Kidney Exchange with Immunosuppressants

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Abstract

We investigate the implications of introducing immunosuppressants to the kidney transplant problem. Immunosuppressants relax immunological constraints, allowing patients to receive transplants from any donor. For each compatibility profile, we select a set of patients receiving immunosuppressants and a matching between patients and donors. To increase compatible transplants, we propose to use immunosuppressants as a part of kidney exchange program. We make modifications of the top-trading cycles rule for a better use of immunosuppressants. These solutions satisfy Pareto efficiency, monotonicity, and maximal improvement.

JEL classification Numbers: C78, D47

Keywords: immunosuppressants, kidney exchange, top-trading cycles rules, Pareto efficiency, monotonicity, maximal improvement

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1. Introduction

When a patient suffers from end-stage renal disease and has to receive a kidney transplant, several options are available depending on the immunological compatibility of the patient with her own donor.[1] If the patient is compatible with the donor, a direct transplant within this pair can be performed. Otherwise, the patient has to look for other ways to receive a transplant. One option is that the patient is registered on a waitlist to receive a transplant from a deceased donor. Another option, developed in the last decade, is to participate in a kidney exchange program where patients swap their donors to form compatible pairs (Roth et al., 2004). Unfortunately, the possibility of receiving transplants from deceased donors or through exchanges is quite limited relative to the increasing number of patients in the waitlist, as illustrated in Table 1 of data from the KONOS (Korean Network for Organ Sharing) program.

Recent developments in immunosuppressive protocols have introduced a new option for transplants from *incompatible* donors. Immunosuppressants (suppressants, for short) have been commonly used to relax minor immunological constraints for compatible transplants. Since 1980s, they have been developed to eliminate blood-type compatibility constraints (Alexander et al., 1987) and more recently, they are being used to eliminate *all* immunological compatibility constraints – blood-type, tissue-type, and positive crossmatch – that patients might have against donors (Gloor et al. (2003), Kawai et al. (2008), and Montgomery et al. (2011)). If a patient uses a suppressant, she becomes compatible with *any* donor, so is able to receive a transplant even from an incompatible donor, which we call an *incompatible kidney transplant*.

To receive an incompatible transplant, a patient has to take rituximab, an immunosuppressive drug inactivating a certain part of white blood cells, and should undergo a plasmapheresis treatment to remove some antibodies from the blood. Intravenous immunoglobulin (IVIG) is also added to this procedure to protect the patient from potential infections. Although a precise timing of this procedure and the dose of drugs may vary, incompatible transplants have been reported quite successful. The long-term survival rate of ABO-incompatible transplants is shown equivalent to that of compatible transplants.[2] The performance of tissue-type incom-

[1] Immunological compatibility is mostly determined by biological characteristics of patients and donors, such as ABO blood types, tissue (Human Leukocyte Antigen; HLA) types, and the crossmatch. The ABO blood type is determined by the inherited antigenic substances on the surface of red blood cells. For example, if a patient's antigen is type A, then her blood type is type A and her antibody is type B; if a patient's antigen is type AB, then blood type is AB and she has no antibody. A patient with antibody of type X cannot receive a transplant from a donor with type X antigen. For example, if a person's blood type is A, then her antibody is type B, and thus, she cannot receive a transplant from any donor having a type B antigen, namely, blood types B and AB. Similarly, if a person's blood type is O, then her antibodies are types A and B; therefore, she cannot receive a transplant from any donor of blood types A, B, or AB. The tissue (HLA) type is determined by a patient's and a donor's HLAs, which are proteins on the surface of cells that are responsible for immunological responses. If the patient and the donor have the same HLAs, they are called an *identical match*, which is rare between unrelated persons because the number of possible combinations of HLAs is very large. For more information, see the Genetics Home Reference website, provided by the U.S. National Library of Medicine, at http://ghr.nlm.nih.gov/geneFamily/hla. A patient's antibodies and a donor's HLAs also determine the "crossmatch": If the crossmatch is positive, then the patient's antibodies react to the donor's HLAs, thereby making a transplant unsuccessful.

[2] According to the KONOS Annual Report in 2016, the five-year survival rate of ABO-incompatible living-

Table 1: Kidney transplantation in Korea.

Year	Patients in waitlists	Total transplants	Transplants from deceased donors	Transplants from living donors
2009	4,769	1,238	488	750
2010	5,857	1,287	491	796
2011	7,426	1,639	680	959
2012	9,245	1,788	768	1,020
2013	11,381	1,761	750	1,011
2014	14,477	1,808	808	1,000
2015	16,011	1,891	901	990
2016	18,912	2,236	1,059	1,177

Table 2: Three types of living-donor kidney transplants in Korea.

Year	Transplants from living donors	Direct transplants of ABOc pairs	Direct transplants of ABOi pairs	Exchange transplants
2009	750	675 (90.0%)	35 (4.7%)	40 (5.3%)
2010	796	689 (86.6%)	78 (9.8%)	29 (3.6%)
2011	959	828 (86.3%)	113 (11.8%)	18 (1.9%)
2012	1,020	827 (81.1%)	193 (18.9%)	0 (0.0%)
2013	1,011	795 (78.6%)	212 (21.0%)	4 (0.4%)
2014	1,000	783 (78.3%)	212 (21.2%)	5 (0.5%)
2015	990	772 (78.0%)	208 (21.0%)	10 (1.0%)
2016	1,177	901 (76.6%)	272 (23.1%)	4 (0.3%)

(ABOc means blood-type compatible; ABOi means blood-type incompatible.)

patible transplants or transplants with positive crossmatch is also reported quite satisfactory and is regarded as a good alternative (Kawai et al. (2008), Montogomery et al. (2011), Laging et al. (2014) for incompatible tissue-type transplants and Gloor et al. (2003), Thielke et al. (2009), and Jin et al. (2012) for transplants with positive crossmatch).

In recent years, the number of patients using suppressants has increased in many countries. In Korea, for example, the proportion of blood-type incompatible kidney transplants has increased from 4.7 percent to 23.1 percent of the total living-donor transplants during 2009-2016, as shown in Table 2.[3] In contrast, the proportion of transplants through kidney exchanges has decreased from 5.3 percent to nearly 0 percent during the same period. The proportion of com-

donor kidney transplants is 95.5 percent and that of ABO-compatible living-donor kidney transplants is 96.7 percent. Other papers show similar results: Takahashi et al. (2004), Tyden et al. (2007), Montgomery et al. (2012), and Kong et al. (2013).

[3]This sharp increase is partly because the NHIS of Korea has covered a large fraction of the total cost of suppressants since 2009. Patients pay a small share of the total cost, as low as 20 percent depending on their medical conditions. The cost of suppressants in Sweden, Germany, and Japan is also covered by the public health insurance to some extent.

patible transplants has also decreased by more than 10 percent. As can be seen, suppressants have largely replaced other types of living-donor transplants in Korea.

As the number of patients using suppressants increases, the expenditure of the National Health Insurance Service (NHIS) to subsidize these incompatible transplants also increases. Given the situation, it is natural to ask how suppressants are currently being used and if there is a better way to use suppressants to facilitate transplants.

In this paper, we propose to use suppressants as a part of kidney exchange program. For an illustration, consider an example in which compatibility is determined only by ABO blood type. A pair consists of a patient and a donor, X-Y, where the patient's blood type is X and the donor's type is Y. Suppose that there are three pairs: A-B, B-AB, and O-AB. Due to the immunological constraints, a patient of type A can receive a transplant only from a donor of type A or O, a patient of type B can receive a transplant only from a donor of type B or O, and a patient of type O can receive a transplant only from a donor of type O.

Because each patient is incompatible with her own donor, in the absence of suppressants, each of them should receive a transplant from someone else. In a kidney exchange program, on the other hand, the patients swap their donors to form compatible pairs. Unfortunately, in this example, no such exchanges are possible, because the patients in A-B and O-AB pairs are incompatible with all three donors.

Now suppose that two patients can use suppressants. One easy way to use suppressants is to choose any two patients as recipients and have them receive transplants directly from their own donors. In the example, for instance, the patients in B-AB and O-AB pairs can be provided with suppressants and receive direct transplants from their donors, in which case, the patient in the remaining A-B pair does not receive a transplant. Suppressants are currently used for such incompatible transplants *within* pairs in South Korea, as summarized in the fourth column of Table 2.

However, there is a better way to use suppressants, which enables all patients to receive transplants. Indeed, the pairs A-B and B-AB form a "chain" in a sense that the donor in the A-B pair is compatible with the patient in the B-AB pair, while the remaining patient of type A and the remaining donor of type AB are not compatible. Such a chain can be viewed as a trading cycle with a "missing link": If the donor in the B-AB pair and the patient in the A-B pair were compatible, these pairs would have formed a cycle along which they could swap donors and form compatible pairs. Providing a suppressant to the patient of type A fills in this missing link and transforms the chain into a trading cycle between A-B and B-AB. Then, the patient of type B receives a transplant from the donor of type B and the patient of type A receives a transplant from the donor of type AB, the latter transplant being made possible by the use of a suppressant. The remaining suppressant can now be provided to the patient in the O-AB pair so that she receives an incompatible transplant from her own donor.

A key feature of our proposal is that patient-donor pairs, who *become* compatible through

the use of suppressants, still participate in the kidney exchange pool. Note that in the example, when the patient in the A-B pair uses a suppressant, she can receive a transplant directly from her own donor, and so need not participate in the exchange program. Nevertheless, the participation of this pair can eventually benefit all participants: the patient in the B-AB pair now receives a compatible transplant and the patients in the A-B and O-AB pairs receive incompatible transplants. Provided that it does not make a significant difference from whom a patient receives a transplant when using a suppressant, the pair A-B should be willing to participate in the exchange program, especially given that the patient in this pair has to use a suppressant anyway to receive a transplant. Such an "altruistic" motive of compatible pairs has also been studied in the standard kidney exchange context (Sönmez and Ünver (2014), Roth et al. (2005), and Gentry et al. (2007)).[4]

We begin with kidney exchange model without suppressants as a benchmark. We define two Top-Trading Cycles rules (TTCs) associated with a priority ordering over patients.[5] Both rules are defined by means of an algorithm in which patient-donor pairs form trading cycles in each step. In our setting, however, there can be multiple overlapping cycles because preferences are coarse. We propose two different ways of choosing trading cycles among them.

We next extend the model by introducing suppressants. For a better understanding of efficient use of suppressants, we assume that at most K patients can use the suppressants. For each compatibility profile of patients and donors, we determine which patients are to receive the suppressants. We update the compatibility profile accordingly, as the recipients of suppressants become compatible with any other donors. Based on this new profile, we choose matchings between patients and donors.

As a minimal requirement, we first consider efficiency. *Pareto efficiency* is defined for matching as standard: For each compatibility profile, there should be no other matching that makes all patients weakly better off and at least one patient strictly better off for any given allocation of suppressants.

We also define efficiency for the assignment of suppressants, based on the idea that suppressants should be used so as to maximize the transplants. We introduce two variants of this idea depending on how we define "maximal" transplants. Consider two groups of potential

[4]The term "altruistic pairs" in Sönmez and Ünver (2014) refers to compatible pairs who participate in a kidney exchange even though a direct transplant between themselves is possible. Because participation is voluntary, our proposal further prevents a negative externality that tissue-type suppressants may have on kidney exchange programs. As Sönmez and Ünver (2013) have observed, when tissue-type suppressants become available, the shortage of donors of a particular blood type – usually, blood type O – can get even worse. This is because type O donors are blood-type compatible with all patients and therefore appear in the exchange pool only when they are tissue-type incompatible with their own patients. Therefore, as tissue-type suppressants become available, these donors can be crowded out from the exchange program. In our proposal, on the other hand, all donors in incompatible pairs stay in the pool even after their patients use any types of suppressants.

[5]Shapley and Scarf (1974) propose TTC for a general model with indivisible goods. Roth et al. (2004) develop TTC for the kidney exchange problem by considering chains formed with patients on the waitlist. As in Roth et al. (2004), we impose no constraint on the size of cycles in kidney exchanges. For more discussion on the size of exchanges, see Roth et al. (2007) and Saidman et al. (2006).

recipients. Identify the sets of patients receiving transplants when each group is provided with suppressants. *Maximal improvement* requires that if the first group enables more transplants than the other in terms of *set inclusion*, then the latter group should not be chosen as recipients of suppressants. *Cardinally maximal improvement* modifies this requirement by comparing the *total number of transplants* that they facilitate, rather than comparing them in terms of set inclusion.

In addition to efficiency, we consider a requirement that no patient be made worse off by the availability of suppressants. We regard this requirement as fairness, as it says that no participant has to be penalized when this new technology is introduced to the transplant system. If any patient gets worse off than before, while some others benefit, she would find it unfair. This is especially so because no patient is responsible for the development of transplant technologies. This idea, often referred to as "solidarity" or "monotonicity", has been studied extensively in various resource allocation problems.[6] We call this requirement "monotonicity" and introduce its two variants. *Strong monotonicity* requires that all patients be weakly better off after suppressants are introduced, *no matter who receives them*. In contrast, *monotonicity* requires that there be *some* way of allocating suppressants so that all patients become weakly better off.

We check the compatibility of these requirements. Our first result is that Pareto efficiency and strong monotonicity are incompatible. The second result is that even if strong monotonicity is weakened to monotonicity, we cannot satisfy it together with Pareto efficiency and cardinally maximal improvement.

In view of these impossibilities, we weaken cardinally maximal improvement to maximal improvement and ask if this weaker requirement is compatible with Pareto efficiency and monotonicity. To show that it is, we introduce two solutions by modifying TTCs defined earlier. Each solution operates in four steps. First, apply TTC to the initial compatibility profile, assuming that no one uses suppressant. Identify the set of patients who receive no transplant. Second, modify the initial priority ordering so that the patients identified in the previous step have lower priorities than the other patients. Third, choose the recipients of suppressants. This is done by selecting cycles and chains according to the modified priority ordering. The patients at the head of these chains are provided with suppressants. Finally, update the compatibility profile and apply TTC associated with the modified priority ordering to this profile.

As this solution satisfies monotonicity, any patient who could initially receive a compatible transplant in the exchange pool will receive a transplant in the presence of suppressants – either compatible or incompatible transplant. We refine monotonicity to guarantee that such patient

[6]For a detailed survey on solidarity or monotonicity requirements, see Thomson (2013). The underlying idea of these requirements is that all agents' welfare has to be affected in the same direction – either all worse off or all better off – when there is an exogenous change in the economy. The exogenous changes can be variable populations ("population monotonicity"), a change in the available resource ("resource monotonicity"), an agent's preference change ("welfare domination under preference replacement"), or the introduction of a new technology that we consider in this paper.

receives a *compatible* transplant afterward. We show that a subsolution of the aforementioned TTC solution satisfies this refined requirement.

The literature on kidney exchange stems from the seminal work by Roth et al. (2004) and most papers have taken the compatibility profile as a fixed primitive of the problem. More recently, several possible changes in compatibility profile are taken into considerations in accordance with technological advances. A recent paper by Andersson and Kratz (2016) deepens our understanding of suppressants from a different perspective, complementing our analysis. When suppressants are used to relax blood-type incompatibility, but not tissue-type incompatibility, they provide a way to minimize the use of suppressants, while maximizing the total number of transplants. Sönmez et al. (2016) is also closely related to ours, as they consider a "blood subtyping" technology that enables transplants between certain incompatible blood-types. This technology is different from suppressants, however, in that it is used for *all* patients to identify more detailed biological characteristics once adopted. They analyze the welfare impact of this technology on kidney exchanges by calculating a possible negative externality.

The rest of this paper is organized as follows. Section 2 introduces the standard kidney exchange model without suppressants and defines two versions of TTC. Section 3 extends the model by introducing suppressants and establishes our two impossibility results. Section 4 defines two modifications of TTC and studies their properties. Section 5 presents a refinement of monotonicity and the related results. Section 6 investigates the implications of pairwise exchanges for this problem. Section 7 contains a few concluding remarks.

2. Kidney Exchange Model without Immunosuppressants

A finite set N is a pool of patient-donor pairs. Let n be the number of pairs in N. Each pair i consists of patient i and donor i. A patient is either compatible or incompatible with a donor depending on immunological characteristics. Patient i's preference R_i is dichotomous over N: She prefers pairs whose donors are compatible with her to the other remaining pairs; all pairs with compatible donors are equally desirable and so are all pairs with incompatible donors. For simplicity, we specify pairs with compatible donors in each patient's preference list. A kidney exchange problem, or simply a **problem**, is defined as a preference profile $R = (R_i)_{i \in N}$.[7]

Each pair is given a certain priority according to a linear ordering over N.[8] We denote this linear ordering by $\succ$ and write $i \succ j$ if and only if patient i has a higher priority than patient j.

We introduce a graph whose nodes are the pairs in N. A graph is a collection of directed arcs between the nodes. If patient i is compatible with donor j, we draw a directed arc $j \to i$

[7] Bogomolnaia and Moulin (2004) study dichotomous preferences in a general matching context. They examine randomized matchings to achieve efficiency, fairness, and strategic requirements when only two-way exchanges are allowed.

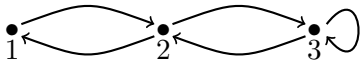
[8] For a detailed discussion on these priorities, see Roth et al. (2005).

to represent a possible transplant from donor j to patient i. We allow self-directed arcs for patients who are compatible with their own donors. Each problem is then represented as a graph.

A list of distinct pairs $i_1, \ldots, i_k$ forms a **cycle** if $i_1 \to i_2 \to \cdots \to i_k \to i_1$. Note that the smallest possible cycle is a self-cycle $i \to i$. Similarly, a list of distinct pairs $i_1, \ldots, i_k$ forms a **chain** if $i_1 \to i_2 \to \cdots \to i_k$ and patient i_1 is incompatible with donor i_k, i.e., $\neg(i_k \to i_1)$. Given a chain $i_1 \to i_2 \to \cdots \to i_k$, patient i_1 is called the **head** of this chain. We say that a collection of cycles are **jointly feasible** if no pair appears in more than one cycle. Similarly, a collection of cycles and chains are **jointly feasible** if no pair appears in more than one chain or cycle. For each R, let $\mathcal{C}(R)$ be the collection of all sets of jointly feasible cycles and let $\mathcal{H}(R)$ be the collection of all sets of jointly feasible cycles and chains.

Example 1. Consider the following problem with three pairs:

R_1	R_2	R_3
2	1, 3	2, 3



Let $R = (R_1, R_2, R_3)$. Each patient is compatible with the donors listed in her preference list. There are three cycles, $1 \to 2 \to 1$, $2 \to 3 \to 2$, and $3 \to 3$. The first and the third cycles are jointly feasible, so the set consisting of these cycles is in $\mathcal{C}(R)$. There are four chains, 1, 2, $1 \to 2 \to 3$, and $3 \to 2 \to 1$. The chain composed of pair 1 and the cycle composed of pairs 2 and 3 are jointly feasible, so the set consisting of this chain and this cycle is in $\mathcal{H}(R)$.

A matching specifies which patient receives a transplant from which donor. Each patient can only be matched to a compatible donor. Otherwise, she remains unmatched, receiving no transplant. Let $\mathcal{M}(R)$ be the set of matchings for each R. A patient (weakly) prefers a matching to another if and only if she (weakly) prefers the donor matched at the former to the donor matched at the latter.

A matching rule selects a set of matchings for each problem. Let φ be a generic matching rule. A matching rule is *essentially single-valued* if all matchings chosen for each problem are equally desirable for all patients. In other words, patients receiving transplants are the same across all matchings chosen for each problem.

A matching is *Pareto efficient* at R if there is no other matching in $\mathcal{M}(R)$ that is weakly preferred by all patients and is strictly preferred by at least one patient. A matching rule is *Pareto efficient* if it selects *Pareto efficient* matchings for each problem. We now define Top-Trading Cycles (TTC) rules adapted to this model.[9]

[9]Roth et al. (2005) define this rule for pairwise exchanges in the standard kidney exchange problem. For other related studies on TTC rules with weak preferences, see Jaramillo and Manjunath (2012), Alcalde-Unzu and Molis (2011), and Saban and Sethuraman (2013).

Top-trading cycles rule associated with $\succ$ (simply, $TTC_{\succ}$):

Let $\mathcal{C}_0$ be the collection of all sets of jointly feasible cycles. If there is none, all patients remain unmatched. Otherwise, proceed to the following step.

Step t (≥ 1): In $\mathcal{C}_{t-1}$, identify all sets of jointly feasible cycles including the patient with the t-th highest priority at $\succ$. If there is such a set, let $\mathcal{C}_t$ be the collection of all these sets. Otherwise, let $\mathcal{C}_t \equiv \mathcal{C}_{t-1}$.

This process terminates at Step n. For each set in $\mathcal{C}_n$, all patients in the cycles of this set are matched to the donors along the directed arcs. All other patients remain unmatched.

Note that this rule can be viewed as a "sequential priority" rule, since we first identify all matchings at which the patient with the highest priority receives a transplant, then all matchings at which the patient with the second highest priority does, and so on.[10] If exchanges can only be made between two pairs, such a sequential priority rule is *Pareto efficient* and it also maximizes the number of transplants. If we allow more than two-way exchanges as in this paper, however, *Pareto efficient* matchings do not necessarily maximize the number of transplants. Given this observation, we formulate another version of the TTC rule counting the number of transplants.

Top-trading cycles rule maximizing the number of transplants (simply, $\overline{TTC}_{\succ}$):

Let $\bar{\mathcal{C}}_0$ be the collection of all sets of jointly feasible cycles maximizing the number of transplants. If there is none, all patients remain unmatched. Otherwise, proceed to the steps described above when defining $TTC_{\succ}$.

Proposition 1. For each priority ordering $\succ$, $TTC_{\succ}$ and $\overline{TTC}_{\succ}$ are essentially single-valued and Pareto efficient.

Proof. It is straightforward from the definitions that $TTC_{\succ}$ and $\overline{TTC}_{\succ}$ are *essentially single-valued.* Suppose, by contradiction, that $TTC_{\succ}$ is not *Pareto efficient.* Then, there are a problem and a matching chosen by $TTC_{\succ}$ for this problem, from which further Pareto improvement can be made. That is, there is a matching at which all patients who receive transplants at $TTC_{\succ}$ also receive transplants and at least one patient newly receives a transplant. Among all patients newly receiving transplants, choose the one with the highest priority under $\succ$. From the definition, $TTC_{\succ}$ should have chosen cycles at which this patient receives a transplant, a contradiction.

Suppose, by contradiction, that $\overline{TTC}_{\succ}$ is not *Pareto efficient.* By the same argument, there is a matching at which all patients who receive transplants at $\overline{TTC}_{\succ}$ also receive transplants

[10] Other versions of sequential priority rules in kidney exchanges, see Roth et al. (2005) and Nicolò and Rodríguez-Álvarez (2017).

and at least one patient newly receives a transplant. From the definition, $\overline{TTC}_{\succ}$ should have chosen matchings maximizing the number of transplants, a contradiction. □

3. Kidney Exchange Model with Immunosuppressants

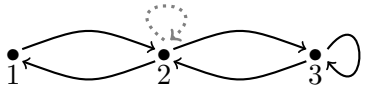
We introduce suppressants to the standard kidney exchange model. If patient i receives a suppressant, she becomes compatible with all donors, including her own. We denote by $\overline{R}_i$ the preference of patient i after receiving a suppressant. For each $S \subseteq N$, let $\overline{R}_S \equiv (\overline{R}_i)_{i \in S}$. For each problem R and each $S \subseteq N$, let $R(S) \equiv (\overline{R}_S, R_{-S})$ denote the preference profile derived from R when patients in S receive suppressants. All definitions in the previous section carry over to this setting by replacing R with $R(S)$.

Example 2. (Example 1 continued) The problem in Example 1 is given as follows:

R_1	R_2	R_3
2	1, 3	2, 3

Now suppose that patient 2 uses a suppressant. Then, the problem changes into:

R_1	$\overline{R}_2$	R_3
2	1, 2, 3	2, 3



Now, there is one additional cycle, $2 \to 2$, while a chain composed of pair 2 disappears. All other three cycles and three chains remain the same.

For a better understanding of efficient use of suppressants, we introduce K as an upper bound of the number of patients receiving suppressants. If $K = 0$, this model coincides with the standard kidney exchange model.

A **solution** is now defined as a pair of a **recipient choice rule**, which selects at most K recipients of suppressants for each problem, and a **matching rule**, which selects matchings after updating these recipients' preferences. Let (σ, φ) be a generic solution where σ is a recipient choice rule and φ is a matching rule. For each problem R, patients in $\sigma(R)$ are provided with suppressants and the problem changes to $R(\sigma(R))$. The resulting matchings are given as $\varphi(R(\sigma(R))$. Let $\varphi^{\sigma}(R) \equiv \varphi(R(\sigma(R))$ be the set of matchings chosen by the solution.

As before, a matching rule φ is *Pareto efficient* if it selects *Pareto efficient* matchings for each problem. Note that *Pareto efficiency* says nothing about how we assign suppressants. It simply requires no further improvement from the matchings chosen by a matching rule for each given set of recipients. A solution (σ, φ) is *Pareto efficient* if for each problem R, φ selects *Pareto efficient* matchings for $R(\sigma(R))$.

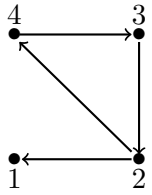
Our next requirement says that, no matter which recipient choice rule we have, a matching rule should make all patients weakly better off after suppressants are introduced.[11]

> **Strong monotonicity**: For each non-negative integer K, each problem R, and each recipient choice rule σ, each matching in $\varphi^{\sigma}(R)$ is weakly preferred to each matching in $\varphi(R)$ by all patients.

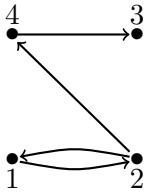
This requirement is convincing especially when we cannot choose a particular recipient set, for example, when patients individually decide whether they use suppressants or not. This is in fact the current practice in South Korea: any incompatible pairs receive direct transplants *from their own donors* by using suppressants. This practice satisfies *strong monotonicity*, whereas $TTC_{\succ}$ and $\overline{TTC}_{\succ}$ do not.

Example 3. ($TTC_{\succ}$ and $\overline{TTC}_{\succ}$ violate *strong monotonicity*.) Consider two problems with four pairs defined as follows:

R_1	R_2	R_3	R_4
2	3	4	2



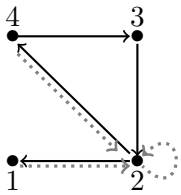
R'_1	R'_2	R'_3	R'_4
2	1	4	2

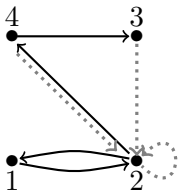


Let $1 \succ 2 \succ 3 \succ 4$ for both problems. Since there is only one cycle for each problem, $TTC_{\succ}(R)$ and $\overline{TTC}_{\succ}(R')$ match the patients and the donors as follows:

$$TTC_{\succ}(R): \begin{bmatrix} \text{patient 1 – unmatched} \\ \text{patient 2 – donor 3} \\ \text{patient 3 – donor 4} \\ \text{patient 4 – donor 2} \end{bmatrix} \qquad \overline{TTC}_{\succ}(R'): \begin{bmatrix} \text{patient 1 – donor 2} \\ \text{patient 2 – donor 1} \\ \text{patient 3 – unmatched} \\ \text{patient 4 – unmatched} \end{bmatrix}$$

Now suppose that $K = 1$ and consider a choice rule choosing patient 2 for both problems. As patient 2's preference changes to $\overline{R}_2$, the graphs change into:





[11] The two monotonicity requirements can be compared with "welfare-dominance under preference replacement" in allocation problems. For a complete survey on this requirement, see Thomson (1999). It requires when a person changes her preferences, all the remaining people be affected in the same direction. Since the use of suppressants by a group of patients change their preferences, our requirements share the same spirit as welfare-domination under preference replacement, but there is no direct logical relation between them. In our requirements, first, at most K patients may change their preference at the same time. Second, when patient i's preference changes, it always changes to $\overline{R}_i$. Third, when preferences change, all patients, including those who use suppressants, should be affected in the same direction. Lastly, all patients should be affected in a particular direction – they are made weakly better off.

For these new profiles $R(\{2\})$ and $R'(\{2\})$, $TTC_{\succ}$ and $\overline{TTC}_{\succ}$ choose the following matchings:

$$TTC_{\succ}(R(\{2\})) : \begin{bmatrix} \text{patient 1 – donor 2} \\ \text{patient 2 – donor 1} \\ \text{patient 3 – unmatched} \\ \text{patient 4 – unmatched} \end{bmatrix} \quad \overline{TTC}_{\succ}(R'(\{2\})) : \begin{bmatrix} \text{patient 1 – unmatched} \\ \text{patient 2 – donor 3} \\ \text{patient 3 – donor 4} \\ \text{patient 4 – donor 2} \end{bmatrix}$$

Under $TTC_{\succ}$, patients 3 and 4 are made worse off. Under $\overline{TTC}_{\succ}$, patient 1 is made worse off.

This observation generalizes to our first impossibility result.

Proposition 2. No matching rule jointly satisfies Pareto efficiency and strong monotonicity.[12]

Proof. The proof is by means of an example with three pairs. Consider the following preferences of three patients:

R_1	R_2	R'_2	R_3
2	1	3	2

Let $R \equiv (R_1, R_2, R_3)$ and $R' \equiv (R_1, R'_2, R_3)$. Let φ be a *Pareto efficient* matching rule. Suppose first that $K = 0$. By *Pareto efficiency*:

$$\varphi(R) : \begin{bmatrix} \text{patient 1 – donor 2} \\ \text{patient 2 – donor 1} \\ \text{patient 3 – unmatched} \end{bmatrix} \qquad \varphi(R') : \begin{bmatrix} \text{patient 1 – unmatched} \\ \text{patient 2 – donor 3} \\ \text{patient 3 – donor 2} \end{bmatrix}$$

Now suppose that $K = 1$. Consider a recipient choice rule choosing patient 2 for both problems. Then, both problems change to $(R_1, \overline{R}_2, R_3)$. To make everyone weakly better off at this new problem than at R, we should have $\varphi(R)$ for the new problem. Similarly, to make everyone weakly better off at the new problem than at R', we should have $\varphi(R')$ for the new problem. Altogether, both $\varphi(R)$ and $\varphi(R')$ should be chosen for the new problem. However, patient 1 is worse off at $\varphi(R')$ than at $\varphi(R)$ and patient 3 is worse off at $\varphi(R)$ than at $\varphi(R')$, a contradiction to *strong monotonicity*. □

Proposition 2 implies that the current practice in South Korea necessarily violates *Pareto efficiency* as it satisfies *strong monotonicity*.

Before we proceed, let us take a closer look at Example 3 above. At R, suppose that patient 1, instead of patient 2, is provided with a suppressant. Then, patient 1 newly forms a self-cycle, while the existing cycle $2 \to 4 \to 3 \to 2$ still remains the same. Similarly, at $\hat{R}$, suppose that patient 4, instead of patient 2, is provided with a suppressant. Then, patients 3

[12] Since a matching rule φ may not be single-valued, there is another way to define *strong monotonicity*: For each non-negative integer K, each problem R, and each recipient choice rule σ, a matching in $\varphi^{\sigma}(R)$ is weakly preferred to a matching in $\varphi(R)$ by all patients. This alternative definition is weaker than *strong monotonicity* and incompatible with *Pareto efficiency* and *essentially single-valuedness*.

and 4 form a new cycle $3 \to 4 \to 3$, while the existing cycle $1 \to 2 \to 1$ still remains the same. Summarizing, in this example, there is a way to choose a recipient to make all patients weakly better off than before. Given this observation, we now ask if it is possible to make all patients weakly better off with a particular recipient choice rule. Consider a solution (σ, φ).

> **Monotonicity**: For each non-negative integer K and each problem R, each matching in $\varphi^{\sigma}(R)$ is weakly preferred to each matching in $\varphi(R)$ by all patients.

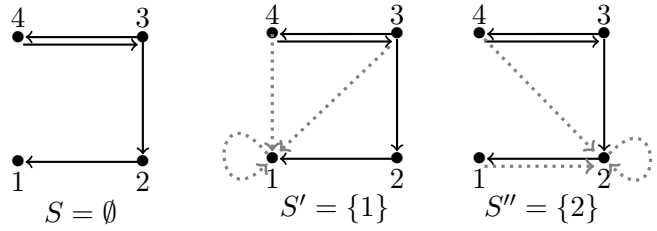
We note that *monotonicity* is trivially satisfied by many solutions. For example, consider a solution (σ, φ) with an *essentially single-valued* matching rule φ and a recipient choice rule σ that never chooses any patient. Then, $\varphi = \varphi^{\sigma}$ and the solution (σ, φ) satisfies *monotonicity*. Obviously, this is not an effective way of using suppressants, so it is reasonable to require that such cases be prevented.

We thereby formulate another efficiency requirement pertaining to the assignment of suppressants. It says that the recipients of suppressants should be chosen so that the set of patients receiving transplants is maximal in terms of set inclusion.

> **Maximal Improvement**: For each non-negative integer and each problem, consider two potential groups of recipients. If the first group results in a set of transplants that properly includes a set of transplants that the second group results in, the second group should not be chosen by σ.

Example 4. (Illustration of Maximal Improvement) Consider the following problem with four pairs and $K = 1$. Consider three potential sets of recipients, S, S', and S'':

R_1	R_2	R_3	R_4
2	3	4	3



Consider a solution (σ, φ) satisfying *maximal improvement* where φ is a matching rule choosing all *Pareto efficient* matchings. For each set of recipients, the matching rule chooses the following matching:

$$\varphi(R(\{1\})) : \begin{bmatrix} \text{patient 1 – donor 1} \\ \text{patient 2 – unmatched} \\ \text{patient 3 – donor 4} \\ \text{patient 4 – donor 3} \end{bmatrix} \qquad \varphi(R(\{2\})) : \begin{bmatrix} \text{patient 1 – donor 2} \\ \text{patient 2 – donor 1} \\ \text{patient 3 – donor 4} \\ \text{patient 4 – donor 3} \end{bmatrix}$$

Also, $\varphi(R)$ is the same as $\varphi(R(\{1\}))$ except that patient 1 remains unmatched. When no patient uses a suppressant, patients 3 and 4 receive transplants. When patient 1 is provided

with a suppressant, patients 1, 3, and 4 receive transplants. When patient 2 is provided with a suppressant, all patients receive transplants. Therefore, the recipient choice rule σ should not choose $\emptyset$ and $\{1\}$ for this problem.

As discussed in Section 2, we can also consider the number of transplants in formulating this requirement. Then, the aforementioned requirement can be defined as follows.

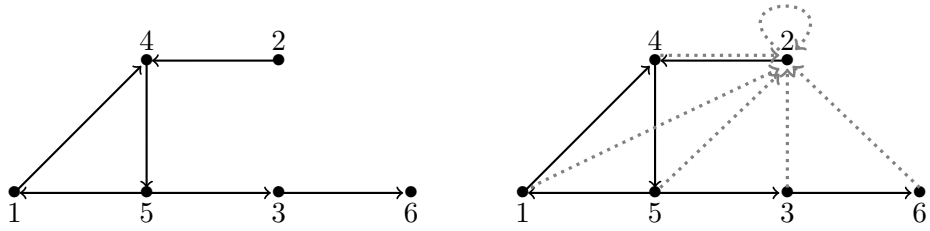
> **Cardinally Maximal Improvement**: For each non-negative integer and each problem, consider two potential groups of recipients. If the first group results in a greater number of transplants than the second group, the second group should not be chosen by σ.

From the definition, *cardinally maximal improvement* implies *maximal improvement*. Unfortunately, we have the second impossibility result if we impose *cardinally maximal improvement*.

Proposition 3. No solution jointly satisfies Pareto efficiency, monotonicity, and cardinally maximal improvement.

Proof. The proof is by means of an example with six pairs. Consider the following problem and consider a solution (σ, φ) satisfying the three requirements:

R_1	R_2	R_3	R_4	R_5	R_6
5	$\emptyset$	5	$1, 2$	4	3



Suppose first that $K = 0$. By *Pareto efficiency*, pairs 1, 4, and 5 form an exchange cycle and these patients are matched with the donors along the directed arcs. Next, suppose that $K = 1$. To satisfy *cardinally maximal improvement*, patient 2 has to use a suppressant and all pairs except for pair 1 form a cycle. The resulting matchings are:

$$\varphi(R): \begin{bmatrix} \text{patient 1 – donor 5} \\ \text{patient 2 – unmatched} \\ \text{patient 3 – unmatched} \\ \text{patient 4 – donor 1} \\ \text{patient 5 – donor 4} \\ \text{patient 6 – unmatched} \end{bmatrix} \quad \varphi(R(\{2\})): \begin{bmatrix} \text{patient 1 – unmatched} \\ \text{patient 2 – donor 6} \\ \text{patient 3 – donor 5} \\ \text{patient 4 – donor 2} \\ \text{patient 5 – donor 4} \\ \text{patient 6 – donor 3} \end{bmatrix}$$

Since patient 1 is made worse off, *monotonicity* is violated. □

In view of impossibility results in Propositions 2 and 3, we weaken *cardinally maximal improvement* to *maximal improvement* and ask whether it is compatible with *Pareto efficiency* and *monotonicity*.

4. Top-Trading Cycles Solutions with Immunosuppressants

We propose two solutions based on the TTC rules defined in Section 2. We first determine the recipients of suppressants and then choose matchings between patients and donors. Consider a problem.

Extended top-trading cycles solution (simply, $eTTC_{\succ}$):

Step 1. Apply $TTC_{\succ}$ to the problem and identify the set of pairs whose patients do not receive transplants at a resulting matching. Denote this set by $\bar{N}$ and let $N \setminus \bar{N}$ be the set of pairs whose patients receive transplants.

Step 2. Let $\succ^*$ be the priority ordering induced from $\succ$ such that all pairs in $\bar{N}$ have lower priorities than those in $N \setminus \bar{N}$, while the relative priorities within $\bar{N}$ and $N \setminus \bar{N}$, respectively, remain the same as in $\succ$.

Step 3. Let $\mathcal{H}_0$ be the collection of all sets of jointly feasible cycles and at most K chains.[13] Proceed to the following steps.

Substep t (≥ 1): In $\mathcal{H}_{t-1}$, identify all sets of jointly feasible cycles and at most K chains including the patient with the t-th highest priority at $\succ^*$. If there is such a set, let $\mathcal{H}_t$ be the collection of all these sets. Otherwise, let $\mathcal{H}_t \equiv \mathcal{H}_{t-1}$.

This process terminates at Substep n. Choose a set in $\mathcal{H}_n$ and choose the patients at the head of the chains in this set to be the recipients of suppressants.

Step 4. Update these recipients' preferences and apply $TTC_{\succ^*}$ to the new preference profile.

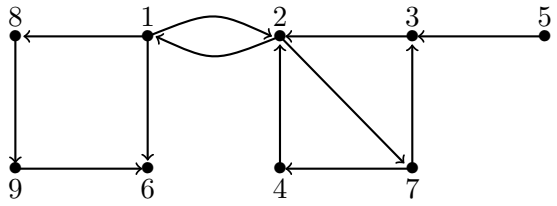
In Step 1, we apply $TTC_{\succ}$ to the initial preference profile and find the set of patients $\bar{N}$ who do not receive transplants at the step. Note that this is not a final allocation but a temporary allocation used in deriving $\succ^*$. Since $TTC_{\succ}$ is *essentially single-valued*, the sets of patients receiving transplants remain the same across all matchings selected by $TTC_{\succ}$. In Step 2, we modify the initial priority ordering $\succ$ to $\succ^*$. As long as the pairs in $N \setminus \bar{N}$ have higher priorities than those in $\bar{N}$, the cycles or chains including these patients will be selected when applying $TTC_{\succ^*}$ later. This guarantees that these patients receive transplants even after the suppressants are used. In Step 3, we determine the recipients of suppressants. This is

[13] Such $\mathcal{H}_0$ is always non-empty. This is because (i) patients outside $\bar{N}$ form cycles among themselves and (ii) patients in $\bar{N}$ do not form cycles, but only chains, among themselves. (If there were any cycle among patients in $\bar{N}$, such a cycle should have been chosen in Step 1 of the algorithm, making these patients not belong to $\bar{N}$.) Therefore, we can choose the cycles from (i) and at most K chains from (ii).

done by selecting cycles and at most K chains according to the modified priority ordering and assigning suppressants to the patients at the head of these chains. In the last step, we update the preference profile and apply TTC associated with the modified priority ordering. It is straightforward from the definition that $eTTC_{\succ}$ is *essentially single-valued*. Therefore, the sets of patients receiving transplants remain the same across all matchings selected by $eTTC_{\succ}$ for each problem.

Example 5. ($eTTC_{\succ}$) Consider the following problem with nine pairs:

R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8	R_9
2	$1,3,4$	$5,7$	7	$\emptyset$	$1,9$	2	1	8



Suppose that $1 \succ 2 \succ \cdots \succ 9$. There are three cycles $1 \to 2 \to 1$, $2 \to 7 \to 4 \to 2$, and $2 \to 7 \to 3 \to 2$. Among these, $TTC_{\succ}$ chooses the first cycle including the pair with the highest priority, which is pair 1. The resulting matching is:

$$\begin{bmatrix} \text{patient 1 – donor 2} \\ \text{patient 2 – donor 1} \\ \text{other patients – unmatched} \end{bmatrix}$$

So, $\bar{N} = \{3, 4, \ldots, 9\}$ and we modify $\succ$ into $\succ^*$, which happens to coincide with $\succ$.

(1) Suppose that at most one patient can receive a suppressant ($K = 1$). We identify the collection of all sets of jointly feasible cycles and a chain, including pair 1, and then among them, we identify the sets including pair 2. Among them, again, we identify the sets including pair 3: $\{5 \to 3 \to 2 \to 1 \to 8 \to 9 \to 6\}$ and $\{7 \to 3 \to 2 \to 1 \to 8 \to 9 \to 6\}$. Among these, we next identify the sets including pair 4. Since there is none, we move on to pair 5 and choose the first set. We choose the patient at the head of this chain, patient 5, to be a recipient of suppressant. We then update patient 5's preference and apply $TTC_{\succ^*}$ to this new profile. A resulting matching is determined along a cycle, $5 \to 3 \to 2 \to 1 \to 8 \to 9 \to 6 \to 5$. All patients in the cycle are matched to the donors along the directed arcs. All other patients remain unmatched.

(2) Suppose instead that at most two patients can receive suppressants ($K = 2$). As above, we identify the set of cycles and at most two chains according to the modified priority ordering $\succ^*$. We end up with the following two sets: $\{5 \to 3 \to 2 \to 1 \to 8 \to 9 \to 6,\ \ 7 \to 4\}$ and $\{7 \to 4 \to 2 \to 1 \to 8 \to 9 \to 6,\ \ 5 \to 3\}$. Suppose that we choose the first set. Then,

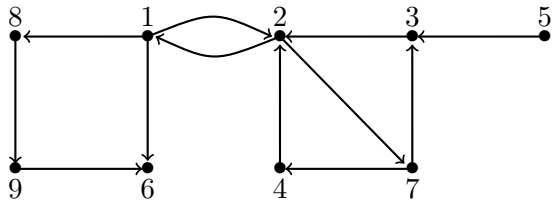
patients 5 and 7 are the heads of the chains in this set and we choose them to be recipients of suppressants. We then update these patients' preferences and apply $TTC_{\succ *}$ to this new profile. The resulting matching is determined along the two cycles, $5 \to 3 \to 2 \to 1 \to 8 \to 9 \to 6 \to 5$ and $7 \to 4 \to 7$. Suppose instead that we choose the second set. Then, patients 5 and 7 are the heads of the chains in this set and we choose them to be recipients of suppressants. The resulting matching is determined along the two cycles, $7 \to 4 \to 2 \to 1 \to 8 \to 9 \to 6 \to 7$ and $5 \to 3 \to 5$. For both sets, all patients receive transplants, confirming that $eTTC_{\succ}$ is *essentially single-valued*.

We can also define a version of the extended top-trading cycles solution by adapting $\overline{TTC}_{\succ}$.

> **Extended top-trading cycles solution maximizing the number of transplants** (simply, $e\overline{TTC}_{\succ}$):
>
> **Step 1.** Apply $\overline{TTC}_{\succ}$ to the problem and identify the set of pairs whose patients do not receive transplants at a resulting matching. Denote this set by $\bar{N}$ and let $N \setminus \bar{N}$ be the set of pairs whose patients receive transplants.
>
> **Steps 2 to 4.** All these steps are exactly the same as above when defining $eTTC_{\succ}$.

Again, it is straightforward from the definition that $e\overline{TTC}_{\succ}$ is *essentially single-valued*.

Example 6. ($e\overline{TTC}_{\succ}$) Consider the problem in Example 5:



Among the three cycles, $\overline{TTC}_{\succ}$ chooses the cycle maximizing the number of transplants; $2 \to 7 \to 4 \to 2$ and $2 \to 7 \to 3 \to 2$. Since there are more than one, the second cycle is chosen since it includes the patient with a higher priority at $\succ$, which is patient 3. The resulting matching is:

$$\begin{bmatrix} \text{patient 2 – donor 3} \\ \text{patient 3 – donor 7} \\ \text{patient 7 – donor 2} \\ \text{other patients – unmatched} \end{bmatrix}$$

So, $\bar{N} = \{1, 4, 5, 6, 8, 9\}$ and we modify $\succ$ into $\succ^*$ such that $2 \succ^* 3 \succ^* 7 \succ^* 1 \succ^* 4 \succ^* 5 \succ^* 6 \succ^* 8 \succ^* 9$.

(1) Suppose that at most one patient can receive a suppressant ($K = 1$). We identify the collection of all sets of jointly feasible cycles and a chain, including pair 2, and then among

them, identify the sets including pair 3. Among them, again, we identify the sets including pair 7, and so on. Then, we end up with the following two sets: $\{1 \to 8 \to 9 \to 6,\ 2 \to 7 \to 3 \to 2\}$ and $\{7 \to 3 \to 2 \to 1 \to 8 \to 9 \to 6\}$. Suppose that we choose the first set. Patient 1 is at the head of a chain in this set, so we choose her to be a recipient of suppressant. We then update this patient's preference and apply $TTC_{\succ^*}$ to the new profile. A resulting matching is determined along two cycles, $1 \to 8 \to 9 \to 6 \to 1$ and $2 \to 7 \to 3 \to 2$. All patients in the cycles are matched to the donors along the directed arcs. All other patients remain unmatched. (2) Suppose that at most two patients can receive suppressants ($K = 2$). As above, we identify the collection of sets of jointly feasible cycles and at most two chains according to the modified priority ordering $\succ^*$. We end up with the following two sets: $\{5 \to 3 \to 2 \to 1 \to 8 \to 9 \to 6,\ 7 \to 4\}$ and $\{7 \to 4 \to 2 \to 1 \to 8 \to 9 \to 6,\ 5 \to 3\}$. Depending on which set we choose, we obtain two resulting matchings, as shown in (2) of Example 5.

We now show that these two solutions satisfy the aforementioned requirements.

Theorem 1. $eTTC_{\succ}$ and $e\overline{TTC}_{\succ}$ satisfy Pareto efficiency, monotonicity, and maximal improvement.

Proof. (Pareto efficiency) For each problem, $eTTC_{\succ}$ chooses the recipients of suppressants and update the preference profile, to which $TTC_{\succ^*}$ is applied. By Proposition 1, the resulting matchings are *Pareto efficient* at the updated preference profile. The same argument applies to show that $e\overline{TTC}_{\succ}$ is *Pareto efficient*.

(Monotonicity) Note that at each matching, no patient who receives a transplant can be made strictly better off and no patient who does not receive a transplant can be made strictly worse off. Therefore, for each problem R, it is enough to show that all patients who received transplants at $TTC_{\succ}(R)$ still receive transplants at $eTTC_{\succ}(R)$. Suppose, by contradiction, that there is a patient who receives transplant at $TTC_{\succ}(R)$, but not at $eTTC_{\succ}(R)$. If there is more than one such patient, then choose the one with the highest priority at $\succ^*$. Call this patient i^*.

Since patient i^* received a transplant at $TTC_{\succ}(R)$, she is in one of the jointly feasible cycles that $TTC_{\succ}$ chooses. When suppressants are assigned to the recipients, all these jointly feasible cycles still remain jointly feasible at the new compatibility profile. Note also that all patients receiving transplants at $TTC_{\succ}(R)$ have higher priorities than all the remaining patients at $\succ^*$. Therefore, any patient whose priority is higher than i^* at $\succ^*$ must have received a transplant at $TTC_{\succ}(R)$.

Case 1. If all patients receiving transplants at $eTTC_{\succ}(R)$ have higher priorities than patient i^* at $\succ^*$, then these patients are the ones who received transplants at $TTC_{\succ}(R)$. However, this contradicts Step 3 of the $eTTC_{\succ}$ algorithm: a set of cycles and chains should have been chosen so that patient i^* is also included together with these patients. There does exist such a set: for example, the set of jointly feasible cycles that $TTC_{\succ}(R)$ chooses.

Case 2. If there is a patient receiving a transplant at $eTTC_{\succ}(R)$ who has a lower priority than patient i^* at $\succ^*$, then this again contradicts Step 3 of the $eTTC_{\succ}$: a set of cycles and chains should have been chosen so that patient i^* is included ahead of the patient with a lower priority. There does exist such a set: for example, the set of jointly feasible cycles that $TTC_{\succ}(R)$ chooses.

Altogether, there should be no such patient i^*, completing the proof. The same argument applies to show that $e\overline{TTC}_{\succ}$ is *monotonic*.

(Maximal improvement) Suppose, by contradiction, that for some problem R, there is a set S of patients whose use of suppressants results in a matching with more transplants than a matching in $eTTC_{\succ}(R)$ in terms of set inclusion. Let H be a set of cycles and chains that results in this matching at $R(S)$. Then, all patients who receive transplants at $eTTC_{\succ}(R)$ also receive transplants when S is chosen. There should also be a patient who receives a transplant when S is chosen, but not at $eTTC_{\succ}(R)$. If there is more than one such patient, then choose the one with the highest priority at $\succ^*$. Call this patient i^*.

Case 1. If all patients receiving transplants at $eTTC_{\succ}(R)$ have higher priority than patient i^* at $\succ^*$, then this contradicts Step 3 of the $eTTC_{\succ}$: a set of cycles and chains should have been chosen so that patient i^* is also included together with these patients. There does exist such a set: for example, H.

Case 2. If there is a patient receiving a transplant at $eTTC_{\succ}(R)$ with a lower priority than patient i^* at $\succ^*$, then this again contradicts Step 3 of the $eTTC_{\succ}$: a set of cycles and chains should have been chosen so that patient i^* is included ahead of the patient with a lower priority. There does exist such a set: for example, H.

Therefore, there should be no such patient i^*, completing the proof. The same argument applies to show that $e\overline{TTC}_{\succ}$ satisfies *maximal improvement*. □

5. Compatible Transplants versus Incompatible Transplants

The TTC solutions in the previous section satisfy the desirable properties that we consider, but they do not make a distinction between compatible transplants and incompatible transplants in welfare configuration. These solutions are constructed given the fact that both transplants significantly increase patients' survival rate to a comparable level, as we explained in Introduction.

However, patients may have a "finer" preference over the two types of transplants due to, for instance, a higher medical cost to use suppressants.[14] If this is the case, then *monotonicity* has to be modified to make sure that everyone is made weakly better off. The following refinement guarantees that any patient who could initially receive a *compatible* transplant still receives a *compatible* transplant:

[14]The medical cost for suppressants varies across different countries and medical insurance systems.

Monotonicity*: For each non-negative integer K and each problem R, (i) each matching in $\varphi^\sigma(R)$ is weakly preferred to each matching in $\varphi(R)$ by all patients and (ii) any patient who receives a transplant in a matching in $\varphi(R)$ is not in $\sigma(R)$.

Consider the patients who receive transplants in the absence of suppressants. Condition (i) is exactly what *monotonicity* requires: these patients should receive transplants, either compatible or incompatible. That is, these patients should be either in cycles or in chains chosen for a matching when suppressants are used. Condition (ii) says that all patients, who could receive *compatible* transplants in the absence of suppressants, should not be chosen to use suppressants for incompatible transplants. In other words, none of these patients should be the head of a chain.

It is straightforward from the definition that *monotonicity** implies *monotonicity*. To achieve this requirement, we make a small modification on Substep n of Step 3 in the definition of $eTTC_\succ$:

Step 3. Let $\mathcal{H}_0$ be the collection of all sets of jointly feasible cycles and at most K chains. Proceed to the following steps.

Substep t (≥ 1): In $\mathcal{H}_{t-1}$, identify all sets of jointly feasible cycles and at most K chains including the patient with the t-th highest priority at $\succ^*$. If there is such a set, let $\mathcal{H}_t$ be the collection of all these sets. Otherwise, let $\mathcal{H}_t \equiv \mathcal{H}_{t-1}$.

This process terminates at Substep n. *Choose a set in $\mathcal{H}_n$ such that no pair in $N \setminus \bar{N}$ is the head of a chain in the set.* Choose the patients at the head of the chains in this set to be the recipients of suppressants.

We denote this solution by $eTTC^*_\succ$. The difference between $eTTC_\succ$ and $eTTC^*_\succ$ is in the choice of recipients of suppressants: $eTTC_\succ$ chooses any set of cycles and chains from $\mathcal{H}_n$ in Substep n, whereas $eTTC^*_\succ$ chooses a particular one from $\mathcal{H}_n$ such that no pair in $N \setminus \bar{N}$ is a head of a chain. The existence of such a particular set is the key to the proof of the following result.

Theorem 2. $eTTC^*_\succ$ satisfies monotonicity*. Moreover, for each problem, the set of patients who receive transplants under $eTTC^*_\succ$ is the same as that under $eTTC_\succ$.

Proof. Consider Substep n of Step 3 under $eTTC^*_\succ$. From the definition, $\mathcal{H}_n$ is the collection of *all* sets of cycles and at most K chains that include the same set of pairs. It is sufficient to show that there is $H \in \mathcal{H}_n$ in which no pair in $N \setminus \bar{N}$ is the head of a chain.

Choose any $H \in \mathcal{H}_n$. If no pair in $N \setminus \bar{N}$ is the head of a chain in H, we are done. Otherwise, let i^* be a pair in $N \setminus \bar{N}$ who is the head of a chain in H. Let $i^* \to i_1 \to \cdots \to i_k$ be this chain. We show that we can rearrange the cycles and chains of H into another set in $\mathcal{H}_n$ in such a way that i^* is not the head of a chain any more.

Choose a set of jointly feasible cycles chosen in Step 1 of $eTTC^*_{\succ}$. Since $i^* \in N \setminus \bar{N}$, there is a cycle including i^* in this set. Let j^* be the pair that points to i^* along this cycle (it is possible that $j^* = i^*$ in case $i^* \to i^*$). There are two cases.

Case 1. j^* is one of the pairs in $i^* \to i_1 \to \cdots \to i_k$. Let $j^* = i_m$ for some $m \in \{1, \ldots, k\}$. Then, rearrange this chain into a cycle $i^* \to i_1 \to \cdots \to i_m \to i^*$ and form another chain $i_{m+1} \to \cdots \to i_k$, while keeping all other chains and cycles as in H. Then, the resulting set of cycles and chains is still in $\mathcal{H}_n$ and i^* is now in a cycle.

Case 2. j^* is not one of the pairs in $i^* \to i_1 \to \cdots \to i_k$. Since $j^* \in N \setminus \bar{N}$, j^* also has to receive a transplant at $eTTC_{\succ}(R)$. Therefore, she is either in a cycle or a chain of H.

Subcase 1. If j^* is in a cycle of H, say $j^* \to j_1 \to \cdots \to j_l \to j^*$, then, rearrange this cycle and the chain $i^* \to i_1 \to \cdots \to i_k$ into a chain $j_1 \to j_2 \to \cdots \to j^* \to i^* \to i_1 \to \cdots \to i_k$, while keeping all other chains and cycles as in H. Then, the resulting set of cycles and chains is still in $\mathcal{H}_n$ and i^* is not the head of a chain.

Subcase 2. If j^* is in a chain of H, say $j_1 \to \cdots \to j_l \to j^* \to j_{l+1} \to \cdots \to j_t$. Reorganize this chain and the chain $i^* \to i_1 \to \cdots \to i_k$ into two chains $j_1 \to j_2 \to \cdots \to j^* \to i^* \to i_1 \to \cdots \to i_k$ and $j_{l+1} \to \cdots \to j_t$, while keeping all other chains and cycles as in H.[15] Then, the resulting set of cycles and chains is still in $\mathcal{H}_n$ and i^* is not the head of a chain.

If there is any pair in $N \setminus \bar{N}$ who is again the head of a chain in the resulting set of cycles and chains, we repeat the same process as above for that pair. In the end, we obtain a set in $\mathcal{H}$ such that no pair in $N \setminus \bar{N}$ is the head of a chain of the set.[16]

The second statement follows immediately from the fact that $\mathcal{H}_n$ is a collection of sets of cycles and chains including the same pairs, whose patients receive transplants. □

We can also modify *Pareto efficiency* and *maximal improvement*. Denote by R^* the finer preference profile induced from R.[17] A matching is *Pareto efficient* at R^* if there is no further Pareto improvement from the matching under R^*. We say that a solution (σ, φ) is *Pareto efficient** if for each R, each $\mu \in \varphi^\sigma(R)$ is *Pareto efficient* at R^*.

[15] Patient j^* can be in a self-cycle in Subcase 1. Patient j^* can also be at the head of a chain in H in Subcase 2. If so, j^* newly becomes the head of a chain as we rearrange chains and/or cycles. Since j^* is in $N \setminus \bar{N}$ and is now at the head of a chain, we repeat the same process for j^*.

[16] As we repeat this process, i^* never becomes the head of a chain again, and therefore, we only need to repeat this process at most n times. The reason is as follows. After rearranging chains and/or a cycle, j^* gets connected to i^* and there can be at most one pair who newly becomes the head of a chain. From the construction, such a new head appears only when j^* points to this pair in H, while j^* points to i^* in the cycle chosen in Step 1 of the $eTTC^*_{\succ}$ algorithm. We apply this observation to i^*. Since j^* is connected to i^* in the rearranged cycle or chain, for i^* to be a new head as we repeat the process, there has to be another pair k^* to which j^* points in the cycle chosen in Step 1 of the $eTTC^*_{\succ}$ algorithm. However, j^* cannot point to two distinct pairs in this cycle.

[17] More precisely, for each $i \in N$, R^*_i is induced from R_i as follows: patient i prefers (initially) compatible pairs to the pairs that become compatible by using suppressants, which she prefers to being unmatched. Note that *Pareto efficiency* defined in Section 3 uses $R(\sigma(R))$ in evaluating a matching: there is no difference between the initially compatible pairs and the pairs that become compatible by using suppressants.

Interestingly, there is no direct logical relation between *Pareto efficiency** and *Pareto efficiency*.[18] These requirements, however, remain equivalent as long as a solution satisfies *maximal improvement*.

Proposition 4. Suppose that a solution satisfies *maximal improvement*. Then, the solution is *Pareto efficient** if and only if it is *Pareto efficient*.

Proof. Let (σ, φ) be a solution. Consider a profile R and the set of recipients $\sigma(R)$. Since the solution satisfies *maximal improvement*, a patient is chosen to use a suppressant if and only if she is matched to an incompatible donor. Suppose that each matching $\mu \in \varphi^\sigma(R)$ is *Pareto efficient* at R^*. We show that it is also *Pareto efficient* at $R(\sigma(R))$. Suppose not. Then, there is $\mu' \in \mathcal{M}(R(\sigma(R)))$ that Pareto dominates μ at $R(\sigma(R))$. The set of patients can be decomposed into seven groups receiving

under μ		under μ'
(1) compatible transplants	$\to$	compatible transplants
(2) compatible transplants	$\to$	incompatible transplants
(3) incompatible transplants	$\to$	compatible transplants
(4) incompatible transplants	$\to$	incompatible transplants
(5) no transplants	$\to$	compatible transplants
(6) no transplants	$\to$	incompatible transplants
(7) no transplants	$\to$	no transplants

where at least one patient belongs to (5) or (6). Since μ is *Pareto efficient* at R^*, there should be at least one patient who is worse off (under finer preferences) when switching from μ to μ'. That is, there is at least one patient who belongs to (2). Denote this patient by i^*. Since patient i^* receives an incompatible transplant at $\mu' \in \mathcal{M}(R(\sigma(R)))$, we should have $i^* \in \sigma(R)$. However, $i^* \in \sigma(R)$ receives a compatible transplant at $\mu \in \varphi^\sigma(R)$, which contradicts *maximal improvement*.

On the other hand, suppose that each $\mu \in \varphi^\sigma(R)$ is *Pareto efficient* at $R(\sigma(R))$. We show that it is also *Pareto efficient* at R^*. Suppose not. Then, there is $\mu' \in \mathcal{M}(R(\sigma(R)))$ that Pareto dominates μ at R^*. The set of patients can be decomposed into six groups receiving

[18] Consider possible Pareto improvements under $R(\sigma(R))$ and under R^* from a given matching μ. An improvement under $R(\sigma(R))$ may not be an improvement under R^*. Suppose that one patent who receives a compatible transplant at μ now receives an incompatible transplant and one other patient who used to receive no transplant at μ now receives a transplant (either compatible or incompatible), while all other patients' allocation remains as in μ. This is an improvement under $R(\sigma(R))$ but not under R^*. Conversely, an improvement under R^* may not be an improvement under $R(\sigma(R))$. Suppose that one patient who receives an incompatible transplant at μ now receives a compatible transplant, while all other patients' allocation remains as in μ. This is an improvement under R^*, but not under $R(\sigma(R))$.

under μ		under μ'
(1) compatible transplants	$\rightarrow$	compatible transplants
(2) incompatible transplants	$\rightarrow$	compatible transplants
(3) incompatible transplants	$\rightarrow$	incompatible transplants
(4) no transplants	$\rightarrow$	compatible transplants
(5) no transplants	$\rightarrow$	incompatible transplants
(6) no transplants	$\rightarrow$	no transplants

where at least one patient belongs to (2), (4) or (5). If a patient belongs to (4) or (5), it contradicts μ being *Pareto efficient*. If there is a patient i belonging to (2), then $i \in \sigma(R)$ since patient i receives an incompatible transplant at μ. However, it is possible to make this patient receive *compatible* transplants while keeping all other patients' welfare unchanged, by switching from μ to μ'. As the suppressant could have been used for other patients receiving no transplants to increase the transplants, it contradicts *maximal improvement*. □

Lastly, *maximal improvement* can also be modified to *maximal improvement** as follows: Consider two groups of potential recipients. Identify the set of patients receiving *incompatible* transplants and the set of patients receiving *compatible* transplants respectively when each group is provided with suppressants. If the first group enables more compatible transplants and more incompatible transplants than the other in terms of set inclusion, then the latter group should not be chosen as recipients of suppressants. It is easy to check that *maximal improvement** is implied by *maximal improvement*. Therefore, $eTTC^*_{\succ}$ also satisfies *Pareto efficiency** and *maximal improvement**.

6. Immunosuppressants for Pairwise Exchanges

In this section, we ask what happens to this problem when patients can only make pairwise exchanges due to physical and geographical restrictions in operating transplants.[19] When only pairwise exchanges are feasible, the pairs in the pool can only form cycles and chains composed of at most two pairs. Under this constraint, we cannot achieve full efficiency that we define in Section 2. However, we can instead define *constrained Pareto efficiency* of a matching: there should be no further Pareto improvement subject to this restriction.

We can easily modify our TTC solutions as follows: In Step 1, the participants can only form cycles composed of two pairs. Step 2 remains the same, while Step 3 is modified so that the cycles and chains can only be composed of at most two pairs. These modified TTC

[19] Pairwise exchanges are extensively studied in matching and kidney exchange literature: see Bogomolnaia and Moulin (2004) and Roth et al. (2005). Weighted graphs can also be used to identify these matchings when priorities are counted in: Okumura (2014) and Andersson and Kratz (2016). There are other attempts, on the other hand, to relax restrictions on the size of kidney exchange: for instance, Ausubel and Morrill (2014).

solutions trivially satisfy *constrained Pareto efficiency*, *monotonicity*, and *cardinally maximal improvement*.

With pairwise exchanges, we can discuss more about the total number of transplants. To avoid trivial cases, let us assume that the constraint of K suppressants is binding (that is, all of K suppressants are assigned to the patients to maximize transplants). We calculate the upper bound of the increase of transplants. We say that a matching is "cardinally maximal" if it maximizes the number of transplants.

Proposition 5. Suppose that K patients use suppressants. Compared to any cardinally maximal matching in the absence of suppressants, the number of transplants increases by $2K$ at most.

Proof. For each matching μ, let $N(\mu)$ be the set of pairs who are matched at μ. Suppose that μ and μ' are cardinally maximal matchings in the absence of suppressants and in the presence of suppressants, respectively. We claim that $|N(\mu')| \leq |N(\mu)| + 2K$. Suppose, by contradiction, that $|N(\mu')| > |N(\mu)| + 2K$. At μ', there are K exchanges made possible through the use of suppressants. These exchanges are either between two pairs or within individual pairs and at least one pair in each of these exchanges uses a suppressant. The number of pairs that are matched through such exchanges are at most $2K$. Now, consider all remaining exchanges other than these K exchanges. They are made between pairs that do not use any suppressant. Therefore, they could also have been formed in the absence of suppressants. The number of pairs involving these exchanges is at least $|N(\mu')| - 2K$, which is greater than $|N(\mu)|$. This contradicts μ being a cardinally maximal matching in the absence of suppressant. □

The impossibility in Proposition 2 remains valid under this restriction: *Pareto efficiency* and *strong monotonicity* conflict with each other. However, we now obtain a positive result regarding *cardinally maximal improvement*.[20]

Proposition 6. There is a solution satisfying *constrained Pareto efficiency*, *monotonicity*, and *cardinally maximal improvement* when exchanges can only be made pairwise.

Proof. For each matching μ, let $N(\mu)$ be the set of pairs who are matched at μ. Let us call a chain composed of two pairs a **2-chain** and a chain composed of one pair a **1-chain**. Let us introduce a solution (σ, φ) defined by the following algorithm:

> **Step 1.** For each R, identify the set of all matchings maximizing the number of matched pairs in the absence of suppressants:
>
> $$M^* \equiv \{\mu \in \mathcal{M}(R) : |N(\mu)| \geq |N(\mu')| \text{ for each } \mu' \in \mathcal{M}(R)\}^{21}$$

[20] Proposition 6 still holds with *monotonicity**.

[21] M^* can be identified by means of the well-known Gallai-Edmond decomposition. Roth et al. (2005) explain in detail the structure of cardinally maximal matchings for this problem.

Step 2. For each $\mu \in M^*$, remove $N(\mu)$ from the pool and consider the compatibility graph restricted to the remaining pairs. Choose a set of jointly feasible 2-chains that maximizes the number of pairs included in the chains. Denote this set by C.[22]

Step 3. If there are more than K number of 2-chains in C, choose any K chains among them and let $\sigma(R)$ be the set of heads of these chains. Otherwise, choose all the chains in C and set them aside from the pool. Among the remaining ones, choose pairs as many as K minus the number of these 2-chains; then set them aside as 1-chains. Let $\sigma(R)$ be the set of heads of all chosen chains.

Step 4. Let $\mu^* \in \text{argmax}_{\mu \in M^*} |\{i \in N : i \in N(\mu) \text{ or } i \text{ belongs to a chain chosen in Step 3}\}|$. Let $\varphi(R) \equiv \mu^*$ be the matching in the absence of suppressants. Let $\sigma(R)$ be the set of heads of the chains chosen in Step 3 for μ^*. Let $\varphi^\sigma(R)$ be the matching formed by the chains and cycles identified for μ^*.

By the definition, μ^* will be chosen in the absence of suppressants and it remains the same even when suppressants become available. In addition to μ^*, we choose 2-chains and 1-chains and then transform them into cycles by using suppressants. Since the pairs matched at μ^* in the absence of suppressants will be matched to the same pairs in the presence of suppressants, *monotonicity* is trivially satisfied. It is easy to check that no further Pareto improvement can be made from what this solution chooses.

We lastly show that it satisfies *cardinally maximal improvement*. We verify that there are no other set of recipients and no other matching that enable more transplants than what (σ, φ) chooses. Suppose otherwise: For some R, there are a set $S \subseteq N$ of K recipients and a matching $\mu' \in \mathcal{M}(R(S))$ such that the number of transplants at μ' is greater than the number of transplants at the matching chosen by (σ, φ). If there are several such S and μ', then choose those that maximize the number of transplants. Let μ be the matching chosen by (σ, φ). As explained above, μ is made of 2-cycles of μ^* and 2-chains and 1-chains that are added later on account of μ^*.

For each matching, let us define "mutually compatible" exchanges to be the exchanges made between pairs that do not use any suppressant (that is, both pairs of this exchange receive compatible transplants). Let T be the set of pairs making mutually compatible exchanges at μ and T' be the set at μ'. Note that by the definition of the solution, $T = N(\mu^*)$ where μ^* is identified in Step 4 above.

Claim. $|T| = |T'|$.

Since $|T| = |N(\mu^*)|$, if this claim holds, the mutually compatible exchanges at μ' is one of the cardinally maximal matchings identified in Step 1. Since μ^* is chosen to maximize the total

[22] This can be done by using a similar process of Step 1 and then using the Gallai-Edmond decomposition. Modify the graph restricted to $N \setminus N(\mu)$ as follows: For each $i, j \in N \setminus N(\mu)$, if $i \to j$, then add an additional arc $j \to i$ to the graph. In this modified graph, identify the set of matchings maximizing the number of matched pairs.

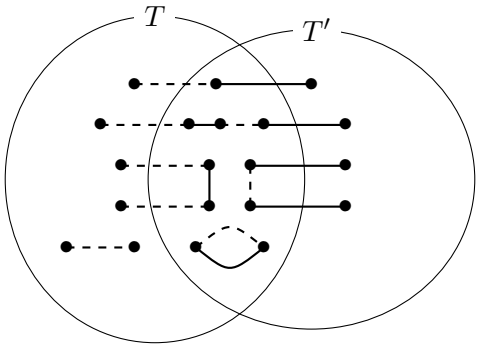


Figure 1: Mutually compatible transplants at μ and μ'

number of transplants among all such cardinally maximal matchings, it is impossible to have more transplants at μ' than at μ, completing the proof.

Proof of Claim. We define a graph representing the mutually compatible exchanges as follows. The nodes are the pairs in N; If two pairs make a mutually compatible exchange at μ, draw a (undirected) dashed arc between them; If the two pairs make a mutually compatible exchange at μ', draw a (undirected) solid arc between them (see Figure 1).

Note that each node in $T \setminus T'$ forms exactly one dashed arc and each node in $T' \setminus T$ forms exactly one solid arc, while each node in $T \cap T'$ forms both. Note that there is no solid arc between the pairs in $T' \setminus T$. If there were any, μ^* plus the matching between these two pairs will make even more transplants than μ^* in Step 1 of the definition, violating μ^* being a cardinally maximal matching in the absence of suppressants. Therefore, each node in T' can only form a solid arc with another node in T.

Next we define an "alternating path" from pair i_1 to pair i_k as an ordered list of distinct arcs between i_1 and i_2, i_2 and i_3, $\cdots$, and i_{k-1} and i_k, where a solid arc is followed by a dashed arc and a dashed arc is followed by a solid arc along the list. If there are no additional distinct arcs from i_k and toward i_1, we say that the alternating path is "complete" with two end nodes, i_1 and i_k. If $i_k = i_1$, we say that the alternating path forms an "alternating cycle". If a node belongs to $(T \setminus T') \cup (T' \setminus T)$, then it cannot be a part of alternating cycle, because the node forms only one arc, either dashed or solid. Therefore, if a path forms an alternating cycle, all pairs of the path should be in $T \cap T'$ (see the cycle in Figure 1).

Now, consider a complete path. If it forms an alternating cycle, the numbers of transplants under μ and μ' in the path are the same, because the numbers of dashed arcs and the number of solid arcs are the same. If it does not form an alternating cycle, there are three cases.

(1) If both end nodes of the path belong to $T' \setminus T$ (one of the bracket-shaped paths in Figure 1), then μ^* is not a cardinally maximal matching in the absence of suppressants, a contradiction: keeping all other dashed arcs the same, match the nodes of the path along the solid arcs, instead of matching them along the dashed arcs. This will increase the number of

transplants by one, because the first and the last arcs of the path are solid and the path is alternating.

(2) If both end nodes of the path belong to $T \setminus T'$ (the other bracket-shaped paths in Figure 1), then symmetrically, μ' is not a cardinally maximal matching in the presence of suppressants, a contradiction: keeping all other solid arcs the same, match the nodes of the path along the dashed lines, instead of matching them along the solid arcs. This will either increase the number of transplants, or keep some suppressants unused, which can be used for other pairs.

(3) If one end node of the path belongs to $T \setminus T'$ and the other belongs to $T' \setminus T$, then the numbers of transplants along the path at μ and μ' are the same, because the first arc is solid and the last arc is dashed (or vice versa), while the path is alternating.

Summarizing, each path in the graph should have one end node in $T \setminus T'$ and the other end node in $T' \setminus T$, preserving the same number of transplants under μ and μ' in the path. The proof of claim is completed by applying this observation to all paths. ■ □

7. Conclusion

In this paper, we take a first step to investigate the implications of introducing suppressants to the kidney exchange problem. We propose several requirements for assigning suppressants and matching patients to donors. We introduce two extended TTC solutions and show that they satisfy Pareto efficiency, monotonicity, and maximal improvement. We also study a refinement of monotonicity.

There remain several interesting questions. First, we may think of different procedures of assigning suppressants, instead of assigning K suppressants all at once as we do in this paper. For instance, suppose that we assign suppressants sequentially, one by one. Each time, we apply the extended TTC solution to assign one unit of suppressant and let all patients who receive transplants leave the pool. We show in an example (which we defer to the appendix) that no patient is better off and that some patients may end up worse off under sequential assignment than under simultaneous assignment.

Second, we can adapt the deferred acceptance (DA) solution to our setting. Since there is a single priority ordering over patients, the DA solution can be defined as follows: Among all sets of jointly feasible cycles and at most K chains, choose ones including a patient with the highest priority; among the resulting collections, choose ones including a patient with the second highest priority; and so on. From what we obtain, we choose the patients at the head of chains to be recipients of suppressant and we let patients receive kidneys from donors along the directed arcs in the cycles and chains. Note that there is no pre-matching step such as Steps 1 and 2 of $eTTC_{\succ}$. From DA, we obtain a "stable" assignment: If a patient does not receive a transplant, then either (i) all patients with lower priorities do not receive transplants,

or (ii) all available suppressants are assigned to patients with higher priorities. This solution also satisfies *Pareto efficiency* and *maximal improvement*. However, it violates *monotonicity*, which can easily be verified with the example in the proof of Proposition 3 after switching the labels of patients 1 and 6 except for the priority ordering.

Lastly, there remains the participation issue. According to Thoerem 2, the recipients chosen by $eTTC^*_{\succ}$ are those who should use suppressants anyway to receive transplants. They would easily accept to use suppressants, but they may still prefer transplants received directly from their own donors, declining to remain in the exchange pool. As their participation benefits other patients, it would be reasonable to introduce an incentive to promote their participation. For example, they could be provided with higher priorities when they participate in exchanges again in case of transplant failure.

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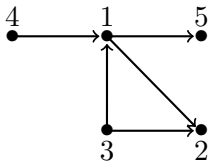
Appendix. Sequential assignment versus simultaneous assignment

We present an example to show that no patient is better off and that some patients may end up worse off under sequential assignment than under simultaneous assignment. Consider the following problem:

Let $1 \succ 2 \succ \cdots \succ 5$. Suppose that at most two patients can use suppressants.

(1) Simultaneous assignment: We apply $eTTC_{\succ}$ by setting $K = 2$. There are several sets of feasible chains that include all patients. Let us choose $\{4 \to 1 \to 5, \ \ 3 \to 2\}$ among others. Then, patients 3 and 4 are provided suppressants and the patients in these chains receive transplants along the two cycles, $4 \to 1 \to 5 \to 4$ and $3 \to 2 \to 3$. All patients receive transplants.

R_1	R_2	R_3	R_4	R_5
$3, 4$	$1, 3$	$\emptyset$	$\emptyset$	1



(2) Sequential assignment: We first apply $eTTC_{\succ}$ by setting $K = 1$. There is only one set of feasible chains chosen according to $\succ$, which is $\{3 \to 1 \to 2\}$. Then, patient 3 is provided with a suppressant and the patients in this chain receive transplants along the cycle $3 \to 1 \to 2 \to 3$. We apply $eTTC_{\succ}$ by setting $K = 1$ to the remaining patients. There is only one set of feasible chain chosen according to $\succ$, which is $\{4\}$. Then, patient 4 is provided with a suppressant and receives a transplant from his own donor. Patient 5 ends up with not receiving a transplant.