Implementation of Efficient Investments in Mechanism Design*

Kentaro Tomoeda†

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Abstract

This paper investigates when we can eliminate investment inefficiency in mechanism design under quasi-linear utility. We show that when agents invest only before participating in the mechanism, inefficient investment equilibria cannot be ruled out for some cost types whenever an allocatively efficient social choice function is implemented. We then consider when agents invest before *and after* participating in the mechanism. When *ex post* investments are possible and an allocatively constrained-efficient social choice function is implemented, efficient investments can be fully implemented in subgame-perfect equilibria if and only if the social choice function is *commitment-proof* (a weaker requirement than strategy-proofness). We also show that efficient investments and allocations are implementable even given a budget-balance requirement. Our positive result continues to hold in the incomplete information setting when the social choice function to be implemented is strategy-proof.

Keywords: investment efficiency, full implementation, mechanism design, commitment, *ex post* investment

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†University of Technology Sydney, email: Kentaro.Tomoeda@uts.edu.au.

1 Introduction

Can an auction, like the spectrum auction, be designed to induce efficient investments as well as efficient allocations? A standard assumption in the mechanism design literature is that the values that the participants get out of the possible outcomes are exogenously given. In many real-life applications however, there are opportunities to invest in the values of the outcomes outside of the mechanism. In the spectrum auction, telecom companies make investments in new technologies or build base stations in anticipation of winning the spectrum licenses. In a procurement auction, participating firms make efforts to reduce the cost of production in preparation for bidding (Tan, 1992; Bag, 1997; Arozamena and Cantillon, 2004). Moreover, the firms in these auctions not only make *ex ante* investments but also make further investments if they win the auction (Piccione and Tan, 1996). These investments endogenously form the valuations of the outcomes that are determined by the auction. At the same time, the incentives of both *ex ante* and *ex post* investments are affected by the structure of the allocation mechanism. Therefore, to seek an efficient mechanism, we should take account of the efficiency of the investments it induces, in addition to its standard efficiency within the mechanism.

The goal of this paper is to analyze when we can *fully* implement efficient investments, i.e., under what mechanisms every equilibrium of the investment game will be efficient for any cost types of agents.[1] To do this, we consider a general mechanism design model with quasi-linear utility. This includes several important applications such as auctions, matching with transfers and the provision of public goods. The valuation functions of agents at the market clearing stage are endogenously determined. We examine the following two environments: (i) agents invest only before participating in the mechanism, and (ii) they invest before and after the mechanism is run. In either setting, we characterize the social choice functions for which efficient allocations and investments are fully implementable. The main results are summarized as follows: first, with only *ex ante* investments, we show that efficient investments are not implementable for any allocatively efficient social choice function (Theorem 1). Next, allowing for *ex post* investments, we show that a new concept of *commitment-proofness* is sufficient and necessary for implementing efficient investments when an allocatively efficient social choice function is implemented (Theorem 2).

Furthermore, we consider budget balance, which is often required in the provision of public goods. In this setting, we show that there exists a commitment-proof, allocatively

[1] When we simply say "implementation" in this paper, this refers to full implementation. See Definition 3 and 6 for the mathematical expressions.

efficient and budget-balanced social choice function (Proposition 1). This implies that even with a budget-balance requirement, it is always possible to implement efficient investments and efficient allocations at the same time.

We also extend the model to the incomplete information setting where agents are unsure about the cost types of other agents. In this environment, we show that if a strategy-proof social choice function is implemented in perfect Bayesian equilibria (PBE), efficient investments are also implementable in PBE (Proposition 2).

We would like to highlight two main points of our research question. First, we seek mechanisms for which *every* equilibrium of the investment game becomes socially efficient. This advances the traditional question of the *existence* of an efficient investment equilibrium, which has been examined by several papers. Rogerson (1992) showed that when agents make pre-mechanism investments, there is a socially efficient investment equilibrium for any Bayesian incentive compatible and allocatively efficient mechanisms. In the context of information acquisition (Milgrom, 1981; Obara, 2008), Bergemann and Välimäki (2002) indicate the link between *ex ante* efficiency and strategy-proofness; the VCG mechanism ensures *ex ante* efficiency under private values. Second, unlike most of the models in the incomplete-contracts literature (Grossman and Hart, 1986; Hart and Moore, 1988, 1990; Aghion et al., 1994), we assume that the designer does not know the cost functions of agents for their investments.[2] Therefore, we consider mechanisms for which investment efficiency is satisfied in equilibrium for any possible cost types of agents. Hatfield et al. (2016) take this approach, and show that strategy-proofness is sufficient and necessary to ensure the existence of an efficient investment equilibrium for all cost functions. Putting these two points together however, the literature has not given a general answer to the question of when we can ensure investment efficiency in every equilibrium for any cost types of agents.

Even though efficient investments are achieved in equilibrium under strategy-proof mechanisms (Hatfield et al., 2016), with only *ex ante* investments, there may exist another inefficient equilibrium for some cost types. Many authors in the literature pointed out this problem in a particular example, but they have not developed a general result.[3] Consider an example where telecom firms are competing for a spectrum license, and suppose they know the competitors' cost functions for investments. When investments are observable, the *ex ante* investment may work as a commitment device even for a firm whose investment is more

[2]Among agents, we assume that the cost types are complete information in the main model of the paper. In Section 6, we consider the case of incomplete information.

[3]For example, see Example 4 of Hatfield et al. (2016). This motivated the spectrum auction example which will be introduced in the next section.

costly than others. If it is the only firm that makes an investment, at the market clearing stage, this firm may value the license more highly than any other firm does because the cost of investment has been sunk. Therefore, there is an equilibrium at which this inefficient firm makes a lot of costly *ex ante* investments and deters its competitors from investing. This role of *ex ante* investment has also been studied as an entry-deterring behavior for an incumbent firm in an oligopolistic market (Spence, 1977, 1979; Salop, 1979; Dixit, 1980). This intuition is generalized by our first result: when agents invest only before participating in the mechanism, inefficient investment equilibria cannot be ruled out whenever an allocatively efficient social choice function is implemented (Theorem 1).

In order to eliminate such investment inefficiency, we consider a setting where agents can invest before *and after* participating in the mechanism. In many applications, agents make further investments after the market clearing stage to maximize the value of the outcome realized in the mechanism. In the context of bidding for government contracts, firms invest in cost reduction once they are selected by the government to perform the task (McAfee and McMillan, 1986; Laffont and Tirole, 1986, 1987). For simplicity, we model investment as an explicit choice of valuation functions. *Ex ante* and *ex post* investments are modeled in the following way. First, agents choose their own valuation functions over the outcomes prior to the mechanism. The cost of each valuation function is determined by each agent's cost type. In the main model of our paper, we assume that the cost types are complete information among agents, but not known to the designer. These *ex ante* investments are irreversible, but after participating in the mechanism, agents may make further investments by revising their valuations to more costly ones. Our main theorem characterizes allocatively efficient social choice functions for which investment efficiency is guaranteed in every equilibrium for any cost types: given that an allocatively constrained-efficient social choice function is implemented, commitment-proofness of the social choice function is sufficient and necessary for implementing efficient investments in every subgame-perfect equilibrium (Theorem 2).

We introduce a new concept of a social choice function called *commitment-proofness*, which is illustrated in the following scenario. Suppose that before the mechanism is run, each participant of a mechanism can use a (hypothetical) commitment device which changes her valuations of the outcomes in an arbitrary way. If the commitment device is free, agents would improve their valuations as they like. But consider the following cost incurred by the use of such devices: if an agent were to increase the valuations of the outcomes, she must pay the maximum increment out of all increments for the possible outcomes. Commitment-proofness requires that no agent be willing to make use of such commitment devices.[4] Commitment-proofness is an abstract concept defined in this way, and Theorem 2 reveals that this is indeed sufficient and necessary for achieving efficiency of investments in our model.

Then, how does the introduction of *ex post* investment circumvent the impossibility result in Theorem 1? First, as we assume no externality of investments, investment efficiency is achieved by any allocatively efficient mechanism if agents make no *ex ante* investments. Therefore, we need to find the conditions under which no agent has the incentive to make positive *ex ante* investments for any cost type.[5] Consider the firm whose investment is more costly than others in the spectrum auction explained above. Suppose that no other firms invest *ex ante*. The values of the spectrum license for these efficient firms would be low if there were no *ex post* investment opportunities. But now the value for each of them should be equal to the maximum net profit from the license inclusive of the cost of future investment because any firm would make the optimal investment *ex post* upon winning the auction. Thus, in order for the inefficient firm to win, it needs to beat its competitors who value the license more than its potential profit. To completely suppress the incentive of this firm to win out by investing *ex ante*, there must be a certain amount of payment for the license. Commitment-proofness of social choice functions characterizes such transfer payments that are sufficient and necessary for suppressing the incentives to invest *ex ante* in a general environment. In this way, the information of firms' cost types is partly revealed by the presence of *ex post* investment, and commitment-proofness eliminates the incentives for using *ex ante* investment as a commitment device.

In our model, the difficulty of implementing efficient investments stems from the combination of the following assumptions: (i) investments are not verifiable, (ii) investments are irreversible, and (iii) the agents' cost types are not known to the mechanism designer. First, if investments were verifiable to a third party, they could just be part of the outcome of mechanisms and the standard implementation theory applies. However, investment behaviors are usually difficult to describe; they are multi-dimensional and they involve the expenditure of time and effort as well as the expenditure of money (Hart, 1995). These non-contractible investments have also been a central concern in the hold-up problems (Klein et al., 1978; Williamson, 1979, 1983; Grossman and Hart, 1986; Hart and Moore, 1988, 1990). Second, if investments were reversible, the efficiency of allocations would not be affected by the choice

[4] As we will show in Section 4, this property is weaker than the well-known strategy-proofness condition.

[5] In the main model, we introduce a (slight) time discounting between two investment stages so that given that the allocation rule is efficient, investment efficiency is achieved only when no agents make costly *ex ante* investments.

of *ex ante* investments. Therefore, we could apply mechanisms proposed by the standard implementation theory. Finally, if the designer knew the agents' cost types, he would be able to identify the first-best outcome in our model. And since investments do not have any externalities, the efficient level of investment is chosen by agents if the outcome is fixed to the first best.

Unlike related papers that analyze specific mechanisms such as the first-price auction and the second-price auction (Tan, 1992; Piccione and Tan, 1996; Stegeman, 1996; Bag, 1997; Arozamena and Cantillon, 2004), we consider the entire space of social choice functions. Also, we focus on the equilibrium analysis of the investment game outside of the mechanism. That is, the analysis of the game within the mechanism to implement a social choice function is set apart from the discussion. This is because we know that a large class of social choice functions are implementable under complete information. The strongest result in the implementation literature is that any social choice function can be implemented by an extensive form mechanism in subgame-perfect equilibria under quasi-linear utility and complete information environments (Moore and Repullo, 1988; Maskin and Tirole, 1999). Even with static mechanisms, it is shown that a large class of social choice functions are virtually implementable (Abreu and Matsushima, 1992). We take these positive results as given, and give a complete classification of allocatively efficient mechanisms which implement efficient investments as well.[6] In order to detect whether a specific mechanism implements efficient investments from our results, one can simply check if it implements a commitment-proof and allocatively efficient social choice function.

There is also large literature on investment incentives in bargaining and two-sided matching (Gul, 2001; Cole et al., 2001a,b; Felli and Roberts, 2002; de Meza and Lockwood, 2010; Mailath et al., 2013; Nöldeke and Samuelson, 2015). These papers usually do not model the possibility of *ex post* investments. This is because they focus on how agents bargain over the surplus of investments in the market clearing stage and their utility does not reflect their future investments. Therefore, our positive theorem may not be applied to their settings. That being said, our first impossibility theorem applies to many of their models. Moreover, since the externalities of investments are often allowed in the literature, it is even more difficult to eliminate inefficient investment equilibria due to coordination failure. For this reason, this literature has not tackled the question of full implementation, which is our central concern.

The rest of the paper is organized as follows. In Section 2, we explain a numerical

[6] In the extension to the incomplete information environment, we need a tigher condition for a social choice function to be implemented. See Section 6 for more details.

example of the spectrum auction to provide intuition for our main results. Section 3 introduces the formal model and defines implementability of efficient investments. In Section 4, commitment-proofness is introduced, and the impossibility results without *ex post* investments and the possibility results with *ex post* investments are presented. A budget-balance requirement is considered in Section 5. In Section 6, we extend the main model to the incomplete information setting and provide one positive result. Section 7 concludes. All proofs are in Appendix A.

2 Example: Spectrum Auction

Before introducing the general model, we provide intuition for our main theorems (Theorem 1 and 2) using a simple example of an auction. Consider a situation where two firms, A and B, are competing for a single spectrum license. The spectrum license is sold in the second-price sealed-bid auction. (We also consider another mechanism in the last part of the section.) The potential value of the spectrum license is in $[0, 10]$. Each firm $i = \mathrm{A}, \mathrm{B}$ makes investments to determine its own value a^i of the license outside the auction mechanism. Here, we model the investment behavior as the explicit choice of a value from the interval $[0, 10]$.[7] In order to realize $a^{\mathrm{A}}, a^{\mathrm{B}} \in [0, 10]$, each firm incurs the cost of investment which is represented by the following cost functions:

$$c^{\mathrm{A}}(a^{\mathrm{A}}) = \frac{1}{6}(a^{\mathrm{A}})^2,$$
$$c^{\mathrm{B}}(a^{\mathrm{B}}) = \frac{1}{4}(a^{\mathrm{B}})^2.$$

Cost functions are common knowledge between firms, but not known to the mechanism designer.[8] We also assume that investments are observable among agents (but not verifiable). Therefore, the information is complete between firms in the games which will be defined below.

First, consider efficient investments and allocation which maximize the sum of each firm's profit from the license inclusive of the cost of investments (i.e., the social welfare). If firm A

[7]This means that there is no externality for investments. We assume this in the general model as well. See Matsushima and Noda (2016) for the case where investments have arbitrary externality effects on other agents' valuations.

[8]We only consider these cost functions in this example, but the potential set of cost types is assumed to be large in the main model.

obtains the license, the optimal investment would be

$$\arg\max_{a^{\mathrm{A}}\in[0,10]}\Big\{-\frac{1}{6}(a^{\mathrm{A}})^2+a^{\mathrm{A}}\Big\}=3.$$

The maximum net profit for firm A in this case is

$$\max_{a^{\mathrm{A}}\in[0,10]}\Big\{-\frac{1}{6}(a^{\mathrm{A}})^2+a^{\mathrm{A}}\Big\}=\frac{3}{2}.$$

Similarly, for firm B, the optimal investment would be

$$\arg\max_{a^{\mathrm{B}}\in[0,10]}\Big\{-\frac{1}{4}(a^{\mathrm{B}})^2+a^{\mathrm{B}}\Big\}=2.$$

The maximum net profit for firm B in this case is

$$\max_{a^{\mathrm{B}}\in[0,10]}\Big\{-\frac{1}{4}(a^{\mathrm{B}})^2+a^{\mathrm{B}}\Big\}=1.$$

Since there is a single license, it is clear that only one of the firms should make a positive investment to achieve investment efficiency. Therefore, the unique profile of efficient investments is $(a^{*\mathrm{A}}, a^{*\mathrm{B}}) = (3, 0)$ and we should allocate the license to firm A. The maximum social welfare is $\frac{3}{2}$.

Now we define the investment stage as a game between these two firms, and examine whether every equilibrium of the investment game achieves efficiency. The following two settings are considered: [1] firms make investments only before participating in the mechanism, and [2] they make investments before and after participating in the mechanism. We analyze the second-price auction in both cases, and also analyze another mechanism in the second setting. We consider subgame-perfect equilibria in which no agent uses a weakly dominated strategy in this section.[9]

[1] Second-price auction with only *ex ante* investments.

In this case, we model the *ex ante* investment stage as a simultaneous move game where each firm chooses its own valuation.[10] The timeline of the investment and the auction is as follows:

[9] We use this solution concept only in this example to simplify the discussion. In the general model, we employ standard subgame-perfect equilibria for implementation.

[10] Our main results do not heavily rely on the simultaneity of investments. For example, the inefficient equilibrium in the first setting is also achieved when firm B moves first. In addition, the efficiency result in the second setting under the second-price auction is robust to the sequential moves of firms because firm B would not want to invest whatever the sequence of the move is.

1. Each firm $i = \text{A, B}$ chooses its own valuation a^i from $[0, 10]$ simultaneously. The cost of investment $c^i(a^i)$ is paid.

2. They participate in the second-price auction given the valuations $(a^{\text{A}}, a^{\text{B}})$.

First, we consider the auction stage. The unique undominated Nash equilibrium is that each firm bids its own valuation truthfully. Since the valuations of the license for firms are $(a^{\text{A}}, a^{\text{B}})$, firm $i \in \{A, B\}$ whose valuation is higher than the other, i.e., $a^i \geq a^j$ where $j \neq i$, wins the license and pays a^j in the unique equilibrium. Therefore, given the equilibrium of the second-price auction, for any choice of investments $(a^{\text{A}}, a^{\text{B}}) \in [0, 10]^2$, the net profit of firm $i = \text{A, B}$ is written as

$$-c^i(a^i) + (a^i - a^j)\mathbb{1}_{\{a^i \geq a^j\}}$$

where j is the other firm.[11]

Next, let us analyze the equilibrium of the investment stage. First, it is easy to see that the socially efficient investments $(a^{*\text{A}}, a^{*\text{B}}) = (3, 0)$ are achieved in equilibrium. Consider another investment profile $(a^{\text{A}}, a^{\text{B}}) = (0, 2)$ where firm A makes no investment and firm B chooses 2 *ex ante*. Consider firm A's incentive given $a^{\text{B}} = 2$. If firm A wins the auction, the payment in the second-price auction would be 2, which exceeds the maximum net profit of $\frac{3}{2}$ for firm A;

$$-\frac{1}{6}(a^{\text{A}})^2 + (a^{\text{A}} - 2)\mathbb{1}_{\{a^{\text{A}} \geq 2\}} \leq \frac{3}{2} - 2 < 0$$

for any $a^A \in [0, 10]$. Thus, firm A does not have the incentive to win the auction by making a positive investment. For firm B, it is clear that choosing 2 is optimal given that firm A does not make any investments because B will obtain the license in the auction. Therefore, this profile $(a^{\text{A}}, a^{\text{B}}) = (0, 2)$ is an undominated Nash equilibrium of the *ex ante* investment game. However, this is not an efficient investment profile because it gives less social welfare than $(a^{*\text{A}}, a^{*\text{B}}) = (3, 0)$. Thus, we can conclude that there is a socially inefficient undominated subgame-perfect equilibrium. □

This is an example where the second-price auction failed to fully implement efficient investments. Unfortunately, we show that not only the second-price auction but any other

[11] $\mathbb{1}$ is an indicator function. For any proposition p, $\mathbb{1}_{\{p\}}$ is defined by

$$\mathbb{1} = \begin{cases} 1 & \text{if } p \text{ is true,} \\ 0 & \text{otherwise.} \end{cases}$$

mechanism fails to implement efficient investments in the general model when there are no *ex post* investment opportunities and the allocation is selected efficiently (Theorem 1). Next, let us consider what will happen with *ex post* investments when the same second-price auction is used.

[2-1] Second-price auction with *ex ante* and *ex post* investments.

When *ex post* investments are possible, another investment stage for revising their own valuations is added after the mechanism. The timeline of the investment and the auction in this case is:

1. Each firm $i =$ A, B chooses its own valuation a^i from $[0, 10]$ simultaneously. The cost of investment $c^i(a^i)$ is paid.

2. They participate in the second-price auction.

3. Each firm $i =$ A, B again chooses its own valuation $\bar{a}^i$ from $[a^i, 10]$. The cost of additional investment $c^i(\bar{a}^i) - c^i(a^i)$ is paid.

We assume the irreversibility of investments; $\bar{a}^i$ can be only chosen from $[a^i, 10]$. Also, the cost function is assumed to be unchanged over time so that for a fixed total amount $\bar{a}^i$, the total cost of investment is $c^i(\bar{a}^i)$ and choosing any *ex ante* investments $a^i \in [0, \bar{a}^i]$ is indifferent if the allocation is fixed. However, since we consider an auction mechanism to determine the allocation, *ex ante* choices matter as they affect the outcome of the auction. The net profit of firm $i =$ A, B is written as

$$-c^i(a^i) + (\bar{a}^i - p)\mathbb{1}_{\{i \text{ wins the auction}\}} - (c^i(\bar{a}^i) - c^i(a^i))$$

where p is the payment in the auction, whose equilibrium value will be computed below.

Although the investment game is different from the first setting, efficient investments and allocation are unchanged; firm A should obtain the license and it makes investments $(a^{*\mathrm{A}}, \bar{a}^{*\mathrm{A}}) \in [0, 10]^2$ such that $a^{*\mathrm{A}} \leq \bar{a}^{*\mathrm{A}} = 3$. Firm B should not make any investment, i.e., $(a^{*\mathrm{B}}, \bar{a}^{*\mathrm{B}}) = (0, 0)$.

The equilibrium is solved by backward induction. Consider firm A's optimal strategy in the *ex post* investment stage. Given any *ex ante* valuation choice $a^{\mathrm{A}} \in [0, 10]$, the profit from the license in the last stage is

$$\bar{a}^{\mathrm{A}} - \left(c^{\mathrm{A}}(\bar{a}^{\mathrm{A}}) - c^{\mathrm{A}}(a^{\mathrm{A}})\right).$$

Thus, it makes further investment only when it obtains the license and $a^{\rm A}$ is less than 3. The optimal *ex post* investment strategy given $a^{\rm A}$ is

$$\bar{a}^{\rm A} = \begin{cases} \max\{3, a^{\rm A}\} & \text{if firm A obtains the license,} \\ a^{\rm A} & \text{otherwise.} \end{cases}$$

Similarly, firm B's optimal *ex post* investment strategy given $a^{\rm B}$ is

$$\bar{a}^{\rm B} = \begin{cases} \max\{2, a^{\rm B}\} & \text{if firm B obtains the license,} \\ a^{\rm B} & \text{otherwise.} \end{cases}$$

Next, let us analyze the second-price auction. Again, in the unique undominated Nash equilibrium, the firm with the higher willingness to pay should win and it pays the other firm's valuation. Let $b^i(a^i)$ be the value of the license in the auction stage when firm i chooses a^i *ex ante*. The following two things should be noted in calculating it; (i) $b^i(a^i)$ takes account of the optimal strategy in the *ex post* stage, and (ii) the cost of *ex ante* investment is sunk. For each $a^{\rm A} \in [0, 10]$, it is

$$b^{\rm A}(a^{\rm A}) = \max_{\bar{a}^{\rm A} \in [a^{\rm A}, 10]} \left\{ \bar{a}^{\rm A} - \left(c^{\rm A}(\bar{a}^{\rm A}) - c^{\rm A}(a^{\rm A}) \right) \right\} = \begin{cases} \frac{3}{2} + \frac{1}{6}(a^{\rm A})^2 & \text{if } a^{\rm A} \in [0, 3) \text{ and} \\ a^{\rm A} & \text{if } a^{\rm A} \in [3, 10], \end{cases}$$

and for each $a^{\rm B} \in [0, 10]$,

$$b^{\rm B}(a^{\rm B}) = \max_{\bar{a}^{\rm B} \in [a^{\rm B}, 10]} \left\{ \bar{a}^{\rm B} - \left(c^{\rm B}(\bar{a}^{\rm B}) - c^{\rm B}(a^{\rm B}) \right) \right\} = \begin{cases} 1 + \frac{1}{4}(a^{\rm B})^2 & \text{if } a^{\rm B} \in [0, 2) \text{ and} \\ a^{\rm B} & \text{if } a^{\rm B} \in [2, 10]. \end{cases}$$

Intuitively, when firm i's initial investment a^i is greater than the optimal value 3, $b^i(a^i)$ is equal to a^i as there is no further investment. If a^i is less than the optimal value 3, $b^i(a^i)$ is increasing in a^i exactly by the amount of $c^i(a^i)$ because more *ex ante* investment means less cost of additional investment when the license is awarded to the firm. Under the truth-telling equilibrium of the second-price auction, if firm A wins the license, the payment will be $b^{\rm B}(a^{\rm B})$ and vice versa.

Given these equilibrium strategies, we can analyze the first investment stage. Consider firm B's incentive. If it wins the license in the second-price auction, the payment is at least $\frac{3}{2}$ because $b^{\rm A}(a^{\rm A}) \geq \frac{3}{2}$ holds for any $a^{\rm A} \in [0, 10]$. However, since the maximum net profit from the spectrum license is 1 for firm B, it does not have the incentive to win by choosing $a^{\rm B} > \frac{3}{2}$;

$$\max\{2, a^{\rm B}\} - b^{\rm A}(a^{\rm A}) - \frac{1}{4}\left(\max\{2, a^{\rm B}\} \right)^2 \leq 1 - \frac{3}{2} < 0.$$

Therefore, firm B refrains from making investments in equilibrium, and chooses $a^{*\mathrm{B}} = 0$. Since firm A always wins the auction with the payment $b^{\mathrm{B}}(0) = 1$, it is indifferent to choose any investments $(a^{*\mathrm{A}}, \bar{a}^{*\mathrm{A}})$ such that $a^{*\mathrm{A}} \leq \bar{a}^{*\mathrm{A}} = 3$. Finally, $a^{*\mathrm{A}} = 0$ is the unique undominated Nash equilibrium strategy for firm A.[12] Therefore, investment efficiency is achieved in any undominated subgame-perfect equilibrium. □

Now allowing for *ex post* investments, any equilibrium achieves investment efficiency in the second-price auction. Why did this become possible? Intuitively, with only *ex ante* investments, if firm A has not made any investment, it will only bid zero in the second-price auction and firm B will choose an investment $a^{\mathrm{B}} = 2$ to maximize its profit. Furthermore, firm A will optimally choose not to make any investment given $a^{\mathrm{B}} = 2$ because firm B will be able to bid aggresively in the second-price auction. On the other hand, with *ex post* investments, firm A's bid in the second-price auction will be at least $\frac{3}{2}$ because firm A can make a profit when firm B bids less than $\frac{3}{2}$. Now, since firm B's payment would be at least $\frac{3}{2}$ should it win the auction, it cannot make a profit from any positive investment.

However, under other mechanisms, allowing *ex post* investment does not necessarily solve the problem. More importantly, this is not because the mechanism fails to allocate the license efficiently, but because an inefficient investment equilibrium exists even though the mechanism always selects an efficient allocation (according to the valuations in the auction stage).

To introduce such an example of a mechanism, we review the literature of (subgame-perfect) implementation. A seminal paper by Moore and Repullo (1988) showed that under complete information and quasi-linear utility functions, any social choice function is subgame-perfect implementable. This implies that by their mechanism, we can implement an efficient allocation rule with any transfer rule. Consider here one such mechanism: a Moore-Repullo mechanism which always chooses an efficient allocation according to $(b^{\mathrm{A}}(a^{\mathrm{A}}), b^{\mathrm{B}}(a^{\mathrm{B}}))$ and does not impose any transfers.[13]

[2-2] The zero-transfer Moore-Repullo mechanism with *ex ante* and *ex post* investments.

The timeline of the investment game is the same as in the previous case [2-1]. The

[12] In the general model, we introduce a strict time discounting and pin down the unique optimal investment $(a^{*\mathrm{A}}, \bar{a}^{*\mathrm{A}}) = (0, 3)$.

[13] In some countries such as Japan, spectrum licenses are still allocated to firms for free once they are screened by the government. Although this process is not a mechanism, if the government correctly observes the valuations $(b^{\mathrm{A}}(a^{\mathrm{A}}), b^{\mathrm{B}}(a^{\mathrm{B}}))$, it is exactly the social choice function implemented by this Moore-Repullo mechanism.

second-price auction is replaced by the following mechanism.

Stage 1:

1-1. Firm A announces its own valuation $\bar{b}^{\mathrm{A}}$.

1-2. Firm B decides whether to challenge firm A's announcement $\bar{b}^{\mathrm{A}}$.

If firm B does not challenge it, go to stage 2.

If firm B challenges, firm A pays 20 to the mechanism designer. Firm B receives 20 if the challenge is successful, but pays 20 to the mechanism designer if it is a failure. Whether it is a success or a failure is determined by the following game: The license is sold in the second-price auction. Firm B chooses some $\bar{b}^{\mathrm{B}}$ to submit to the auction and a positive value $\eta > 0$, and asks firm A to choose one of them:

(i) submitting any value,

(ii) submitting $\bar{b}^{\mathrm{A}}$ and receiving an additional transfer η.

The challenge is successful only if firm A picks (i). Stop.

Stage 2: Same as stage 1, but the roles of A and B are switched.

Stage 3: If there are no challenges in stage 1 and 2, the license is given for free to firm i such that $\bar{b}^i \geq \bar{b}^j$ where j is the other firm.

Given the optimal strategies in the *ex post* investment stage, for any profile of *ex ante* investments $(a^{\mathrm{A}}, a^{\mathrm{B}})$, it is shown that the unique subgame-perfect equilibrium of this mechanism is such that each firm i = A, B announces its true valuation $b^i(a^i)$, and no firm challenges the other firm's claim (Moore and Repullo, 1988). The intuitive reason is that in the challenge phase, the other firm j can choose some $\bar{b}^j$ and $\eta > 0$ so that the challenge is successful (firm i optimally chooses (i)) whenever the announcement $\bar{b}^i$ of firm i is different from $b^i(a^i)$. Also, the other firm's challenge would never be successful when the announcement is truthful since (ii) is always chosen by a truthful firm. Therefore, the allocation is always determined efficiently and no transfer is imposed in equilibrium.

Consider firm B's incentive in the first investment stage. Now firm B has the incentive to invest more than firm A as long as A's investment is socially efficient, i.e., $a^{\mathrm{A}} \leq 3$. This is because the price of the license is zero in the mechanism and firm B would still earn a positive profit by winning the auction: for some $a^{\mathrm{B}} \in (3, 4)$,

$$\max\{2, a^{\mathrm{B}}\} - 0 - \frac{1}{4}\big(\max\{2, a^{\mathrm{B}}\}\big)^2 > 0.$$

Actually, there is a mixed strategy equilibrium in which $a^{\mathrm{B}} > 0$ occurs with a positive probability. Thus, efficient investments are not implemented by this allocatively efficient Moore-Repullo mechanism with no transfers. □

In the second-price auction with *ex post* investments, firm B could not make a profit by investing $a^{\mathrm{B}} = 2$ because the price of the license was greater than $\frac{3}{2}$. However, in this zero-payment mechanism, $a^{\mathrm{B}} = 2$ remains profitable because firm B does not pay anything in the auction. This shows that the range of the price of the license is critical for inducing the right incentive for firm B. Suppose that the allocation is always efficiently determined, and that firm A does not make any *ex ante* investment, i.e., $a^{\mathrm{A}} = 0$. Then, firm B would lose the auction when choosing $a^{\mathrm{B}} = 0$, but would win the auction if it chooses $a^{\mathrm{B}} = 2$. In order to prevent firm B from choosing 2, the price p of the license when firms choose $(a^{\mathrm{A}}, a^{\mathrm{B}}) = (0, 2)$ *ex ante* should satisfy

$$0 \geq b^{\mathrm{B}}(2) - c^{\mathrm{B}}(2) - p \iff p \geq 1.$$

Obviously, the second-price auction satisfied this condition, but the Moore-Repullo mechanism with no transfers violated it. This idea of choosing a right transfer rule can be applied to more general environments. Our main contribution is to find a sufficient and necessary condition of a social choice function to make no agent inclined to invest *ex ante* in the general model.

3 General Model

There is a finite set I of agents and a finite set Ω of alternatives. Assume $|I| \geq 2$ and $|\Omega| \geq 2$. A valuation function of agent $i \in I$ is $v^i : \Omega \to \mathbb{R}$. The valuation function is endogenously determined by each agent's investment decision as described below. The set of possible valuation functions is $V^i \equiv \times_{\omega \in \Omega}[0, \alpha^{i,\omega}]$ such that $\alpha^{i,\omega} > 0$ for any $\omega \in \Omega$.[14] Denote the profile of the sets of valuations by $V \equiv \times_{i \in I} V^i$. We assume that investments are not verifiable to a third party. Therefore, a mechanism chooses an alternative and transfers, but cannot choose agents' investment behaviors directly. We will explain how we model the mechanism stage (i.e., social choice functions and mechanisms) later in this section.

Each agent makes an investment decision to determine her own valuation over alternatives. We take a shortcut of modeling investment as an explicit choice of a valuation function. For each valuation function, the cost of investment is determined by a cost function $c^i : V^i \times \Theta^i \to \mathbb{R}_+$ where Θ^i is a set of cost types of agent i. Denote the profile of the sets

[14] We make this assumption in order for V to be the domain of social choice functions even when we allow for *ex post* investments. This is not, however, a strong restriction. Because of a great flexibility in the set of possible cost types, we can find a cost type for whom some valuations are extremely costly. Therefore, any subset of V can be the set of valuation functions that are virtually feasible for some cost type of an agent.

of cost types by $\Theta \equiv \times_{i \in I} \Theta^i$. Without loss of generality, the cost of investment is assumed to be non-negative, and for each $\theta^i \in \Theta^i$, there is $v^i \in V^i$ such that $c^i(v^i, \theta^i) = 0$. We also assume that the set of cost types Θ is rich in the sense that any cost vector is possible. More formally, for any agent $i \in I$ and any function $r : V^i \to \mathbb{R}_+$ such that there is $v^i \in V^i$ with $r(v^i) = 0$, there exists a cost type $\theta^i \in \Theta^i$ such that $r(\cdot) = c^i(\cdot, \theta^i)$.

In the main model of this paper, we assume complete information of cost types $\theta \in \Theta$ among agents.[15] However, the mechanism designer does not know their realized cost types. He only knows the structure of the investment game, i.e., the set I of agents, the set Ω of alternatives, the set V of possible valuation functions, the cost functions c and the set Θ of possible cost types. Thus, the goal of the mechanism designer is to implement efficient investments and allocations for any possible cost types $\theta \in \Theta$. We also assume that chosen valuation functions are observable among agents, but neither verifiable nor observable to the designer.

There are two investment stages: before and after participating in the mechanism. We model each investment stage as a simultaneous move game by all agents. Assume that the investment is irreversible; if agent i with cost type θ^i chooses $v^i \in V^i$ first, she can only choose a valuation function from the set $\{\bar{v}^i \in V^i | c^i(\bar{v}^i, \theta^i) \geq c^i(v^i, \theta^i)\}$ in the second investment stage.[16] To clarify, the timeline of the investment game induced by a mechanism is:

0. Agents observe their cost types $\theta \in \Theta$.
1. Each agent makes a prior investment by choosing a valuation function $v^i \in V^i$ simultaneously.
2. Agents participate in a mechanism.
3. After the mechanism is run, each agent can make an additional investment, i.e., each agent chooses a valuation function from $\{\bar{v}^i \in V^i | c^i(\bar{v}^i, \theta^i) \geq c^i(v^i, \theta^i)\}$.

The *ex ante* utility function of an agent has the following three components: the valuation functions she chooses in the first and the second investment stages, the cost function and a discount factor. Let $\beta \in (0, 1]$ be a discount factor which discounts the utility realized in

[15] We relax this assumption in Section 6.

[16] The essential assumption is actually that the cost of *ex ante* investment is sunk, rather than the (physical) irreversibility of an investment itself. However, we maintain the assumption of irreversibility since it keeps the analysis simple and easy to understand.

the second stage and later.[17] For an alternative $\omega \in \Omega$, a transfer vector $t \equiv (t^i)_{i \in I} \in \mathbb{R}^I$ and an investment schedule $(v^i, \bar{v}^i) \in (V^i)^2$ where v^i is the valuation function chosen before the mechanism and $\bar{v}^i$ is the final valuation function, the *ex ante* utility of agent i with cost type θ^i is defined by:

$$-c^i(v^i, \theta^i) + \beta\Big[\bar{v}^i(\omega) - t^i - \big(c^i(\bar{v}^i, \theta^i) - c^i(v^i, \theta^i)\big)\Big]. \tag{1}$$

In the first stage, only the cost $c^i(v^i, \theta^i)$ of *ex ante* investment is paid. In the second stage, the outcome (ω, t) of the mechanism is evaluated by the final valuation function $\bar{v}^i$. And in the last stage, the additional cost $c^i(\bar{v}^i, \theta^i) - c^i(v^i, \theta^i) \geq 0$ of revising the valuation function is paid.[18] Throughout the paper, we consider this quasi-linear utility function, i.e., utility to be perfectly transferable.

When agents face the mechanism in the second stage, the cost of investment made in the first stage has been sunk. Moreover, for a rational agent, an alternative $\omega \in \Omega$ is evaluated by a valuation function which is the optimal choice of the *ex post* investment. Therefore, we can define the valuations of agents *at the time of the mechanism* in the following way using the notation b^{θ^i, v^i} for any cost type $\theta^i \in \Theta^i$ and the prior investment $v^i \in V^i$.

Definition 1. *The valuation function* $b^{\theta^i, v^i} : \Omega \to \mathbb{R}$ *at the time of the mechanism* given a cost type $\theta^i \in \Theta^i$ and a valuation function $v^i \in V^i$ is defined by

$$b^{\theta^i, v^i}(\omega) = \max_{\bar{v}^i \in \{\tilde{v}^i \in V^i | c^i(\tilde{v}^i, \theta^i) \geq c^i(v^i, \theta^i)\}} \Big\{\bar{v}^i(\omega) - c^i(\bar{v}^i, \theta^i)\Big\} + c^i(v^i, \theta^i)$$

for each $\omega \in \Omega$. Let $b^{\theta, v} \equiv (b^{\theta^i, v^i})_{i \in I}$.

The equation is taken from the second term of equation (1), and takes account of each agent's optimal *ex post* investment choice given the cost type. Given a prior investment $v^i \in V^i$ and an alternative $\omega \in \Omega$, the optimal choice of the *ex post* investment should be $\bar{v}^i \in V^i$ which maximizes the net value $\bar{v}^i(\omega) - c^i(\bar{v}^i, \theta^i)$ among the set of feasible valuation functions, which is $\{\tilde{v}^i \in V^i | c^i(\tilde{v}^i, \theta^i) \geq c^i(v^i, \theta^i)\}$.[19] By the assumption of V and the richness

[17] There is no time discounting between the mechanism stage and the *ex post* investment stage. But this is without loss of generality because the set of cost types is rich.

[18] Here, we assume that the same cost function is used for both investment stages. Some of the main results, however, still hold when the cost functions differ across time. For example, the sufficiency part of our possibility theorem (Theorem 2) holds as long as the *ex post* cost function is weakly lower than the *ex ante* cost function.

[19] If the cost of *ex ante* investments is refundable, the valuation function at the time of the mechanism only shifts by a constant for any choice of *ex ante* investment (since the first term of $b^{\theta^i, v^i}(\omega)$ would then be

of Θ, it is easy to show that the set of all possible valuation functions at the time of the mechanism is also V, i.e., $\{b \in \mathbb{R}^\Omega | \exists \theta^i \in \Theta^i \text{ and } v^i \in V^i \text{ such that } b^{\theta^i, v^i} = b\} = V^i$ for each $i \in I$.

A social choice function $h : V \to \Omega \times \mathbb{R}^I$ is defined as a mapping from V, which is the potential set of valuation functions at the time of the mechanism, to the set Ω of alternatives and the set $\mathbb{R}^I$ of transfer vectors. A social choice function $h \equiv (h_\omega, h_t)$ has the following two components; $h_\omega : V \to \Omega$ is called an allocation rule and $h_t : V \to \mathbb{R}^I$ is called a transfer rule. The transfer rule for each agent is denoted by $h_t^i : V \to \mathbb{R}$ and $h_t(b) = (h_t^i(b))_{i\in I}$ for any $b \in V$.

We are interested in whether efficient investments are fully implementable in subgame-perfect equilibria when an allocatively efficient social choice function is implemented. In this paper, we focus on the analysis of an investment game induced by a social choice function, and do not explicitly consider mechanisms to implement the social choice function. Although we do not discuss whether a specific social choice function is implementable, the literature has shown several positive results under complete information. For example, Moore and Repullo (1988) showed that any social choice function is subgame-perfect implementable by their extensive form mechanism under transferable utility and complete information environments.[20] Even with static mechanisms, Abreu and Matsushima (1992) showed that a large class of social choice functions are virtually implementable. Therefore, we take these positive theorems as given, and simply consider the entire space of social choice functions in this paper. We leave the equilibrium analysis within a mechanism outside the scope of the paper, and concentrate on finding the properties of social choice functions which enable us to implement efficient investments.

To introduce the implementability of efficient investments, we first define a subgame-perfect equilibrium of the investment game induced by a given social choice function. Let $\mathcal{M}^i$ be the set of all mappings from V to V^i. Let $\mathcal{M} \equiv \times_{i\in I}\mathcal{M}^i$.

Definition 2. A profile of investment strategies $(v^*, \mu^*) \in V \times \mathcal{M}$ is a *subgame-perfect equilibrium (SPE) of the investment game* at cost types $\theta \in \Theta$ given a social choice function $h : V \to \Omega \times \mathbb{R}^I$ and a discount factor $\beta \in (0, 1]$ if for each $i \in I$,

fixed). This means that concepts such as allocative efficiency (defined shortly) are not essentially affected by the *ex ante* investment behaviors. Therefore, we focus on the non-trivial cases where *ex ante* investment is irreversible.

[20]To use the Moore-Repullo mechanism, the utility of agents must be uniformly bounded as the amount of penalty used in this mechanism must be large enough. This is possible in our setup because V is a compact set.

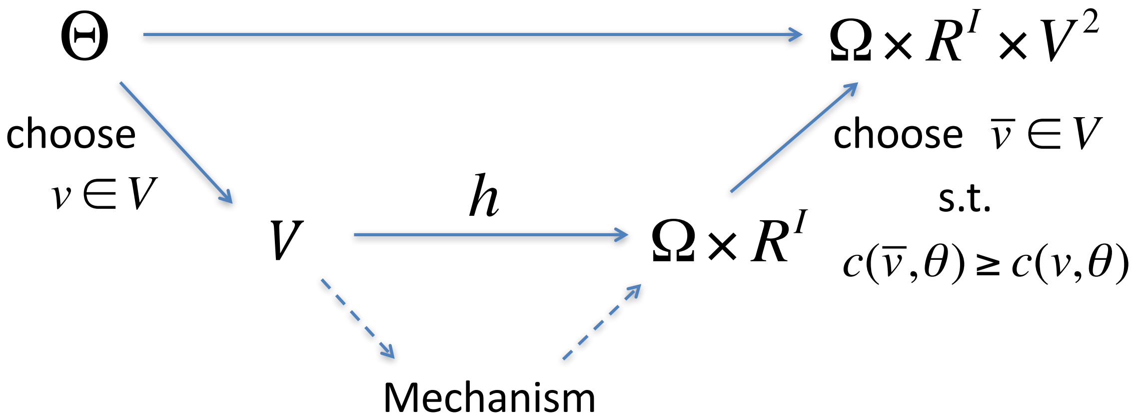

Figure 1: The structure of a social choice function and the investment game.

1. $\mu^{*i}(v) \in \underset{\bar{v}^i \in \{\tilde{v}^i \in V^i | c^i(\tilde{v}^i,\theta^i) \geq c^i(v^i,\theta^i)\}}{\arg\max} \Big\{ \bar{v}^i(h_\omega(b^{\theta,v})) - c^i(\bar{v}^i, \theta^i) \Big\}$ for any $v \in V$, and

2. $v^{*i} \in \underset{v^i \in V^i}{\arg\max} \Big\{ - c^i(v^i, \theta^i) +$

$$\beta \Big[\mu^{*i}(v^i, v^{*-i})(h_\omega(b^{\theta,v^i,v^{*-i}})) - h_t^i(b^{\theta,v^i,v^{*-i}}) - \big(c^i(\mu^{*i}(v^i, v^{*-i}), \theta^i) - c^i(v^i, \theta^i) \big) \Big] \Big\}$$

hold. Let $SPE(\theta, h, \beta) \equiv \{(v, \bar{v}) \in V^2 | \ \exists$ SPE (v, μ) at θ given h and β s.t. $\mu^i(v) = \bar{v}^i \ \forall \ i \in I\}$ denote the set of all valuation functions that are on the equilibrium paths of the investment game at cost types θ given a social choice function h and a discount factor β.

The first condition of an SPE is the optimality in the *ex post* investment stage. Since the investment does not have an externality, this is simply an individual maximization problem. The second condition requires that v^* form a Nash equilibrium of the first stage investment game at cost types θ, given the optimal *ex post* investment strategy μ^* and the social choice function h.

Using the set of valuation functions on the equilibrium paths of the investment game, subgame-perfect implementability of efficient investments is defined. Given that a social choice function h is implemented by some mechanism, efficient investments are said to be subgame-perfect implementable if for any cost types θ, the set of all SPE valuation functions of the investment game at θ given h and β coincides with the set of investment schedules which maximize the total utility of agents net of cost of investments at θ given h and β.

Definition 3. Given a social choice function $h : V \to \Omega \times \mathbb{R}^I$ and a discount factor $\beta \in (0, 1]$, *efficient investments are subgame-perfect implementable* if for any profile of cost types $\theta \in \Theta$,

$$SPE(\theta, h, \beta)$$
$$= \underset{(v,\bar{v})\in\{(p,q)\in V^2 | c^i(q^i,\theta^i)\geq c^i(p^i,\theta^i) \forall i\in I\}}{\arg\max} \sum_{i\in I} \Big\{ -c^i(v^i,\theta^i) + \beta \Big[\bar{v}^i(h_\omega(b^{\theta,v})) - \big(c^i(\bar{v}^i,\theta^i) - c^i(v^i,\theta^i)\big)\Big]\Big\}.$$

Next, we define the properties of social choice functions. There are two versions of allocative efficiency. The first definition of allocative efficiency is standard: the allocation rule chooses an alternative to maximize the sum of the valuation of agents. A social choice function $h : V \to \Omega \times \mathbb{R}^I$ is *allocatively efficient* if for any $b \in V$,

$$h_\omega(b) \in \underset{\omega'\in\Omega}{\arg\max} \sum_{i\in I} b^i(\omega').$$

Our main theorem (Theorem 2) holds for a weaker notion of allocative efficiency, which is called allocative constrained-efficiency. This guarantees allocative efficiency within a certain subset of alternatives.

Definition 4. A social choice function $h : V \to \Omega \times \mathbb{R}^I$ is *allocatively constrained-efficient for* $\Omega' \subseteq \Omega$ *with* $\Omega' \neq \emptyset$ if for any $b \in V$, the allocation rule satisfies

$$h_\omega(b) \in \underset{\omega'\in\Omega'}{\arg\max} \sum_{i\in I} b^i(\omega').$$

Note that Ω' in the definition above can be a singleton set. Thus a constant social choice function $\bar{h} : V \to \Omega \times \mathbb{R}^I$ such that $\bar{h}_\omega(b) = \bar{\omega} \in \Omega$ for any $b \in V$ also satisfies allocative constrained-efficiency for $\Omega' \equiv \{\bar{\omega}\}$. We also say that an allocation rule $h_\omega : V \to \Omega$ is allocatively (constrained-) efficient if a social choice function $h \equiv (h_\omega, h_t)$ is allocatively (constrained-) efficient.

As mentioned in the introduction, a new concept called commitment-proofness plays a crucial role in our possibility theorem (Theorem 2). Since it needs a careful explanation, we will defer the definition of commitment-proofness to subsection 4.2 where we begin to discuss the possibility of implementing efficient investments.

4 Implementation of Efficient Investments

4.1 Impossibility without Ex Post Investments

In the literature, it is often assumed that investments are made only before the mechanism is run. In such a situation, Rogerson (1992) and Hatfield et al. (2016) showed that we can find an efficient equilibrium of the investment game given allocatively efficient and strategy-proof social choice functions. But at the same time, another inefficient equilibrium exists in many examples. This is due to the fact that the *ex ante* investment stage may incentivize some agents to make more investments than at the efficient level. To see if this observation can be generalized, we consider the implementability of efficient investments without the post-mechanism investments in our model. For this purpose, we need to redefine the implementability of efficient investments for this environment accordingly.

When *ex post* investments are not possible, the investment game induced by a social choice function is a one-shot game which takes place prior to the mechanism. Thus, the equilibrium concept we employ in the investment game reduces to a Nash equilibrium in this case.

Definition 5. A profile of investments $v^* \in V$ is a *Nash equilibrium of the ex ante investment game* at cost types $\theta \in \Theta$ given a social choice function $h : V \to \Omega \times \mathbb{R}^I$ and a discount factor $\beta \in (0, 1]$ if for each $i \in I$,

$$v^{*i} \in \arg\max_{v^i \in V^i} \Big\{ - c^i(v^i, \theta^i) + \beta \Big[v^i(h_\omega(v^i, v^{*-i})) - h_t^i(v^i, v^{*-i}) \Big] \Big\}$$

holds. Let $NE(\theta, h, \beta) \subseteq V$ denote the set of all Nash equilibria of the *ex ante* investment game at cost types θ given a social choice function h and a discount factor β.

Implementability of efficient investments is redefined by the set of Nash equilibria of the *ex ante* investment game. In this environment, investment efficiency requires that the total utility of agents be maximized given that agents cannot revise their original choices of valuation functions after the mechanism stage.

Definition 6. Given a social choice function $h : V \to \Omega \times \mathbb{R}^I$ and a discount factor $\beta \in (0, 1]$, *efficient ex ante investments are Nash implementable* if for any profile of cost types $\theta \in \Theta$,

$$NE(\theta, h, \beta) = \arg\max_{v \in V} \sum_{i \in I} \Big\{ - c^i(v^i, \theta^i) + \beta v^i(h_\omega(v)) \Big\}.$$

The question is whether efficient *ex ante* investments are Nash implementable given certain social choice functions. Unfortunately, the result is negative when we require allocative efficiency; for any allocatively efficient social choice function, there is a profile of cost types at which there exists an inefficient equilibrium of the *ex ante* investment game.

Theorem 1. *Given any allocatively efficient social choice function $h : V \to \Omega \times \mathbb{R}^I$ and any discount factor $\beta \in (0, 1]$, there exists a profile of cost types $\theta \in \Theta$ such that an inefficient*

Nash equilibrium of the ex ante investment game exists, which means that efficient ex ante investments are not Nash implementable.

We show Theorem 1 by considering the following two cases: when the social choice function h is strategy-proof and when it is not. Here strategy-proofness plays a key role because *ex post* investments are not possible and hence the model has the same structure as those considered by Rogerson (1992) and Hatfield et al. (2016). Therefore, there exists an efficient Nash equilibrium of the *ex ante* investment game if h is strategy-proof, and there may not if it is not strategy-proof.

When h is not strategy-proof, the logic follows Theorem 1 and 2 of Hatfield et al. (2016), who show that for an allocatively efficient social choice function h, if agent i's privately optimal choice of valuation always maximizes the social welfare given other agents' valuations, then h must be strategy-proof for i. Therefore, when it is not strategy-proof, we can construct a profile of cost types at which, given other agents' valuations, the privately optimal *ex ante* investment choice for agent i does not achieve investment efficiency.[21]

On the other hand, for any strategy-proof social choice function, the logic of the second-price auction example in Section 2 applies. Thus, we can always construct a case where an inefficient investment equilibrium exists in addition to the efficient one. This is because the *ex ante* investment stage could give commitment power to agents whose cost types are different. Once some agent makes a large investment, then other more efficient agents may refrain from making investments as it is costly to compete with them in the mechanism. Hence, the mechanism allows them to achieve a socially inefficient outcome in equilibrium. In the next subsection, we introduce *commitment-proofness* to eliminate such incentives when further investments are possible.

4.2 Commitment-proofness

The previous subsection demonstrated that inefficient equilibria cannot be ruled out if there are no *ex post* investment opportunities. Now, we seek the possibility of implementation when *ex post* investments are possible. When investments are possible both *ex ante* and *ex post*, there are two opposing forces for the implementability of efficient investments. The *ex post* investment stage helps to achieve it by allowing agents to reflect the information of their cost types onto the valuations at the time of the mechanism. As we saw in Theorem 1

[21] Note that the cost functions constructed in the proof are slightly different from Hatfield et al. (2016) because the cost of investment in our model is non-negative whereas it is not assumed as such in their paper.

however, the *ex ante* investment stage does the opposite by preventing us from extracting the information of their cost types. Which of these two forces dominates the other depends on the characteristics of the social choice function to be implemented. To answer this question, we introduce a new concept of a social choice function called *commitment-proofness*.

Definition 7. A social choice function $h : V \to \Omega \times \mathbb{R}^I$ is *commitment-proof* if for any $i \in I$, $b \in V$, $\tilde{b}^i \in V^i$ and $x \geq 0$ such that $\tilde{b}^i(\omega) \leq b^i(\omega) + x$ for all $\omega \in \Omega$,

$$\tilde{b}^i(h_\omega(\tilde{b}^i, b^{-i})) - h^i_t(\tilde{b}^i, b^{-i}) - x \leq b^i(h_\omega(b)) - h^i_t(b). \tag{2}$$

The concept of commitment-proofness involves a manipulation of an agent's true valuation through a certain commitment behavior given that the social choice function is implemented. Conceptually, this is distinct from a misreport of valuations when the social choice function is regarded as a direct mechanism, but there is indeed a close relationship with the strategy-proofness condition. This point will be demonstrated shortly. Consider a (hypothetical) commitment device, by which an agent i changes her valuations of the outcomes in an arbitrary way before participating in the mechanism. If agent i were to increase the valuations of the outcomes from b^i to $\tilde{b}^i$, she must pay at least the maximum increment out of all increments for the possible outcomes $\omega \in \Omega$. The non-negative value x exactly represents the cost of using such commitment devices. Equation (2) requires that agent i (and any other agents) should not be able to benefit from such a commitment under h.

The following example gives a numerical illustration of a commitment $(\tilde{b}^i, x)$ for a given b^i, and shows how we tell whether a particular social choice function is commitment-proof.

Example 1. Consider an auction with a single item and two bidders. Let $I = \{i, j\}$ and $\Omega = \{\omega^i, \omega^j\}$ where ω^i and ω^j each represent the alternatives where i and j obtain the item respectively. Consider the effective domain of social choice functions: $\{a\mathbb{1}_{\{\omega=\omega^i\}} | a \in [0, 20]\}$ for i and $\{a\mathbb{1}_{\{\omega=\omega^j\}} | a \in [0, 20]\}$ for j.

Suppose that the original valuation function (at the time of the mechanism) of agent i is $b^i \in V^i$ such that

$$b^i(\omega^i) = 10,$$
$$b^i(\omega^j) = 0.$$

Consider $x = 5$ and another valuation function (at the time of the mechanism) $\tilde{b}^i \in V^i$ such that

$$\tilde{b}^i(\omega^i) = 15,$$
$$\tilde{b}^i(\omega^j) = 0.$$

These x and $\tilde{b}^i$ satisfy the condition that $\tilde{b}^i(\omega) \leq b^i(\omega) + x$ for all $\omega \in \Omega$. Thus, $(\tilde{b}^i, x)$ is one of the commitments given b^i.

Suppose agent j's valuation function is fixed to $b^j \in V^j$ such that

$$b^j(\omega^i) = 0,$$
$$b^j(\omega^j) = 11.$$

Consider the following two social choice functions:

1. The second-price auction h^{SPA} which gives the item to who values it most and has the winner pay the other agent's value, and
2. The half-price auction h^{half} which gives the item to who values it most and has the winner pay the half of her own value.

We examine whether the equation (2) holds for the example of valuation functions $b^i, \tilde{b}^i$ and b^j above.

[1] Under h^{SPA}, the RHS of equation (2) is 0 because agent i loses the auction. On the LHS, i wins the auction when her true valuation is $\tilde{b}^i$, and the utility from the auction is $\tilde{b}^i(h^{SPA}_\omega(\tilde{b}^i, b^j)) - h^{SPA,i}_t(\tilde{b}^i, b^j) = 15 - 11 = 4$. However, including the cost of commitment $x = 5$, we have

$$\tilde{b}^i(h^{SPA}_\omega(\tilde{b}^i, b^j)) - h^{SPA,i}_t(\tilde{b}^i, b^j) - x = -1 < 0 = b^i(h^{SPA}_\omega(b)) - h^{SPA,i}_t(b).$$

Thus, equation (2) holds for this example of valuation functions.[22]

[2] Under h^{half}, the RHS of equation (2) is again 0 for the same reason. On the LHS, i wins the auction when her true valuation is $\tilde{b}^i$, and the utility from the auction is $\tilde{b}^i(h^{half}_\omega(\tilde{b}^i, b^j)) - h^{half,i}_t(\tilde{b}^i, b^j) = 15 - 7.5 = 7.5$. Then, even with the cost of commitment $x = 5$, we have

$$\tilde{b}^i(h^{half}_\omega(\tilde{b}^i, b^j)) - h^{half,i}_t(\tilde{b}^i, b^j) - x = 2.5 > 0 = b^i(h^{half}_\omega(b)) - h^{half,i}_t(b).$$

Therefore, we know that the half-price auction h^{half} is not commitment-proof. □

Commitment-proofness is defined as a property of a social choice function and is not directly related to the structure of the investment game. Our main theorem establishes a strong connection between this concept and the implementability of efficient investments:

[22] Indeed, it is shown that this holds for any other valuation functions concerned in the definition of commitment-proofness and that the second-price auction is commitment-proof.

commitment-proofness is sufficient and necessary for implementing efficient investments in SPE. Intuitively, for any cost types, it will be shown that the cost of *ex ante* investment corresponds to the cost of commitment (x) in the definition of commitment-proofness. Thus, no agent has the incentive to make a costly investment before the mechanism is run, and investment efficiency is achieved. As we will see in more detail in the next subsection, commitment-proofness works as a dividing ridge for understanding the interaction of two investment stages: (only) when commitment-proof social choice functions are implemented, the role of the *ex post* investment stage outweighs that of the *ex ante* investment stage.

As mentioned above, commitment-proofness has an interesting relationship with the more well-known strategy-proofness: any strategy-proof social choice function is commitment-proof. To see this, we first define strategy-proofness. A social choice function $h : V \to \Omega \times \mathbb{R}^I$ is strategy-proof if for any $i \in I$, $b \in V$ and $\tilde{b}^i \in V^i$,

$$b^i(h_\omega(\tilde{b}^i, b^{-i})) - h_t^i(\tilde{b}^i, b^{-i}) \leq b^i(h_\omega(b)) - h_t^i(b)$$

Showing that commitment-proofness is implied by strategy-proofness is straightforward: for any $i \in I$, $b \in V$, $\tilde{b}^i \in V^i$ and $x \geq 0$ such that $\tilde{b}^i(\omega) \leq b^i(\omega) + x$ for all $\omega \in \Omega$,

$$\tilde{b}^i(h_\omega(\tilde{b}^i, b^{-i})) - h_t^i(\tilde{b}^i, b^{-i}) - x \leq b^i(h_\omega(\tilde{b}^i, b^{-i})) - h_t^i(\tilde{b}^i, b^{-i}) \leq b^i(h_\omega(b)) - h_t^i(b),$$

where the first inequality follows from the definition of $\tilde{b}^i$, and the second inequality holds from the strategy-proofness of h. Commitment-proofness concerns the agents' behavior to manipulate their own valuations outside the mechanism, rather than their misreports in the mechanism. Nonetheless, the fact that commitment-proofness is weaker than strategy-proofness implies that the consequence of commitments considered in this definition is translated into a type of misreports when the social choice function is regarded as a direct mechanism.

From this relationship, we know that the VCG auction, which is known to be strategy-proof, satisfies commitment-proofness. The VCG social choice function h^{VCG} is defined as follows: for any $b \in V$,

$$h_\omega^{VCG}(b) \in \arg\max_{\omega \in \Omega} \sum_{i \in I} b^i(\omega),$$
$$h_t^{VCG,i}(b) = \max_{\omega \in \Omega} \sum_{j \in I \setminus \{i\}} b^j(\omega) - \sum_{j \in I \setminus \{i\}} b^j(h_\omega^{VCG}(b)) \text{ for any } i \in I.$$

The second-price auction is a special case of the VCG auction, so it is also commitment-proof.

Since commitment-proofness is weaker than strategy-proofness, there exists a non-strategy-proof social choice function which is commitment-proof. Consider a class of social choice functions $h^\alpha : V \to \Omega \times \mathbb{R}^I$ parameterized by $\alpha \in [0, 1)$ such that the alternative is efficiently chosen and the payment is a convex combination of the VCG payment and each agent's own valuation from the alternative itself: for any $b \in V$,

$$h^\alpha_\omega(b) \in \arg\max_{\omega \in \Omega} \sum_{i \in I} b^i(\omega),$$
$$h^{\alpha,i}_t(b) = \alpha \Big\{ \max_{\omega \in \Omega} \sum_{j \in I \setminus \{i\}} b^j(\omega) - \sum_{j \in I \setminus \{i\}} b^j(h^\alpha_\omega(b)) \Big\} + (1-\alpha) b^i(h^\alpha_\omega(b)) \text{ for any } i \in I$$

for some $\alpha \in [0, 1)$. This h^α is not strategy-proof because for some valuations of other agents, an agent will be strictly better off by decreasing her report of valuation without changing the alternative chosen by h^α. However, this is shown to be commitment-proof. The first part of the payment is exactly the VCG payment, and we know that the VCG social choice function satisfies equation (2). Regarding the second part of the payment, it is easy to see that for any $i \in I$, $b \in V$, $\tilde{b}^i \in V^i$ and $x \geq 0$ such that $\tilde{b}^i(\omega) \leq b^i(\omega) + x$ for all $\omega \in \Omega$,

$$\tilde{b}^i(h_\omega(\tilde{b}^i, b^{-i})) - h^i_t(\tilde{b}^i, b^{-i}) - x = -x \leq 0 = b^i(h_\omega(b)) - h^i_t(b)$$

holds. Therefore, equation (2) is satisfied when the transfer rule is a convex combination of these two, and hence h^α is commitment-proof.

4.3 Possibility with Ex Ante and Ex Post Investments

Now we formally present the possibility theorem in our original model. In what follows, we demonstrate how commitment-proofness allows us to implement efficient investments when *ex post* investments are possible.

First, for the purpose of the main theorem, we prove the following lemma.

Lemma 1. *For any agent $i \in I$ and any cost type $\theta^i \in \Theta^i$,*

$$c^i(v^i, \theta^i) \geq \max_{\omega \in \Omega} \Big\{ b^{\theta^i, v^i}(\omega) - b^{\theta^i, v^{0i}}(\omega) \Big\}$$

holds for $v^i \in V^i$ and $v^{0i} \in V^i$ such that $c^i(v^{0i}, \theta^i) = 0$.

Proof: From the definition of the valuation at the time of the mechanism,

$$\begin{aligned} b^{\theta^i, v^{0i}}(\omega) &= \max_{\bar{v}^i \in V^i} \Big\{ \bar{v}^i(\omega) - c^i(\bar{v}^i, \theta^i) \Big\} \\ &\geq \max_{\bar{v}^i \in \{\tilde{v}^i \in V^i | c^i(\tilde{v}^i, \theta^i) \geq c^i(v^i, \theta^i)\}} \Big\{ \bar{v}^i(\omega) - c^i(\bar{v}^i, \theta^i) \Big\} \\ &= b^{\theta^i, v^i}(\omega) - c^i(v^i, \theta^i) \end{aligned}$$

holds for any $\omega \in \Omega$. Thus, we have $c^i(v^i, \theta^i) \geq \max_{\omega \in \Omega} \left\{ b^{\theta^i, v^i}(\omega) - b^{\theta^i, v^{0i}}(\omega) \right\}$. $\square$

This lemma shows that the cost of changing the original valuation $b^{\theta^i, v^{0i}}$ with least costly *ex ante* investment v^{0i} to another valuation b^{θ^i, v^i} with some *ex ante* investment v^i is at least as large as the maximum element of the difference between $b^{\theta^i, v^{0i}}$ and b^{θ^i, v^i}. This is useful when we connect the definition of commitment-proofness to the structure of the investment game in the following theorem.

The next result is the main theorem of this paper which identifies when efficient investments are implementable.

Theorem 2. *Suppose that a social choice function $h : V \to \Omega \times \mathbb{R}^I$ is allocatively constrained-efficient for some $\Omega' \subseteq \Omega$ with $\Omega' \neq \emptyset$. Given the social choice function h, efficient investments are subgame-perfect implementable for any discount factor $\beta \in (0, 1)$ if and only if h is commitment-proof.*

The proof consists of the following two parts: (i) commitment-proofness of h as sufficient for implementing efficient investments, and (ii) it also being necessary. First, we characterize the set of SPE when h is commitment-proof. We show that under commitment-proof social choice functions, no agent has the incentive to make a costly investment *ex ante* for any cost type. This is because the cost of any (costly) investment corresponds to x in the definition of commitment-proofness as shown in Lemma 1, and every agent i prefers to have the valuation $b^{\theta^i, v^{0i}}$ with least costly *ex ante* investment at the mechanism stage. And we show that any such SPE maximizes the social welfare when the social choice function h is allocatively constrained-efficient. For the necessity part, we show that if h is not commitment-proof for agent i, there is a cost type at which agent i has the incentive to make a costly investment *ex ante*, which is socially inefficient. Therefore, we conclude that only under commitment-proof social choice functions, the incentive for making a commitment through *ex ante* investment is completely suppressed by the presence of the *ex post* investment stage, and efficient investments are implemented.

Regarding the two distinct features of our main result that (i) inefficient investment equilibria are eliminated when (ii) post-mechanism investments are allowed, Piccione and Tan (1996) provided a closely related result in the literature. They analyze a procurement auction in which firms make R&D investments prior to the auction and the firm that wins the procurement contract exerts an additional effort to reduce costs. One of the main results of their paper is that the full-information solution (in which investments and alternative are efficient) can be uniquely implemented by the first-price and second-price auctions when the R&D technology exhibits decreasing returns to scale. Although the model is similar to ours, the focus of their theorem is different. Their result determines the structure of cost functions which enable unique implementation under those two common auction rules. On the other hand, we characterize the set of social choice functions for which efficient investments are implementable. Also, our cost functions allow any arbitrary heterogeneity among agents, which is not allowed in Piccione and Tan (1996), but we assume a certain relationship between *ex ante* and *ex post* cost functions. (See footnote 18.) Since we do not analyze the equilibrium of specific mechanisms such as the first-price auction, it would be an interesting direction to analyze such mechanisms and see how the result relates to Piccione and Tan (1996).

In the rest of the section, we provide two examples to show the importance of (i) β being strictly less than one and (ii) the allocative constrained-efficiency of h in Theorem 2.

First, a strict time discounting plays an important role. Although commitment-proofness implies implementability of efficient investments for any β which is arbitrarily close to one, it does not when β is exactly one.[23] Intuitively, this is because when β is one, there are cases where the choice between investing *ex ante* and *ex post* is indifferent and there exists an equilibrium in which more than one agents chooses costly *ex ante* investments, which is socially inefficient. We provide an example where given $\beta = 1$ and a VCG social choice function (see subsection 4.2 for the definition), which is allocatively efficient and strategy-proof, efficient investments are not implementable in SPE.

Observation 1. *Given the VCG social choice function $h^{VCG} : V \to \Omega \times \mathbb{R}^I$ and $\beta = 1$, efficient investments are not subgame-perfect implementable.*

Example 2. Let $\{i, j\} \subseteq I$ and $\{\omega_1, \omega_2\} \subseteq \Omega$. Consider the following sets of valuations:

$$
\begin{aligned}
V^i &= \{b^i, \tilde{b}^i\}, \\
V^j &= \{b^j, \tilde{b}^j\}, \\
V^k &= \{0\} \text{ for any } k \in I \setminus \{i, j\}
\end{aligned}
$$

where

$$
\begin{aligned}
&b^i(\omega_1) = b^j(\omega_1) = 5,\ b^i(\omega_2) = b^j(\omega_2) = 4,\ b^i(\omega) = b^j(\omega) = 0 \text{ for any } \omega \in \Omega \setminus \{\omega_1, \omega_2\} \\
&\tilde{b}^i(\omega_1) = \tilde{b}^j(\omega_1) = 0,\ \tilde{b}^i(\omega_2) = \tilde{b}^j(\omega_2) = 6,\ \tilde{b}^i(\omega) = \tilde{b}^j(\omega) = 0 \text{ for any } \omega \in \Omega \setminus \{\omega_1, \omega_2\}.
\end{aligned}
$$

[23] Note that the necessity of commitment-proofness in Theorem 2 still holds for $\beta = 1$.

Consider the following cost types $\theta \in \Theta$:

$$c^i(b^i, \theta^i) = c^j(b^j, \theta^j) = 0,$$
$$c^i(\tilde{b}^i, \theta^i) = c^j(\tilde{b}^j, \theta^j) = 2,$$
$$c^k(0, \theta^k) = 0 \text{ for any } k \in I \setminus \{i, j\}.$$

Since the only choice of valuation is 0 for any $k \in I \setminus \{i, j\}$, we can ignore these agents. Given a VCG social choice function h^{VCG}, the most efficient investment schedules of agents i and j is $((b^i, b^j), (b^i, b^j))$. This is because it achieves the maximum social welfare $\beta(5+5) = \beta 10 = 10$ as h^{VCG} chooses ω_1 for (b^i, b^j), and the cost of (b^i, b^j) is zero for $\theta \in \Theta$.

Next, consider an investment strategy $(\tilde{b}^k, \mu^k) \in V^k \times \mathcal{M}^k$ for each agent $k = i, j$ where μ^k is the optimal *ex post* investment strategy. First, because $c^k(\tilde{b}^k, \theta^k) > c^k(b^k, \theta^k)$ for each agent $k = i, j$,

$$\mu^k(\tilde{b}^k, b^{-k}) = \tilde{b}^k$$

holds for any $b^{-k} \in V^{-k}$. Thus, given the *ex ante* investment $\tilde{b}^k$, the valuation at the time of the mechanism is also $\tilde{b}^k$.

Suppose that agent j takes this investment strategy $(\tilde{b}^j, \mu^j) \in V^j \times \mathcal{M}^j$, and consider agent i's incentive. When she chooses b^i in the first stage, since $b^i(\omega) \geq \tilde{b}^i(\omega) - c^i(\tilde{b}^i, \theta^i)$ holds for any $\omega \in \Omega$, the valuation at the time of the mechanism is

$$b^{\theta^i, b^i}(\omega) = \max_{\bar{v}^i \in \{b^i, \tilde{b}^i\}} \left\{ \bar{v}^i(\omega) - c^i(\bar{v}^i, \theta^i) \right\} = b^i(\omega)$$

for each $\omega \in \Omega$. In this case, the outcome of the social choice function should be

$$h^{VCG}_\omega(b^i, \tilde{b}^j, 0) = \omega_2, \text{ and}$$
$$h^{VCG,i}_t(b^i, \tilde{b}^j, 0) = 0.$$

The total utility of agent i would be $4\beta = 4$. On the other hand, when she chooses $\tilde{b}^j$ in the first stage, the outcome of the social choice function will be

$$h^{VCG}_\omega(\tilde{b}^i, \tilde{b}^j, 0) = \omega_2, \text{ and}$$
$$h^{VCG,i}_t(\tilde{b}^i, \tilde{b}^j, 0) = 0.$$

The total utility of agent i would be $6\beta - 2 = 4$. Since these choices are indifferent, choosing $\tilde{b}^i$ in the first stage can be a best response for agent i. Therefore, the same logic applies to agent j, and $\{(\tilde{b}^k, \mu^k) \in V^k \times \mathcal{M}^k\}_{k=i,j}$ constitutes an SPE of the investment game. But since $((\tilde{b}^i, \tilde{b}^j), (\tilde{b}^i, \tilde{b}^j))$ gives a social welfare of 8, which is less than under $((b^i, b^j), (b^i, b^j))$, efficient investments are not subgame-perfect implementable given h^{VCG} and $\beta = 1$. $\square$

As a second observation, the sufficiency of commitment-proofness in Theorem 2 no longer holds if the social choice function is not allocatively constrained-efficient for any $\Omega' \subseteq \Omega$. The next example demonstrates that efficient investments are not implementable given a strategy-proof (and hence, commitment-proof) social choice function which is not allocatively constrained-efficient.

Observation 2. *There is a strategy-proof social choice function $h : V \to \Omega \times \mathbb{R}^I$ which is not allocatively constrained-efficient for any $\Omega' \subseteq \Omega$ such that efficient investments are not subgame-perfect implementable given h and some $\beta \in (0, 1)$.*

Example 3. Let $\{i, j\} \subseteq I$ and $\{\omega_1, \omega_2\} \subseteq \Omega$. Consider a social choice function $h : V \to \Omega \times \mathbb{R}^I$ such that for any $b \in V$,

$$h_\omega(b) \in \arg\max_{\omega \in \Omega} \left\{ b^i(\omega) \right\},$$
$$h_t^k(b) = 0 \text{ for any } k \in I.$$

This means that the best alternative for agent i is always chosen and no transfer is made under h. This h is strategy-proof because i does not have the incentive to manipulate her type and j's report does not affect the outcome. It is clear that h is not allocatively constrained-efficient because other agents' valuations are not taken into account. Consider the following sets of valuations:

$$V^i = \{b^i, \tilde{b}^i\},$$
$$V^j = \{b^j\},$$
$$V^k = \{0\} \text{ for any } k \in I \setminus \{i, j\}$$

where

$$b^i(\omega_1) = 5,\ b^i(\omega_2) = 4,\ b^i(\omega) = 0 \text{ for any } \omega \in \Omega \setminus \{\omega_1, \omega_2\}$$
$$\tilde{b}^i(\omega_1) = 5,\ \tilde{b}^i(\omega_2) = 6,\ \tilde{b}^i(\omega) = 0 \text{ for any } \omega \in \Omega \setminus \{\omega_1, \omega_2\}$$
$$b^j(\omega_1) = 0,\ b^j(\omega_2) = 3,\ b^j(\omega) = 0 \text{ for any } \omega \in \Omega \setminus \{\omega_1, \omega_2\}.$$

Also consider the following cost types $\theta \in \Theta$:

$$c^i(b^i, \theta^i) = 0,\ c^i(\tilde{b}^i, \theta^i) = 3,$$
$$c^j(b^j, \theta^j) = 0,$$
$$c^k(0, \theta^k) = 0 \text{ for any } k \in I \setminus \{i, j\}.$$

Since the only choice of valuation is 0 for any $k \in I \setminus \{i, j\}$, we can ignore these agents. For j, the only choice of valuation is b^j.

Consider the optimal choice for agent i in the second investment stage. If i chooses b^i *ex ante*, since $b^i(\omega) > \tilde{b}^i(\omega) - c^i(\tilde{b}^i, \theta^i)$ holds for any $\omega \in \Omega$, her optimal choice of *ex post* valuation is b^i. If i chooses $\tilde{b}^i$ *ex ante*, then the only valuation she can choose afterwards is $\tilde{b}^i$ because $c^i(\tilde{b}^i, \theta^i) > c^i(b^i, \theta^i)$. In either case, when the same valuation is taken *ex ante* and *ex post*, the valuation at the time of the mechanism is also the same valuation. To summarize, agent i's optimal *ex post* investment strategy and the valuation at the time of the mechanism is as follows:

Ex Ante Valuation	Valuation at the Mechanism	Optimal *Ex Post* Valuation
b^i	b^i	ω_1: b^i ω_2: b^i
$\tilde{b}^i$	$\tilde{b}^i$	ω_1: $\tilde{b}^i$ ω_2: $\tilde{b}^i$

Thus, we can compare two investment choices b^i and $\tilde{b}^i$ of agent i in the first stage to analyze the investment efficiency and the equilibrium.

First, we show that $\tilde{b}^i$ gives higher social welfare than b^i for sufficiently large $\beta \in (0, 1)$. Given j's valuation b^j, the social welfare when i chooses $\tilde{b}^i$ is

$$-3 + \beta(6 + 3) = 9\beta - 3.$$

The social welfare when i chooses b^i is

$$0 + \beta(5 + 0) = 5\beta.$$

Since the former is larger for $\beta > \frac{3}{4}$, choosing $\tilde{b}^i$ is socially efficient, and choosing b^i is not for such β.

Next, consider the incentive of agent i. Given j's valuation b^j, compare the utility of i when she chooses $\tilde{b}^i$ and b^i in the first stage. When i chooses $\tilde{b}^i$, her utility is $6\beta - 3$ whereas it is 5β when i chooses b^i. Since

$$6\beta - 3 < 5\beta \text{ for any } \beta \in (0, 1),$$

agent i chooses b^i. Thus, (b^i, b^i) is agent i's on-path valuation of an SPE in the investment game, but it is not efficient for $\beta > \frac{3}{4}$. Therefore, efficient investments are not subgame-perfect implementable given h and such β. □

5 Budget Balance

In this section, we consider an additional requirement of social choice functions: budget balance. This is especially important in the provision of public goods as the cost must be covered by the participants of the mechanism. We still consider the same general model as in Section 3. Mathematically, budget balance means that the sum of the transfers must be equal to zero:

Definition 8. A social choice function $h : V \to \Omega \times \mathbb{R}^I$ is *budget-balanced* if

$$\sum_{i \in I} h_t^i(b) = 0$$

for any $b \in V$.

Budget balance can be considered as part of allocative efficiency if the transfer collected by the mechanism designer is regarded as the loss of welfare. In this environment, it is known that there is no social choice function that is strategy-proof, allocatively efficient and budget-balanced (Green and Laffont, 1977; Hölmstrom, 1979; Walker, 1980). Therefore, when only *ex ante* investment is possible, it is impossible to even ensure the existence of efficient investment equilibria if we require budget balance and allocatively efficiency of the social choice function (Hatfield et al., 2016). However, we can show that commitment-proofness is compatible with these two properties, i.e., there is a social choice function which is commitment-proof, allocatively efficient and budget-balanced.

Proposition 1. *For any efficient allocation rule $h_\omega : V \to \Omega$, there exists a transfer rule $h_t : V \to \mathbb{R}^I$ such that $h = (h_\omega, h_t)$ is commitment-proof and budget-balanced.*

Proposition 1 is shown by proposing a specific transfer rule h_t: for any agent $i \in I$, h_t^i is defined by

$$h_t^i(b) = b^i(h_\omega(b)) - \frac{1}{n} \sum_{i \in I} b^i(h_\omega(b)).$$

In this transfer rule, the maximized social welfare is equally divided to all agents. Consider the definition of commitment-proofness. Under this transfer rule, the value $\tilde{b}^i(h_\omega(\tilde{b}^i, b^{-i})) - h_t^i(\tilde{b}^i, b^{-i})$ from the social choice function h under type $\tilde{b}^i$ increases from the original value $b^i(h_\omega(b)) - h_t^i(b)$ under type b^i by only $\frac{1}{n}$ of the increment of the social welfare. On the other hand, since x satisfies $x \geq \max_{\omega \in \Omega}\{\tilde{b}^i(\omega) - b^i(\omega)\}$, x should be larger than the increment of social welfare. Therefore, the equation of commitment-proofness is satisfied under this transfer rule. It is easy to see that this h is not strategy-proof because agents have the incentive to underreport their valuations to reduce the payment.

By the result of Theorem 2, we obtain the following corollary: with *ex post* investments, budget balance does not preclude the implementation of efficient investments.

Corollary 1. *There exists an allocatively efficient and budget-balanced social choice function $h : V \to \Omega \times \mathbb{R}^I$ such that efficient investments are subgame-perfect implementable given h and any discount factor $\beta \in (0, 1)$.*

6 Extension: Incomplete Information

In the main model, we assumed complete information of the cost types among agents. In fact, complete information is not necessary for characterizing efficient investments in Theorem 2 because making the least costly *ex ante* investment is the strictly dominant strategy as long as a commitment-proof social choice function is implemented. In incomplete information environments, we would rather need a tighter condition for implementing social choice functions within a mechanism since not every social choice function is implementable (even with extensive form mechanisms). Below, we will review the class of social choice functions which can be fully implemented in perfect Bayesian equilibria, and develop a possibility theorem.

We consider the case where each agent i knows her own cost type $\theta^i \in \Theta^i$, but may be unsure about other agents' cost types $\theta^{-i} \equiv (\theta^j)_{j \in I \setminus \{i\}}$. They have a common prior on Θ, denoted by p. Conditional on knowing her own cost type θ^i, agent i's posterior distribution over $\Theta^{-i} \equiv \times_{j \in I \setminus \{i\}} \Theta^j$ is denoted $p(\cdot|\theta^i)$. For simplicity, we assume that $V^i \subseteq \mathbb{R}^\Omega$ and Θ^i are both finite sets in this subsection.[24] The prior belief is diffuse, i.e., $p(\theta) > 0$ for any $\theta \in \Theta$, and $p(\cdot|\theta^i)$ is computed by Bayes' rule.

In this Bayesian setting, the investment strategies are defined in the following way. The set of *ex ante* investment strategies for agent i is the set of all mappings from Θ^i to V^i, denoted by Σ^i. The set of *ex post* investment strategies for agent i is the set of all mappings from $V \times \Omega \times \Theta^i$ to V^i, denoted by $\bar{\mathcal{M}}^i$. First, we define a perfect Bayesian equilibrium of the investment game given that a social choice function h is implemented. In this environment, the domain of a social choice function is defined as B where B^i be the set of possible

[24]In one of the extensions, Duggan (1998) argues that the finiteness of these sets is not necessary for implementing Bayesian incentive compatible social choice functions. However, we assume finiteness as it makes the model simpler and would not change the main argument of our result.

valuation functions at the mechanism stage, i.e., $B^i \equiv \{b^i \in \mathbb{R}^\Omega | \exists(\theta^i, v^i) \in \Theta^i \times V^i$ such that $b^i = b^{\theta^i, v^i}\}$.

Definition 9. A profile of investment strategies $(\sigma^*, \mu^*) \in \Sigma \times \bar{\mathcal{M}}$ is a *perfect Bayesian equilibrium (PBE) of the investment game* given a social choice function $h : B \to \Omega \times \mathbb{R}^I$ and a discount factor $\beta \in (0, 1]$ if for each $i \in I$ and $\theta^i \in \Theta^i$,

1. $\mu^{*i}(v, \omega, \theta^i) \in \underset{\bar{v}^i \in \{\tilde{v}^i \in V^i | c^i(\tilde{v}^i, \theta^i) \geq c^i(v^i, \theta^i)\}}{\arg\max} \Big\{ \bar{v}^i(\omega) - c^i(\bar{v}^i, \theta^i) \Big\}$ for any $v \in V$ and $\omega \in \Omega$, and

2. $$\sigma^{*i}(\theta^i) \in \arg\max_{v^i \in V^i} \Big\{ - c^i(v^i, \theta^i) + \beta \textstyle\sum_{\theta^{-i} \in \Theta^{-i}} p(\theta^{-i}|\theta^i) \Big[\mu^{*i}(v^i, \sigma^{*-i}(\theta^{-i}), h_\omega(b^{\theta^i, v^i}, b^{-i}), \theta^i)(h_\omega(b^{\theta^i, v^i}, b^{-i})) - h^i_t(b^{\theta^i, v^i}, b^{-i}) - c^i(\mu^{*i}(v^i, \sigma^{*-i}(\theta^{-i}), h_\omega(b^{\theta^i, v^i}, b^{-i}), \theta^i), \theta^i) + c^i(v^i, \theta^i)\Big]\Big\}$$

   where $b^{-i} \equiv b^{\theta^{-i}, \sigma^{*-i}(\theta^{-i})}$

hold. Let $PBE(h, \beta) \subseteq \Sigma \times \bar{\mathcal{M}}$ denote the set of all PBE of the investment game given h and β.

Note that in the *ex post* investment stage, we do not need to specify the beliefs of agents because other agents' cost types are irrelevant to their decisions in this stage.

Now let us turn to the implementability of the social choice function itself. Several papers have identified the class of social choice functions that can be implemented by extensive form mechanisms in this Bayesian setting (Brusco, 1995; Bergin and Sen, 1998; Duggan, 1998; Baliga, 1999). They differ in the generality of the model and the equilibrium concept (PBE or sequential equilibrium), but it is common that they require Bayesian incentive compatibility. Moreover, in the simplest model of Duggan (1998) where quasi-linear preferences are employed, he shows that any Bayesian incentive compatible social choice function is implementable in Perfect Bayesian equilibria and sequential equilibria.[25] Thus, we simply require Bayesian incentive compatibility in the mechanism stage instead of explicitly analyzing extensive form mechanisms in our paper.

Our model has an *ex ante* investment stage, and the belief system at the mechanism stage is endogenously formed by the observations of *ex ante* investments and the agents' investment strategies. Let $p(\theta^{-i}|\theta^i, v, \sigma)$ be agent i's belief on θ^{-i} given her own cost type,

[25]Duggan (1998) also requires value-measurability for implementation, in the sense that chosen outcomes do not change unless at least one agent's preference does, but this is automatically satisfied by the social choice functions we consider in this paper.

observed *ex ante* investments and all agents' investment strategies. This is computed by Bayes' rule:

$$p(\theta^{-i}|\theta^i, v, \sigma) \equiv \begin{cases} \frac{p(\theta^{-i}|\theta^i)\Pi_{j\in I\setminus\{i\}}\mathbb{1}_{\{\sigma^j(\theta^j)=v^j\}}}{\sum_{\tilde{\theta}^{-i}\in\Theta^{-i}} p(\tilde{\theta}^{-i}|\theta^i)\Pi_{j\in I\setminus\{i\}}\mathbb{1}_{\{\sigma^j(\tilde{\theta}^j)=v^j\}}} & \text{if } \sum_{\tilde{\theta}^{-i}\in\Theta^{-i}} p(\tilde{\theta}^{-i}|\theta^i)\Pi_{j\in I\setminus\{i\}}\mathbb{1}_{\{\sigma^j(\tilde{\theta}^j)=v^j\}} > 0, \\ \text{any probability distribution over } \Theta^{-i} & \text{otherwise.} \end{cases}$$

Bayesian incentive compatibility of a social choice function is defined for each information set at the mechanism stage.

Definition 10. A social choice function $h : B \to \Omega \times \mathbb{R}^I$ is *Bayesian incentive compatible at* $v \in V$ *given* $\sigma \in \Sigma$ if for any $i \in I$, $\theta^i \in \Theta^i$, $\tilde{\theta}^i \in \Theta^i$ and $\tilde{v}^i \in V^i$,

$$\sum_{\theta^{-i}\in\Theta^{-i}} p(\theta^{-i}|\theta^i, v, \sigma)\Big[b^{\theta^i,v^i}(h_\omega(b^{\theta,v})) - h^i_t(b^{\theta,v})\Big]$$
$$\geq \sum_{\theta^{-i}\in\Theta^{-i}} p(\theta^{-i}|\theta^i, v, \sigma)\Big[b^{\theta^i,v^i}(h_\omega(b^{\tilde{\theta}^i,\tilde{v}^i}, b^{\theta^{-i},v^{-i}})) - h^i_t(b^{\tilde{\theta}^i,\tilde{v}^i}, b^{\theta^{-i},v^{-i}})\Big].$$

In our model, we assume that the mechanism designer does not observe the agents' investment decisions. Therefore, in the mechanism, each agent is asked to report her *ex ante* investment v^i in addition to her cost type θ^i. The definition of Bayesian incentive compatibility here reflects this fact. Moreover, because the designer does not know their *ex ante* investments v, the social choice function to be implemented must satisfy Bayesian incentive compatibility for any possible beliefs at the mechanism stage.[26] Then the implementability of efficient investments is defined by the following two conditions: (i) the social choice function must be Bayesian incentive compatible given the PBE investment strategy in every information set at the mechanism stage, and (ii) the set of PBE of the investment game coincides with the set of efficient investments.

Definition 11. Given a discount factor $\beta \in (0, 1]$, *a social choice function* $h : B \to \Omega \times \mathbb{R}^I$ *and efficient investments are implementable in PBE* if

1. h is Bayesian incentive compatible at any $v \in V$ given any $\sigma^* \in \Sigma$ such that σ^* is part of a PBE, and
2. $PBE(h, \beta) = \underset{(\sigma,\mu)\in\Sigma\times\bar{\mathcal{M}}}{\arg\max} \sum_{\theta\in\Theta} p(\theta) \sum_{i\in I} \Big\{ - c^i(\sigma^i(\theta^i), \theta^i)$
$+\beta\Big[\mu^i(\sigma(\theta), h_\omega(b^{\theta,\sigma(\theta)}), \theta^i)(h_\omega(b^{\theta,\sigma(\theta)})) - c^i(\mu^i(\sigma(\theta), h_\omega(b^{\theta,\sigma(\theta)}), \theta^i), \theta^i) + c^i(\sigma^i(\theta^i), \theta^i)\Big]\Big\}.$

[26]Duggan (1998) also considers global implementation, in which the mechanism designer elicits the information of agents' belief system and implements different social choice functions depending on their belief system. But we do not consider it in our paper and leave it for future research.

In Proposition 2, we show that if the social choice function is strategy-proof, efficient investments and allocations are implementable in PBE. The definition of strategy-proofness for social choice functions in this section is as follows: a social choice function $h : B \to \Omega \times \mathbb{R}^I$ is strategy-proof if for any $i \in I$, $b \in B$ and $\tilde{b}^i \in B^i$,

$$b^i(h_\omega(\tilde{b}^i, b^{-i})) - h_t^i(\tilde{b}^i, b^{-i}) \leq b^i(h_\omega(b)) - h_t^i(b).$$

Proposition 2. *Suppose that a social choice function $h : B \to \Omega \times \mathbb{R}^I$ is strategy-proof and allocatively constrained-efficient for some $\Omega' \subseteq \Omega$ with $\Omega' \neq \emptyset$. Given any discount factor $\beta \in (0, 1)$, h and efficient investments are implementable in PBE.*

In this setting, the condition for implementing the same social choice function for every information set at the mechanism stage is quite tight because Bayesian incentive compatibility must be satisfied in every information set. Proposition 2 shows that strategy-proofness, which implies Bayesian incentive compatibility for any beliefs of agents, also ensures that efficient investments are fully implementable in PBE in the investment game.

7 Concluding Remarks

Our main result shows that when *ex post* investments are possible, commitment-proofness is equivalent to the implementability of efficient investments for allocatively efficient social choice functions. This has the following two implications. First, whenever it is possible, the mechanism should be run sufficiently before the actual production or consumption is carried out. This allows agents to reflect the information of their cost types onto their valuations at the mechanism stage through the optimal behavior in the *ex post* investment stage. Otherwise, according to Theorem 1, we cannot eliminate the possibility of inefficient equilibria. Second, commitment-proofness of the mechanism is essential. This ensures that no agent has the incentive to commit to having a different valuation in the mechanism by making prior investments. Moreover, commitment-proofness is not a restrictive concept since it is much weaker than the strategy-proofness condition.

In this paper, we have made strong assumptions on the technology of investments. In particular, (i) the investment technology has no uncertainty and (ii) the *ex post* investment is strictly less costly than *ex ante*.[27] These make the analysis simple because agents defer investments to the *ex post* stage unless they have the incentive to make a commitment. In

[27] Although out result still holds for some systematic changes of cost functions after the mechanism (see footnote 18), we do not know what would happen when the *ex post* cost function is uncertain *ex ante*.

future research, these assumptions can be relaxed. Agents may make uncertain *ex ante* investments, which introduces a new source of uncertainty to the model. Moreover, some positive *ex ante* investment could be socially efficient when *ex ante* investment cannot be easily substituted by *ex post* investment. These extensions would make a closer connection between our paper and Piccione and Tan (1996) or other papers on information acquisition (Bergemann and Välimäki, 2002; Obara, 2008). Under these settings, we hope to obtain conditions of social choice functions or cost structures under which efficient investments are implementable.

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Appendix A Proofs of the Main Results

A.1 Proof of Theorem 1

Consider any arbitrary allocatively efficient social choice function $h : V \to \Omega \times \mathbb{R}^I$. We will examine two cases where h is not strategy-proof and h is strategy-proof. In the former case, we find a profile of cost types at which an inefficient Nash equilibrium exists in the investment game, exploiting the equation that strategy-proofness of h is violated. In the latter case, we show that a simple auction has multiple equilibria in the investment game and one of them is less efficient than the other for any strategy-proof h.

[1] When h is not strategy-proof. Since the social choice function h is not strategy-proof, there are $i \in I$, $v \in V$ and $\tilde{v}^i \in V^i$ such that

$$v^i(h_\omega(\tilde{v}^i, v^{-i})) - h_t^i(\tilde{v}^i, v^{-i}) > v^i(h_\omega(v)) - h_t^i(v). \tag{3}$$

Consider a profile of cost types $\theta \in \Theta$ such that

$$\begin{aligned}
&c^i(v^i, \theta^i) = \max\Big\{0, \beta\Big[v^i(h_\omega(v)) - h_t^i(v) - \big(\tilde{v}^i(h_\omega(\tilde{v}^i, v^{-i})) - h_t^i(\tilde{v}^i, v^{-i})\big)\Big]\Big\},\\
&c^i(\tilde{v}^i, \theta^i) = \max\Big\{0, \beta\Big[\tilde{v}^i(h_\omega(\tilde{v}^i, v^{-i})) - h_t^i(\tilde{v}^i, v^{-i}) - \big(v^i(h_\omega(v)) - h_t^i(v)\big)\Big]\Big\},\\
&c^j(v^j, \theta^j) = 0 \text{ for all } j \in I \setminus \{i\},\\
&\sup_{p^i \in V^i \setminus \{v^i, \tilde{v}^i\}, p^{-i} \in V^{-i}, \omega \in \Omega, \gamma \in \{0,1\}} \Big\{ - c^i(p^i, \theta^i) + \beta\Big[p^i(\omega) - \gamma h_t^i(p)\Big]\Big\}\\
&< \inf_{p^i \in \{v^i, \tilde{v}^i\}, p^{-i} \in V^{-i}, \omega \in \Omega, \gamma \in \{0,1\}} \Big\{ - c^i(p^i, \theta^i) + \beta\Big[p^i(\omega) - \gamma h_t^i(p)\Big]\Big\}, \text{ and}\\
&\sup_{p^j \in V^j \setminus \{v^j\}, p^{-j} \in V^{-j}, \omega \in \Omega, \gamma \in \{0,1\}} \Big\{ - c^j(p^j, \theta^j) + \beta\Big[p^j(\omega) - \gamma h_t^j(p)\Big]\Big\}\\
&< \inf_{p^{-j} \in V^{-j}, \omega \in \Omega, \gamma \in \{0,1\}} \Big\{\beta\Big[v^j(\omega) - \gamma h_t^j(v^j, p^{-j})\Big]\Big\} \text{ for all } j \in I \setminus \{i\}.
\end{aligned}$$

We can find such cost type θ in Θ because of the richness condition of Θ. Note that

$$c^i(v^i, \theta^i) - c^i(\tilde{v}^i, \theta^i) = \beta\Big[v^i(h_\omega(v)) - h_t^i(v) - \big(\tilde{v}^i(h_\omega(\tilde{v}^i, v^{-i})) - h_t^i(\tilde{v}^i, v^{-i})\big)\Big]$$

always holds. The last two conditions ensure that any equilibrium and any efficient investment profile should be in $\{v^i, \tilde{v}^i\} \times \{v^{-i}\}$. Thus, we only need to analyze which of v^i and $\tilde{v}^i$ agent i chooses, and which of them is more efficient.

First, consider i's incentive for choosing between v^i and $\tilde{v}^i$. The total utility from choosing v^i when the valuations of other agents are v^{-i} is

$$-c^i(v^i, \theta^i) + \beta\Big[v^i(h_\omega(v)) - h_t^i(v)\Big],$$

and that from choosing $\tilde{v}^i$ is

$$-c^i(\tilde{v}^i, \theta^i) + \beta\Big[\tilde{v}^i(h_\omega(\tilde{v}^i, v^{-i})) - h_t^i(\tilde{v}^i, v^{-i})\Big].$$

The difference is

$$\begin{aligned}
& -c^i(v^i, \theta^i) + \beta\Big[v^i(h_\omega(v)) - h_t^i(v)\Big] - \Big\{ - c^i(\tilde{v}^i, \theta^i) + \beta\Big[\tilde{v}^i(h_\omega(\tilde{v}^i, v^{-i})) - h_t^i(\tilde{v}^i, v^{-i})\Big]\Big\} \\
= \; & \beta\Big[v^i(h_\omega(v)) - h_t^i(v) - \big(\tilde{v}^i(h_\omega(\tilde{v}^i, v^{-i})) - h_t^i(\tilde{v}^i, v^{-i})\big)\Big] - \Big(c^i(v^i, \theta^i) - c^i(\tilde{v}^i, \theta^i)\Big) \\
= \; & 0.
\end{aligned}$$

Therefore, v^i and $\tilde{v}^i$ are indifferent for agent i, and both v and $(\tilde{v}^i, v^{-i})$ are Nash equilibria of the investment game.

Next, compare the social welfare between v and $(\tilde{v}^i, v^{-i})$. For v, the sum of utility of all agents is

$$\sum_{j \in I} \Big\{ - c^j(v^j, \theta^j) + \beta v^j(h_\omega(v))\Big\} = -c^i(v^i, \theta^i) + \beta \sum_{j \in I} v^j(h_\omega(v)).$$

And for $(\tilde{v}^i, v^{-i})$, the sum of utility of all agents is

$$-c^i(\tilde{v}^i, \theta^i) + \beta\Big[\tilde{v}^i(h_\omega(\tilde{v}^i, v^{-i})) + \sum_{j \in I\setminus\{i\}} v^j(h_\omega(\tilde{v}^i, v^{-i}))\Big].$$

The difference of these two is:

$$\begin{aligned}
& -c^i(v^i, \theta^i) + \beta \sum_{j \in I} v^j(h_\omega(v)) + c^i(\tilde{v}^i, \theta^i) - \beta\Big[\tilde{v}^i(h_\omega(\tilde{v}^i, v^{-i})) + \sum_{j \in I\setminus\{i\}} v^j(h_\omega(\tilde{v}^i, v^{-i}))\Big] && (4) \\
\geq \; & \beta\Big[\sum_{j \in I} v^j(h_\omega(\tilde{v}^i, v^{-i})) - \tilde{v}^i(h_\omega(\tilde{v}^i, v^{-i})) - \sum_{j \in I\setminus\{i\}} v^j(h_\omega(\tilde{v}^i, v^{-i}))\Big] && (5) \\
& -(c^i(v^i, \theta^i) - c^i(\tilde{v}^i, \theta^i)) && (6) \\
= \; & \beta\Big[v^i(h_\omega(\tilde{v}^i, v^{-i})) - \tilde{v}^i(h_\omega(\tilde{v}^i, v^{-i}))\Big] - (c^i(v^i, \theta^i) - c^i(\tilde{v}^i, \theta^i)) && (7) \\
> \; & \beta\Big[v^i(h_\omega(v)) - h_t^i(v) + h_t^i(\tilde{v}^i, v^{-i}) - \tilde{v}^i(h_\omega(\tilde{v}^i, v^{-i}))\Big] - (c^i(v^i, \theta^i) - c^i(\tilde{v}^i, \theta^i)) && (8) \\
= \; & 0, && (9)
\end{aligned}$$

in which the inequality in (5) follows from the allocative efficiency of h; the inequality in (8) follows from equation (3). Therefore, $(\tilde{v}^i, v^{-i})$ is not an efficient investment profile although it is supported by a Nash equilibrium. Hence, there is an inefficient equilibrium of the investment game, and efficient *ex ante* investments are not Nash implementable given h.

[2] When h is strategy-proof. We consider a slight modification of Example 4 of Hatfield et al. (2016): an auction where two agents bid for a single good. Consider any social

choice function h which is allocatively efficient and strategy-proof. Suppose $\{i,j\} \subseteq I$ and $\{\omega^i, \omega^j\} \subseteq \Omega$. By the assumption of V, for some $\alpha > 0$,

$$\begin{aligned}
&\{a\mathbb{1}_{\{\omega=\omega^i\}} : a \in [0,\alpha]\} \subseteq V^i, \\
&\{a\mathbb{1}_{\{\omega=\omega^j\}} : a \in [0,\alpha]\} \subseteq V^j, \text{ and} \\
&0 \in V^k \text{ for any } k \in I \setminus \{i,j\}
\end{aligned}$$

hold. Here ω^i and ω^j each represent the alternatives where i and j obtain the item respectively. Consider the following cost types $\theta \in \Theta$ such that

$$\begin{aligned}
&c^i(a\mathbb{1}_{\{\omega=\omega^i\}}, \theta^i) = \frac{5}{3\alpha}\beta a^2, \text{ for any } a \in [0,\alpha], \\
&c^j(a\mathbb{1}_{\{\omega=\omega^j\}}, \theta^j) = \frac{5}{2\alpha}\beta a^2, \text{ for any } a \in [0,\alpha], \\
&c^k(0,\theta^k) = 0 \text{ for all } k \in I \setminus \{i,j\}, \\
&\sup_{p^i \in V^i \setminus \{a\mathbb{1}_{\{\omega=\omega^i\}} : a \in [0,\alpha]\}, p^{-i} \in V^{-i}, \omega \in \Omega, \gamma \in \{0,1\}} \Big\{ - c^i(p^i, \theta^i) + \beta\Big[p^i(\omega) - \gamma h^i_t(p)\Big]\Big\} \\
&< \inf_{p^i \in \{a\mathbb{1}_{\{\omega=\omega^i\}} : a \in [0,\alpha]\}, p^{-i} \in V^{-i}, \omega \in \Omega, \gamma \in \{0,1\}} \Big\{ - c^i(p^i, \theta^i) + \beta\Big[p^i(\omega) - \gamma h^i_t(p)\Big]\Big\}, \\
&\sup_{p^j \in V^j \setminus \{a\mathbb{1}_{\{\omega=\omega^j\}} : a \in [0,\alpha]\}, p^{-j} \in V^{-j}, \omega \in \Omega, \gamma \in \{0,1\}} \Big\{ - c^j(p^j, \theta^j) + \beta\Big[p^j(\omega) - \gamma h^j_t(p)\Big]\Big\} \\
&< \inf_{p^j \in \{a\mathbb{1}_{\{\omega=\omega^j\}} : a \in [0,\alpha]\}, p^{-j} \in V^{-j}, \omega \in \Omega, \gamma \in \{0,1\}} \Big\{ - c^j(p^j, \theta^j) + \beta\Big[p^j(\omega) - \gamma h^j_t(p)\Big]\Big\}, \text{ and} \\
&\sup_{p^k \in V^k \setminus \{0\}, p^{-k} \in V^{-k}, \omega \in \Omega, \gamma \in \{0,1\}} \Big\{ - c^k(p^k, \theta^k) + \beta\Big[p^k(\omega) - \gamma h^k_t(p)\Big]\Big\} \\
&< \inf_{p^{-k} \in V^{-k}, \omega \in \Omega, \gamma \in \{0,1\}} \Big\{ - \beta\gamma h^k_t(0, p^{-k})\Big\} \text{ for all } k \in I \setminus \{i,j\}.
\end{aligned}$$

The last three conditions ensure that any equilibrium and any efficient investment profile should be in $\{a\mathbb{1}_{\{\omega=\omega^i\}} : a \in [0,\alpha]\} \times \{a\mathbb{1}_{\{\omega=\omega^j\}} : a \in [0,\alpha]\} \times \{0\}^{V \setminus \{i,j\}}$. Thus, we can focus on the investment choices of agents i and j.[28]

First, consider efficient investment profiles under this allocatively efficient h. It is clear that only one of agents i and j should make a positive investment. If agent i obtains the item, the optimal choice of valuation should be

$$\arg\max_{a \in [0,\alpha]} \beta\Big\{ - \frac{5}{3\alpha}a^2 + a\Big\} = \frac{3}{10}\alpha.$$

[28]The parameters are equivalent to the example in Section 2 if $\alpha = 10$.

If agent j obtains it, the optimal choice of valuation should be

$$\arg\max_{a\in[0,\alpha]} \beta\Big\{-\frac{5}{2}a^2+a\Big\}=\frac{1}{5}\alpha.$$

The social welfare achieved by $(\frac{3}{10}\alpha\mathbb{1}_{\{\omega=\omega^i\}},0)$ is

$$\beta\Big\{-\frac{3}{20}\alpha+\frac{3}{10}\alpha\Big\}=\frac{3}{20}\alpha\beta$$

and the social welfare achieved by $(0,\frac{1}{5}\alpha\mathbb{1}_{\{\omega=\omega^i\}})$ is

$$\beta\Big\{-\frac{1}{10}\alpha+\frac{1}{5}\alpha\Big\}=\frac{1}{10}\alpha\beta.$$

Thus, $(\frac{3}{10}\alpha\mathbb{1}_{\{\omega=\omega^i\}},0)$ is the unique investment profile of i and j which maximizes the social welfare.

Then consider the other investment profile $(0,\frac{1}{5}\alpha\mathbb{1}_{\{\omega=\omega^j\}})$, and show that it is a Nash equilibrium of the investment game. First, it is clear that the valuation of agent j is a best response to i's choice 0 because it maximizes the value of the item. Next, given $\bar{v}^j\equiv\frac{1}{5}\alpha\mathbb{1}_{\{\omega=\omega^j\}}$,

$$\begin{aligned}
&\arg\max_{v^i\in V^i}\Big\{-c^i(v^i,\theta^i)+\beta v^i(h_\omega(v^i,\bar{v}^j))+\beta\bar{v}^j(h_\omega(v^i,\bar{v}^j))\Big\}\\
=\;&\arg\max_{v^i\in V^i}\Big\{-\frac{1}{\beta}c^i(v^i,\theta^i)+v^i(h_\omega(v^i,\bar{v}^j))+\bar{v}^j(h_\omega(v^i,\bar{v}^j))\Big\}\\
=\;&0
\end{aligned}$$

holds. This is because given agent j's valuation $\bar{v}^j=\frac{1}{5}\alpha\mathbb{1}_{\{\omega=\omega^j\}}$, the equation is maximized when agent j obtains the item and agent i does not make any investments (the value of the second equation becomes $\frac{1}{5}$, which cannot be achieved by any positive valuation of agent i since the sum of the first two terms do not exceed $\frac{3}{20}$). Since h is allocatively efficient and strategy-proof, $h^i_t(\cdot,\bar{v}^j)$ is written as a Groves function (Green and Laffont, 1977):

$$h^i_t(v^i,\bar{v}^j)=g(\bar{v}^j)-\bar{v}^j(h_\omega(v^i,\bar{v}^j)).$$

Hence,

$$\begin{aligned}
&\arg\max_{v^i\in V^i}\Big\{-c^i(v^i,\tilde{\theta}^i)+v^i(h_\omega(v^i,\bar{v}^j))-h^i_t(v^i,\bar{v}^j)\Big\}\\
=\;&\arg\max_{v^i\in V^i}\Big\{-c^i(v^i,\tilde{\theta}^i)+v^i(h_\omega(v^i,\bar{v}^j))-g(\bar{v}^j)+\bar{v}^j(h_\omega(v^i,\bar{v}^j))\Big\}\\
=\;&\arg\max_{v^i\in V^i}\Big\{-c^i(v^i,\tilde{\theta}^i)+v^i(h_\omega(v^i,\bar{v}^j))+\bar{v}^j(h_\omega(v^i,\bar{v}^j))\Big\}
\end{aligned}$$

should hold for any cost type $\tilde{\theta}^i \in \Theta^i$. Thus, we have

$$\begin{aligned} & \arg\max_{v^i \in V^i} \Big\{ -c^i(v^i,\theta^i) + \beta v^i(h_\omega(v^i,\bar{v}^j)) - \beta h^i_t(v^i,\bar{v}^j) \Big\} \\ = \ & \arg\max_{v^i \in V^i} \Big\{ -\frac{1}{\beta} c^i(v^i,\theta^i) + v^i(h_\omega(v^i,\bar{v}^j)) - h^i_t(v^i,\bar{v}^j) \Big\} \\ = \ & \arg\max_{v^i \in V^i} \Big\{ -\frac{1}{\beta} c^i(v^i,\theta^i) + v^i(h_\omega(v^i,\bar{v}^j)) + \bar{v}^j(h_\omega(v^i,\bar{v}^j)) \Big\} \\ = \ & 0. \end{aligned}$$

This means that 0 is the best response for agent i, and hence $(0, \frac{1}{5}\alpha \mathbb{1}_{\{\omega=\omega^j\}})$ is a Nash equilibrium of the investment game. However, this does not achieve investment efficiency given h because it is less efficient than $(\frac{3}{10}\alpha \mathbb{1}_{\{\omega=\omega^i\}}, 0)$. Therefore, there is an inefficient equilibrium of the investment game, which means that efficient *ex ante* investments are not Nash implementable given h.

A.2 Proof of Theorem 2

We fist show that when h is allocatively constrained-efficient for some $\Omega' \subseteq \Omega$ with $\Omega' \neq \emptyset$, any efficient investment profile $(v,\bar{v}) \in V^2$ at cost types $\theta \in \Theta$ is such that for each $i \in I$, (i) $c^i(v^i,\theta^i) = 0$ and (ii) $\bar{v}^i$ is chosen optimally given the outcomes of h. Next, the sufficiency of commitment-proofness for subgame-perfect implementation is proved by showing that such profiles of valuations are exactly the set of on-path valuations of SPE of the investment game given h when it is commitment-proof. Finally, we will show the necessity of commitment-proofness of h by finding a profile of cost types at which the set of SPE valuation functions of the investment game differs from the set of efficient investments for some $\beta \in (0,1)$ if h is not commitment-proof.

[1] Efficient Investments. First, take any $\beta \in (0,1)$ and $\theta \in \Theta$, and fix them. For any $i \in I$, let V^{0i} be the set of all valuation functions in V^i whose costs are zero, i.e., $V^{0i} \equiv \{v^i \in V^i | c^i(v^i,\theta^i) = 0\}$. Let $V^0 \equiv \times_{i\in I} V^{0i}$. Moreover, for each $i \in I$, let $M^i : V \twoheadrightarrow V^i$ be the optimal choice correspondence of *ex post* valuations under h for each profile of valuation functions $v \in V$ chosen before the mechanism:

$$M^i(v) \equiv \arg\max_{\bar{v}^i \in \{\tilde{v}^i \in V^i | c^i(\tilde{v}^i,\theta^i) \geq c^i(v^i,\theta^i)\}} \Big\{ \bar{v}^i(h_\omega(b^{\theta,v})) - c^i(\bar{v}^i,\theta^i) \Big\}.$$

We will show that the set of efficient investment schedules is

$$\{(p,q) \in V^2 | p \in V^0 \text{ and } q^i \in M^i(p) \text{ for all } i \in I\}.$$

Take any profile of valuations $(v^0, \bar{v}^*)$ from this set:

$$(v^0, \bar{v}^*) \in \{(p,q) \in V^2 | p \in V^0 \text{ and } q^i \in M^i(p) \text{ for all } i \in I\}.$$

Also take another profile of valuations $(v, \bar{v})$ with a costly prior investment:

$$(v, \bar{v}) \in \{(p,q) \in V^2 | p \notin V^0 \text{ and } c^i(q^i, \theta^i) \geq c^i(p^i, \theta^i) \text{ for all } i \in I\}.$$

It is shown that the social welfare given h under $(v, \bar{v})$ is strictly less than that under $(v^0, \bar{v}^*)$:

$$\begin{aligned}
& \sum_{i \in I} \Big\{ - c^i(v^i, \theta^i) + \beta \Big[\bar{v}^i(h_\omega(b^{\theta,v})) - \big(c^i(\bar{v}^i, \theta^i) - c^i(v^i, \theta^i) \big) \Big] \Big\} && (10) \\
= & \sum_{i \in I} \Big\{ \beta \Big[\bar{v}^i(h_\omega(b^{\theta,v})) - c^i(\bar{v}^i, \theta^i) \Big] - (1-\beta) c^i(v^i, \theta^i) \Big\} && (11) \\
< & \sum_{i \in I} \beta \Big[\bar{v}^i(h_\omega(b^{\theta,v})) - c^i(\bar{v}^i, \theta^i) \Big] && (12) \\
\leq & \sum_{i \in I} \beta b^{\theta^i, v^{0i}}(h_\omega(b^{\theta,v})) && (13) \\
\leq & \sum_{i \in I} \beta b^{\theta^i, v^{0i}}(h_\omega(b^{\theta,v^0})) && (14) \\
= & \sum_{i \in I} \beta \Big[\bar{v}^{*i}(h_\omega(b^{\theta,v^0})) - c^i(\bar{v}^{*i}, \theta^i) \Big], && (15)
\end{aligned}$$

in which the last equation (15) is the social welfare given h under $(v^0, \bar{v}^*)$. The inequality in (12) holds because $c^i(v^i, \theta^i) > 0$ for some $i \in I$ and $\beta < 1$; the inequality in (13) follows from the definition of $b^{\theta^i, v^{0i}}$; the inequality in (14) follows from the fact that h is allocatively constrained-efficient; the equality of (15) follows from the definition of $b^{\theta^i, v^{0i}}$ and $\bar{v}^{*i}$. Moreover, consider when V^0 is not a singleton set. For any prior investments $v^0, \tilde{v}^0 \in V^0$, the valuations b^{θ, v^0} and $b^{\theta, \tilde{v}^0}$ at the time of the mechanism are the same, and hence, the outcomes of the social choice function should also be the same. Therefore, any profile of valuations in $\{(p,q) \in V^2 | p \in V^0 \text{ and } q^i \in M^i(p) \text{ for all } i \in I\}$ maximizes the social welfare given h. Thus, the set of efficient investment schedules is characterized by

$$\{(p,q) \in V^2 | p \in V^0 \text{ and } q^i \in M^i(p) \text{ for all } i \in I\}.$$

[2] Sufficiency of commitment-proofness. We still fix arbitrary $\beta \in (0,1)$ and cost types $\theta \in \Theta$. We will show that when h is commitment-proof, the set of valuation functions that are on the equilibrium path of some SPE of the investment game given h is exactly

$$\{(p,q) \in V^2 | p \in V^0 \text{ and } q^i \in M^i(p) \text{ for all } i \in I\}.$$

Take any agent $i \in I$ and $v^{-i} \in V^{-i}$, and consider i's incentive for investments when the valuation functions of other agents at the time of the mechanism are fixed to $b^{-i} \equiv b^{\theta^{-i},v^{-i}}$.

First, consider the second stage investment strategy $\mu^{*i} : V \to V^i$ in any SPE. It is straightforward that μ^{*i} should satisfy the following:

$$\mu^{*i}(v) \in M^i(v) \text{ for any } v \in V.$$

Thus any valuation functions of agent i that are on the equilibrium path should be in the set:

$$\{(p^i, q^i) \in (V^i)^2 | p^i \in V^i \text{ and } q^i \in M^i(p^i, v^{-i})\}.$$

Next, fix the optimal second stage investment strategy $\mu^{*i} : V \to V^i$ and consider two different valuation choices in the first stage. Take any $v^{0i} \in V^{0i}$ and any $v^i \notin V^{0i}$. We can show that v^{0i} gives a strictly higher utility than v^i for agent i. To see this, the *ex ante* utility from (v^i, μ^{*i}) given v^{-i} and $b^{\theta^{-i},v^{-i}}$ is written as:

$$
\begin{aligned}
& -c^i(v^i, \theta^i) && (16)\\
& +\beta\Big[\mu^{*i}(v^i, v^{-i})(h_\omega(b^{\theta^i,v^i}, b^{-i})) - h_t^i(b^{\theta^i,v^i}, b^{-i}) - \big(c^i(\mu^{*i}(v^i, v^{-i}), \theta^i) - c^i(v^i, \theta^i)\big)\Big] && (17)\\
= \; & \beta\Big[\mu^{*i}(v^i, v^{-i})(h_\omega(b^{\theta^i,v^i}, b^{-i})) - h_t^i(b^{\theta^i,v^i}, b^{-i}) - c^i(\mu^{*i}(v^i, v^{-i}), \theta^i)\Big] - (1-\beta)c^i(v^i, \theta^i) && (18)\\
< \; & \beta\Big[\mu^{*i}(v^i, v^{-i})(h_\omega(b^{\theta^i,v^i}, b^{-i})) - h_t^i(b^{\theta^i,v^i}, b^{-i}) - c^i(\mu^{*i}(v^i, v^{-i}), \theta^i)\Big] && (19)\\
= \; & \beta\Big[b^{\theta^i,v^i}(h_\omega(b^{\theta^i,v^i}, b^{-i})) - h_t^i(b^{\theta^i,v^i}, b^{-i}) - c^i(v^i, \theta^i)\Big] && (20)\\
\leq \; & \beta\Big[b^{\theta^i,v^i}(h_\omega(b^{\theta^i,v^i}, b^{-i})) - h_t^i(b^{\theta^i,v^i}, b^{-i}) - \max\Big\{0, \max_{\omega\in\Omega}\big\{b^{\theta^i,v^i}(\omega) - b^{\theta^i,v^{0i}}(\omega)\big\}\Big\}\Big] && (21)\\
\leq \; & \beta\Big[b^{\theta^i,v^{0i}}(h_\omega(b^{\theta^i,v^{0i}}, b^{-i})) - h_t^i(b^{\theta^i,v^{0i}}, b^{-i})\Big] && (22)\\
= \; & \beta\Big[\mu^{*i}(v^{0i}, v^{-i})(h_\omega(b^{\theta^i,v^{0i}}, b^{-i})) - h_t^i(b^{\theta^i,v^{0i}}, b^{-i}) - c^i(\mu^{*i}(v^{0i}, v^{-i}), \theta^i)\Big], && (23)
\end{aligned}
$$

in which the last equation (23) is the *ex ante* utility from (v^{0i}, μ^{*i}) given v^{-i} and $b^{\theta^{-i},v^{-i}}$. The inequality in (19) holds because $c^i(v^i, \theta^i) > 0$ and $\beta < 1$; the equality in (20) follows from the definition of b^{θ^i,v^i}; the inequality in (21) follows from Lemma 1; the inequality in (22) follows from the fact that h is commitment-proof; and the equality in (23) follows from the definition of $b^{\theta^i,v^{0i}}$. Moreover, consider when V^{0i} is not a singleton set. As long as μ^{*i} is taken in the second stage, any valuation functions in V^{0i} give exactly the same utility from the calculation above. Therefore, V^{0i} is the set of best responses of agent i in the first investment stage to any $b^{-i} \in \mathbb{R}^{\Omega\times(|I|-1)}$.

Given the optimal investment strategies in the second stage, for any prior investment $v^j \in V^{0j}$, agent j should have the same valuation at the time of the mechanism. Therefore, the utility agent i obtains from choosing any investment in V^{0i} is unchanged as long as v^j is taken from V^{0j} for any $j \in I \setminus \{i\}$. From this argument, the SPE of the investment game is characterized by

$$\{(v^*, \mu^*) \in V \times \mathcal{M} | v^* \in V^0 \text{ and } \mu^{*i}(v) \in M^i(v) \text{ for all } i \in I \text{ and } v \in V\}.$$

Therefore, $SPE(\theta, h, \beta) = \{(p, q) \in V^2 | p \in V^0 \text{ and } q^i \in M^i(p) \text{ for all } i \in I\}$ and it coincides with the set of efficient investment schedules.

[3] Necessity of commitment-proofness. Consider a social choice function h which is allocatively constrained-efficient for some $\Omega' \subseteq \Omega$ with $\Omega' \neq \emptyset$ but is not commitment-proof. Since h is allocatively constrained-efficient, for any given cost types $\theta \in \Theta$, the set of efficient investment schedules is

$$\{(p, q) \in V^2 | p \in V^0 \text{ and } q^i \in M^i(p) \text{ for all } i \in I\}$$

by the first argument. We will show that for some $\theta \in \Theta$, there is an SPE whose on-path valuations are not in $\{(p, q) \in V^2 | p \in V^0 \text{ and } q^i \in M^i(p) \text{ for all } i \in I\}$.

First, since h is not commitment-proof, there are $i \in I$, $b \in V$ and $\tilde{b}^i \in V^i$ such that

$$\tilde{b}^i(h_\omega(\tilde{b}^i, b^{-i})) - h_t^i(\tilde{b}^i, b^{-i}) - \Big(b^i(h_\omega(b)) - h_t^i(b)\Big) > \max\Big\{0, \max_{\omega \in \Omega}\big\{\tilde{b}^i(\omega) - b^i(\omega)\big\}\Big\}. \tag{24}$$

Consider the following profile of cost types $\theta \in \Theta$ such that

$$\begin{aligned}
&c^i(b^i, \theta^i) = 0,\\
&c^i(\tilde{b}^i, \theta^i) = \begin{cases} \max_{\omega \in \Omega}\Big\{\tilde{b}^i(\omega) - b^i(\omega)\Big\} & \text{if } \max_{\omega \in \Omega}\Big\{\tilde{b}^i(\omega) - b^i(\omega)\Big\} > 0,\\ \delta & \text{otherwise,}\end{cases}\\
&c^j(b^j, \theta^j) = 0 \text{ for all } j \in I \setminus \{i\},\\
&\sup_{p^i \in V^i \setminus \{b^i, \tilde{b}^i\}, p^{-i} \in V^{-i}, \omega \in \Omega, \gamma \in \{0,1\}} \Big\{ - c^i(p^i, \theta^i) + \beta\Big[p^i(\omega) - \gamma h_t^i(p)\Big]\Big\}\\
&< \inf_{p^i \in \{b^i, \tilde{b}^i\}, p^{-i} \in V^{-i}, \omega \in \Omega, \gamma \in \{0,1\}} \Big\{ - c^i(p^i, \theta^i) + \beta\Big[p^i(\omega) - \gamma h_t^i(p)\Big]\Big\}, \text{ and}\\
&\sup_{p^j \in V^j \setminus \{b^j\}, p^{-j} \in V^{-j}, \omega \in \Omega, \gamma \in \{0,1\}} \Big\{ - c^j(p^j, \theta^j) + \beta\Big[p^j(\omega) - \gamma h_t^j(p)\Big]\Big\}\\
&< \inf_{p^{-j} \in V^{-j}, \omega \in \Omega, \gamma \in \{0,1\}} \Big\{\beta\Big[b^j(\omega) - \gamma h_t^j(b^j, p^{-j})\Big]\Big\} \text{ for all } j \in I \setminus \{i\}
\end{aligned}$$

where $\delta > 0$. The last two conditions ensure that for any subgame-perfect equilibrium and any efficient investment profile, b and $(\tilde{b}^i, b^{-i})$ are the only candidates for the first stage investments. Thus, we only need to analyze which of b^i and $\tilde{b}^i$ agent i chooses prior to the mechanism, and which of them is more efficient.

Agent i has two choices b^i and $\tilde{b}^i$. First, consider her optimal choice in the second investment stage. When i chooses $\tilde{b}^i$ prior to the mechanism, since $c^i(\tilde{b}^i, \theta^i) > c^i(b^i, \theta^i)$, the optimal choice of a valuation function in the *ex post* stage is $\tilde{b}^i$ for any $\omega \in \Omega$ because it is the unique choice for her. Thus, the valuation at the time of the mechanism is

$$b^{\theta^i, \tilde{b}^i}(\omega) = \left\{\tilde{b}^i(\omega) - c^i(\tilde{b}^i, \theta^i)\right\} + c^i(\tilde{b}^i, \theta^i) = \tilde{b}^i(\omega)$$

for each $\omega \in \Omega$. On the other hand, when i chooses b^i prior to the mechanism, in the *ex post* stage, she can still choose from $\{b^i, \tilde{b}^i\}$ because b^i is a costless valuation. However, by the construction of the cost function, we can see that

$$b^i(\omega) \geq \tilde{b}^i(\omega) - c^i(\tilde{b}^i, \theta^i)$$

for any $\omega \in \Omega$. Thus, the valuation at the time of the mechanism is

$$b^{\theta^i, b^i}(\omega) = \max_{\bar{v}^i \in \{b^i, \tilde{b}^i\}} \left\{\bar{v}^i(\omega) - c^i(\bar{v}^i, \theta^i)\right\} = b^i(\omega)$$

for each $\omega \in \Omega$. To summarize, agent i's optimal investment strategy and the valuation at the time of the mechanism is as follows:

Ex Ante Valuation	Valuation at the Mechanism	Optimal *Ex Post* Valuation
b^i	b^i	for any ω: b^i (or $\tilde{b}^i$ if $b^i(\omega) = \tilde{b}^i(\omega) - c^i(\tilde{b}^i)$)
$\tilde{b}^i$	$\tilde{b}^i$	for any ω: $\tilde{b}^i$

Given this optimal strategy in the second stage, we can compare the first stage investments b^i and $\tilde{b}^i$. Consider agent i's incentive for first stage investment given the valuations b^{-i} of other agents. The total utility of agent i when choosing an investment $\tilde{b}^i$ is

$$-c^i(\tilde{b}^i, \theta^i) + \beta\left[\tilde{b}^i(h_\omega(\tilde{b}^i, b^{-i})) - h_t^i(\tilde{b}^i, b^{-i})\right]$$

and when choosing an investment b^i, it is

$$\beta\left[b^i(h_\omega(b)) - h_t^i(b)\right].$$

The difference of these two is calculated as:

$$\begin{aligned}
& -c^i(\tilde{b}^i, \theta^i) + \beta\left[\tilde{b}^i(h_\omega(\tilde{b}^i, b^{-i})) - h_t^i(\tilde{b}^i, b^{-i})\right] - \beta\left[b^i(h_\omega(b)) - h_t^i(b)\right] \\
= \; & -(1-\beta)c^i(\tilde{b}^i, \theta^i) + \beta\left[\tilde{b}^i(h_\omega(\tilde{b}^i, b^{-i})) - h_t^i(\tilde{b}^i, b^{-i}) - c^i(\tilde{b}^i, \theta^i)\right] - \beta\left[b^i(h_\omega(b)) - h_t^i(b)\right] \\
= \; & -(1-\beta)c^i(\tilde{b}^i, \theta^i) \\
& +\beta\left[\tilde{b}^i(h_\omega(\tilde{b}^i, b^{-i})) - h_t^i(\tilde{b}^i, b^{-i}) - \left(b^i(h_\omega(b)) - h_t^i(b)\right) - \max\left\{\delta, \max_{\omega \in \Omega}\left\{\tilde{b}^i(\omega) - b^i(\omega)\right\}\right\}\right] \\
> \; & 0,
\end{aligned}$$

in which $c^i(\tilde{b}^i, \theta^i) = \max\left\{\delta, \max_{\omega\in\Omega}\left\{\tilde{b}^i(\omega) - b^i(\omega)\right\}\right\}$ holds for sufficiently small $\delta > 0$, and the final inequality holds from equation (24) when we take β sufficiently close to 1 and $\delta > 0$ sufficiently small. Therefore, $\tilde{b}^i$ is chosen in the first investment stage in an SPE of this investment game. However for this profile of cost types θ, since h is allocatively constrained-efficient, (b, b) is a profile of efficient investment schedules because $c^k(b^k) = 0$ for all $k \in I$ and $\arg\max_{\bar{v}^i \in V^i}\left\{\bar{v}^i(\omega) - c^i(\bar{v}^i, \theta^i)\right\} = b^i(\omega)$ for any $\omega \in \Omega$. Thus, efficient investments are not subgame-perfect implementable at θ given h and this β.

A.3 Proof of Proposition 1

For any efficient allocation rule h_ω, consider the following transfer rule h_t which divides the maximum sum of valuations equally among all agents:

$$h_t^i(b) = b^i(h_\omega(b)) - \frac{1}{n}\sum_{i\in I} b^i(h_\omega(b)). \tag{25}$$

It is clear that h is budget-balanced. It suffices to show that h is commitment-proof. Consider any $i \in I$, $b \in V$, $\tilde{b}^i \in V^i$ and $x \geq 0$ such that $\tilde{b}^i(\omega) \leq b^i(\omega) + x$ for all $\omega \in \Omega$. We will show:

$$\tilde{b}^i(h_\omega(\tilde{b}^i, b^{-i})) - h_t^i(\tilde{b}^i, b^{-i}) - x \leq b^i(h_\omega(b)) - h_t^i(b)$$

for this transfer rule (25). Since $x \geq \max\left\{0, \max_{\omega\in\Omega}\left\{\tilde{b}^i(\omega) - b^i(\omega)\right\}\right\}$ holds,

$$\begin{aligned}
& \text{(RHS) - (LHS)} \\
\geq\ & \left[b^i(h_\omega(b)) - h_t^i(b)\right] - \left[\tilde{b}^i(h_\omega(\tilde{b}^i, b^{-i})) - h_t^i(\tilde{b}^i, b^{-i})\right] + \max\left\{0, \max_{\omega\in\Omega}\left\{\tilde{b}^i(\omega) - b^i(\omega)\right\}\right\} \\
=\ & \frac{1}{n}\sum_{i\in I} b^i(h_\omega(b)) - \frac{1}{n}\left\{\tilde{b}^i(h_\omega(\tilde{b}^i, b^{-i})) + \sum_{j\in I\setminus\{i\}} b^j(h_\omega(\tilde{b}^i, b^{-i}))\right\} + \max\left\{0, \max_{\omega\in\Omega}\left\{\tilde{b}^i(\omega) - b^i(\omega)\right\}\right\} \\
=\ & -\frac{1}{n}\left\{\tilde{b}^i(h_\omega(\tilde{b}^i, b^{-i})) + \sum_{j\in I\setminus\{i\}} b^j(h_\omega(\tilde{b}^i, b^{-i})) - \sum_{i\in I} b^i(h_\omega(b))\right\} + \max\left\{0, \max_{\omega\in\Omega}\left\{\tilde{b}^i(\omega) - b^i(\omega)\right\}\right\} \\
=\ & -\frac{1}{n}\left\{\tilde{b}^i(h_\omega(\tilde{b}^i, b^{-i})) - b^i(h_\omega(\tilde{b}^i, b^{-i})) + \sum_{i\in I} b^i(h_\omega(\tilde{b}^i, b^{-i})) - \sum_{i\in I} b^i(h_\omega(b))\right\} \\
& + \max\left\{0, \max_{\omega\in\Omega}\left\{\tilde{b}^i(\omega) - b^i(\omega)\right\}\right\} \\
\geq\ & -\frac{1}{n}\left\{\tilde{b}^i(h_\omega(\tilde{b}^i, b^{-i})) - b^i(h_\omega(\tilde{b}^i, b^{-i}))\right\} + \max\left\{0, \max_{\omega\in\Omega}\left\{\tilde{b}^i(\omega) - b^i(\omega)\right\}\right\} \\
\geq\ & -\frac{1}{n}\max\left\{0, \max_{\omega\in\Omega}\left\{\tilde{b}^i(\omega) - b^i(\omega)\right\}\right\} + \max\left\{0, \max_{\omega\in\Omega}\left\{\tilde{b}^i(\omega) - b^i(\omega)\right\}\right\} \\
=\ & \frac{n-1}{n}\max\left\{0, \max_{\omega\in\Omega}\left\{\tilde{b}^i(\omega) - b^i(\omega)\right\}\right\} \\
\geq\ & 0.
\end{aligned}$$

The second inequality holds from the allocative efficiency of h. Therefore, this h is commitment-proof and the proof is done.

A.4 Proof of Proposition 2

First, it is easy to characterize the socially efficient investment strategies in this incomplete information environment. Since investment can depend on the agent's own cost type and h is allocatively constrained-efficient for some $\Omega' \subseteq \Omega$ with $\Omega' \neq \emptyset$, the same argument as the complete information case applies to this case. In other words, the socially efficient investment strategies $\sigma \in \Sigma$ and $\mu \in \mathcal{M}$ are characterized by

1. $\mu^i(v, \omega, \theta^i) \in \underset{\bar{v}^i \in \{\tilde{v}^i \in V^i | c^i(\tilde{v}^i, \theta^i) \geq c^i(v^i, \theta^i)\}}{\arg\max} \left\{ \bar{v}^i(\omega) - c^i(\bar{v}^i, \theta^i) \right\}$ for any $v \in V$ and $\omega \in \Omega$, and
2. $\sigma^i(\theta^i) \in \{v^i \in V^i | c^i(v^i, \theta^i) = 0\}$

for each $i \in I$ and $\theta^i \in \Theta^i$.

As the first condition on μ coincides with the one for PBE, we only need to show that the second condition on σ characterizes PBE when h is allocatively constrained-efficient and strategy-proof. Consider any agent $i \in I$ and her cost type $\theta^i \in \Theta^i$. Every agent's *ex post* investment strategy is fixed to the PBE strategy μ^*, and take any arbitrary *ex ante* investment strategies $\sigma^{-i} \in \Sigma^{-i}$ of other agents. Take any $v^{0i} \in \{v^i \in V^i | c^i(v^i, \theta^i) = 0\}$ and any $v^i \notin \{v^i \in V^i | c^i(v^i, \theta^i) = 0\}$. We can show that v^{0i} gives a strictly higher utility than v^i for agent i of cost type θ^i. To see this, the *ex ante* utility from (v^i, μ^{*i}) given σ^{-i} and

$b^{\theta^{-i},\sigma^{-i}(\theta^{-i})}$ is written as:

$$
\begin{aligned}
& \beta \sum_{\theta^{-i}\in\Theta^{-i}} p(\theta^{-i}|\theta^{i})\Big[\mu^{*i}(v^{i},\sigma^{-i}(\theta^{-i}),h_{\omega}(b^{\theta^{i},v^{i}},b^{\theta^{-i},\sigma^{-i}(\theta^{-i})}),\theta^{i})(h_{\omega}(b^{\theta^{i},v^{i}},b^{\theta^{-i},\sigma^{-i}(\theta^{-i})})) && (26)\\
& -h_{t}^{i}(b^{\theta^{i},v^{i}},b^{\theta^{-i},\sigma^{-i}(\theta^{-i})})-c^{i}(\mu^{*i}(v^{i},\sigma^{-i}(\theta^{-i}),h_{\omega}(b^{\theta^{i},v^{i}},b^{\theta^{-i},\sigma^{-i}(\theta^{-i})}),\theta^{i}),\theta^{i})\Big] && (27)\\
& -(1-\beta)c^{i}(v^{i},\theta^{i}) && (28)\\
< \;& \beta \sum_{\theta^{-i}\in\Theta^{-i}} p(\theta^{-i}|\theta^{i})\Big[\mu^{*i}(v^{i},\sigma^{-i}(\theta^{-i}),h_{\omega}(b^{\theta^{i},v^{i}},b^{\theta^{-i},\sigma^{-i}(\theta^{-i})}),\theta^{i})(h_{\omega}(b^{\theta^{i},v^{i}},b^{\theta^{-i},\sigma^{-i}(\theta^{-i})})) && (29)\\
& -h_{t}^{i}(b^{\theta^{i},v^{i}},b^{\theta^{-i},\sigma^{-i}(\theta^{-i})})-c^{i}(\mu^{*i}(v^{i},\sigma^{-i}(\theta^{-i}),h_{\omega}(b^{\theta^{i},v^{i}},b^{\theta^{-i},\sigma^{-i}(\theta^{-i})}),\theta^{i}),\theta^{i})\Big] && (30)\\
= \;& \beta \sum_{\theta^{-i}\in\Theta^{-i}} p(\theta^{-i}|\theta^{i})\Big[b^{\theta^{i},v^{i}}(h_{\omega}(b^{\theta^{i},v^{i}},b^{\theta^{-i},\sigma^{-i}(\theta^{-i})}))-h_{t}^{i}(b^{\theta^{i},v^{i}},b^{\theta^{-i},\sigma^{-i}(\theta^{-i})})-c^{i}(v^{i},\theta^{i})\Big] && (31)\\
\leq \;& \beta \sum_{\theta^{-i}\in\Theta^{-i}} p(\theta^{-i}|\theta^{i})\Big[b^{\theta^{i},v^{i}}(h_{\omega}(b^{\theta^{i},v^{i}},b^{\theta^{-i},\sigma^{-i}(\theta^{-i})}))-h_{t}^{i}(b^{\theta^{i},v^{i}},b^{\theta^{-i},\sigma^{-i}(\theta^{-i})}) && (32)\\
& -\max\Big\{0,\max_{\omega\in\Omega}\big\{b^{\theta^{i},v^{i}}(\omega)-b^{\theta^{i},v^{0i}}(\omega)\big\}\Big\}\Big] && (33)\\
\leq \;& \beta \sum_{\theta^{-i}\in\Theta^{-i}} p(\theta^{-i}|\theta^{i})\Big[b^{\theta^{i},v^{0i}}(h_{\omega}(b^{\theta^{i},v^{0i}},b^{\theta^{-i},\sigma^{-i}(\theta^{-i})}))-h_{t}^{i}(b^{\theta^{i},v^{0i}},b^{\theta^{-i},\sigma^{-i}(\theta^{-i})})\Big] && (34)\\
= \;& \beta \sum_{\theta^{-i}\in\Theta^{-i}} p(\theta^{-i}|\theta^{i})\Big[\mu^{*i}(v^{0i},\sigma^{-i}(\theta^{-i}),h_{\omega}(b^{\theta^{i},v^{0i}},b^{\theta^{-i},\sigma^{-i}(\theta^{-i})}),\theta^{i})(h_{\omega}(b^{\theta^{i},v^{0i}},b^{\theta^{-i},\sigma^{-i}(\theta^{-i})})) && (35)\\
& -h_{t}^{i}(b^{\theta^{i},v^{0i}},b^{\theta^{-i},\sigma^{-i}(\theta^{-i})})-c^{i}(\mu^{*i}(v^{0i},\sigma^{-i}(\theta^{-i}),h_{\omega}(b^{\theta^{i},v^{0i}},b^{\theta^{-i},\sigma^{-i}(\theta^{-i})}),\theta^{i}),\theta^{i})\Big], && (36)
\end{aligned}
$$

in which the last equation (35)-(36) is the *ex ante* utility from (v^{0i},μ^{*i}) given σ^{-i} and $b^{\theta^{-i},\sigma^{-i}(\theta^{-i})}$. The inequality in (29) holds because $c^{i}(v^{i},\theta^{i})>0$ and $\beta<1$; the equality in (31) follows from the definition of $b^{\theta^{i},v^{i}}$; the inequality in (32) follows from Lemma 1; the inequality in (34) follows from the fact that h is commitment-proof as it is strategy-proof; and the equality in (35) follows from the definition of $b^{\theta^{i},v^{0i}}$. Therefore, for any strategies σ^{-i} of other agents, the optimal *ex ante* investment strategy for agent i of cost type θ^{i} is characterized by the condition $\sigma^{i}(\theta^{i})\in\{v^{i}\in V^{i}|c^{i}(v^{i},\theta^{i})=0\}$. This implies that the set of PBE coincides with the set of efficient investment strategies and the proof is done.