, and Cournot games with a dominant firm.

Keywords. Ordinal potentials · smooth games · strategic complements and substitutes · semipositive matrices

JEL-Codes. C6 Mathematical Methods · Programming Models · Mathematical and Simulation Modeling; C72 – Noncooperative Games; D43 Oligopoly and Other Forms of Market Imperfection; D72 Political Processes: Rent-Seeking, Lobbying, Elections, Legislatures, and Voting Behavior

*) The idea for this paper originated during an inspiring conversation with Nikolai Kukushkin.
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1. Introduction

When a strategic game admits a *potential*, players' preferences may be conveniently summarized in a single objective function (Rosenthal, 1973; Monderer and Shapley, 1996a). Potentials of different types have been identified for a large variety of games. Moreover, the underlying methods have been found useful for the analysis of oligopolistic markets (Slade, 1994), learning processes (Monderer and Shapley, 1996b; Fudenberg and Levine, 1998; Young, 2004), population dynamics (Sandholm, 2001, 2009; Cheung, 2014), the robustness of equilibria (Frankel et al., 2003; Morris and Ui, 2005; Okada and Tercieux, 2012), the decomposition of games (Candogan et al., 2011), imitation strategies (Duersch et al., 2012), dynamics in near-potential games (Candogan et al., 2013a, 2013b), the existence of Nash equilibrium (Voorneveld, 1997; Kukushkin, 1994, 2011), solution concepts (Peleg et al., 1996; Tercieux and Voorneveld, 2010), games with monotone best-response selections (Huang, 2002; Dubey et al., 2006; Jensen, 2010), supermodular and zero-sum games (Brânzei et al., 2003), and even issues in mechanism design (Jehiel et al., 2008).

Both exact and ordinal variants of the concept have been considered in the literature.[1] In the case of finite strategy spaces, complete characterizations are known for exact and (generalized) ordinal potential games. Exact potential games admit a convenient characterization also in the important class of smooth games, i.e., in the class of games with Euclidean strategy spaces and twice continuously differentiable payoff functions. For instance, a smooth game with one-dimensional interval strategy spaces admits an exact potential if and only if the Jacobian of that game, i.e., the matrix of cross-derivatives of players' payoff functions, is globally symmetric. For the class of infinite ordinal potential games, a useful characterization has been established by Voorneveld and Norde (1997). Specifically, a game admits an ordinal potential if and only there are no weak improvement cycles and an order condition is satisfied.[2] However, as far as we know, no *differentiable* characterization has been available up to this point for the classes

[1] For a real-valued function on the set of strategy profiles to be an *exact* potential (a *weighted* potential), the difference in a player's payoff resulting from a unilateral change of her strategy must equal precisely (up to a positive factor) the corresponding difference in the potential. For a potential to be *ordinal*, any strict gain in a player's payoff resulting from a unilateral change of her strategy must be reflected by a strict gain in the potential and, unless the ordinal potential is *generalized*, vice versa.

[2] For a rigorous statement of this important result, we refer the reader to Voorneveld and Norde (1997).

of ordinal or generalized potential games. This has been a highly undesirable situation because the ordinal notions are of considerable conceptual interest.[3]

The present paper addresses this long-standing issue by studying the local feasibility of a generalized ordinal potential in a small neighborhood of an interior Cournot-Nash equilibrium (or, more generally, in a small neighborhood of any interior strategy profile at which at least two first-order conditions hold).[4] To this end, we consider an *arbitrary* cyclic path that is contained in a small open neighborhood of the equilibrium. In the simplest case, the path runs along the edges of a small rectangular box that contains the equilibrium at its center. In general, however, the path need not be centered, it may even be zig-zagging, crossing itself, or forming a complicated knot. By shrinking the path to infinitesimal size, we identify conditions on the slopes of players' local best-response functions such that each player's payoff is strictly increasing over the respective edges of the path that reflect her changes in strategy. Since a strict improvement cycle is impossible in a generalized ordinal potential game (Voorneveld, 1997), this approach indeed delivers a set of tight necessary conditions for the existence of a generalized ordinal potential in a wide class of games with continuous strategy spaces.

For example, it will be recalled that a smooth n-player game with interval strategy spaces is a weighted potential game if and only if there exist positive weights $w_1 > 0, ..., w_n > 0$ such that[5]

$$w_i \frac{\partial^2 u_i(x_N)}{\partial x_j \partial x_i} = w_j \frac{\partial^2 u_j(x_N)}{\partial x_i \partial x_j} \qquad (i, j \in \{1, ..., n\}, i \neq j) \tag{1}$$

holds at any strategy profile x_N. Below, it will be shown that, at any interior Cournot-Nash equilibrium x_N^* of any smooth generalized ordinal potential n-player game, provided that cross-derivatives do not vanish, there exist positive weights $w_1(x_N^*) > 0, ..., w_n(x_N^*) > 0$ such that

$$w_i(x_N^*) \frac{\partial^2 u_i(x_N^*)}{\partial x_i \partial x_j} = w_j(x_N^*) \frac{\partial^2 u_j(x_N^*)}{\partial x_j \partial x_i} \qquad (i, j \in \{1, ..., n\}, i \neq j). \tag{2}$$

[3]Monderer and Shapley (1996a, p. 135) wrote: "Unlike (weighted) potential games, ordinal potential games are not easily characterized. We do not know of any useful characterization, analogous to the one given in (4.1), for differentiable ordinal potential games." Since then, the problem has apparently remained open. See, e.g., the recent surveys by Mallozzi (2013), González-Sánchez and Hernández-Lerma (2016), or Lã et al. (2016).

[4]As will be explained, local conditions such as those considered in the present paper cannot be sharpened by considering additional strategy profiles.

[5]Here, x_i and $u_i = u_i(x_N)$ denote player i's strategy and payoff function, respectively, where $x_N = (x_1, ..., x_n)$ is the corresponding strategy profile. The notation will be introduced more carefully in Section 2.

Thus, any smooth generalized ordinal potential game satisfies a local analogue of the global differentiable property of a weighted potential game at any interior equilibrium.[6] Moreover, as will also be seen, this condition implies rather tight restrictions in specific classes of games. In this sense, a (partial) differentiable characterization of generalized ordinal potential games with continuous strategy spaces is obtained.

The analysis starts by considering strict improvement cycles that involve two players only. For this case, the existence of a generalized ordinal potential is shown to imply that the product of the slopes of any two players' mutual local best-response functions (or, more generally, the product of the corresponding cross-derivatives) at any interior regular equilibrium must be nonnegative.[7] Thus, borrowing the terminology familiar from contributions such as Bulow et al. (1985), Amir (1996), Dubey et al. (2006), and Monaco and Sabarwal (2016), we obtain as our first main necessary condition that the game must exhibit pairwise strategic substitutes or complements at any interior regular equilibrium. The strict improvement cycle, provided it can be constructed, may then run either clockwise or counterclockwise around the equilibrium, depending on whether the horizontal (vertical) player's local best-response function is strictly increasing (strictly declining) or strictly declining (strictly increasing). As an illustration of its usefulness, it will be shown that the criterion is tight in a class of probabilistic all-pay contests.

The criterion is then sharpened by considering strict improvement cycles that involve more than two players. In the abstract, the existence of a particular strict improvement cycle is shown to correspond to the *semipositivity* (Fiedler and Pták, 1966; Johnson et al., 1994) of a matrix that is constructed from the Jacobian by replacing all diagonal entries by zero and by multiplying all entries above the diagonal with negative one.[8] Exploiting the specific structure of the problem at hand, the semipositivity condition is then reformulated in more explicit

[6] The assumption that cross-derivatives do not vanish is indeed needed. To see this, consider the two-player ordinal potential game with payoffs $u_1(x_1, x_2) = -(x_1 + x_2)^2$ and $u_2(x_1, x_2) = -(x_1 + x_2)^6$. Then, at any Cournot-Nash equilibrium x_N^*, the cross-derivatives are given by $\partial^2 u_1(x_N^*)/\partial x_2 \partial x_1 = -2$ and $\partial^2 u_2(x_N^*)/\partial x_1 \partial x_2 = 0$, in conflict with relationship (2).

[7] We call an interior Nash equilibrium *regular* if the local second-order conditions hold strictly at the equilibrium point. In a neighborhood of a regular Nash equilibrium, one may define local best-response functions, which allows a more intuitive discussion of some of the findings of this paper. Apart from the expositional simplification, however, the regularity assumption is not crucial for the analysis.

[8] Semipositivity generalizes the concept of a P-matrix (Gale and Nikaidô, 1965). The relevant elements of the theory of semipositive matrices will be reviewed in the next section.

terms, such as the invertibility of the modified Jacobian and polynomial constraints on the slopes of players' local best-response functions. Moreover, useful additional conditions are derived by either renaming players, or by flipping around individual strategy spaces (Vives, 1990; Amir, 1996). In particular, this leads to our second main condition in the case of one-dimensional strategy spaces, viz. a set of equality constraints that must be satisfied by the slopes of players' local best-response functions (or alternatively, by the cross-derivatives of players' payoff functions) at any interior regular equilibrium of any generalized ordinal potential game with at least three players. The two main conditions are then combined and rephrased into the simple cross-derivative condition stated above.

The remainder of this paper is structured as follows. Section 2 contains preliminaries. The first main necessary condition is derived in Section 3. Section 4 deals with the general geometry of strict improvement cycles involving more than two players. Section 5 discusses the second main necessary condition. Specific classes of games are considered in Section 6. Section 7 discusses extensions. Section 8 concludes. All proofs have been relegated to an Appendix.

2. Preliminaries

2.1 Games with continuous strategy spaces

A *game* Γ is defined by a set of players $N = \{1, ..., n\}$, a strategy space X_i for each $i \in N$, and a payoff function $u_i : X_N \equiv X_1 \times ... \times X_n \to \mathbb{R}$ for each $i \in N$. The game Γ will be called *smooth* (e.g., Vives, 1999) if X_i is a subset of some Euclidean space and u_i is twice continuously differentiable in the interior of X_N, for any $i \in N$. For expositional simplicity, the analysis will subsequently focus on the case in which $X_i \subseteq \mathbb{R}$ for all $i \in N$.[9] Clearly, under this condition, marginal payoffs $\upsilon_i \equiv \partial u_i / \partial x_i$ are well-defined in the interior of X_N for all $i \in N$.

By a *(Cournot-Nash) equilibrium* of a game Γ, we mean a strategy profile $x_N^* \equiv (x_1^*, ..., x_n^*) \in X_N$ such that $u_i(x_i^*, x_{-i}^*) \geq u_i(x_i, x_{-i}^*)$ for any $i \in N$ and for any $x_i \in X_i$, where $x_{-i}^* = (x_1^*, ..., x_{i-1}^*, x_{i+1}^*, ..., x_n^*)$ is the profile comprised of the strategies chosen by the opponents of player i, so that $x_{-i}^* \in X_{-i} \equiv X_1 \times ... \times X_{i-1} \times X_{i+1} \times ... \times X_n$. An equilibrium x_N^* of a smooth game Γ will be called *interior* if x_i^* is an interior point of X_i for all $i \in N$. At an interior equi-

[9] The case of multi-dimensional strategy spaces will be dealt with in Section 7.

librium x_N^*, the first-order necessary condition associated with player i's optimization problem implies

$$\upsilon_i(x_i^*, x_{-i}^*) = \frac{\partial u_i(x_i^*, x_{-i}^*)}{\partial x_i} = 0 \qquad (i \in N). \tag{3}$$

An equilibrium x_N^* will be called *regular* if, in addition,

$$\frac{\partial \upsilon_i(x_i^*, x_{-i}^*)}{\partial x_i} = \frac{\partial^2 u_i(x_i^*, x_{-i}^*)}{\partial x_i^2} < 0 \qquad (i \in N). \tag{4}$$

Consider an interior regular equilibrium x_N^*. Then, as a direct consequence of the implicit function theorem, the equation $\upsilon_i(x_i, x_{-i}) = 0$ defines a continuously differentiable function $\beta_i \equiv \beta_i(x_{-i}) \equiv \beta_i(x_{-i}; x_N^*)$ that maps any vector x_{-i} from a small open neighborhood $U \subseteq X_{-i}$ of x_{-i}^* to a strategy $\beta_i(x_{-i}) \in X_i$ such that $\upsilon_i(\beta_i(x_{-i}), x_{-i}) = 0$. We will refer to $\beta_i(\cdot; x_N^*)$ as player i's *local best-response function* around x_N^*.[10] For any other player $j \neq i$, we will refer to

$$\sigma_{ij} \equiv \sigma_{ij}(x_N^*) = \frac{\partial \beta_i(x_{-i}^*)}{\partial x_j} = -\frac{\partial \upsilon_i(x_N^*)/\partial x_j}{\partial \upsilon_i(x_N^*)/\partial x_i} = -\frac{\partial^2 u_i(x_N^*)/\partial x_j \partial x_i}{\partial^2 u_i(x_N^*)/\partial x_i^2} \tag{5}$$

as the *slope* of player i's local best-response function with respect to player j.

2.2 Potentials and potential games

The following well-known definitions do not require differentiability.[11] A game Γ is an *exact potential game* if there exists a function $P : X_N \to \mathbb{R}$, referred to as an *exact potential* of Γ, such that

$$u_i(x_i, x_{-i}) - u_i(\widehat{x}_i, x_{-i}) = P(x_i, x_{-i}) - P(\widehat{x}_i, x_{-i}) \tag{6}$$

for any $i \in N$, $x_i \in X_i$, $\widehat{x}_i \in X_i$, and $x_{-i} \in X_{-i}$. A game Γ is called a *weighted potential game* if there exist positive factors $w_1 > 0, ..., w_n > 0$ as well as a function $P : X_N \to \mathbb{R}$, referred to as a *weighted potential* of Γ, such that

$$u_i(x_i, x_{-i}) - u_i(\widehat{x}_i, x_{-i}) = w_i(P(x_i, x_{-i}) - P(\widehat{x}_i, x_{-i})) \tag{7}$$

[10] This function may actually correspond to player i's global best-response function (e.g., if $u_i(x_i, x_{-i}^*)$ is strictly quasiconcave in x_i). However, this is not assumed.

[11] In fact, as shown by Voorneveld (1997) in response to a question raised by Peleg et al. (1996), an ordinal potential game with continuous payoff functions need not possess a continuous ordinal potential function. However, this does not constitute a problem for the present analysis because the necessary conditions derived below do not impose any continuity assumption on the generalized ordinal potential.

for any $i \in N$, $x_i \in X_i$, $\widehat{x}_i \in X_i$, and $x_{-i} \in X_{-i}$. Next, Γ is an *ordinal potential game* if condition (6) in the definition of an exact potential game is replaced by

$$u_i(x_i, x_{-i}) > u_i(\widehat{x}_i, x_{-i}) \text{ if and only if } P(x_i, x_{-i}) > P(\widehat{x}_i, x_{-i}). \tag{8}$$

Finally, Γ is a *generalized ordinal potential game* if (6) is replaced by the even weaker condition

$$u_i(x_i, x_{-i}) > u_i(\widehat{x}_i, x_{-i}) \text{ implies } P(x_i, x_{-i}) > P(\widehat{x}_i, x_{-i}). \tag{9}$$

In the latter two cases, the function P is called an *ordinal potential* or *generalized ordinal potential*, respectively, of the game Γ. Any exact potential is a weighted potential, any weighted potential is an ordinal potential, and any ordinal potential is a generalized ordinal potential. However, a generalized ordinal potential game need not, in general, be an ordinal potential game, an ordinal potential game need not, in general, be a weighted potential game, and a weighted potential game need not, in general, be an exact potential game.

Smooth exact potential games with intervals as strategy spaces may be conveniently characterized in terms of the cross derivatives of players' payoff functions.

Lemma 1 (Monderer and Shapley, 1996a). *Consider a smooth game Γ in which strategy spaces are intervals. Then Γ is an exact potential game if and only if*

$$\frac{\partial^2 u_i(x_N)}{\partial x_j \partial x_i} = \frac{\partial^2 u_j(x_N)}{\partial x_i \partial x_j} \qquad (i, j \in N, j \neq i; x_N \in X_N). \tag{10}$$

The extension of Lemma 1 to weighted potential games is immediate. However, as discussed in the Introduction, an analogous characterization for ordinal games, specifically geared toward the class of games with continuous strategy spaces, has apparently not been known so far. A general necessary condition may be formulated in terms of the following concept. A *strict improvement cycle* for a game Γ (of length L) is a finite sequence of strategy profiles

$$... \to x_N^0 \to x_N^1 \to ... \to x_N^{L-1} \to ... \tag{11}$$

in X_N with the property that, for any $l = 0, ..., L-1$, there is a player $i \equiv \iota(l) \in N$ such that $x_{-i}^{l+1} = x_{-i}^l$ and $u_i(x_i^{l+1}, x_{-i}^l) > u_i(x_i^l, x_{-i}^l)$, where x_N^L should be read as x_N^0.

We then have the following useful result.

Lemma 2 (Voorneveld, 1997). *A generalized ordinal game does not admit any strict improvement cycle.*[12]

2.3 Strategic substitutes and complements

Let x_N be an interior strategy profile (e.g., an interior Cournot-Nash equilibrium) in a smooth game Γ. Then, Γ will be said to exhibit *strategic complements (strategic substitutes) at x_N* if $\partial u_i(x_N)/\partial x_j \partial x_i \geq 0$ (≤ 0) for any two players i and j with $j \neq i$. Fix two players i and j with $j \neq i$. We will say that Γ exhibits *strategic complements (strategic substitutes) between i and j at x_N* if $\partial u_i(x_N)/\partial x_j \partial x_i \geq 0$ (≤ 0) and $\partial u_j(x_N)/\partial x_i \partial x_j \geq 0$ (≤ 0). Finally, we will say that Γ exhibits *pairwise strategic complements or substitutes at x_N* if Γ exhibits, for any two players i and j with $j \neq i$, either strategic complements between i and j at x_N or strategic substitutes between i and j at x_N.

2.4 Semipositivity

Consider a vector $\lambda_N = (\lambda_1, ..., \lambda_n)^T \in \mathbb{R}^n$, where the superscript T indicates transposition, as usual. We will write $\lambda_N > 0$ $(\lambda_N \geq 0)$ if all entries of λ_N are positive (nonnegative), i.e., if $\lambda_i > 0$ $(\lambda_i \geq 0)$ for all $i = 1, ..., n$. The following definition goes back at least to Fiedler and Pták (1966).

Definition 1. *A square matrix $A \in \mathbb{R}^{n \times n}$ is called semipositive if there exists a vector $\lambda_N \geq 0$ such that $A\lambda_N > 0$.*

In the definition, we may obviously replace the weak inequality by a strict one, using a simple perturbation argument. Thus, semipositivity amounts to the condition that (the interior of) the convex cone generated by the columns of A intersects the positive orthant $\mathbb{R}^n_{++} = \{z_N \in \mathbb{R}^n : z_N > 0\}$. Along these lines, semipositivity may be seen to correspond to a straightforward feasibility condition in linear programming.

[12] For games in which strategy spaces can be totally ordered, *local potentials* (Morris, 1999; Frankel et al., 2003; Morris and Ui, 2005; Okada and Tercieux, 2012) generalize exact potentials by requiring an inequality condition that relaxes equation (7) in several ways. Since Lemma 2 does not apply, this extension lies outside of our present focus. However, a characterization of local potential maximizers in smooth games can be found in Morris (1999, p. 28).

Call a square matrix $A \in \mathbb{R}^{n \times n}$ *inverse nonnegative* if its matrix inverse A^{-1} exists and all entries of A^{-1} are nonnegative. The following lemma provides a very useful recursive characterization of semipositivity.

Lemma 3 (Johnson et al., 1994). *A square matrix $A \in \mathbb{R}^{n \times n}$ is semipositive if and only if at least one of the following two conditions holds:*

(i) A is inverse nonnegative;

(ii) there exists $m \in \{1, ..., n-1\}$ and a submatrix $\widehat{A} \in \mathbb{R}^{n \times m}$ obtained from A via deletion of $n - m$ columns, such that all $m \times m$ submatrices of $\widehat{A}$ are semipositive.

3. The first necessary condition

3.1 Statement of the result

In this section, we derive the following simple yet apparently not widely known condition that is necessary for the existence of a generalized ordinal potential in a game with continuous strategy spaces.

Proposition 1. *Suppose that the smooth game Γ admits a generalized ordinal potential. Then, at any regular interior Cournot-Nash equilibrium x_N^*, necessarily*

$$\sigma_{ij}(x_N^*) \cdot \sigma_{ji}(x_N^*) \geq 0 \tag{12}$$

for any two players $i \in N$ and $j \in N$ with $i \neq j$.

Thus, any generalized ordinal potential game with continuous strategy spaces necessarily exhibits pairwise strategic substitutes or complements at any interior regular equilibrium.

It is important to note that the respective slopes of the local best-response functions are required to satisfy the inequality only at the profile x_N^* itself, rather than, say, in an open neighborhood of the equilibrium. This is not a weakness of our result but ultimately owed to the flexibility of the ordinal concept. In fact, as may be seen from the illustration given at the end of this section, there are examples of ordinal potential games (viz. symmetric contests) for which the mutual cross-derivatives $\partial^2 u_i(x_N)/\partial x_j \partial x_i$ and $\partial^2 u_j(x_N)/\partial x_i \partial x_j$ have different signs almost everywhere on the set of strategy profiles (viz. off the hyperplane defined through $x_i = x_j$), even though condition (12) is certainly satisfied at the unique equilibrium.

It is similarly important to note that, in games with more than two players, condition (12) requires only pairwise strategic complements or substitutes. Therefore, unless the game is sum-aggregative (see Section 6 for a definition), the conclusion of Proposition 1 is less stringent than the property that the game exhibits either strategic complements or strategic substitutes at any interior regular equilibrium. Again, this should not come as a surprise because, e.g., flipping around the strategy space of one of three players, say, may certainly destroy the property of strategic complements or strategic substitutes, but does not change the property of being a generalized ordinal game.

The conclusion of Proposition 1 is quite immediate when Γ actually admits an *exact* (or even weighted) potential. Indeed, in this case, Lemma 1 implies that $\partial^2 u_i / \partial x_j \partial x_i = \partial^2 u_j / \partial x_i \partial x_j$ holds in the interior of X_N. Therefore, at any regular interior equilibrium x_N^*,

$$\sigma_{ij}(x_N^*) \cdot \sigma_{ji}(x_N^*) = \left(-\frac{\partial^2 u_i(x_N^*)/\partial x_j \partial x_i}{\partial^2 u_i(x_N^*)/\partial x_i^2} \right) \cdot \left(-\frac{\partial^2 u_j(x_N^*)/\partial x_i \partial x_j}{\partial^2 u_j(x_N^*)/\partial x_j^2} \right) \tag{13}$$

$$= \frac{\left(\partial^2 u_i(x_N^*)/\partial x_j \partial x_i\right)^2}{\left(\partial^2 u_i(x_N^*)/\partial x_i^2\right) \cdot \left(\partial^2 u_j(x_N^*)/\partial x_j^2\right)} \geq 0, \tag{14}$$

consistent with Proposition 1.

For a less obvious example, consider the interesting class of multiplicatively separable aggregative games for which an ordinal potential has been constructed explicitly (Kukushkin, 1994; Nocke and Schutz, 2016). Thus, assume that payoffs admit the representation $u_i(x_N) = x_i \cdot \phi(x_N)$ for all $i \in N$, where $\phi : X_N \to \mathbb{R}$ is an arbitrary twice continuously differentiable function that does not depend on the player i. Then, at any interior regular equilibrium x_N^*, player j's optimality condition implies

$$x_j^* \cdot \frac{\partial \phi(x_N^*)}{\partial x_j} + \phi(x_N^*) = 0, \tag{15}$$

so that

$$x_j^* \cdot \sigma_{ij}(x_N^*) = x_j^* \cdot \left\{ -\frac{\partial^2 u_i(x_N^*)/\partial x_j \partial x_i}{\partial^2 u_i(x_N^*)/\partial x_i^2} \right\} \tag{16}$$

$$= x_j^* \cdot \left\{ -\frac{x_i^* \cdot (\partial^2 \phi(x_N^*)/\partial x_j \partial x_i) + (\partial \phi(x_N^*)/\partial x_j)}{\partial^2 u_i(x_N^*)/\partial x_i^2} \right\} \tag{17}$$

$$= -\frac{x_i^* \cdot x_j^* \cdot (\partial^2 \phi(x_N^*)/\partial x_j \partial x_i) - \phi(x_N^*)}{\partial^2 u_i(x_N^*)/\partial x_i^2}. \tag{18}$$

Thus, noting the symmetry of the numerator in (18) with respect to i and j, we arrive at $\sigma_{ij}(x_N^*) \cdot \sigma_{ji}(x_N^*) \geq 0$, consistent with Proposition 1.

However, if inequality (12) fails to hold for any two players at any regular interior equilibrium, then there cannot exist a generalized ordinal potential for Γ. In particular, a smooth game with an interior regular equilibrium that exhibits, in the strategic interaction between two players, an increasing reaction curve for one player and a decreasing reaction curve for the other player is *never* a generalized ordinal potential game. A classic example is the mixed oligopoly model by Singh and Vives (1984), in which one firm chooses a price, and the other firm chooses a quantity. Another famous example is quantity competition between a dominant firm and several fringe firms (Bulow et al., 1985). Many further examples, taken from diverse areas such as law enforcement, business strategy, and citizen protests, for instance, can be found in Tombak (2006) and Monaco and Sabarwal (2016).

3.2 Outline of proof

To understand why Proposition 1 holds true, consider Figure 1. Here, keeping the strategy profile $x_{-i,j}^* = (x_1^*, ..., x_{i-1}^*, x_{i+1}^*, ..., x_{j-1}^*, x_{j+1}^*, ..., x_n^*)$ fixed, player i's local best-response function $\beta_i = \beta_i(x_j, x_{-i,j}^*)$ around x_N^* is strictly *increasing* in player j's strategy x_j, and player j's local best-response function $\beta_j = \beta_j(x_i, x_{-i,j}^*)$ around x_N^* is strictly *decreasing* in player i's strategy x_i. Therefore, $\sigma_{ij}(x_N^*) \cdot \sigma_{ji}(x_N^*) < 0$, and the necessary condition fails. And indeed, for $\varepsilon > 0$ small enough, the finite sequence starting at the upper left corner and running clockwise around the square,

$$... \rightarrow (x_i^* - \varepsilon, x_j^* + \varepsilon, x_{-i,j}^*) \rightarrow (x_i^* + \varepsilon, x_j^* + \varepsilon, x_{-i,j}^*) \rightarrow \tag{19}$$

$$\rightarrow (x_i^* + \varepsilon, x_j^* - \varepsilon, x_{-i,j}^*) \rightarrow (x_i^* - \varepsilon, x_j^* - \varepsilon, x_{-i,j}^*) \rightarrow ...,$$

constitutes a strict improvement cycle, as will be explained now. To start with, consider the strategy change corresponding to the upper side of the square. Then, with ε small, player i's payoff is first increasing (over a longer section of the side) and then decreasing (over a shorter section of the side).

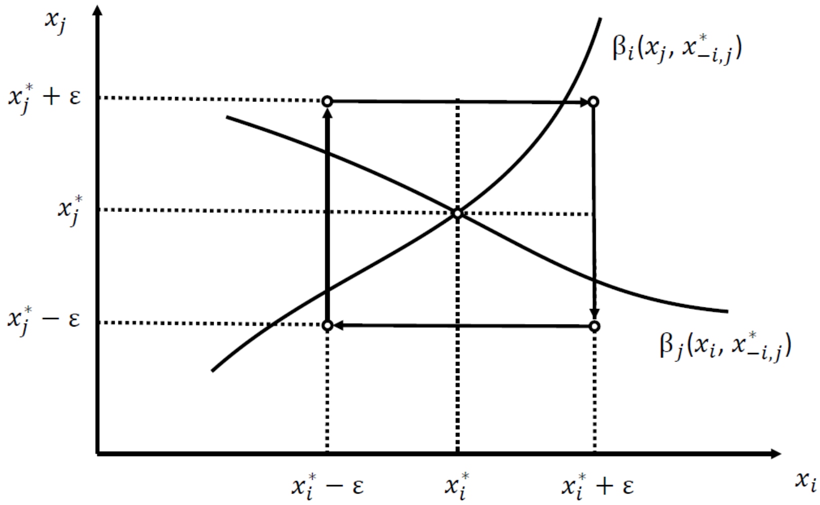

Figure 1. Constructing a strict improvement cycle involving two players.

The point to note is now that, as a consequence of smoothness of payoffs at the Cournot-Nash equilibrium, player i's payoff function along the upper side of the square may be approximated arbitrarily well by a parabola opening downwards, provided the square is small enough. As the parabola is symmetric around its peak, the payoff difference for player i, when switching from strategy $x_i - \varepsilon$ to $x_i + \varepsilon$, will be overall positive.[13] Similar considerations apply to the remaining three sides of the square. In fact, at the bottom side, there is no trade-off because player i's marginal payoff is always negative there. Thus, in sum, one may construct a strict improvement cycle that leads around the equilibrium. As seen in the previous section, however, this is incompatible with the existence of a generalized ordinal potential.

3.3 Discussion

Proposition 1 may be further strengthened by focusing on the conditions that are actually used in the proof. For example, the interiority assumption in Proposition 1 can be easily relaxed.

[13]There is a minor technical subtlety here in so far that the payoff difference approaches zero as ε goes to zero. However, as shown in the Appendix with the help of a careful limit consideration, the payoff difference approaches zero *from above* since the corresponding cross-derivative is positive. This turns out to be sufficient to settle the trade-off for a sufficiently small but still positive ε.

What matters is that those players that are involved in the strict improvement cycle use an interior strategy. Second, players that are not involved in the strict improvement circle may use any strategy, even a suboptimal one.[14] Further, the regularity assumption may be dropped entirely provided that the conditions on the slopes introduced above and later in the paper are replaced by the corresponding conditions on the cross-derivatives of players' payoff functions. For instance, in a two-player game, the necessary condition that $\sigma_{12}(x_N^*) \cdot \sigma_{21}(x_N^*) \geq 0$ holds at any regular interior equilibrium x_N^* may be replaced by the somewhat more stringent, but also maybe less vivid condition that $(\partial^2 u_1(x_N^*)/\partial x_2 \partial x_1) \cdot (\partial^2 u_2(x_N^*)/\partial x_1 \partial x_2) \geq 0$ holds at any interior (i.e., not necessarily regular) equilibrium x_N^*. Thus, the regularity assumption is purely expositional.[15] Then, the restriction to one-dimensional strategy spaces can be easily relaxed, essentially because a finite sequence that is a strict improvement cycle remains a strict improvement cycle when players are granted more strategic flexibility. In fact, as will be explained in Section 7, the existence of a generalized ordinal potential in a smooth game with multi-dimensional strategy spaces leads to implications that are much stronger than those discussed so far (because there is more freedom for constructing strict improvement cycles in higher dimensions). Next, the criterion applies more generally to any strategy profile at which the first-order conditions for all players are satisfied. Thus, rather than a global maximum, the individual player's problem may have a local maximum, local minimum, or inflection point at x_N^*. Finally, the game Γ actually need not be smooth. It suffices that the payoff functions of the involved players are twice continuously differentiable at the critical point under consideration.

However, no further strengthening of the results is possible from considering strategy profiles that are not local equilibria between at least two players. The reason is that, if at most one player's marginal condition holds at some x_N^*, then there are no strict improvement cycles locally at x_N^*. To the contrary, it is then always feasible to construct locally an ordinal potential by

[14] These two generalizations will be illustrated below.

[15] However, the use of slopes rather than cross-derivatives is suggested also by the analysis of sufficient conditions, which is not part of the present paper, though.

exploiting the strict monotonicity of $n-1$ payoff functions.[16] Relatedly, the consideration of more complicated paths (in which players may move more than twice), does not lead to more stringent conditions than those reported in the present paper. Indeed, while it may indeed be easier to achieve a strict gain in payoff on a single non-centered path segment, shrinking the path down to infinitesimal size necessarily leads to a system of linear inequalities on second-order derivatives only (cf. also the proof of Lemma 4 given in the Appendix). However, as will be discussed later in the analysis, a simple search model actually covers all possible (nonzero) slope combinations consistent with the necessary conditions. Along these lines, the conditions identified in the present paper will be seen to be actually equivalent to the absence of any *local* strict improvement cycle (defined by the requirement that it remains a strict improvement cycle even after being shrunk by any factor $\varepsilon \in (0,1)$ via a pantograph fixed at x_N^*).[17]

3.4 An illustration

In the n-player *lottery contest* with valuations $V_1 > 0, ..., V_n > 0$, player i's payoff is given by

$$u_i(x_1, ..., x_n) = \frac{x_i}{x_1 + ... + x_n} V_i - x_i, \tag{21}$$

where we assume that $X_1 = ... = X_n = [0, \infty)$.[18] It follows from a general result of Szidarovszky and Okuguchi (1997) that this game has a unique (yet not necessarily interior) equilibrium $x_N^* = (x_1^*, ..., x_n^*)$.

Rather than applying our criterion to the n-player equilibrium, we will consider an equilibrium in the two-player game between arbitrary players $i \in N$ and $j \in N$ with $j \neq i$, assuming that all remaining players remain passive. This actually strengthens our criterion.[19] So consider

[16] E.g., in the two-player case, if $\partial u_1(x_N^*)/\partial x_1 = 0 < \partial u_2(x_N^*)/\partial x_2$, then

$$P(x_N) = u_1(x_N) + x_2 \cdot \max\{2\,|\partial u_1(x_N^*)/\partial x_2|\,, 1\} \tag{20}$$

is an ordinal potential in a small neighborhood of x_N^*. Similar constructions can be used to cover the cases where either (i) there are more than two players, or (ii) the marginal payoffs of all players are non-zero at x_N^*.

[17] Strict improvement cycles that are not local in this sense are discussed in the extensions section.

[18] If $x_1 + ... + x_n = 0$, then we assume $u_i = \frac{1}{n}$.

[19] E.g., in a three-player contest with almost identical heterogeneous valuations, the respective slopes of the local best-response functions at the unique interior equilibrium are all negative. Thus, in that case, a direct application of Proposition 1 would not yield any valuable conclusions.

a profile $x_N^\# = (x_i^\#, x_j^\#, x_{-i,j}^\#) \in X_N$ such that the following conditions hold:

$$u_i(x_i^\#, x_j^\#, x_{-i,j}^\#) \geq u_i(x_i, x_j^\#, x_{-i,j}^\#) \qquad (x_i \in X_i), \tag{22}$$

$$u_j(x_j^\#, x_i^\#, x_{-i,j}^\#) \geq u_j(x_j, x_i^\#, x_{-i,j}^\#) \qquad (x_j \in X_i), \tag{23}$$

$$x_{-i,j}^\# = (0, ..., 0) \in \mathbb{R}^{n-2}. \tag{24}$$

In the bilateral game between players i and j, equilibrium efforts are given by the well-known expressions (cf. Konrad, 2009)

$$x_i^\# = \frac{V_i^2 V_j}{(V_i + V_j)^2} \text{ and } x_j^\# = \frac{V_i V_j^2}{(V_i + V_j)^2}. \tag{25}$$

From

$$\frac{\partial^2 u_i(x_N^\#)}{\partial x_i^2} = -\frac{2x_j^\# V_i}{(x_i^\# + x_j^\#)^3} < 0, \tag{26}$$

and an analogous inequality for player j, we see that the equilibrium is regular. Moreover, the slope of player i's local best-response function is given by

$$\sigma_{ij}(x_N^\#) = -\frac{\partial^2 u_i(x_N^\#)}{\partial x_j \partial x_i} \cdot \left(\frac{\partial^2 u_i(x_N^\#)}{\partial x_i^2} \right)^{-1} = \frac{x_i^\# - x_j^\#}{2x_j^\#} = \frac{V_i - V_j}{2V_j}. \tag{27}$$

An analogous expression may be derived for player j. We therefore see that the necessary condition $\sigma_{ij}(x_N^\#) \cdot \sigma_{ji}(x_N^\#) \geq 0$ holds if and only if

$$-\frac{(V_i - V_j)^2}{4V_i V_j} \geq 0, \tag{28}$$

or equivalently, if and only if $V_i = V_j$. Thus, if valuations are strictly heterogeneous in the sense that at least two valuations differ, then the n-player contest introduced above does not allow a generalized ordinal potential.[20]

On the other hand, the lottery contest with homogenous valuations $V \equiv V_1 = ... = V_n$ belongs to the beforementioned class of multiplicatively separable aggregative ordinal potential games. Specifically, the function

$$P(x_N) = x_1 \cdot ... \cdot x_n \cdot \left\{ \frac{V}{x_1 + ... + x_n} - 1 \right\} \qquad (x_N > 0) \tag{29}$$

[20] As will becomes clear later, the same conclusion holds under the much more flexible assumptions of Dixit (1987).

is an ordinal potential for the lottery contest in the interior of the strategy space. In that sense, our criterion is not only necessary but also sufficient in the considered class of contests.

4. Strict improvement cycles involving more than two players[21]

4.1 The role of semipositivity

In this section, we will discuss the geometry of strict improvement cycles that involve more than two players.

To fix ideas, the initial focus will be on a particular path in which players 1 through n consecutively raise their respective strategies, and subsequently lower their strategies, following the same order. Figure 2 illustrates a path of this kind for the case of three players. In contrast to the case of cycles that involve two players only, it turns out that more stringent necessary conditions are obtained when allowing for a rectangular-shaped box with edges that are not necessarily of equal length.

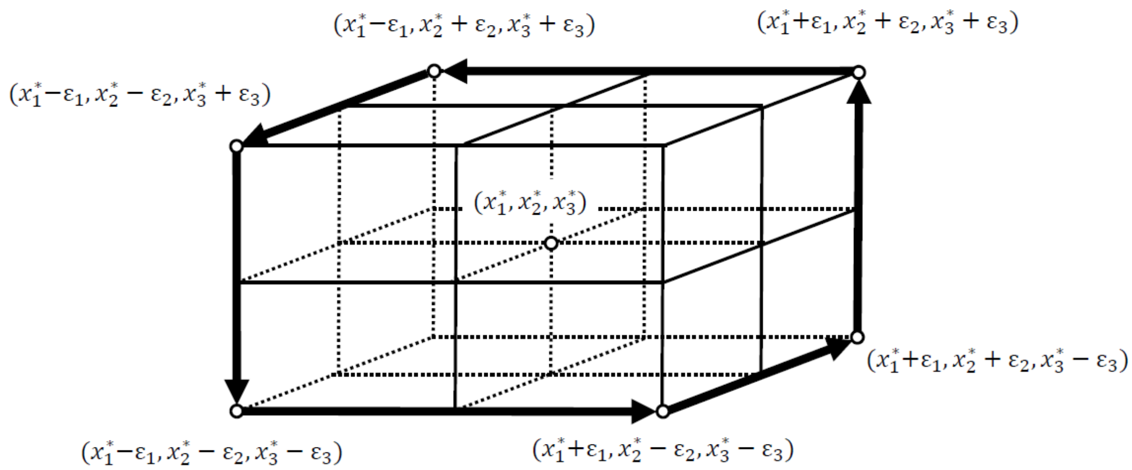

Figure 2. Constructing a circular improvement path involving three players.

An analysis of the conditions necessary and sufficient for the described path to constitute a strict improvement cycle leads to the following observation.

Lemma 4. *Suppose that the smooth game Γ admits a generalized ordinal potential. Then, at*

[21] This section is more technical than the rest of the paper and could be skipped in a first reading.

any regular interior Cournot-Nash equilibrium x_N^*, the matrix of "signed slopes"

$$J \equiv J(x_N^*) = \begin{pmatrix} 0 & -\sigma_{12}(x_N^*) & \cdots & -\sigma_{1n}(x_N^*) \\ \sigma_{21}(x_N^*) & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -\sigma_{n-1n}(x_N^*) \\ \sigma_{n1}(x_N^*) & \cdots & \sigma_{nn-1}(x_N^*) & 0 \end{pmatrix} \tag{30}$$

cannot be semipositive.

Thus, by logical contraposition, if the matrix J defined through (30) happens to be semipositive at some regular interior equilibrium, then Γ does not admit a generalized ordinal potential.

In the sequel, we will take two more steps so as to develop Lemma 4 into our most general result for the case of one-dimensional strategy spaces. First, note that the conclusion of Lemma 4, i.e., that J is <u>not</u> semipositive, may certainly be replaced by the less stringent conclusion that J is <u>not</u> inverse nonnegative. As a matter of fact, this simplification will not weaken our criterion at all, essentially because case (ii) of Lemma 3 corresponds to a situation in which one may construct a strict improvement cycle with less than n players involved.

4.2 Permutations of the player set and flipped strategy spaces

Second, recall that Lemma 4 looks at one particular path only. Alternative paths, corresponding to additional necessary conditions, may be constructed, e.g., by either (i) changing the order in which players change their respective strategies, or by (ii) flipping around the natural order of individual strategy spaces. While the first concept is rather specific to the problem at hand, the second concept (i.e., flipping around the natural order of an individual strategy space) is familiar from the theory of the oligopoly, where it has been used, in particular, to convert a two-player Cournot game with strategic substitutes into a supermodular game (Vives, 1990; Amir, 1996).

Formally, let $\pi : N \to N$ be an arbitrary bijection of the set of players. Then the natural ordering $1, 2, ..., n$ in which the set N is run through twice in the construction of the strict improvement cycle is permuted such that the strategy change of player i takes place at position $\pi(i)$ rather than at position i. In other words, when π^{-1} denotes the inverse of π, player $\pi^{-1}(1)$ moves first, and player $\pi^{-1}(n)$ last. Below, $\pi = \text{id}$ will refer to the identity mapping on N, and $\pi = (i_1 i_2 ... i_m)$ to the round-robin permutation that maps i_1 to i_2, i_2 to i_3, ..., i_{m-1} to i_m, and i_m

back to i_1, leaving all remaining players unchanged. Further, denote by $F \subseteq N$ the arbitrary subset of players for which the strategy space is flipped around.

It turns out that both of these operations, including combinations thereof, may be conveniently implemented by a set of pairwise sign changes applied to the slopes of players' local best-response functions. More precisely, the matrix J introduced in the statement of Lemma 4 may be replaced, without affecting the validity of the lemma, by any matrix

$$J^{(\pi,F)} \equiv J^{(\pi,F)}(x_N^*) = \begin{pmatrix} 0 & -\sigma_{12}^{(\pi,F)}(x_N^*) & \cdots & -\sigma_{1n}^{(\pi,F)}(x_N^*) \\ \sigma_{21}^{(\pi,F)}(x_N^*) & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -\sigma_{n-1n}^{(\pi,F)}(x_N^*) \\ \sigma_{n1}^{(\pi,F)}(x_N^*) & \cdots & \sigma_{nn-1}^{(\pi,F)}(x_N^*) & 0 \end{pmatrix}, \tag{31}$$

where the entries off the diagonal are given by the formula[22]

$$\sigma_{ij}^{(\pi,F)}(x_N^*) = (-1)^{I_{\{i\in F\}}+I_{\{j\in F\}}} \cdot \frac{\text{sgn}(\pi(j)-\pi(i))}{\text{sgn}(j-i)} \cdot \sigma_{ij}(x_N^*) \qquad (i,j \in N, j \neq i). \tag{32}$$

From equation (32), it is easy to see that flipping around all of the players' individual strategy spaces does not lead to any new condition, i.e., $J^{(\text{id},N)} = J$. Moreover, a circular shift of the players forward by one position is equivalent to flipping around player n's strategy space only, i.e., $J^{((12...n),\varnothing)} = J^{(\text{id},\{n\})}$. Taking account of such redundancies, however, a total of $(n-1)!2^{n-1}$ independent conditions remain. Some of these will prove useful below.

4.3 A more general result

Recall that a *principal submatrix* of a square matrix $A \in \mathbb{R}^{n\times n}$ is a submatrix $\widetilde{A} \in \mathbb{R}^{m\times m}$, for some $m \in \{1,...,n\}$, that is obtained from A by deleting $n-m$ pairs of corresponding rows and columns. In particular, A is a principal submatrix of itself. Summarizing the discussion so far, we arrive at the following extension of Proposition 1.

Lemma 5. *Suppose that the smooth game Γ admits a generalized ordinal potential. Then, at any regular interior Cournot-Nash equilibrium x_N^*, the matrix $J = J(x_N^*)$ defined through (30) does not possess an inverse nonnegative principal submatrix. Moreover, the same is true if J is replaced by any matrix $J^{(\pi,F)} = J^{(\pi,F)}(x_N^*)$ where players have been renamed using an arbitrary*

[22] Here and elsewhere in the paper, sgn denotes the *sign* function, satisfying $\text{sgn}(d) = +1$ if $d > 0$, $\text{sgn}(d) = 0$ if $d = 0$, and $\text{sgn}(d) = -1$ if $d < 0$. Moreover, I denotes the *indicator* function, satisfying $I_{\{i\in F\}} = 1$ if $i \in F$ and $I_{\{i\in F\}} = 0$ otherwise (similarly for $I_{\{j\in F\}}$).

bijection $\pi : N \to N$, and where an arbitrary subset $F \subseteq N$ of individual strategy spaces have been flipped around.

Illustrations of how Lemma 5 sharpens the conclusion of Proposition 1 will be given below. The following example shows, however, that the conclusion of Lemma 5 boils down to the conclusion of Proposition 1 in the case of two players (i.e., the earlier restriction to strict improvement cycles running over the edges of a square was indeed innocuous).

Example 1. It is obvious that the two one-dimensional principal submatrices of

$$J = \begin{pmatrix} 0 & -\sigma_{12}(x_N^*) \\ \sigma_{21}(x_N^*) & 0 \end{pmatrix} \tag{33}$$

are not inverse nonnegative. The condition that J itself is inverse nonnegative is that J is nonsingular, with all entries of the inverse matrix being nonnegative. Because of

$$J^{-1} = \frac{1}{\sigma_{12}(x_N^*)\sigma_{21}(x_N^*)} \begin{pmatrix} 0 & \sigma_{12}(x_N^*) \\ -\sigma_{21}(x_N^*) & 0 \end{pmatrix}, \tag{34}$$

this means $\sigma_{12}(x_N^*)\sigma_{21}(x_N^*) < 0$ and $\sigma_{21}(x_N^*) > 0 > \sigma_{12}(x_N^*)$.[23] By flipping around the strategy space of exactly one of the two players (i.e., by letting $F = \{1\}$ or $F = \{2\}$, and $\pi = \mathrm{id}$), or alternatively by changing the order of moves (i.e., $F = \varnothing$ and $\pi = (12)$), one may assure oneself that the sign condition on the individual slopes may be dropped without loss. Thus, we return to the criterion captured by Proposition 1, viz. that the inequality $\sigma_{12}(x_N^*)\sigma_{21}(x_N^*) < 0$ is incompatible with the existence of a generalized ordinal potential for Γ.

5. The second necessary condition

5.1 Statement of the result

While Lemma 5 is quite general, it is also desirable to know less stringent conditions that can be applied more readily to specific games. In this section, we therefore derive a second set of conditions that are necessary for the existence of a generalized ordinal potential in a given game with continuous strategy spaces. In contrast to the pairwise strategic substitutes or

[23] Indeed, suppose that J is inverse nonnegative with $\sigma_{12}(x_N^*)\sigma_{21}(x_N^*) > 0$. Then, since all entries of J^{-1} must be nonnegative, $\sigma_{12}(x_N^*) \geq 0$ and $\sigma_{21}(x_N^*) \leq 0$, so that $\sigma_{12}(x_N^*)\sigma_{21}(x_N^*) \leq 0$, which is impossible.

complements condition appearing in Proposition 1, the conditions introduced in the following result impose restrictions on the slopes of local best-response functions (or, more generally, cross-derivatives) of at least three players.

Proposition 2. *Suppose that the smooth game* Γ *with more than two players admits a generalized ordinal potential. Then, at any regular interior Cournot-Nash equilibrium* x_N^**, and for any set* $\{i_1, i_2, i_3\} \subseteq N$ *of pairwise different players,*

$$\sigma_{i_1 i_2}(x_N^*) \cdot \sigma_{i_2 i_3}(x_N^*) \cdot \sigma_{i_3 i_1}(x_N^*) = \sigma_{i_2 i_1}(x_N^*) \cdot \sigma_{i_3 i_2}(x_N^*) \cdot \sigma_{i_1 i_3}(x_N^*). \tag{35}$$

Moreover, for any set of pairwise distinct players $\{i_1, ..., i_m\} \subseteq N$ *with* $m \geq 4$*,*

$$\begin{aligned} &\sigma_{i_1 i_2}(x_N^*) \cdot \sigma_{i_2 i_3}(x_N^*) \cdot ... \cdot \sigma_{i_{m-1} i_m}(x_N^*) \cdot \sigma_{i_m i_1}(x_N^*) \\ &= \sigma_{i_2 i_1}(x_N^*) \cdot \sigma_{i_3 i_2}(x_N^*) \cdot ... \cdot \sigma_{i_m i_{m-1}}(x_N^*) \cdot \sigma_{i_1 i_m}(x_N^*), \end{aligned} \tag{36}$$

provided that $\sigma_{i_1 i_3}(x_N^*) \neq 0, ..., \sigma_{i_1 i_{m-1}}(x_N^*) \neq 0$ *and* $\sigma_{i_3 i_1}(x_N^*) \neq 0, ..., \sigma_{i_{m-1} i_1}(x_N^*) \neq 0$.

Thus, the product of pairwise slopes of the local best-response functions over an arbitrary cycle of three players remains unchanged if the order in which the cycle is run through is reversed. Moreover, this result extends to cycles of length four and beyond provided that a certain slopes of players' local best-response functions are all nonzero.[24]

For exact potential games, the conclusion of Proposition 2 may be checked directly. For instance, for an exact potential game with $n = 3$ players, Lemma 1 implies

$$\begin{aligned} &\sigma_{12}(x_N^*) \cdot \sigma_{23}(x_N^*) \cdot \sigma_{31}(x_N^*) \\ &= \frac{\partial^2 u_1(x_N^*)/\partial x_2 \partial x_1}{\partial^2 u_1(x_N^*)/\partial x_1^2} \cdot \frac{\partial^2 u_2(x_N^*)/\partial x_3 \partial x_2}{\partial^2 u_2(x_N^*)/\partial x_2^2} \cdot \frac{\partial^2 u_3(x_N^*)/\partial x_1 \partial x_3}{\partial^2 u_3(x_N^*)/\partial x_3^2} &(37)\\ &= \frac{\partial^2 u_2(x_N^*)/\partial x_1 \partial x_2}{\partial^2 u_1(x_N^*)/\partial x_1^2} \cdot \frac{\partial^2 u_3(x_N^*)/\partial x_2 \partial x_3}{\partial^2 u_2(x_N^*)/\partial x_2^2} \cdot \frac{\partial^2 u_1(x_N^*)/\partial x_3 \partial x_1}{\partial^2 u_3(x_N^*)/\partial x_3^2} &(38)\\ &= \sigma_{21}(x_N^*) \cdot \sigma_{32}(x_N^*) \cdot \sigma_{13}(x_N^*), &(39) \end{aligned}$$

as claimed. Obviously, this argument extends in a straightforward way to more than three players and likewise to the case of weighted potential games.

[24]For ordinal potentials that are sufficiently well-behaved, as in the subsequently listed examples, the assumption that certain slopes do not vanish is obsolete. In general, however, it seems that the assumption cannot be easily dropped.

For the class of ordinal potential games in which payoffs are given by $u_i(x_N) = x_i \cdot \phi(x_N)$, it is again the symmetry relationship

$$x_j^* \cdot \frac{\partial^2 u_i(x_N^*)}{\partial x_i^2} \cdot \sigma_{ij}(x_N^*) = x_i^* \cdot \frac{\partial^2 u_j(x_N^*)}{\partial x_j^2} \cdot \sigma_{ji}(x_N^*) \qquad (i, j \in N, j \neq i) \tag{40}$$

derived in equations (16-18) that allows the same conclusion.

Proposition 2 shows that these properties hold, more generally, for any generalized ordinal potential game. The result is actually somewhat unexpected, because the rather inflexible equality constraints (35-36) follow from a set of assumptions that are entirely of an ordinal nature.

A way to reformulate and summarize the conclusions of Propositions 1 and 2, essentially without losing any mileage,[25] is the following, beforementioned result.

Corollary 1. *Consider an interior regular Cournot-Nash equilibrium x_N^* in a smooth game Γ such that all slopes $\{\sigma_{ij}(x_N^*) : i, j \in N \text{ s.t. } i \neq j\}$ are nonzero. If Γ admits a generalized ordinal potential, then there exist positive weights $w_1(x_N^*) > 0, ..., w_n(x_N^*) > 0$ such that*

$$\sigma_{ij}(x_N^*) w_i(x_N^*) = \sigma_{ji}(x_N^*) w_j(x_N^*) \qquad (i, j \in N, j \neq i). \tag{41}$$

Thus, as discussed in the Introduction, the existence of an ordinal potential implies a local property that is reminiscent of the global condition for a weighted potential game.

5.2 Illustrations

We will illustrate Proposition 2 and Corollary 1 with the help of two additional examples. These are a search model of the Diamond-type, which extends an example in Milgrom and Roberts (1990), and a model of horizontally differentiated price competition. The first example will also allow us to settle an earlier question regarding strict improvement cycles that do not simply follow the edges of a rectangular box.

Consider first the following search model. Each of n players $i = 1, ..., n$ chooses a search effort $x_i \geq 0$ at costs $C_i(x_i)$, and receives a payoff

$$u_i(x_1, ..., x_n) = \left(\textstyle\sum_{j \neq i} \alpha_{ij} x_i x_j \right) + b_i x_i - C_i(x_i), \tag{42}$$

[25] That is, Corollary 1 implies both Proposition 1 and Proposition 2 in the case of nonzero slopes.

where $\alpha_{ij} \neq 0$ measures player i's expected benefit (or damage) resulting from a random encounter with player j, and $b_i \in \mathbb{R}$ is player i's intrinsic marginal valuation of search effort. Note that it is not assumed here that the game is supermodular. We shall assume, however, that the cost functions are quadratic for all players, i.e., that $C_i(x_i) = c_i x_i^2$ for some $c_i > 0$. Suppose that an interior Cournot-Nash equilibrium exists. Then, by Corollary 1, the search game admits a generalized ordinal potential only if there are factors $w_1 > 0, ..., w_n > 0$ such that $\alpha_{ij} w_i = \alpha_{ji} w_j$ for all i and j with $j \neq i$. However, that condition is equivalent to the existence of a weighted potential. Hence, given that being a weighted potential game is more stringent than being a generalized ordinal potential game, the condition of Corollary 1 is actually seen to be tight also in this case.

Relatedly, returning to the discussion adjourned at the end of Section 3, for any given interior strategy profile x_N^*, and arbitrary nonzero slopes $\{\sigma_{ij}(x_N^*) \neq 0 : j \neq i\}$, the profile x_N^* is easily seen to be a regular Cournot-Nash equilibrium in the search model for parameters $\alpha_{ij} = \sigma_{ij}(x_N^*)$, $b_i = x_i^* - \Sigma_{j \neq i} \sigma_{ij} x_j^*$, and $c_i = \frac{1}{2}$. Therefore, the conditions on the slopes of the local best-response functions obtained in Corollary 1 cannot be tightened any further in this specific class of games. Since the consideration of local strict improvement cycles of arbitrary shape can only lead to slope conditions that apply regardless of the specific game at hand, this implies that the consideration of local strict improvement cycles of arbitrary shape in arbitrary smooth games indeed does not lead to additional insights over those already obtained.[26]

Next, consider the following model of Bertrand-style competition between n firms $i = 1, ..., n$ with differentiated products. Suppose that each firm $i \in N$ chooses a price x_i (keeping the notation for convenience), and subsequently sells a quantity

$$q_i(x_N) = Q_i - s_i x_i + \sum_{j \neq i} \theta_{ij} x_j, \tag{43}$$

where $Q_i > 0$, $s_i > 0$, and $\theta_{ij} \neq 0$ are parameters. Firm i's production cost is represented by a convex and twice continuously differentiable function γ_i. Thus, firm i's profit reads

$$u_i(x_N) = x_i q_i(x_N) - \gamma_i(q_i(x_N)). \tag{44}$$

[26] In fact, it follows now from the proofs that, if a local strict improvement cycle of any shape and for any number of players exists, then there will also be a strict improvement cycle that is rectangular-shaped and that involves at most three players.

The conditions for the existence of an exact potential may be derived in a straightforward way. Specifically, the cross-derivative of firm i's profit with respect to firm j is given by

$$\frac{\partial^2 u_i(x_N)}{\partial x_j \partial x_i} = \theta_{ij}(1 + s_i \gamma_i''(q_i(x_N))). \tag{45}$$

Hence, from Lemma 1, the price-setting game admits an exact potential if and only if

$$\theta_{ij}(1 + s_i \gamma_i'') = \theta_{ji}(1 + s_j \gamma_j'') \qquad (i, j \in N; j \neq i). \tag{46}$$

In particular, all cost functions need to be (at most) quadratic. Even though this is a classic example, little was known about the possibility of a generalized ordinal potential. Suppose that the price-setting game allows an interior equilibrium x_N^*. Then, from Proposition 2, we obtain a necessary condition that is less stringent than (46), viz. that

$$\theta_{i_1 i_2} \theta_{i_2 i_3} \theta_{i_3 i_1} = \theta_{i_2 i_1} \theta_{i_3 i_2} \theta_{i_1 i_3} \tag{47}$$

holds for any set $\{i_1, i_2, i_3\}$ of pairwise different firms. Thus, if price externalities are generic (in the sense that equation (47) fails to hold for some triplet $\{i_1, i_2, i_3\}$ of pairwise different firms, then the price-setting game with more than two firms does not admit a generalized ordinal potential. Clearly, the same conclusion may be drawn if any of the analogues of equation (47) fails to hold for any $m \geq 4$.

6. Some specific classes of games

The purpose of this section is it to characterize the restrictions that our necessary conditions impose in three specific classes of games. By *necessary conditions*, we mean here throughout the strongest-form necessary conditions summarized in Lemma 5.

6.1 Sum-aggregative games

In a *sum-aggregative game* (e.g., Corchón, 1994), each player i's payoff function may be written as $u_i(x_i, x_{-i}) = U_i(x_i, \overline{x}_{-i})$ for some function U_i on $X_i \times \mathbb{R}$, where $\overline{x}_{-i} = \sum_{j \neq i} x_j$. Examples include Cournot games and contests (such as the one considered above). We claim that the consideration of strict improvement cycles involving any number of players yields no conclusions on top of what Proposition 1 would deliver. To see this, note that, at any profile x_N from the interior of X_N, the payoff representation of the sum-aggregative game implies

$$\frac{\partial^2 u_i(x_N)}{\partial x_j \partial x_i} = \frac{\partial^2 u_i(x_N)}{\partial x_k \partial x_i} \tag{48}$$

for any set $\{i, j, k\}$ of pairwise different players. Therefore, at any interior regular equilibrium x_N^*, the slopes of the local best-response functions satisfy $\sigma_{ij}(x_N^*) \equiv \sigma_i(x_N^*)$ for any two players i and j with $j \neq i$. Consequently, the matrix $J = J(x_N^*)$ defined through (30) attains the particular form

$$J = \begin{pmatrix} 0 & -\sigma_1(x_N^*) & -\sigma_1(x_N^*) & \cdots & -\sigma_1(x_N^*) \\ \sigma_2(x_N^*) & 0 & -\sigma_2(x_N^*) & \cdots & -\sigma_2(x_N^*) \\ \sigma_3(x_N^*) & \sigma_3(x_N^*) & 0 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & -\sigma_{n-1}(x_N^*) \\ \sigma_n(x_N^*) & \sigma_n(x_N^*) & \cdots & \sigma_n(x_N^*) & 0 \end{pmatrix}. \tag{49}$$

To see under what conditions this matrix is semipositive, it clearly suffices to restrict attention to the case where $\sigma_i(x_N^*) \neq 0$ for all $i \in N$. Since we are interested in conclusions that go beyond those of Proposition 1, we may even assume that $\sigma_i(x_N^*)\sigma_j(x_N^*) > 0$ for any two players i and j with $j \neq i$. But then, all slopes are nonzero and of the same sign, so that J may be rescaled into a skew-symmetric matrix by multiplying it from the left with a positive diagonal matrix. Thus, J cannot be semipositive.[27] Moreover, the conclusion of skew-symmetry does not change when we permute the player set or flip around individual strategy spaces.

The discussion may be summarized as follows.

Corollary 2. *A smooth sum-aggregative game* Γ *satisfies the necessary conditions for the*

[27] Indeed, a skew-symmetric matrix is never semipositive. To see this, suppose that $A \in \mathbb{R}^{n \times n}$ is semipositive. Then, there exists $\lambda_N \in \mathbb{R}^n$ with $\lambda_N > 0$ such that $A\lambda_N > 0$. Hence, if A is also skew-symmetric, $A^T = -A$, so that $A^T \lambda_N = -A\lambda_N < 0$. However, by the Theorem of the Alternative (Johnson et al., 1994, Th. 2.9), this is impossible.

existence of a generalized ordinal potential if and only if Γ *exhibits, at any interior regular equilibrium, either strategic complements or strategic substitutes.*

6.2 Symmetric games

A game Γ is *symmetric* if all players have the same strategy space $X \equiv X_1 = ... = X_N$, and if, for any permutation $\pi : N \to N$ of the player set, payoffs satisfy

$$u_i(x_1, ..., x_n) = u_{\pi(i)}(x_{\pi(1)}, ..., x_{\pi(n)}) \qquad (i \in N, (x_1, ..., x_n) \in X_N). \tag{50}$$

Furthermore, a Cournot-Nash equilibrium $x_N^* = (x_1^*, ..., x_n^*)$ is *symmetric* if $x_1^* = ... = x_N^*$. Smooth symmetric games admit at least one symmetric equilibrium under standard assumptions (Moulin, 1986, p. 115).[28] Therefore the following observation may be useful.

Corollary 3. *In any smooth symmetric game* Γ*, the necessary conditions for the existence of a generalized ordinal potential are automatically satisfied at any symmetric Cournot-Nash equilibrium.*

For instance, it is known that any symmetric game with one-dimensional strategy spaces and best-response functions that have a slope globally strictly above negative one admits at most one Cournot-Nash equilibrium (Vives, 1999). Since the equilibrium is necessarily symmetric in that case, such games satisfy our necessary conditions as well.

6.3 Zero-sum games

As usual, we call a two-player game Γ *zero-sum* if $u_1(x_N) + u_2(x_N) = 0$ for all $x_N \in X_N$. For this case, Proposition 1 yields the following noteworthy implication.

Corollary 4. *A smooth two-player zero-sum game* Γ *satisfies the necessary conditions for the existence of a generalized ordinal potential if and only if* $\sigma_{12}(x_N^*) = \sigma_{21}(x_N^*) = 0$ *at any interior regular saddle point* x_N^* *of* Γ*.*

Thus, if players' reaction curves always intersect at a right angle that is aligned with the coordinate system (as it is the case in a symmetric two-player zero-sum game, for instance), then

[28] However, there are also large classes of economically relevant symmetric games that admit only asymmetric pure-strategy Nash equilibria (cf. Amir et al., 2010).

the necessary conditions for the existence of a generalized ordinal potential hold. Conversely, no generalized ordinal potential is feasible in a two-player zero-sum game if the tangents to the players' reaction curves are not parallel to the coordinate axes at any point of intersection.

Corollary 4 extends to two-player games that are strategically zero-sum in the sense of Moulin and Vial (1978), i.e., to games such that $u_1(x_N) + u_2(x_N)$ is additively separable in x_1 and x_2. In particular, this is an alternative way to look at the contest example discussed above.

7. Extensions

7.1 Multi-dimensional strategy spaces

Below, we will briefly summarize the adaptions that need to be made to accommodate multi-dimensional strategy spaces. In fact, as it turns out, the extension of the necessary conditions is mainly a matter of notation.

In the case of one-dimensional strategy spaces, the most general condition for the existence of a strict improvement cycle required the existence of a vector $\lambda_N = (\lambda_1, ..., \lambda_n)^T \in \mathbb{R}^n$ with $\lambda_N > 0$ such that $J\lambda_N > 0$, where J denotes as before the matrix of "signed slopes" introduced in Lemma 4. Taking account of the possibility of flipping around any subset of individual strategy spaces, yet keeping the natural ordering of the players, this condition is equivalent to the existence of a vector $\lambda_N = (\lambda_1, ..., \lambda_n)^T \in \mathbb{R}^n$ (all components of which are necessarily nonzero) such that

$$\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix} \begin{pmatrix} 0 & -\frac{\partial^2 u_1(x_N^*)}{\partial x_2 \partial x_1} & \cdots & -\frac{\partial^2 u_1(x_N^*)}{\partial x_n \partial x_1} \\ \frac{\partial^2 u_2(x_N^*)}{\partial x_1 \partial x_2} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -\frac{\partial^2 u_{n-1}(x_N^*)}{\partial x_n \partial x_{n-1}} \\ \frac{\partial^2 u_n(x_N^*)}{\partial x_1 \partial x_n} & \cdots & \frac{\partial^2 u_n(x_N^*)}{\partial x_{n-1} \partial x_n} & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix} < 0. \tag{51}$$

The benefit of this reformulation is that condition (51) easily extends to the case of multi-dimensional strategy spaces. To see this, suppose that now $X_i \subseteq \mathbb{R}^{d_i}$ for $i = 1, ..., n$, where d_i denotes the dimension of player i's strategy space. Thus, each player $i \in N$ chooses a vector $(x_i^{(1)}, ..., x_i^{(d_i)}) \in X_i$, such that her payoff function u_i has a total of $\dim X_N = d_1 + ... + d_n$

arguments. Then, to obtain the multi-dimensional analogue of the semipositivity condition, one replaces each scalar λ_i (i.e., the length of the respective side of the rectangular box) by a vector $\Lambda_i = (\lambda_i^{(1)}, ..., \lambda_i^{(d_1)}) \in \mathbb{R}^{d_i}$ (i.e., a direction of change for the unilateral change of strategy by player i in the strict improvement cycle), and correspondingly, each cross derivative $(\partial^2 u_i(x_N^*)/\partial x_j \partial x_i)$ by a matrix $H_{ij} \in \mathbb{R}^{d_i \times d_j}$ collecting the cross-derivatives of player i's payoff function u^i with respect to any pair consisting of one of player i's and one of player j's choice variables. The component $\lambda_i(\partial^2 u_i(x_N^*)/\partial x_j \partial x_i)\lambda_j$ resulting from the left-hand side of (51) must then be replaced by $(\Lambda_i)^T H_{ij} \Lambda_j$.

For example, in the case of $n = 2$ players with $d_1 = d_2 = 2$, the semipositivity condition (allowing for flipped strategy spaces) must be replaced by the condition that there exist vectors $\Lambda_1 = (\lambda_1^{(1)}, \lambda_1^{(2)})^T$ and $\Lambda_2 = (\lambda_2^{(1)}, \lambda_2^{(2)})^T$ such that

$$(\Lambda_1)^T H_{12} \Lambda_2 > 0 \text{ and } (\Lambda_2)^T H_{21} \Lambda_1 < 0, \tag{52}$$

where

$$H_{ij} = \begin{pmatrix} \dfrac{\partial^2 u_i(x_N^*)}{\partial x_j^{(1)} \partial x_i^{(1)}} & \dfrac{\partial^2 u_i(x_N^*)}{\partial x_j^{(2)} \partial x_i^{(1)}} \\ \dfrac{\partial^2 u_i(x_N^*)}{\partial x_j^{(1)} \partial x_i^{(2)}} & \dfrac{\partial^2 u_i(x_N^*)}{\partial x_j^{(2)} \partial x_i^{(2)}} \end{pmatrix} \qquad (i \in N = \{1, 2\}, i \neq j) \tag{53}$$

denotes the beforementioned matrix of cross-derivatives. The corresponding conditions for either more than two involved players or strategy spaces of dimension larger than two may now be found by straightforward extension. Using this notation, the proof of Lemma 4 extends in a straightforward way.

7.2 Non-local strict improvement cycles

The approach of this paper extends to non-local strict improvement cycles, where the role of the interior equilibrium is taken over by a cyclic path along which the generalized ordinal potential stays constant. We illustrate the basic idea with an example, featuring two players and one-dimensional strategy spaces. Suppose that $x_N^- = (x_1^-, x_2^-)$ and $x_N^+ = (x_1^+, x_2^+)$ are two interior strategy profiles such that

$$u_1(x_1^-, x_2^+) = u_1(x_1^+, x_2^+), \tag{54}$$

$$u_2(x_1^+, x_2^+) = u_2(x_1^+, x_2^-), \tag{55}$$

$$u_1(x_1^+, x_2^-) = u_1(x_1^-, x_2^-), \tag{56}$$

$$u_2(x_1^-, x_2^-) = u_2(x_1^-, x_2^+). \tag{57}$$

In other words, the finite sequence

$$... \to (x_1^-, x_2^+) \to (x_1^+, x_2^+) \to (x_1^+, x_2^-) \to (x_1^-, x_2^-) \to ... \tag{58}$$

is a cyclic path along which the player that changes her strategy keeps an unchanged payoff. Consider now a perturbation of the profiles, say

$$\widetilde{x}_N^-(\varepsilon) = (x_1^- + \varepsilon\lambda_1^-, x_2^- + \varepsilon\lambda_2^-), \tag{59}$$

$$\widetilde{x}_N^+(\varepsilon) = (x_1^+ + \varepsilon\lambda_1^+, x_2^+ + \varepsilon\lambda_2^+), \tag{60}$$

for $\varepsilon > 0$ small, and for an arbitrary vector $\lambda^\# = (\lambda_1^-, \lambda_1^+, \lambda_2^-, \lambda_2^+)^T \in \mathbb{R}^4$. Then, for ε sufficiently small, the finite sequence

$$... \to (\widetilde{x}_1^-(\varepsilon), \widetilde{x}_2^+(\varepsilon)) \to (\widetilde{x}_1^+(\varepsilon), \widetilde{x}_2^+(\varepsilon)) \to (\widetilde{x}_1^+(\varepsilon), \widetilde{x}_2^-(\varepsilon)) \to (\widetilde{x}_1^-(\varepsilon), \widetilde{x}_2^-(\varepsilon)) \to ... \tag{61}$$

is a strict improvement cycle provided that the following conditions hold:

$$\lambda_1^+ \frac{\partial u_1(x_1^+, x_2^+)}{\partial x_1} + \lambda_2^+ \frac{\partial u_1(x_1^+, x_2^+)}{\partial x_2} - \lambda_1^- \frac{\partial u_1(x_1^-, x_2^+)}{\partial x_1} - \lambda_2^+ \frac{\partial u_1(x_1^-, x_2^+)}{\partial x_2} > 0 \tag{62}$$

$$\lambda_1^+ \frac{\partial u_2(x_1^+, x_2^-)}{\partial x_1} + \lambda_2^- \frac{\partial u_2(x_1^+, x_2^-)}{\partial x_2} - \lambda_1^+ \frac{\partial u_2(x_1^+, x_2^+)}{\partial x_1} - \lambda_2^+ \frac{\partial u_2(x_1^+, x_2^+)}{\partial x_2} > 0 \tag{63}$$

$$\lambda_1^- \frac{\partial u_1(x_1^-, x_2^-)}{\partial x_1} + \lambda_2^- \frac{\partial u_1(x_1^-, x_2^-)}{\partial x_2} - \lambda_1^+ \frac{\partial u_1(x_1^+, x_2^-)}{\partial x_1} - \lambda_2^- \frac{\partial u_1(x_1^+, x_2^-)}{\partial x_2} > 0 \tag{64}$$

$$\lambda_1^- \frac{\partial u_2(x_1^-, x_2^+)}{\partial x_1} + \lambda_2^+ \frac{\partial u_2(x_1^-, x_2^+)}{\partial x_2} - \lambda_1^- \frac{\partial u_2(x_1^-, x_2^-)}{\partial x_1} - \lambda_2^- \frac{\partial u_2(x_1^-, x_2^-)}{\partial x_2} > 0 \tag{65}$$

In particular, if Γ admits a generalized ordinal potential, then the matrix

$$\nabla(x_N^-, x_N^+) \tag{66}$$

$$= \begin{pmatrix} -\frac{\partial u_1(x_1^-, x_2^+)}{\partial x_1} & \frac{\partial u_1(x_1^+, x_2^+)}{\partial x_1} & 0 & \frac{\partial \{u_1(x_1^+, x_2^+) - u_1(x_1^-, x_2^+)\}}{\partial x_2} \\ 0 & \frac{\partial \{u_2(x_1^+, x_2^-) - u_2(x_1^+, x_2^+)\}}{\partial x_1} & \frac{\partial u_2(x_1^+, x_2^-)}{\partial x_2} & -\frac{\partial u_2(x_1^+, x_2^+)}{\partial x_2} \\ \frac{\partial u_1(x_1^-, x_2^-)}{\partial x_1} & -\frac{\partial u_1(x_1^+, x_2^-)}{\partial x_1} & \frac{\partial \{u_1(x_1^-, x_2^-) - u_1(x_1^+, x_2^-)\}}{\partial x_2} & 0 \\ \frac{\partial \{u_2(x_1^-, x_2^+) - u_2(x_1^-, x_2^-)\}}{\partial x_1} & 0 & -\frac{\partial u_2(x_1^-, x_2^-)}{\partial x_2} & \frac{\partial u_2(x_1^-, x_2^+)}{\partial x_2} \end{pmatrix}$$

must not be semipositive. Moreover, an analogous conclusion is obtained for any matrix derived from $\nabla(x_N^-, x_N^+)$ by multiplying an arbitrary subset of the column vectors with negative one. Thus, an extension to non-local cycles is indeed feasible.[29]

8. Concluding remarks

In this paper, we have identified tight conditions necessary for the existence of a generalized ordinal potential in any given game with continuous strategy spaces and twice continuously differentiable payoff functions. Since every ordinal game is, in particular, a generalized ordinal potential game, the same conditions are equally crucial for the existence of an ordinal potential. In this sense, a (partial) differentiable characterization of these important classes of games has been accomplished.

We have used our criteria to prove the non-existence of generalized ordinal potentials in a variety of classic games, including probabilistic all-pay contests with heterogeneous valuations, mixed oligopoly, and quantity competition with a dominant firm. Parameter constraints have been obtained for a search model and a differentiated Bertrand game with more than two firms. Besides illustrating the usefulness of the conditions, these applications allow to see some of the economic implications of ordinal potential concepts.

Our results imply, in particular, that the class of concave games (Rosen, 1965) is not contained in the class of generalized ordinal potential games.[30] While both concepts impose related restrictions on second-order derivatives, viz. negative quasidefiniteness of the Jacobian in the case of concave games and <u>not</u> semipositivity of the sign-modified Jacobian in the case of generalized ordinal potential games, the relationship is actually rather loose. For instance, in a smooth two-player game with payoff functions that are strictly concave in own strategy, negative

[29] One might speculate whether the kernel of the matrix $\nabla(x_N^-, x_N^+)$ contains information about the isoquants of any ordinal potential. Numerical investigations suggest, however, that unless isoquants are elliptic, $\nabla(x_N^-, x_N^+)$ will be invertible. Intuitively, this means that there typically does not exist a nearby "constant-payoff cycle" of the same length.

[30] Conversely, however, it is well-known that any smooth game admitting a twice continuously differentiable exact potential function whose Hessian is globally negative definite is a concave game. See, e.g., Neyman (1997), Ui (2008), and Hofbauer and Sandholm (2009).

quasidefiniteness of the Jacobian is tantamount to

$$\left| \frac{\partial^2 u_1(x_N)}{\partial x_2 \partial x_1} + \frac{\partial^2 u_2(x_N)}{\partial x_1 \partial x_2} \right| \leq 2 \sqrt{\left| \frac{\partial^2 u_1(x_N)}{\partial x_1^2} \right| \cdot \left| \frac{\partial^2 u_2(x_N)}{\partial x_2^2} \right|}. \tag{67}$$

Thus, mixed signs of mutual cross-derivatives at an interior equilibrium are feasible in a concave game, yet as noted above, not in a generalized ordinal potential game.

Clearly, our findings may also be conducive to the identification of new classes of ordinal potential games. In particular, we have shown that necessary conditions in the strongest form are satisfied by three important classes of games, which may be informally described as (i) sum-aggregative games with either increasing or decreasing best-response functions, (ii) symmetric games in which best-response functions have everywhere slopes strictly exceeding negative one, and (iii) symmetric two-person zero-sum games. This allows for the theoretical possibility that some of these games might indeed admit a generalized ordinal potential.[31]

A somewhat unexpected feature of the analysis is reflected in the equality constraints that apply to smooth generalized ordinal potential games with more than two players. After all, the assumptions driving the equality constraints are of a purely ordinal nature, while the implications are nongeneric in nature. We have no simple intuition for this finding.

Appendix

This Appendix contains the proofs of our results. For the proofs of the known facts summarized above as Lemmas 1 through 3, the reader is referred to Monderer and Shapley (1996a, Th. 4.5), Voorneveld (1997, Lemma 2.1), and Johnson et al. (1994, Cor. 3.5 & Th. 4.3), respectively.

Proof of Proposition 1. By contradiction. Suppose that, at some interior regular equilibrium x_N^*, and for some players i and j with $j \neq i$, we have $\sigma_{ij}(x_N^*) \cdot \sigma_{ji}(x_N^*) < 0$. By renaming players, if necessary, we may assume that $\sigma_{ji}(x_N^*) < 0 < \sigma_{ij}(x_N^*)$. Thus, player i's local best-response

[31] Preliminary research by the author on the construction of ordinal potentials in smooth games strongly suggests that the necessary conditions identified in the present paper are indicative regarding sufficiency as well. However, unfortunately, the matter of sufficiency is highly involved. For example, the pseudo- and best-reply potentials ingeniously constructed in prior work (Huang, 2002; Dubey et al., 2006; Jensen, 2010) need not be generalized ordinal potentials in general. Because of such difficulties, it has to remain feasible for the time being that the restrictions implied by the existence of a generalized ordinal potential are even more restrictive than the necessary conditions identified in the present analysis.

function around x_N^* is strictly increasing in x_j, whereas player j's local best-response function around x_N^* is strictly declining in x_i (i.e., just as shown in Figure 1), with

$$\frac{\partial^2 u_i(x_N^*)}{\partial x_j \partial x_i} > 0 \text{ and } \frac{\partial^2 u_j(x_N^*)}{\partial x_i \partial x_j} < 0. \tag{68}$$

It is claimed now that, for any sufficiently small $\varepsilon > 0$, the payoff difference corresponding to the upper side of the square satisfies

$$\Delta_i^+(\varepsilon) \equiv u_i(x_i^* + \varepsilon, x_j^* + \varepsilon, x_{-i,j}^*) - u_i(x_i^* - \varepsilon, x_j^* + \varepsilon, x_{-i,j}^*) > 0. \tag{69}$$

To prove this, we determine the first and second derivatives of the function $\Delta_i^+(\varepsilon)$, and evaluate at $\varepsilon = 0$. As for the first derivative, one obtains

$$\begin{aligned} \frac{\partial \Delta_i^+(\varepsilon)}{\partial \varepsilon} &= \left\{ \frac{\partial u_i(x_i^* + \varepsilon, x_j^* + \varepsilon, x_{-i,j}^*)}{\partial x_i} + \frac{\partial u_i(x_i^* + \varepsilon, x_j^* + \varepsilon, x_{-i,j}^*)}{\partial x_j} \right\} \\ &\quad - \left\{ -\frac{\partial u_i(x_i^* - \varepsilon, x_j^* + \varepsilon, x_{-i,j}^*)}{\partial x_i} + \frac{\partial u_i(x_i^* - \varepsilon, x_j^* + \varepsilon, x_{-i,j}^*)}{\partial x_j} \right\}. \end{aligned} \tag{70}$$

Evaluating at $\varepsilon = 0$, and subsequently exploiting the necessary first-order condition for player i at the interior equilibrium x_N^*, we find

$$\frac{\partial \Delta_i^+(0)}{\partial \varepsilon} = 2 \cdot \frac{\partial u_i(x_N^*)}{\partial x_i} = 0. \tag{71}$$

Next, consider the second derivative of $\Delta_i^+(\varepsilon)$ at $\varepsilon = 0$, i.e.,

$$\begin{aligned} \frac{\partial^2 \Delta_i^+(0)}{\partial \varepsilon^2} &= \left\{ \frac{\partial^2 u_i(x_N^*)}{\partial x_i^2} + \frac{\partial^2 u_i(x_N^*)}{\partial x_j \partial x_i} + \frac{\partial^2 u_i(x_N^*)}{\partial x_i \partial x_j} + \frac{\partial^2 u_i(x_N^*)}{\partial x_j^2} \right\} \qquad (72) \\ &\quad - \left\{ \frac{\partial^2 u_i(x_N^*)}{\partial x_i^2} - \frac{\partial^2 u_i(x_N^*)}{\partial x_j \partial x_i} - \frac{\partial^2 u_i(x_N^*)}{\partial x_i \partial x_j} + \frac{\partial^2 u_i(x_N^*)}{\partial x_j^2} \right\} \\ &= 2 \cdot \frac{\partial^2 u_i(x_N^*)}{\partial x_j \partial x_i} + 2 \cdot \frac{\partial^2 u_i(x_N^*)}{\partial x_i \partial x_j}. \qquad (73) \end{aligned}$$

Invoking Schwarz's theorem regarding the equality of cross-derivatives for twice continuously differentiable functions, and subsequently using (68), one finds

$$\frac{\partial^2 \Delta_i^+(0)}{\partial \varepsilon^2} = 4 \cdot \frac{\partial^2 u_i(x_N^*)}{\partial x_j \partial x_i} > 0. \tag{74}$$

In sum, (71) and (74) imply that, indeed, $\Delta_i^+(\varepsilon) > 0$ for any sufficiently small $\varepsilon > 0$. Analogous arguments can be used to deal with the other three sides of the square. Specifically, one defines

$$\Delta_j^+(\varepsilon) = u_j(x_j^* - \varepsilon, x_i^* + \varepsilon, x_{-i,j}^*) - u_j(x_j^* + \varepsilon, x_i^* + \varepsilon, x_{-i,j}^*), \tag{75}$$

$$\Delta_i^-(\varepsilon) = u_i(x_i^* - \varepsilon, x_j^* - \varepsilon, x_{-i,j}^*) - u_i(x_i^* + \varepsilon, x_j^* - \varepsilon, x_{-i,j}^*), \tag{76}$$

$$\Delta_j^-(\varepsilon) = u_j(x_j^* + \varepsilon, x_i^* - \varepsilon, x_{-i,j}^*) - u_j(x_j^* - \varepsilon, x_i^* - \varepsilon, x_{-i,j}^*), \tag{77}$$

and now readily verifies that

$$\frac{\partial \Delta_j^+(0)}{\partial \varepsilon} = (-2) \cdot \frac{\partial u_j(x_N^*)}{\partial x_j} = 0, \tag{78}$$

$$\frac{\partial \Delta_i^-(0)}{\partial \varepsilon} = (-2) \cdot \frac{\partial u_i(x_N^*)}{\partial x_i} = 0, \tag{79}$$

$$\frac{\partial \Delta_j^-(0)}{\partial \varepsilon} = 2 \cdot \frac{\partial u_j(x_N^*)}{\partial x_j} = 0, \tag{80}$$

and that

$$\frac{\partial^2 \Delta_j^+(0)}{\partial \varepsilon^2} = (-4) \cdot \frac{\partial^2 u_j(x_N^*)}{\partial x_i \partial x_j} > 0, \tag{81}$$

$$\frac{\partial^2 \Delta_i^-(0)}{\partial \varepsilon^2} = 4 \cdot \frac{\partial^2 u_i(x_N^*)}{\partial x_j \partial x_i} > 0, \tag{82}$$

$$\frac{\partial^2 \Delta_j^-(0)}{\partial \varepsilon^2} = (-4) \cdot \frac{\partial^2 u_j(x_N^*)}{\partial x_i \partial x_j} > 0. \tag{83}$$

It follows that $\Delta_i^+(\varepsilon) > 0$, $\Delta_j^+(\varepsilon) > 0$, $\Delta_i^-(\varepsilon) > 0$, and $\Delta_j^-(\varepsilon) > 0$ all hold for $\varepsilon > 0$ small enough. But then, the finite sequence (19) is a strict improvement cycle, which is incompatible with the existence of a generalized ordinal potential by Lemma 2. □

Proof of Lemma 4. A semipositive matrix remains semipositive after multiplication from the left or right with any positive diagonal matrix (Johnson et al., 1994, p. 267). Therefore, the semipositivity of J is equivalent to the semipositivity of the matrix

$$\overline{J} = \begin{pmatrix} 0 & -\dfrac{\partial^2 u_1(x_N^*)}{\partial x_2 \partial x_1} & \cdots & -\dfrac{\partial^2 u_1(x_N^*)}{\partial x_n \partial x_1} \\ \dfrac{\partial^2 u_2(x_N^*)}{\partial x_1 \partial x_2} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -\dfrac{\partial^2 u_{n-1}(x_N^*)}{\partial x_n \partial x_{n-1}} \\ \dfrac{\partial^2 u_n(x_N^*)}{\partial x_1 \partial x_n} & \cdots & \dfrac{\partial^2 u_n(x_N^*)}{\partial x_{n-1} \partial x_n} & 0 \end{pmatrix}, \tag{84}$$

which is constructed from the Jacobian of Γ by replacing all diagonal entries by zero and by multiplying all entries above the diagonal with negative one. Suppose now that J is semipositive, so that $\overline{J}$ is semipositive as well. Then, by definition, there exists a vector $\lambda_N = (\lambda_1, ..., \lambda_n)^T \in \mathbb{R}$ with $\lambda_N > 0$ such that $\overline{J}\lambda_N > 0$. Consider now the finite sequence

$$\begin{aligned}
... &\rightarrow x_N^{(1,+)}(\varepsilon) = (x_1^* + \lambda_1\varepsilon, x_2^* - \lambda_2\varepsilon, x_3^* - \lambda_3\varepsilon, ..., x_{n-1}^* - \lambda_{n-1}\varepsilon, x_n^* - \lambda_n\varepsilon) \rightarrow \\
&\rightarrow x_N^{(2,+)}(\varepsilon) = (x_1^* + \lambda_1\varepsilon, x_2^* + \lambda_2\varepsilon, x_3^* - \lambda_3\varepsilon, ..., x_{n-1}^* - \lambda_{n-1}\varepsilon, x_n^* - \lambda_n\varepsilon) \rightarrow \\
&\vdots \\
&\rightarrow x_N^{(n,+)}(\varepsilon) = (x_1^* + \lambda_1\varepsilon, x_2^* + \lambda_2\varepsilon, x_3^* + \lambda_3\varepsilon, ..., x_{n-1}^* + \lambda_{n-1}\varepsilon, x_n^* + \lambda_n\varepsilon) \rightarrow \qquad (85) \\
&\rightarrow x_N^{(1,-)}(\varepsilon) = (x_1^* - \lambda_1\varepsilon, x_2^* + \lambda_2\varepsilon, x_3^* + \lambda_3\varepsilon, ..., x_{n-1}^* + \lambda_{n-1}\varepsilon, x_n^* + \lambda_n\varepsilon) \rightarrow \\
&\rightarrow x_N^{(2,-)}(\varepsilon) = (x_1^* - \lambda_1\varepsilon, x_2^* - \lambda_2\varepsilon, x_3^* + \lambda_3\varepsilon, ..., x_{n-1}^* + \lambda_{n-1}\varepsilon, x_n^* + \lambda_n\varepsilon) \rightarrow \\
&\vdots \\
&\rightarrow x_N^{(n,-)}(\varepsilon) = (x_1^* - \lambda_1\varepsilon, x_2^* - \lambda_2\varepsilon, x_3^* - \lambda_3\varepsilon, ..., x_{n-1}^* - \lambda_{n-1}\varepsilon, x_n^* - \lambda_n\varepsilon) \rightarrow ...,
\end{aligned}$$

where $\varepsilon > 0$ is a small constant as before. Figure 2 illustrates this path for $n = 3$, where the rectangular-shaped box has sides of respective length $\varepsilon_i = \lambda_i\varepsilon$ for $i = 1, 2, 3$. It is claimed that, for any $\varepsilon > 0$ sufficiently small, the following four conditions hold:

(i) player 1's payoff at $x_N^{(1,+)}(\varepsilon)$ is strictly higher than at $x_N^{(n,-)}(\varepsilon)$;

(ii) for $i = 2, ..., n$, player i's payoff at $x_N^{(i,+)}(\varepsilon)$ is strictly higher than at $x_N^{(i-1,+)}(\varepsilon)$;

(iii) player 1's payoff at $x_N^{(1,-)}(\varepsilon)$ is strictly higher than at $x_N^{(n,+)}(\varepsilon)$;

(iv) for $i = 2, ..., n$, player i's payoff at $x_N^{(i,-)}(\varepsilon)$ is strictly higher than at $x_N^{(i-1,-)}(\varepsilon)$.

To establish (i), proceed precisely as in the proof of Proposition 1, and consider the first two derivatives of the payoff difference

$$\Delta^{(1,+)}(\varepsilon) = u_1(x_N^{(1,+)}(\varepsilon)) - u_1(x_N^{(n,-)}(\varepsilon)) \qquad (86)$$

$$= u_1(x_1^* + \lambda_1\varepsilon, x_2^* - \lambda_2\varepsilon, x_3^* - \lambda_3\varepsilon, ..., x_{n-1}^* - \lambda_{n-1}\varepsilon, x_n^* - \lambda_n\varepsilon) \qquad (87)$$

$$- u_1(x_1^* - \lambda_1\varepsilon, x_2^* - \lambda_2\varepsilon, x_3^* - \lambda_3\varepsilon, ..., x_{n-1}^* - \lambda_{n-1}\varepsilon, x_n^* - \lambda_n\varepsilon)$$

at $\varepsilon = 0$. The first derivative of $\Delta^{(1,+)}(\varepsilon)$ at $\varepsilon = 0$ is given by

$$\frac{\partial \Delta^{(1,+)}(0)}{\partial \varepsilon} = \left(\lambda_1 \frac{\partial u_1(x_N^*)}{\partial x_1} - \lambda_2 \frac{\partial u_1(x_N^*)}{\partial x_2} - \ldots - \lambda_n \frac{\partial u_1(x_N^*)}{\partial x_n}\right) \tag{88}$$

$$- \left(-\lambda_1 \frac{\partial u_1(x_N^*)}{\partial x_1} - \lambda_2 \frac{\partial u_1(x_N^*)}{\partial x_2} - \ldots - \lambda_n \frac{\partial u_1(x_N^*)}{\partial x_n}\right)$$

$$= 2\lambda_1 \frac{\partial u_1(x_N^*)}{\partial x_1}. \tag{89}$$

Hence, from player 1's first-order condition,

$$\frac{\partial \Delta^{(1,+)}(0)}{\partial \varepsilon} = 0. \tag{90}$$

Next, one considers the second derivative of $\Delta^{(1,+)}(\varepsilon)$ at $\varepsilon = 0$, i.e.,

$$\begin{aligned}
\frac{\partial^2 \Delta^{(1,+)}(0)}{\partial \varepsilon^2} = &\left\{(\lambda_1)^2 \frac{\partial^2 u_1(x_N^*)}{\partial x_1^2} - \lambda_2\lambda_1 \frac{\partial^2 u_1(x_N^*)}{\partial x_2 \partial x_1} - \ldots - \lambda_n\lambda_1 \frac{\partial^2 u_1(x_N^*)}{\partial x_n \partial x_1}\right.\\
&- \lambda_1\lambda_2 \frac{\partial^2 u_1(x_N^*)}{\partial x_1 \partial x_2} + (\lambda_2)^2 \frac{\partial^2 u_1(x_N^*)}{\partial x_2^2} + \ldots + \lambda_n\lambda_2 \frac{\partial^2 u_1(x_N^*)}{\partial x_n \partial x_2}\\
&\vdots\\
&\left.-\lambda_1\lambda_n \frac{\partial^2 u_1(x_N^*)}{\partial x_1 \partial x_n} + \lambda_2\lambda_n \frac{\partial^2 u_1(x_N^*)}{\partial x_2 \partial x_n} + \ldots + (\lambda_n)^2 \frac{\partial^2 u_1(x_N^*)}{\partial x_n^2}\right\}\\
&- \left\{(\lambda_1)^2 \frac{\partial^2 u_1(x_N^*)}{\partial x_1^2} + \lambda_2\lambda_1 \frac{\partial^2 u_1(x_N^*)}{\partial x_2 \partial x_1} + \ldots + \lambda_n\lambda_1 \frac{\partial^2 u_1(x_N^*)}{\partial x_n \partial x_1}\right.\\
&+ \lambda_1\lambda_2 \frac{\partial^2 u_1(x_N^*)}{\partial x_1 \partial x_2} + (\lambda_2^2) \frac{\partial^2 u_1(x_N^*)}{\partial x_2^2} + \ldots + \lambda_n\lambda_2 \frac{\partial^2 u_1(x_N^*)}{\partial x_n \partial x_2}\\
&\vdots\\
&\left.+ \lambda_1\lambda_n \frac{\partial^2 u_1(x_N^*)}{\partial x_1 \partial x_n} + \lambda_2\lambda_n \frac{\partial^2 u_1(x_N^*)}{\partial x_2 \partial x_n} + \ldots + (\lambda_n)^2 \frac{\partial^2 u_1(x_N^*)}{\partial x_n^2}\right\}.
\end{aligned} \tag{91}$$

Collecting terms, one obtains

$$\frac{\partial^2 \Delta^{(1,+)}(0)}{\partial \varepsilon^2} = -2\lambda_1 \left\{\lambda_2 \frac{\partial^2 u_1(x_N^*)}{\partial x_2 \partial x_1} + \lambda_3 \frac{\partial^2 u_1(x_N^*)}{\partial x_3 \partial x_1} + \ldots + \lambda_n \frac{\partial^2 u_1(x_N^*)}{\partial x_n \partial x_1}\right\}. \tag{92}$$

Thus, using $\lambda_1 > 0$, and recalling that the signs in the first of the n inequalities in the system $\overline{J}\lambda_N > 0$ are all negative, one arrives at

$$\frac{\partial^2 \Delta^{(1,+)}(0)}{\partial \varepsilon^2} > 0. \tag{93}$$

It follows that $\Delta^{(1,+)}(\varepsilon) > 0$ for any $\varepsilon > 0$ sufficiently small, which proves (i). To verify claims

(ii) through (iv), define payoff differences

$$\Delta^{(i,+)}(\varepsilon) = u_i(x_N^{(i,+)}(\varepsilon)) - u_i(x_N^{(i-1,+)}(\varepsilon)) \qquad (i = 2, ..., n), \tag{94}$$

$$\Delta^{(1,-)}(\varepsilon) = u_1(x_N^{(1,-)}(\varepsilon)) - u_1(x_N^{(n,+)}(\varepsilon)), \tag{95}$$

$$\Delta^{(i,-)}(\varepsilon) = u_i(x_N^{(i,-)}(\varepsilon)) - u_i(x_N^{(i-1,+)}(\varepsilon)) \qquad (i = 2, ..., n). \tag{96}$$

Using the players' necessary first-order conditions, it is straightforward to validate that

$$\frac{\partial \Delta^{(i,+)}(0)}{\partial \varepsilon} = 0 \qquad (i = 2, ..., n), \tag{97}$$

$$\frac{\partial \Delta^{(1,-)}(0)}{\partial \varepsilon} = 0, \tag{98}$$

$$\frac{\partial \Delta^{(i,-)}(0)}{\partial \varepsilon} = 0 \qquad (i = 2, ..., n). \tag{99}$$

Moreover, for $i = 2, ..., n$, calculations analogous to (91) yield

$$\frac{\partial^2 \Delta^{(i,+)}(0)}{\partial \varepsilon^2} = 2\lambda_i \left\{ \lambda_1 \frac{\partial^2 u_i(x_N^*)}{\partial x_1 \partial x_i} + ... + \lambda_{i-1} \frac{\partial^2 u_i(x_N^*)}{\partial x_{i-1} \partial x_i} \right. \tag{100}$$

$$\left. -\lambda_{i+1} \frac{\partial^2 u_i(x_N^*)}{\partial x_{i+1} \partial x_i} - ... - \lambda_n \frac{\partial^2 u_i(x_N^*)}{\partial x_n \partial x_i} \right\}$$

$$> 0, \tag{101}$$

where the inequality corresponding to player i's strategy change corresponds precisely to the i's inequality in the system $\overline{J}\lambda_N > 0$. Finally, one notes that, since $d(-\varepsilon)^2 = d\varepsilon^2$, it follows that

$$\frac{\partial^2 \Delta^{(i,-)}(0)}{\partial \varepsilon^2} = \frac{\partial^2 \Delta^{(i,+)}(0)}{\partial \varepsilon^2} \qquad (i = 1, ..., n). \tag{102}$$

In sum, this clearly proves (ii) through (iv). Thus, there exists a strict improvement cycle in Γ. Since this is impossible, the lemma follows. □

Proof of Lemma 5. Take a bijection $\pi : N \to N$ and a subset $F \subseteq N$. Suppose that players have been renamed corresponding to π, so that $\pi^{-1}(1)$ moves first and $\pi^{-1}(n)$ moves last, and the strategy spaces of the players in F have been flipped around. Suppose first that $F = \varnothing$. Then, if a strict improvement cycle corresponding to (π, F) can be constructed, Lemma

4 implies that the matrix

$$J^{\pi}(x_N^*) = \begin{pmatrix} 0 & -\sigma_{\pi^{-1}(1)\pi^{-1}(2)}(x_N^*) & \cdots & -\sigma_{\pi^{-1}(1)\pi^{-1}(n)}(x_N^*) \\ \sigma_{\pi^{-1}(2)\pi^{-1}(1)}(x_N^*) & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -\sigma_{\pi^{-1}(n-1)\pi^{-1}(n)}(x_N^*) \\ \sigma_{\pi^{-1}(n)\pi^{-1}(1)}(x_N^*) & \cdots & \sigma^{(\pi,F)}_{\pi^{-1}(n)\pi^{-1}(n-1)}(x_N^*) & 0 \end{pmatrix}, \tag{103}$$

is semipositive. Define the $n \times n$ permutation matrix $\Xi^{\pi} = \{\xi_{ij}\}$ with entries $\xi_{ij} = 1$ if $j = \pi(i)$ and $\xi_{ij} = 0$ if $j \neq \pi(i)$. Then it can be checked that[32]

$$\Xi^{\pi} J^{\pi}(x_N^*)(\Xi^{\pi})^T = J^{(\pi,\varnothing)}(x_N^*). \tag{108}$$

Since semipositivity of a matrix is not affected by multiplication from the right or left with a permutation matrix (Johnson et al., 1994, p. 267), $J^{(\pi,\varnothing)}(x_N^*)$ is semipositive. To complete the proof, it suffices to note that the slope $\sigma_{ij}(x_N^*)$ changes sign when precisely one of the two strategy spaces of players i and j is flipped around. □

Proof of Proposition 2. Fix a regular interior equilibrium x_N^* of the generalized ordinal potential game Γ. To prove the first claim, let $\{i_1, i_2, i_3\}$ be any triplet of pairwise different players. Clearly, one may rename the players such that $i_1 = 1$, $i_2 = 2$, and $i_3 = 3$. Suppose first that Γ exhibits either strategic complements or strategic substitutes at x_N^*. Thus, we assume

[32] E.g., let $\pi = (123)$. Then, with i counting rows and j counting columns,

$$\Xi^{\pi} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{104}$$

Therefore,

$$\Xi^{\pi} J^{\pi}(x_N^*)(\Xi^{\pi})^T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\sigma_{31}(x_N^*) & -\sigma_{32}(x_N^*) \\ \sigma_{13}(x_N^*) & 0 & -\sigma_{12}(x_N^*) \\ \sigma_{23}(x_N^*) & \sigma_{21}(x_N^*) & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{105}$$

$$= \begin{pmatrix} 0 & -\sigma_{12}(x_N^*) & \sigma_{13}(x_N^*) \\ \sigma_{21}(x_N^*) & 0 & \sigma_{23}(x_N^*) \\ -\sigma_{31}(x_N^*) & -\sigma_{32}(x_N^*) & 0 \end{pmatrix} \tag{106}$$

$$= \begin{pmatrix} 0 & -\sigma^{(\pi,\varnothing)}_{12}(x_N^*) & -\sigma^{(\pi,\varnothing)}_{13}(x_N^*) \\ \sigma^{(\pi,\varnothing)}_{21}(x_N^*) & 0 & -\sigma^{(\pi,\varnothing)}_{23}(x_N^*) \\ \sigma^{(\pi,\varnothing)}_{31}(x_N^*) & \sigma^{(\pi,\varnothing)}_{32}(x_N^*) & 0 \end{pmatrix}, \tag{107}$$

consistent with relationship (108).

that either $\sigma_{ij}(x_N^*) \geq 0$ for all $i, j \in \{1, 2, 3\}$ with $j \neq i$ (in the case of strategic complements), or $\sigma_{ij}(x_N^*) \leq 0$ for all $i, j \in \{1, 2, 3\}$ with $j \neq i$ (in the case of strategic substitutes). In this case, one flips around player 2's strategy space, yet leaves unchanged the order in which the finite sequence is run through (i.e., $\pi = \mathrm{id}$ and $F = \{2\}$). Then, by Lemma 5, the modified matrix

$$J_3 \equiv J^{(\mathrm{id},\{2\})}(x_N^*) = \begin{pmatrix} 0 & \sigma_{12}(x_N^*) & -\sigma_{13}(x_N^*) \\ -\sigma_{21}(x_N^*) & 0 & \sigma_{23}(x_N^*) \\ \sigma_{31}(x_N^*) & -\sigma_{32}(x_N^*) & 0 \end{pmatrix} \tag{109}$$

cannot be inverse nonnegative. To prove (35), it suffices to show that the determinant of J_3,

$$|J_3| = \sigma_{12}(x_N^*)\sigma_{23}(x_N^*)\sigma_{31}(x_N^*) - \sigma_{21}(x_N^*)\sigma_{32}(x_N^*)\sigma_{13}(x_N^*), \tag{110}$$

vanishes. To provoke a contradiction, suppose first that $|J_3| > 0$. Then, from the temporary assumption of either strategic complements or strategic substitutes at x_N^*, all the entries of the matrix inverse of J_3,

$$(J_3)^{-1} = \frac{1}{|J_3|} \begin{pmatrix} \sigma_{23}(x_N^*)\sigma_{32}(x_N^*) & \sigma_{13}(x_N^*)\sigma_{32}(x_N^*) & \sigma_{12}(x_N^*)\sigma_{23}(x_N^*) \\ \sigma_{31}(x_N^*)\sigma_{23}(x_N^*) & \sigma_{13}(x_N^*)\sigma_{31}(x_N^*) & \sigma_{21}(x_N^*)\sigma_{13}(x_N^*) \\ \sigma_{21}(x_N^*)\sigma_{32}(x_N^*) & \sigma_{12}(x_N^*)\sigma_{31}(x_N^*) & \sigma_{12}(x_N^*)\sigma_{21}(x_N^*) \end{pmatrix}, \tag{111}$$

are nonnegative, in contradiction to the earlier conclusion that J_3 is not inverse nonnegative. Hence, $|J_3| \leq 0$. Suppose next that $|J_3| < 0$. Then, by running through the same path in the opposite direction (e.g., by letting $\pi = (13)$ and $F = \{2\}$), one obtains from Lemma 5 that

$$J_3' \equiv J^{((13),\{2\})}(x_N^*) = \begin{pmatrix} 0 & -\sigma_{12}(x_N^*) & \sigma_{13}(x_N^*) \\ \sigma_{21}(x_N^*) & 0 & -\sigma_{23}(x_N^*) \\ -\sigma_{31}(x_N^*) & \sigma_{32}(x_N^*) & 0 \end{pmatrix} = -J_3 \tag{112}$$

is not inverse nonnegative. The matrix inverse of J_3' is consequently given by $(J_3')^{-1} = -(J_3)^{-1}$. Hence, in this case, recalling (111) and $|J_3| < 0$, all entries of $(J_3')^{-1}$ are nonnegative, in contradiction to the fact that J_3' is not inverse nonnegative. It follows that $|J_3| = 0$, which proves the first claim in the case where Γ exhibits either strategic complements or strategic substitutes at x_N^*. Next, we drop the assumption that Γ exhibits either strategic complements or strategic substitutes at x_N^*. From Proposition 1, we know, however, that Γ exhibits pairwise strategic complements or substitutes at x_N^*. Hence, up to another renaming of the players, there are only two cases:

(i) Strategic complements at x_N^* between player 1 and each of players 2 and 3, as well as strategic substitutes at x_N^* between players 2 and 3;

(ii) Strategic substitutes at x_N^* between player 1 and each of players 2 and 3, as well as strategic complements at x_N^* between players 2 and 3.

In either case, by flipping around the strategy space of player 1, the game may be transformed into a game that exhibits either strategic substitutes at x_N^* or strategic complements at x_N^*. Since the operation of flipping around individual strategy spaces does not affect the validity of equation (35), we find that the equation indeed holds generally in the case of three players.

The proof for $m \geq 4$ follows now easily by induction. To see this, let $\{i_1, i_2, ..., i_m\}$ be an arbitrary set of pairwise different players. Suppose that the claim holds for any $m' \in \{3, 4, ..., m-1\}$. Then, in particular, a consideration of the two subsets $\{i_1, i_2, ..., i_{m-1}\}$ and $\{i_{m-1}, i_m, i_1\}$ shows that

$$\sigma_{i_1 i_2}(x_N^*) \cdot ... \cdot \sigma_{i_{m-2} i_{m-1}}(x_N^*) \cdot \sigma_{i_{m-1} i_1}(x_N^*) = \sigma_{i_2 i_1}(x_N^*) \cdot ... \cdot \sigma_{i_{m-1} i_{m-2}}(x_N^*) \cdot \sigma_{i_1 i_{m-1}}(x_N^*), \quad (113)$$

$$\sigma_{i_{m-1} i_m}(x_N^*) \cdot \sigma_{i_m i_1}(x_N^*) \cdot \sigma_{i_1 i_{m-1}}(x_N^*) = \sigma_{i_m i_{m-1}}(x_N^*) \cdot \sigma_{i_1 i_m}(x_N^*) \cdot \sigma_{i_{m-1} i_1}(x_N^*). \quad (114)$$

Taking the respective products of the left-hand and right-hand sides of these equations yields

$$\begin{aligned} &\left(\sigma_{i_1 i_2}(x_N^*) \cdot ... \cdot \sigma_{i_{m-1} i_m}(x_N^*) \cdot \sigma_{i_m i_1}(x_N^*)\right) \cdot \left(\sigma_{i_{m-1} i_1}(x_N^*) \cdot \sigma_{i_1 i_{m-1}}(x_N^*)\right) \\ &= \left(\sigma_{i_2 i_1}(x_N^*) \cdot ... \cdot \sigma_{i_{m-1} i_m}(x_N^*) \cdot \sigma_{i_1 i_m}(x_N^*)\right) \cdot \left(\sigma_{i_1 i_{m-1}}(x_N^*) \cdot \sigma_{i_{m-1} i_1}(x_N^*)\right). \end{aligned} \quad (115)$$

By assumption, $\sigma_{i_1 i_{m-1}}(x_N^*) \neq 0$ and $\sigma_{i_{m-1} i_1}(x_N^*) \neq 0$. Hence, eliminating the common nonzero factors, (115) implies

$$\sigma_{i_1 i_2}(x_N^*) \cdot ... \cdot \sigma_{i_{m-1} i_m}(x_N^*) \cdot \sigma_{i_m i_1}(x_N^*) = \sigma_{i_2 i_1}(x_N^*) \cdot ... \cdot \sigma_{i_{m-1} i_m}(x_N^*) \cdot \sigma_{i_1 i_m}(x_N^*), \quad (116)$$

as claimed. This concludes the induction argument, and therefore proves the proposition. □

Proof of Corollary 1. Let x_N^* be an interior regular Nash equilibrium such that $\sigma_{ij}(x_N^*) \neq 0$ for all $i \neq j$. We need to find positive constants $w_1(x_N^*) > 0, ..., w_n(x_N^*) > 0$ such that

$$\sigma_{ij}(x_N^*) w_i(x_N^*) = \sigma_{ji}(x_N^*) w_j(x_N^*) \qquad (i, j \in N, j \neq i). \quad (117)$$

It is claimed that

$$w_i(x_N^*) = (|\sigma_{12}(x_N^*)| \cdot ... \cdot |\sigma_{i-1i}(x_N^*)|) \cdot (|\sigma_{i+1i}(x_N^*)| \cdot ... \cdot |\sigma_{nn-1}(x_N^*)|) \qquad (i \in N) \quad (118)$$

does the job. Clearly, it suffices to check (117) for $i < j$. Splitting the product in the second bracket of (118), and plugging the result into the left-hand side of (117), one obtains

$$\begin{aligned}\sigma_{ij}(x_N^*)w_i(x_N^*) &= (|\sigma_{12}(x_N^*)| \cdot ... \cdot |\sigma_{i-1i}(x_N^*)|) \\ &\cdot \operatorname{sgn}(\sigma_{ij}(x_N^*)) \cdot |\sigma_{i+1i}(x_N^*)| \cdot ... \cdot |\sigma_{jj-1}(x_N^*)| \cdot |\sigma_{ij}(x_N^*)| \\ &\cdot (|\sigma_{j+1j}(x_N^*)| \cdot ... \cdot |\sigma_{nn-1}(x_N^*)|) .\end{aligned} \tag{119}$$

From Proposition 1 and the assumption that slopes do not vanish,

$$\operatorname{sgn}(\sigma_{ij}(x_N^*)) = \operatorname{sgn}(\sigma_{ji}(x_N^*)). \tag{120}$$

Moreover, from Proposition 2,

$$\sigma_{i+1i}(x_N^*) \cdot ... \cdot \sigma_{jj-1}(x_N^*) \cdot \sigma_{ij}(x_N^*) = \sigma_{ii+1}(x_N^*) \cdot ... \cdot \sigma_{j-1j}(x_N^*) \cdot \sigma_{ji}(x_N^*). \tag{121}$$

Plugging (120) and (121) into relationship (119) delivers

$$\begin{aligned}\sigma_{ij}(x_N^*)w_i(x_N^*) &= (|\sigma_{12}(x_N^*)| \cdot ... \cdot |\sigma_{i-1i}(x_N^*)|) \\ &\cdot \operatorname{sgn}(\sigma_{ji}(x_N^*)) \cdot |\sigma_{ii+1}(x_N^*)| \cdot ... \cdot |\sigma_{j-1j}(x_N^*)| \cdot |\sigma_{ji}(x_N^*)| \qquad (122) \\ &\cdot (|\sigma_{j+1j}(x_N^*)| \cdot ... \cdot |\sigma_{nn-1}(x_N^*)|) \\ &= \sigma_{ji}(x_N^*)w_j(x_N^*). \qquad (123)\end{aligned}$$

This proves the claim and, hence, the corollary. □

Proof of Corollary 2. See the text before the corollary. □

Proof of Corollary 3. It suffices to note that, at any symmetric equilibrium x_N^*, the matrix $J^{(\pi,F)}(x_N^*)$ is skew-symmetric for any bijection $\pi : N \to N$ and for any subset $F \subseteq N$. □

Proof of Corollary 4. *(Only if)* From Proposition 1, $\sigma_{12}(x_N^*)\sigma_{21}(x_N^*) \geq 0$. However, from the zero-sum property and Schwarz's theorem,

$$\operatorname{sgn}(\frac{\partial^2 u_1(x_N^*)}{\partial x_2 \partial x_1}) = -\operatorname{sgn}(\frac{\partial^2 u_2(x_N^*)}{\partial x_2 \partial x_1}) = -\operatorname{sgn}(\frac{\partial^2 u_2(x_N^*)}{\partial x_1 \partial x_2}). \tag{124}$$

Hence, $\operatorname{sgn}(\sigma_{12}(x_N^*)) = -\operatorname{sgn}(\sigma_{21}(x_N^*))$, and consequently $\sigma_{12}(x_N^*) = \sigma_{21}(x_N^*) = 0$. *(If)* Immediate. □

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List of symbols (not for publication)

Γ, $N = \{1, ..., n\}$	game, set of players
$X_i \subseteq \mathbb{R}$, $x_i, \widehat{x}_i, x_i^{\#}$	player i's strategy set, typical elements
$u_i : X_N \equiv X_1 \times ... \times X_n \rightarrow \mathbb{R}$	player i's payoff function
$\upsilon_i = \partial u_i / \partial x_i$	marginal payoff
$x_N^* = (x_1^*, ..., x_n^*) \in X_N$	Nash equilibrium
$x_{-i} = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n), X_{-i}$	strategy profile of i's opponents, set
$\beta_i \equiv \beta_i(x_{-i}) \equiv \beta(x_{-i}; x_N^*)$	player i's local best-response function
$U \subseteq X_{-i}$	a small open neighborhood of x_{-i}^*
$\sigma_{ij} = \sigma_{ij}(x_N^*)$	slope of β_i at x_N^* and w.r.t j
$P : X_N \rightarrow \mathbb{R}$	potential function
$... \rightarrow x_N^0 \rightarrow x_N^1 \rightarrow ... \rightarrow x_N^{L-1} \rightarrow ...$	strict improvement cycle (of length L)
$\iota(l)$	player changing x_N^l to x_N^{l+1}
$\lambda_N = (\lambda_1, ..., \lambda_n)^T \in \mathbb{R}^n$	vector of coefficients
$A \in \mathbb{R}^{n \times n}$, $\widehat{A} \in \mathbb{R}^{n \times m}$, $\widetilde{A} \in \mathbb{R}^{m \times m}$	square, sub-, and principal submatrix
$\mathbb{R}_{++}^n = \{z_N \in \mathbb{R}^n : z_N > 0\}$	open positive orthant
A^{-1}	matrix inverse of A
$\phi : X_N \rightarrow \mathbb{R}$	auxiliary function
$x_{-i,j}, X_{-i,j}$	strategy profile (excluding i and j), set
$\varepsilon, \varepsilon_1, \varepsilon_2, \varepsilon_3$	small positive constants
V_i, V	player i's valuation, common valuation
$J \in \mathbb{R}^{n \times n}$	slope matrix
$\pi : N \rightarrow N, \mathrm{id}, (i_1 ... i_m), F \subseteq N$	permutations, set of "flipped" players
$J^{(\pi,F)}, \sigma_{ij}^{(\pi,F)}, \mathrm{sgn}(d), I_{\{i \in F\}}$	signed slope (matrix), sign, indicator
$i_1, ..., i_m$	pairwise distinct players
$\alpha_{ij} \neq 0, b_i \in \mathbb{R}, c_i > 0, C_i(.)$	parameters, cost function
$Q_i > 0, s_i > 0, \theta_{ij} \neq 0, \gamma_i(.)$	parameters, cost function
$U_i(x_i, \overline{x}_{-i}), \sigma_i$	aggregative game payoff, slope
X	symmetric strategy space
$(x_i^{(1)}, ..., x_i^{(d_i)})$	player i's strategy (multi-dimensional)
$H_{ij} \in \mathbb{R}^{d_i \times d_j}$	matrix of cross-derivatives
$\Lambda_i = (\lambda_i^{(1)}, ..., \lambda_i^{(d_1)}) \in \mathbb{R}^{d_i} \backslash \{0\}$	direction
$x_N^- = (x_1^-, x_2^-), x_N^+ = (x_1^+, x_2^+)$	two strategy profiles
$\widetilde{x}_N^-(.), \widetilde{x}_N^+(.)$	further strategy profiles
$\lambda^{\#} = (\lambda_1^-, \lambda_1^+, \lambda_2^-, \lambda_2^+) \in \mathbb{R}^4$	vector
$\nabla(x_N^-, x_N^+) \in \mathbb{R}^{4 \times 4}$	an auxiliary matrix
$\Delta_i^+(.), \Delta_i^-(.), \Delta_j^+(.), \Delta_j^-(.)$	payoff differences (two players)
$\overline{J}, J_3, J_3'$	auxiliary matrices
$x_N^{(i,+)}(.), x_N^{(i,-)}(.)$	points in the strict improvement cycle
$\Delta^{(i,+)}(.), \Delta^{(i,-)}(.)$	payoff differences (n players)
$\Xi^\pi = \{\xi_{ij}\}$	permutation matrix
m'	positive integer used in the induction