Economic distributions and primitive distributions in Monopolistic Competition

Simon P. Anderson and André de Palma*

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Abstract

We link fundamental technological and taste distributions to endogenous economic distributions of prices and firm size (output, profit). We provide constructive proofs to recover the demand structure, mark-ups, and distributions of cost, price, output and profit from just two distributions (or from demand and one distribution). A continuous logit demand model illustrates: exponential (resp. normal) quality-cost distributions generate Pareto (log-normal) economic size distributions. Pareto prices and profits are reconciled through an appropriate quality-cost relation.

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*Department of Economics, University of Virginia, USA, sa9w@virginia.edu; CES-University Paris-Saclay, FRANCE. andre.depalma@ens-cachan.fr. The first author gratefully acknowledges research funding from the NSF. We thank Isabelle Mejean, Julien Monardo, Maxim Engers, Farid Toubal, James Harrigan, and Ariell Resheff for valuable comments, and seminar participants at Hebrew University, Melbourne University, Stockholm University, KU Leuven, Laval, Vrij Universiteit Amsterdam, and Paris Dauphine. This is a revised version of CEPR DP 10748.

1 Introduction

Distributions of economic variables have attracted the interest of economists at least since Pareto (1896). In industrial organization, firm size distributions (measured by output, sales, or profit) have been analyzed, while different studies have looked at the distribution of prices within an industry. Firm sizes (profitability, say) within industries are wildly asymmetric, and frequently involve a long-tail of smaller firms. The idea of the long tail has recently been invoked prominently in studies of Internet Commerce (Anderson, 2006, Elberse and Oberholzer-Gee, 2006), and particular distributions – mainly the Pareto and log-normal – seem to fit the data well in other areas too (see Head, Mayer, and Thoenig, 2014). In international trade, recent advances have enabled studying distributions of sales revenues (see, e.g., Eaton, Kortum, and Kramarz, 2011). The distributions of these "economic" variables are (presumably) jointly determined by the fundamental underlying distributions of tastes and technologies. In this paper we determine the links between the various distributions. We link the economic ones to each other and to the primitive distributions and tastes. Moreover, the primitives can be uncovered from the observed economic distributions.

Philosophically, the paper closest (and complementary) to ours is Mrázová, Neary, and Parenti (2016). These authors also study the relations between equilibrium distributions of sales and mark-ups, the primitive productivity distribution, and (a specific) demand form (although they do not include heterogeneous quality). They are mainly interested in when distributions are in the same ("self-reflecting") class (e.g., when both productivity and sales are log-normal or Pareto). They also provide some empirical analysis of log-normal and Pareto distributions.

We start by deploying a general monopolistic competition model with a continuum of firms (see Thisse and Uschev, 2016, for a review of this literature). We first show how the demand function delivers a mark-up function, and then we show our key converse result that the mark-up (or "pass-through" function of Weyl and Fabinger, 2013) determines the form of the demand function. We next engage these results with constructive proofs to show how cost and price distributions suffice to determine the shape of the economic profit and output distributions and the demand form. Along broader lines, we show when and how any two elements (e.g., two distributions) suffice to deliver all the missing pieces.

Allowing for both quality and cost heterogeneity,[1] we show a three-way relation between two groups of distributions and the quality-to-cost relation: knowing one element from any two of these ties down the third. On one leg, we generate the relation between equilibrium profit dispersion, firm outputs, and the fundamental quality-cost distribution. On a second leg, we show the relation between the cost distribution and equilibrium price dispersion. If we know demand, then knowing any one of the distributions on one leg suffices to determine the others on that leg. Moreover, knowing a distribution from each leg allows us to determine what the relation between cost and quality must be on the third leg. If the demand form is not known, then we show that it can be deduced from observing price, output, and profit distributions (and the cost distribution and the relation between costs and quality can also be determined).

We next develop and deploy a logit model of monopolistic competition.[2] The logit is the

[1] Ironically, Chamberlin (1933) is best remembered for his symmetric monopolistic competition analysis. Yet he went to great length to point out that he believed asymmetry to be the norm, and that symmetry was a very restrictive assumption. We model both quality and production cost differences across firms.

[2] The Logit is an attractive alternative framework to the CES. Anderson, de Palma, and Thisse (1992) have shown that the CES can be viewed as a form of Logit model.

workhorse model in structural empirical IO. Some useful characterization results are that normally distributed quality-costs induce log-normal distributions for profits, and that an exponential distribution of quality-costs leads to a Pareto distribution for profit. Cost heterogeneity alone cannot induce Pareto distributions for both profits and prices. We show by construction for the logit example that the added dimension of quality (and the associated quality-cost relation) can generate Pareto distributions for both, thus allowing sufficient richness to link diverse distribution types.

2 Links between distributions

A continuum of firms produce substitute goods. Each has constant unit production costs, but these differ across firms. With a continuum of firms, each firm effectively faces a monopoly problem where the price choice is independent of the actions of rivals. In this spirit, we allow for a general demand formulation, and show how the primitive (demand and cost distribution) feed through to the endogenous economic distributions and variables.

We first give the demand model, and derive the equilibrium mark-up schedule in Lemma 1 as a function of firm unit cost, c. Our proofs here and beyond are constructive: we derive the relations between distributions and primitives. Theorem 1 shows how the economic distributions are linked to the demand form and the cost distribution. Throughout, we make explicit the appropriate monotonicity conditions.

We shall assume for the exposition that all distributions are absolutely continuous and strictly increasing. As should become apparent, any gaps in a distribution's support will correspond to gaps in supports of the other distributions; the analysis applies piecewise on the interior of the supports. Likewise, mass-points in the interior of the support pose no problem because they correspond to mass points in the other distributions.

2.1 Demand and mark-ups

Assumption 1 *Suppose that demand for a firm charging p is*

$$y = h(p), \tag{1}$$

a positive, strictly decreasing, strictly (-1)-concave, and twice differentiable function.[3]

We suppress for the present the impact of other firms' actions on demand, which would be expressed as aggregate variables in the individual demand function. Under monopolistic competition with a continuum of firms, each firm's individual action has no measurable impact on the aggregate variables.[4] Because we look at the cross-section relation between equilibrium distributions, the actions of other firms are the same across the comparison, and therefore have no bearing on our results. We return to this when we discuss specific examples. The profit for a firm with per unit cost c is $\pi = (p-c)\,h(p) = mh(m+c)$, where $m = p - c$ is its mark-up. With a continuum of firms (monopolistic competition), the equilibrium mark-up satisfies

$$m = -\frac{h(m+c)}{h'(m+c)}. \tag{2}$$

Lemma 1 *Under Assumption 1, the equilibrium mark-up, $\mu(c) > 0$ is the unique continuously differentiable solution to (2), with $\mu'(c) > -1$. $\mu'(c) \geq 0$ if $h(.)$ is log-convex and $\mu'(c) \leq 0$*

[3]This is equivalent to $\frac{1}{h(.)}$ strictly convex, and is a minimal condition ensuring a maximum to profit. See Caplin and Nalebuff (1991) and Anderson, de Palma, and Thisse (1992, p.164) for more on ρ-concave functions; and Weyl and Fabinger (2013) for the properties of pass-through as a function of demand curvature.

[4]For example, the "price index" in the CES model, or the Logit denominator.

if $h(.)$ is log-concave. The associated equilibrium demand, $h^*(c) \equiv h(\mu(c)+c)$, is strictly decreasing and continuously differentiable. The equilibrium profit function, $\pi^*(c)=\mu(c)h^*(c)$, is strictly convex and twice continuously differentiable with $\pi^{*\prime}(c)=-h^*(c)<0$.

Proof. The solution to (2), denoted $\mu(c)$, is uniquely determined (and strictly positive) when the RHS of (2) has slope less than one, as is implied by $h(.)$ being strictly (-1)-concave. Applying the implicit function theorem to (2) shows that

$$\mu'(c)=\frac{-\left(\frac{h(m+c)}{h'(m+c)}\right)'}{1+\left(\frac{h(m+c)}{h'(m+c)}\right)'}>-1, \tag{3}$$

where the denominator is strictly positive under Assumption 1.[5] The numerator is (weakly) positive for h log-convex and (weakly) negative for h log-concave. Let $h^*(c)=h(\mu(c)+c)$ denote the value of $h(.)$ under the profit-maximizing mark-up. Then, $h^*(c)$ is strictly increasing, as claimed, because

$$dh^*(c)/dc=(\mu'(c)+1)h'(\mu(c)+c)<0 \tag{4}$$

and given that $\mu'(c)>-1$. Finally, $\pi^*(c)=\mu(c)h^*(c)$ is strictly decreasing with $\pi^{*\prime}(c)=-h^*(c)<0$ by the envelope theorem. $\pi^*(c)$ is twice continuously differentiable because $h^*(c)$ is continuously differentiable, and strictly convex because $h^*(c)$ is strictly decreasing. ■

Notice that $h^*(c)$ is the inverse marginal revenue curve. Because marginal revenue slopes down strictly, $h^*(c)$ is a continuous function. The result that $\pi^{*\prime}(c)=-h^*(c)$ is the analogue (for monopoly) to Hotelling's Lemma. Notice that the property $\mu'(c)>-1$ is just the standard property that price never goes down as costs increase. As the next Corollary stresses, continuity

[5] When $h(u)$ is strictly (-1)-concave, then $h(u)h''(u)-2[h'(u)]^2<0$, which rearranges to $\left[\frac{h(u)}{h'(u)}\right]'>-1$.

of $\mu'(c)$ implies equilibrium price is a continuously differentiable and, because $\mu'(c) > -1$ by (3) under A1, it is a strictly increasing function of cost.

Corollary 1 *Under Assumption 1, equilibrium price is a strictly increasing and continuously differentiable function of cost, c.*

The key implication of this Corollary and Lemma 1 is that we can rely on monotonic relations between variables, which is crucial in twinning distributions (as we do below). The firms with costs higher than some value c are the same ones that have prices higher than p, an output below y and a profit below π, where the specific values satisfy $\pi = (p - c) h(p)$ (and the mark-up $(p - c)$ satisfies (3)). That is, letting z denote the fraction of firms with profit below some level π, we have

$$1 - F_C(c) = 1 - F_P(p) = F_Y(y) = F_\Pi(\pi) = z. \tag{5}$$

Some characterization results rely on a delineation of the degree of curvature of demand:

Corollary 2 *Under Assumption 1, if demand is strictly log-concave (resp. strictly log-convex), higher cost firms have lower (resp. higher) equilibrium markups ($\mu'(c) < 0$, resp. $\mu'(c) > 0$).*

In the log-concave case, low-cost firms use their advantage in both mark-up and output dimensions. Under log-convexity, low-cost firms exploit the opportunity to capitalize on much larger demand by setting small mark-ups. In both cases though, as per Lemma 1, profits are higher. The only demand function with constant (absolute) mark-up is the exponential (associated to the Logit), which has $h(\cdot)$ log-linear in p, and so $\frac{h(m+c)}{h'(m+c)}$ is constant. For $h(\cdot)$ strictly log-concave, $\mu'(c) < 0$, so firms with higher costs have lower mark-ups in the cross-section of firm types (price pass-through is less than 100%). They also have lower equilibrium outputs. When $h(\cdot)$ is strictly log-convex, the mark-up *increases* with c, so cost pass-through is greater than 100%, which is a hall-mark of CES demands, which have constant elasticity and hence constant relative mark-up.[6]

These properties indicate properties of the price distribution relative to the cost distribution. The price distribution is a *compression* of the cost distribution when h is log-concave, and a *magnification* when h is log-convex, in the simple sense that prices are closer together (or, respectively, farther apart) than costs. The border case (Logit / log-linear demand) has constant mark-ups, so the price distribution mirrors the cost one.

An important special case is when demand is $\rho-$linear (which means that h^ρ is linear). Suppose then that

$$h(.) = (1 + (k-p)\rho)^{1/\rho}, \tag{6}$$

where k is a constant. Then

$$\mu(c) = \frac{1+\rho(k-c)}{1+\rho},$$

which is linear in c.[7] For $\rho = 1$ demand is linear and the standard property is apparent that mark-ups fall fifty cents on the dollar with cost. Log-linearity is $\rho = 0$ (note that $\lim_{\rho\to 0} h(.) = \exp(k-p)$) and delivers a constant mark-up. For $\rho-$linear demands, equilibrium demand is $h^*(c) = \left(\frac{1+\rho(k-c)}{1+\rho}\right)^{1/\rho}$ and then (see (11) below) $\frac{dh^*(c)/dc}{h^*(c)} = \frac{-1}{1+\rho(k-c)} = -\frac{\mu'(c)+1}{\mu(c)} < 0$. Notice that

[6] So a 1% cost rise causes equilibrium price to rise by 1%.

[7] More generally, $\mu'(c) \geq \frac{-\rho}{1+\rho}$ when h is ρ-convex and $\mu'(c) \leq \frac{-\rho}{1+\rho}$ when h is ρ-concave.

$h^*(c)$ is also ρ-linear.

2.2 Equilibrium distributions

The relations above already determine some links between the equilibrium price distribution and the cost distribution and demand. We now show how the other economic distributions are determined and linked in the model.

Our analysis makes extensive use of the following result.

Lemma 2 *Consider two distributions $F_{X_1}(x_1)$ and $F_{X_2}(x_2)$, which are absolutely continuous and strictly increasing on their respective domains. Let X_1 and X_2 be related by a monotone function $X_1 = \xi(X_2)$. Then $F_{X_2}(x_2) = F_{X_1}(\xi(x_2))$ for $\xi(.)$ increasing, and $F_{X_2}(x_2) = 1 - F_{X_1}(\xi(x_2))$ for $\xi(.)$ decreasing.*

Proof. For $\xi(.)$ increasing, $F_{X_1}(x_1) = \Pr(X_1 < x_1) = \Pr(\xi(X_2) < x_1) = \Pr\left(X_2 < \xi^{-1}(x_1)\right) = F_{X_2}\left(\xi^{-1}(x_1)\right)$. Equivalently, $F_{X_2}(x_2) = F_{X_1}(\xi(x_2))$. For $\xi(.)$ decreasing, $F_{X_1}(x_1) = \Pr(\xi(X_2) < x_1) = \Pr\left(X_2 > \xi^{-1}(x_1)\right) = 1 - F_{X_2}\left(\xi^{-1}(x_1)\right)$; equivalently, $F_{X_2}(x_2) = 1 - F_{X_1}(\xi(x_2))$. ■

We can now turn to the equilibrium analysis. Figure 1 illustrates. The upper right panel gives the demand curve, from which we determine the corresponding marginal revenue function. The latter is the key to finding the output distribution from the cost distribution. Notice that $h^*(c)$ defined above determines the equilibrium output (for a firm with per unit cost c) as a function of its cost. As earlier noted, the inverse function, $c = h^{*-1}(y)$ therefore traces out the marginal revenue curve.

The distribution of costs is given in the upper left panel. The negative linear relation between the cost and output distributions is given in the lower left panel: as noted in Lemma

1, higher costs are associated to lower outputs. Therefore, the $z\%$ of firms with costs below c are the $z\%$ of firms with output above $y = h^*(c)$. We hence choose some arbitrary level $z \in (0,1)$ (see (5)). This means that all firm types with cost levels above $c(z) = F_C^{-1}(1-z)$ are the firms with outputs and profits below y and π. That is, $1 - F_C(c) = F_Y(h^*(c))\ (= z)$. The lower right panel therefore connects this relation as the output distribution, $F_Y(y)$. (Notice that in the above argument, only the marginal revenue curve was used from the demand side: as we show later in Section 5, the cost and output distribution determine the marginal revenue, but we then need to integrate up to find demand).

Figure 1 also provides information to determine the price distribution. The upper right panel gives the vertical distance between the marginal revenue and demand, which is the mark-up (which can be expressed as $\mu(c)$), and is thus the vertical shift between cost and price distributions in the upper left panel. It can be constructed simply from the information in the top two panels[8] by drawing across the demand price associated to a marginal revenue - marginal cost intersection. We could also draw in the mark-up distribution in the upper left panel, but have avoided the extra clutter. Notice that (as drawn) the price and cost distributions diverge, as is consistent with Lemma 1 for increasing $\mu(c)$, i.e., log-concave demand.

In summary then, the marginal revenue curve $h^{*-1}(y)$ together with the cost distribution ties down the output distribution (and conversely, for when we shall later be interested). The demand function then finds the price distribution, and therefore relates price and output distributions.

One relation that is missing in the Figure is the profit distribution. But, as Lemma 1 shows,

[8] Hence we were able to give results on the relationships between cost and price distributions at the start of this section without reference to the output distribution.

analogous arguments apply: $\pi^*(c)$ is a decreasing function and so the relation $1 - F_C(c) = F_\Pi(\pi^*(c))\ (= z)$ can be used to construct the profit distribution.

The following result establishes the existence of a unique equilibrium for the monopolistic competition model. Consequently, equilibrium distributions are tied down from the primitives on costs and demand.

Theorem 1 *(existence of unique equilibrium for monopolistic competition model) Let there be a continuum of firms, with demand (1) satisfying Assumption 1. Let F_C be strictly increasing and twice differentiable on its support. Then the distributions F_P, F_Y, and F_Π are strictly increasing and twice differentiable on their supports and given by $F_P(p) = F_C(c(p))$; $F_Y(y) = 1 - F_C(h^{*-1}(y))$; and $F_\Pi(\pi) = 1 - F_C(\pi^{*-1}(\pi))$, where $c(p)$ inverts $p(c)$, $h^{*-1}(y)$ inverts $h^*(c)$, and $\pi^{*-1}(\pi)$ inverts $\pi^*(c)$.*

Proof. Let $p(c)$ denote the equilibrium price for a firm with cost c; from (3) we have $\mu'(c) > -1$ so that $p(c)$ is strictly increasing, and define the inverse relation as $c(p)$, which is strictly increasing. The relation $p(c)$ (and hence its inverse) is determined from $h(.)$ by Lemma 1.

Given F_C, then $F_P(p)$ is determined by $F_P(p) = F_C(c(p))$. Next, consider $F_Y(y)$. By result (4) we know that output $y = h^*(c)$ is a monotonic decreasing function, and so (by Lemma 2) the fraction of firms with output below $y = h^*(c)$ is the fraction of firms with cost above c, so $F_Y(h^*(c)) = 1 - F_C(c)$, or indeed

$$F_Y(y) = \Pr(h^*(C) < y) = \Pr\left(C > h^{*-1}(y)\right) = 1 - F_C\left(h^{*-1}(y)\right). \tag{7}$$

Finally, by Lemma 1 we know that profit $\pi^*(c) = \mu(c)\,h^*(c)$ is a strictly decreasing function, and so the fraction of firms with profit below $\pi^*(c)$ is the fraction of firms with costs above c, so $F_\Pi(\pi^*(c)) = 1 - F_C(c)$, or

$$F_\Pi(\pi) = \Pr(\Pi < \pi) = \Pr(\pi^*(C) < \pi) = \Pr\left(C > \pi^{*-1}(\pi)\right) = 1 - F_C\left(\pi^{*-1}(\pi)\right). \qquad (8)$$

■

The key relation underlying the twinning of distributions is the decreasing relation between cost and output, profit, and price (see Lemma 1 and Corollary 1). A specific cost distribution generates a specific output, profit, and price distribution. Conversely, as we show in the next result, this output or profit distribution could only have been generated from the initial cost distribution. These links are exploited below in Section 3, where we show how the properties of the distributions feed through to each other in terms of their shapes, and we show what are the restrictions among the admissible distributions.

Researchers often impose specific demand functions (such as CES, or logit). Here we forge the (potentially testable) empirical links that are imposed by so doing: Theorem 1 shows that when a specific functional form is imposed for h (as is done in most of the literature), then all the relevant distributions can be found from $F_C(c)$. Furthermore, all distributions can be found from just one of them.

Theorem 2 *Let there be a continuum of firms, with demand (1) satisfying Assumption 1. Consider the set of 4 distributions, $\{F_C, F_P, F_Y, F_\Pi\}$. Suppose that any one is known and is strictly increasing and twice differentiable on its support. Then all other distributions in the set are explicitly recovered and all are strictly increasing and twice differentiable functions on their* *supports.*

Proof. First, F_C was covered in Theorem 1. So consider now F_P. Then $F_C(c) = F_P(p(c))$, where $p(c)$ is the equilibrium price relation, which we showed in Corollary 1 to be continuously differentiable, and both the other distributions are determined from the steps in the proof of Theorem 1 earlier.

Next start with F_Y. Because $h(p)$ is strictly decreasing, then F_P is determined by $F_P(p) = 1 - F_Y(h(p))$. By the argument above, F_C is then determined, and hence so is $F_\Pi(\pi)$.

Finally, suppose that we start with F_Π. By Lemma 1 we know that profit $\pi^*(c) = \mu(c)\, h^*(c)$ is a strictly decreasing function. Therefore $F_C(c)$ is recovered from $F_C(c) = 1 - F_\Pi(\pi^*(c))$. From Theorem 1, F_P is recovered, and so is F_Y. ■

The Theorem says that for any (-1)- concave demand function and any potential economic distribution, there is only one cost distribution that is consistent with the economic distribution. The other economic distributions are likewise pinned down.

Later on we turn our attention to pairs of distributions that are not consistent with the monopolistic competition model; that is, which pairs would indicate violation of the model. Conversely, for admissible pairs, we show how the implicit demand function is determined.

2.3 Atoms and gaps

Some remarks are in order about relaxing the assumptions made in the last two Theorems. There are two main issues with distributions; gaps in the support, and spikes. On the demand side, we address failures of (-1)-concavity.

If $F_C(c)$ has an atom, then $F(p)$ and the other two economic distributions have corresponding atoms of the same size. Likewise, if $F_C(c)$ has a gap, then the three economic distributions have corresponding gaps.

If $h(p)$ has a kink down at some price, while $F_C(c)$ remains continuous, then $F_P(p)$ and $F_Y(y)$ have atoms corresponding to the kink (a range of costs are associated to the same price and output) while the the profit distribution remains continuous.

If $h(p)$ is not (-1)-concave over some range, the corresponding marginal revenue curve slopes up. As a function of c, equilibrium price jumps down (and equilibrium output jumps up) so that $F_P(p)$ and $F_Y(y)$ have corresponding gaps, while $F_\Pi(\pi)$ does not.

Conversely, $F_P(p)$ and $F_Y(y)$ have gaps, while $F_C(c)$ does not, then $h(p)$ is not (-1)-concave over some part of the intervening range, etc. Therefore, such behavior of the distributions can still be consistent with the monopolistic competition model, although not under Assumption 1 and a continuous $F_C(c)$.

3 shapes of things (and inheritance properties)

We take two complementary perspectives on describing how distribution shapes are related to each other. The first is in terms of the degree of concavity that is inherited from other distributions. The second is the relationship between elasticities of distributions and densities. These are crisply expressed via elasticities of the other pertinent economic variables. The latter are expressed as various demand-side statistics.

3.1 Distribution ρ-concavity/convexity inheritance properties

The import of the next result is that we can determine how curvature properties of one distribution carry over to a related one, and vice versa.

Lemma 3 *Consider two functions $F_{X_1}(x_1)$ and $F_{X_2}(x_2)$, which are absolutely continuous and strictly increasing on their respective domains. Let X_1 and X_2 be related by an increasing function $X_1 = \xi(X_2)$. Then:*

a) if $\xi(X_2)$ is concave, $F_{X_2}(x_2) = F_{X_1}(\xi(x_2))$ is a ρ-concave function if $F_{X_1}(x_1)$ is ρ-concave.

b) if $\xi(X_2)$ is convex, $F_{X_2}(x_2) = F_{X_1}(\xi(x_2))$ is a ρ-convex function if $F_{X_1}(x_1)$ is ρ-convex.

If $F_{X_1}(x_1)$ is strictly decreasing on its domain, then

c) if $\xi(X_2)$ is convex, $F_{X_2}(x_2) = F_{X_1}(\xi(x_2))$ is a ρ-concave function if $F_{X_1}(x_1)$ is ρ-concave.

d) if $\xi(X_2)$ is concave, $F_{X_2}(x_2) = F_{X_1}(\xi(x_2))$ is a ρ-convex function if $F_{X_1}(x_1)$ is ρ-convex.

Proof. Lemma 2 shows for $\xi(.)$ increasing that $F_{X_2}(x_2) = F_{X_1}(\xi(x_2))$. Hence $F^{\rho}_{X_2}(x_2) = F^{\rho}_{X_1}(\xi(x_2))$. If $F_{X_1}(x_1)$ is ρ-concave, then $F^{\rho}_{X_1}(\xi(x_2))$ is concave, because it is a concave function of an increasing and concave function. Therefore $F^{\rho}_{X_2}(x_2)$ is concave, and so $F_{X_2}(x_2)$ is ρ-concave. The other relations are proved in a similar manner.[9] ■

The reason for dealing with decreasing F for the two last statements is that we can formulate properties for survivor functions.

The Lemma enables us to bound curvatures, as long as the inside argument obeys the requisite concavity/convexity. For example, take the relation $F_C(c) = F_P(p(c))$, and recall

[9] Succinctly, notice that $F''_{X_2}(x_2) = F''_{X_1}\xi'^2(x_2) + F'_{X_1}\xi''(x_2)$.

that $p(c)$ is increasing. If $p(c)$ is concave, then the Lemma (case (a)) tells us that if F_P is ρ-concave, then so is F_C (and conversely). Case (b) tells us the analogous property for $p(c)$ convex. So, say, if $p(c)$ is linear, as behooves linear demand, and F_P lies between the limits of ρ-concavity and ρ'-convexity (with $\rho \geq \rho'$) then F_C also must lie within the same concavity-convexity limits.[10] Or, indeed, if F_P were log-concave ($\rho = 0$), then so too would be F_C if price were concave in cost. If the relevant $\rho \geq 1$, then a decreasing density f_P implies a decreasing density f_C, and vice versa.

The more interesting relations concern the cases when the distributions are negatively related, e.g., $F_C(c) = 1 - F_\Pi(\pi(c))$. Then we can work with the survivor function $G_\Pi(\pi) = 1 - F_\Pi(\pi)$, which is a decreasing function so that cases (c) and (d) apply. Taking the cost-profit example, we recall $\pi(c)$ is convex, so that case (c) applies here. Then $F_C(c)$ is ρ-concave if $G_\Pi(\pi)$ is ρ-concave. A log-concave profit survivor function implies the cost distribution that would generate such a pattern must be log-concave.

Or take the output distribution and the price survivor function, $G_P(p) = 1 - F_P(p)$. Since $y = h(p)$, the shape of the output distribution is related to the price distribution (and its survivor function) via the concavity or convexity of demand. If demand is concave, we have case (d): the output distribution is ρ-convex if the price survivor function is ρ-convex. Uniform prices are associated to a convex output distribution, which means an increasing output density. This makes sense: half the prices exceed the average one, while the output at the average price is above half the average output for concave demand.

[10]See Anderson and Renault (2005) for more on bounds of ρ-concave and ρ'-convex functions.

3.2 Density and distribution elasticity relations

Economics has several key relations involving elasticities, notably the inverse-elasticity of demand relation with the Lerner index, and the Dorfman-Steiner relations for optimal advertising. The ones we provide below (in the Lemma) are straightforward to derive, but they are quite fundamental for the monopolistic competition setting.

Elasticities of distributions (or survivor functions) are readily calculated from (5) using the relations between the various variables. There are also clean and useful conditions that relate the elasticities of equilibrium densities. These simple formulae show which other elasticities connect the distributions (or survivor functions) and densities, and they are all different aspects of the demand side. For example, the profit density elasticity is related to the cost density elasticity via the elasticities of profit and (inverse) marginal revenue (with respect to unit cost, c), both of which are derived from the fundamental demand form. When the demand side delivers constants for the various elasticities, as it does for constant elasticity demand, then density elasticities are just affine functions of each other (this result is presented in Anderson and de Palma, 2018, for the CES demand model). Moreover, when in turn one of the density elasticities is constant, then they all are constant, as an implication of this property.

The fundamental relations are derived from the following elasticity lemma.

Lemma 4 *Consider two distributions $F_{X_1}(x_1)$ and $F_{X_2}(x_2)$, which are absolutely continuous and strictly increasing on their respective domains. Let X_1 and X_2 be related by a monotone function $X_1 = \xi(X_2)$. Then we have the elasticity relation between densities as $\eta_{f_{X_2}} = \eta_{f_{X_1}} \eta_\xi + \eta_{\xi'}$.*

Proof. For $\xi(.)$ increasing, from Lemma 2, we have $F_{X_2}(x_2) = F_{X_1}(\xi(x_2))$. Differentiation yields $f_{X_2}(x_2) = f_{X_1}(\xi(x_2))\,\xi'(x_2)$; differentiating again gives $f'_{X_2}(x_2) = f'_{X_1}(\xi(x_2))\,(\xi'(x_2))^2 + f_{X_1}(\xi(x_2))\,\xi''(x_2)$. Dividing through the second expression by the first and multiplying both sides by x_2 delivers

$$x_2 \frac{f'_{X_2}(x_2)}{f_{X_2}(x_2)} = \xi'(x_2)\, \frac{f'_{X_1}(\xi(x_2))}{f_{X_1}(\xi(x_2))} \frac{\xi'(x_2)}{\xi(x_2)} x_2 + \frac{\xi''(x_2)}{\xi'(x_2)} x_2;$$

which is the expression given in the Lemma.

For $\xi(.)$ decreasing, from Lemma 2, we have $F_{X_2}(x_2) = 1 - F_{X_1}(\xi(x_2))$. Differentiating, $f_{X_2}(x_2) = -f_{X_1}(\xi(x_2))\,\xi'(x_2)$; and the steps above then again imply the expression given. ■

Assumption 1 imposes several restrictions on the various demand-side elasticities that appear in the density elasticity relations below. In particular, $\eta_h < -1$ is the property that demand must be elastic at in a monopolistic competition setting, mirroring the standard monopoly property. Furthermore, $\eta_{h^*} < 0$ is the property that marginal revenue slopes down. The elasticity of the demand curve slope, $\eta_{h'} = \frac{h"p}{h'}$, has the sign of $-h"$ and so is positive for concave demand, and negative for convex demand. The elasticity of the inverse marginal revenue slope, $\eta_{h^{*\prime}}$, involves third derivatives of demand, though notable benchmarks are that it is zero for linear demand (because marginal revenue is linear) and the constant elasticity case discussed in the next paragraph. Finally, the elasticity of maximized profit (with respect to c), η_π, is particularly interesting. Write this as

$$\eta_\pi = \frac{\pi'(c)}{\pi(c)} c = -\frac{c h^*(c)}{\mu(c)\, h^*(c)} = -\frac{c}{\mu(c)} < 0.$$

The third expression is the ratio of total cost to total profit;[11] and the last one is a variant on the Lerner index (which is $\frac{p-c}{p}$) and here we have the statistic $\frac{p-c}{c}$, which is just the relative mark-up over cost.

In the analysis that follows, the reference point of the constant elasticity demand is useful. Write then $h(p) = p^k$ with $k < -1$[12] so $\eta_h = k < -1$ and $\eta_{h'} = k - 1 < -2$. Furthermore, inverse marginal revenue has the same elasticity as demand,[13] so $\eta_{h^*} = k$; and the profit elasticity is $\eta_{\pi^*} = k + 1 < 0$, namely the demand elasticity plus the (unit) mark-up elasticity.[14]

We can track six pairs of relations, those between price, profit, output, and cost. The latter is the fundamental one, the others are the economic ones induced through the taste side encapsulated in the demand-side elasticities. However, any three of the pairs of relations tie down the rest. We describe the more interesting ones.

3.2.1 price and output

The defining relation between these two is $F_P(p) = 1 - F_Y(h(p))$, Lemma 4 tells us that

$$\eta_{f_P} = \eta_{f_Y}\eta_h + \eta_{h'}. \tag{9}$$

[11] Else $\eta_\pi = -\frac{TC}{TR-TC} = \frac{1}{\frac{TR}{TC}-1}$.

[12] Demand must be elastic or else Assumption 1 is violated: with inelastic demand a firm would produce infinitesimal output.

[13] Because the MR=MC condition yields $\left(\frac{1}{k}+1\right)y^{1/k} = c$.

[14] For the CES demand model, with demand parameter ρ, and equilibrium price $p = c/\rho$, then $k = \frac{1}{\rho-1}$ so $\eta_h = k = \frac{1}{\rho-1} < 0$; $\eta_{h'} = k - 1 = \frac{2-\rho}{\rho-1} < 0$; $\eta_{h^*} = \eta_h = \frac{1}{\rho-1} < 0$; and $\eta_\pi = \frac{\rho}{\rho-1} < 0$. The latter can also be expressed (as per the preceding text) as $-\frac{c}{p-c}$ and recall that $p = c/\rho$. These ρ values concur with those used in the analysis of Anderson and de Palma (2018). [check!]

Recall that the elasticity of the demand slope has shown up elsewhere in pricing formulae (e.g., in Helpman and Krugman, 1985). Note that the demand elasticity on the RHS is negative, so that, ceteris paribus, the effect of the first term on the RHS is to deliver a negative relation between price density elasticity and output density elasticity. Now consider the other term on the RHS. As noted above, $\eta_{h'} = \frac{h"p}{h'}$ has the sign of $-h"$ and so is positive for concave demand, and negative for convex demand. For linear demand, the term disappears. Then we have a benchmark that the price and output densities have *opposite* signs. But, this is apparently not always true otherwise. But we can say, for example, that concave demand implies that decreasing output density drives increasing price density. For convex demand, increasing price density drives decreasing output density. To interpret the negative relation in the benchmark, recall that the low price firms are the high output ones, so we are looking at opposite ends of the distributions/densities effectively. Think about an increasing price density. Then there are more firms with higher prices: translating to the output density, there are fewer firms with lower outputs.

For the constant elasticity demand, (9) reduces to

$$\eta_{f_P} = k\eta_{f_Y} + (k-1)$$

(where we recall that $k < -1$), which implies that an output density $\eta_{f_Y} > -1$ entails an decreasing price density. That is, the price density tends to be decreasing even when the output density is decreasing (though it should not be decreasing by too much). So CES suggests strongly tending for price density to decrease (you would need $\eta_{f_Y} < \rho - 2$ to overturn this).

The linear benchmark suggests the outcome of price and output densities sloping opposite ways, and this is tempered or reinforced by the demand curvature.

As regards the distribution elasticity relation, we have

$$\eta_{F_P} = \eta_{G_Y} \eta_h$$

where η_{G_Y} denotes the elasticity of the survivor function of output (and not the output distribution per se). The equilibrium relation shows how the price distribution is more elastic when equilibrium demand is more elastic, ceteris paribus.

3.2.2 costs and profit; profit and price

The defining distribution relation between profit and cost distributions is $F_{\Pi}(\pi(c)) = 1 - F_C(c)$. This delivers the density relation as

$$\begin{aligned} f_{\Pi}(\pi(c))\, \pi'(c) &= -f_C(c) \\ \text{or } f_{\Pi}(\pi(c))\, h^*(c) &= f_C(c) \end{aligned}$$

and then the elasticity relation immediately follows (or else use Lemma 4) as

$$\eta_{f_{\Pi}} \eta_{\pi} = \eta_{f_C} - \eta_{h^*},$$

or, equivalently,

$$\eta_{f_{\Pi}} = \frac{\mu(c)}{c} \left(-\eta_{f_C} + \eta_{h^*}\right),$$

which tells us that the profit density elasticity is proportional to the difference of the demand-side inverse MR and cost density elasticities. Recalling that η_{h^*} is negative, if $\eta_{f_C} > 0$, then necessarily $\eta_{f_\Pi} < 0$. The contra-positive is that $\eta_{f_\Pi} > 0$ implies $\eta_{f_C} < 0$. Then the profit density is falling quite naturally if the cost density is increasing (or not decreasing too much). It would need a strongly falling cost density (strongly increasing productivity) to overturn the effect.

Now consider the CES version, for which $\mu(c) = c\left(\frac{1}{\rho} - 1\right)$, so that $\eta_{f_\Pi} = -\frac{1-\rho}{\rho}\left(\eta_{f_C} + \eta_{h^*}\right) = -\frac{1-\rho}{\rho}\left(\eta_{f_C} - \frac{1}{1-\rho}\right)$,[15] and that therefore the cost tail parameter gets powered into the profit one by ρ.

In terms of distributions, the elasticity relation is

$$\eta_{F_C} = \eta_{G_\Pi}\eta_{\pi(c)},$$

which can be rewritten as

$$\frac{\mu(c)}{c} = -\frac{\eta_{G_\Pi}}{\eta_{F_C}}.$$

This indicates how the relative mark-up in equilibrium can be found from the distribution shapes.

For profit and price, the analogous density expression is $\eta_{f_\Pi}\eta_\pi = \eta_{f_P} - \eta_h$, or $\eta_{f_\Pi} = \frac{\mu(c)}{c}\left(-\eta_{f_P} + \eta_{h^*}\right)$. The distributional elasticity relation writes $\eta_{F_C} = \eta_{G_\Pi}\eta_{\pi(p)}$.

[15] See also Anderson and de Palma (2018).

3.2.3 output and cost

The defining relation for this pair is $F_Y(h^*(c)) = 1 - F_C(c)$. Then Lemma 4 tells us that[16]

$$\eta_{f_Y}\eta_{h^*} = \eta_{f_C} - \eta_{h^{*\prime}}$$

This is directly comparable to the price-output relation described above (namely $\eta_{f_Y}\eta_h = \eta_{f_P} - \eta_{h'}$). Drawing on that analysis, a linear marginal revenue is a useful benchmark,[17] for which output and cost densities necessarily go in opposite directions. The constant elasticity of demand case is just like the output-price case, given that the parameters are the same for both cases.

Another general link (which is also useful for the next case) is that $\eta_{h^*} = \eta_h \eta_{p(c)}$ where $\eta_{p(c)} = \frac{p'(c)c}{p(c)}$. When the elasticity of demand is constant, $\eta_{f_C} = \eta_{f_P}$ so that $\eta_{p(c)}$ has unit elasticity, concurring with the claim made earlier that then $\eta_{h^*} = \eta_h$.

3.2.4 price and cost; profit and cost; output and cost;

These are all similar relations. First, $F_P(p(c)) = F_C(c)$, so $f_P(p)\,p'(c) = f_C(c)$ and thence

$$\eta_{f_C} = \eta_{p(c)}\eta_{f_P} + \eta_{p'}.$$

The analogous expressions for the other pairs noted in the header are simply given by just replacing the p's by the other variables. For constant elasticity demand, we see the property

[16] Or, indeed, the density relation is $f_Y(h^*(c))\,h^{*\prime}(c) = -f_C(c)$; write this in log form and the elasticity relation follows directly.

[17] This comes from linear demand, but is not limited to that – we can add a rectangular hyperbola to demand and still get a linear marginal revenue.

noted above, $\eta_{f_C} = \eta_{f_P}$, because $\eta_{p(c)} = 1$ and $\eta_{p'} = 0$ (because $p'(c)$ is constant).

4 Equivalences

In the sequel in the following sections, we shall determine how to recover demand (and other distributions) from any pair of distributions, and what restrictions on distributions (if any) must be obeyed in order to satisfy Assumption 1 (that demand is twice continuously differentiable and strictly (-1)-concave). To do so, we shall make use of the results of this section, which form the converse properties to the results of Lemma 1. These are properties that form a stand-alone contribution to the theory of monopolistic competition, and of monopoly, so we collect them together here.

Specifically, Lemma 1 and Corollary 1 show that the demand Assumption 1 (statement (i)) implies the properties (ii) through (iv):

(i) demand is twice continuously differentiable and strictly (-1)-concave;

(ii) the equilibrium mark-up, $\mu(c) > 0$, is a continuously differentiable function with $\mu'(c) > -1$;

(iii) the equilibrium price, $p(c)$, is a continuously differentiable function with $p'(c) > 0$;

(iv) the equilibrium demand, $h^*(c)$, is a continuously differentiable function with $h^{*\prime}(c) < 0$;

(v) the equilibrium profit, $\pi^*(c)$, is strictly convex and twice continuously differentiable, with $\pi^{*\prime}(c) = -h^*(c) < 0$.

Here we show that these are all equivalent statements, so that any one implies the others. Indeed, Corollary 1 already indicates that (ii) and (iii) are equivalent. Likewise, (iv) and (v) are equivalent given the envelope theorem result (the monopoly analogue to Hotelling's Lemma),

$\pi^{*\prime}(c) = -h^*(c) < 0$, shown in Lemma 1. Therefore it remains to prove that (ii) implies (i), and (iv) implies (i). We treat these in turn (in reverse order).

4.1 strictly decreasing marginal revenue implies strictly (-1)-concave demand

First note that $h*(c)$ is strictly decreasing if and only if marginal revenue, $MR(y) > 0$, is strictly decreasing, with both continuously differentiable. This is because these are inverse functions. Next, integrating $MR(y)$ yields total revenue, $TR(y)$, which is therefore twice continuously differentiable (and it is strictly quasi-concave, and monotone increasing for $MR(y)$). Average revenue, $AR(y)$, is then $TR(y)/y$, and this twice continuously differentiable function is inverse demand, $p(y)$. Inverting it yields $h(p)$ as a twice continuously differentiable function. It remains to show that $h(p)$ is strictly (-1)-concave. The proof following the next result concludes the issue.

Lemma 5 *If inverse marginal revenue, $h^*(c)$, is strictly decreasing and continuously differentiable, then demand, $h(p)$, is strictly (-1)-concave and twice continuously differentiable.*

Proof. First note that $h(p)$ is strictly (-1)-concave if and only if $h''h - 2(h')^2 < 0$. Write the inverse demand as $p(y)$ so that $h'(p) = \frac{1}{p'(y)}$ and $h''(p) = -\frac{p''(y)}{(p'(y))^3}$. Then the strict (-1)-concavity condition we are to show becomes

$$p''y + 2p' < 0. \tag{10}$$

Now we want to find $p(y)$, using the steps explained before the Lemma. Let $MR(y)$ denote $h^{*-1}(c)$, i.e., marginal revenue. So then Total Revenue, $TR(y)$ is the integral of $MR(y)$ and equilibrium inverse demand, $p(y)$, is

$$p(y) = \frac{TR(y)}{y} = \frac{\int_0^y MR(u)\,du}{y},$$

and the inverse is $h(p)$. Hence $p'(y) = \frac{yc(y) - \int_0^y c(u)du}{y^2}$ and $p''(y) = \frac{c'(y) - 2\left(yc(y) - \int_0^y c(u)du\right)}{y^3}$. Using these expressions in (10) gives

$$MR'(y) < 0,$$

which follows because $h^{*\prime}(c)$ is continuously differentiable and negative. Q.E.D. ■

Notice that the demand $h(p)$ is only determined up to a constant (from the step where $MR(.)$ is integrated): intuitively, one can always add a rectangular hyperbola to any inverse demand (the rectangular hyperbola has a zero Marginal Revenue) and get the same Marginal Revenue function.

4.2 constructing demand from the mark-up function

Here we show how $\mu(c)$ (with $\mu'(c) > -1$) can be used to find the associated equilibrium demand and demand function, $h(p)$. Equivalently, we could start with a continuously differentiable and strictly increasing relation between equilibrium price and cost, $p(c)$. Our converse result to Lemma 1 indicates how the mark-up function $\mu(c)$ determines the form of inverse marginal revenue, $h^*(c)$, and hence determine the form of $h(p)$.

Lemma 6 *Consider any positive mark-up function $\mu(c)$ for $c \in [\underline{c}, \bar{c}]$ with $\mu'(c) > -1$. Then there exists an equilibrium demand function $h^*(c)$ with $h^{*\prime}(c) < 0$, defined on its support $[\underline{c}, \bar{c}]$ and given by (12), which is unique up to a positive multiplicative factor. The associated primitive* *demand function $h(p)$, given by (13), satisfies Assumption 1 on its support $[\mu(\underline{c}) + \underline{c}, \mu(\bar{c}) + \bar{c}]$. $h(p)$ is log-convex if $\mu'(c) \geq 0$ and log-concave if $\mu'(c) \leq 0$.*

Proof. First note from (2) and (4) that

$$\frac{dh^*(c)/dc}{h^*(c)} = \frac{(\mu'(c)+1)\,h'(\mu(c)+c)}{h(\mu(c)+c)} = -\frac{\mu'(c)+1}{\mu(c)} \equiv g(c) < 0, \tag{11}$$

because $\mu'(c) > -1$ by assumption. Thus $[\ln h^*(c)]' = g(c)$, and so $\ln\left(\frac{h^*(c)}{h^*(\underline{c})}\right) = \int_{\underline{c}}^{c} g(v)\,dv$, or

$$h^*(c) = h^*(\underline{c}) \exp\left(\int_{\underline{c}}^{c} g(v)\,dv\right), \quad c \geq \underline{c}, \tag{12}$$

which determines $h^*(c)$ up to the positive factor $h^*(\underline{c})$; it is strictly decreasing because $g(c) < 0$.

We can now use the inverse marginal revenue function, $h^*(c)$, to back out the demand function, $h(m+c)$, via the following steps. First, define $u \equiv \phi(c) = \mu(c) + c$, which is strictly increasing because $\mu'(c) + 1 > 0$, so the inverse function $\phi^{-1}(\cdot)$ is strictly increasing. Now, $h(u) = h^*\left(\phi^{-1}(u)\right)$ and thus the function $h(\cdot)$ is recovered on the support $u \in [\mu(\underline{c}) + \underline{c}, \mu(\bar{c}) + \bar{c}]$ (cf. Lemma 1). Using (12) with $h(u) = h^*\left(\phi^{-1}(u)\right)$,

$$h(u) = h^*(\underline{c}) \exp\left(\int_{\underline{c}}^{\phi^{-1}(u)} g(v)\,dv\right), \tag{13}$$

and so

$$\frac{h(u)}{h'(u)} = \frac{1}{g\left(\phi^{-1}(u)\right)\left[\phi^{-1}(u)\right]'} = \frac{\phi'(c)}{-\frac{\mu'(c)+1}{\mu(c)}} = -\mu(c),$$

where the middle step follows from (11) with $u = \phi(c)$ and the last step follows because $\phi'(c) = \mu'(c) + 1$. Thus

$$\left[\frac{h(u)}{h'(u)}\right]' = -\frac{\mu'(c)}{\mu'(c)+1} > -1,$$

and so $h(u)$ is strictly (-1)-concave (as shown in footnote 5). Note that $h(.)$ is twice differentiable because $\mu(.)$ was assumed differentiable. ∎

Recalling that $\mu(c) = p(c) - c$ for $c \in [\underline{c}, \bar{c}]$, the restriction used in the Lemma ($\mu'(c) > -1$) is that $p'(c) > 0$ so that *any* arbitrary (differentiable) increasing price function of costs can be associated to a unique demand function that could generate it (up to the multiplicative factor).

The reason that demand is only determined up to a positive factor is simply that multiplying demand by a positive constant does not change the optimal mark-up (when marginal costs are constant, as here). The mark-up function can only determine the demand shape, but not its scale. In conjunction, Lemmas 1 and 6 indicate the property that $\mu'(c) > -1$ if and only if $h(u)$ is strictly (-1)-concave.

The steps in the proof are readily confirmed for the $\rho-$linear example given after Lemma 6. Along with Lemma 1, the results of this section indicate that knowing any of $\mu(c)$, $h^*(c)$, or $h(.)$ suffices to determine them all (up to constants in the first two cases). This constitutes a strong characterization result for monopoly pass-through (see Weyl and Fabinger, 2013, for the state of the art, which deeply engages ρ-concave functions).

Notice that the function $h(\cdot)$ is tied down only on the support corresponding to the domain on which we have information about the equilibrium mark-up value in the market. Outside that support, we know only that $h(\cdot)$ must be consistent with the maximizer $\mu(c)$, which restricts the shape of $h(\cdot)$ to be not "too" convex.

5 Rationalizability of distributions via demand

An old question in consumer theory is whether a demand system can be generated from a set of underlying preferences (see Antonelli, 1898, and the discussion in Mas-Collel, Whinston, and Green, 20xx, p.?). Here we look at whether any arbitrary pair of economic/primitive distributions could be consistent with the monopolistic competition model with demand satisfying A1. Surprisingly, for 4 of the possible pairs of distributions, the answer is affirmative, so that the model places no restrictions (above the twice continuously differentiable assumption we retain for simplicity.) For the other two, we derive the conditions the distributions must satisfy, and in all cases we recover the implied demand function. We start with the key result that enables us to recover demand from the mark-up function.

5.1 Deriving demand from price and profit distributions

We now use Lemma 6 to find a unique demand function satisfying A1 from these distributions. This is quite a surprising result. For example, there exists a demand function that squares Pareto distributions for both prices and profits. In the next sub-sections we do likewise for other distribution pairs.

Note that all other distributions are determined (along with demand) from the original pair.

Theorem 3 *Let the price and profit distributions, F_P and F_Π, be two arbitrary strictly increasing and twice continuously differentiable functions on their supports. Then there exists a strictly (-1)-concave demand function (unique up to a positive constant) that rationalizes these distributions in the monopolistic competition model.*

Proof. Applying the techniques above (see (5)), first write $1 - F_C(c) = 1 - F_P(p) = F_Y(y) = F_\Pi(\pi) = z$. Then we can write $\pi = F_\Pi^{-1}(1 - F_P(p)) = h(p)\mu(p) \equiv \tilde{\pi}(p)$, where $\tilde{\pi}(p)$ therefore denotes the relation between the maximized profit level observed and the value of the corresponding maximizing price (i.e., $\tilde{\pi}(p)$ is the optimizing price delivering the profit level). Recall from the optimal choice of mark-up that $h(p)$ and $\mu(p)$ are related by $\mu(p) = -h(p)/h'(p)$ (see (2)), and so $\tilde{\pi}(p) = -h^2(p)/h'(p)$. Integrating,

$$h(p) = \frac{1}{\int_{\underline{p}}^{p} \frac{dr}{F_\Pi^{-1}(1-F_P(r))} + k}. \tag{14}$$

This determines the demand form up to the positive constant $k = 1/h(\underline{p})$ (in the position in the above formula). Finally, (14) is decreasing in p, and is twice continuously differentiable. Furthermore,

$$\left(\frac{1}{h(p)}\right)' = \frac{1}{F_\Pi^{-1}(1 - F_P(p))}$$

which is strictly increasing because both distributions are strictly increasing. That is $1/h(p)$ is convex and so, equivalently, $h(p)$ is (-1)-concave.

By Theorem 2 all the other distributions are determined. ■

Therefore, after using the price and profit distributions we can define the function (14) and the proof shows that the resulting demand function satisfies Assumption 1 *without any further restrictions*. This means, for example, that a decreasing price density is consistent with an increasing profit density (very many high profit firms and yet very few high price ones). The underlying cost distribution along with demand is what renders these features compatible. As regards the constant k, knowing the demand level at any one point ties down the whole demand function.

We have just shown that there are no restrictions on price and profit distribution shapes, though below we have restrictions on some other pairs of distribution functions that can be combined and be consistent with the monopolistic competition model.

5.2 price and cost

We now determine distributions from each other when there are monotone relations between two variables. Suppose first that price and cost distributions, F_P and F_C, are known. Because mark-ups are necessarily positive, it must be that the price distribution first-order stochastically dominates the cost one. However, we will show that this is the *only* restriction on the distributions. The demand function will ensure that they are compatible, even despite it being (-1)-concave.

We show how to find the implied other economic distributions as well as the demand form and mark-up function: we can find all other elements in the market from just the two distributions. This strong result relies on the monotonic relations between all pairs of variables from Corollary 1. We now show how this works. Because price strictly increases with cost, the price and cost distributions are matched: the fraction of firms with costs below some level c equals the fraction of firms with prices below the price charged by a firm with cost c. This enables us to back out the corresponding mark-up function $\mu(c)$ and then access Lemma 6.

Theorem 4 *Let the cost and price distributions, F_C and F_P be two arbitrary strictly increasing and twice continuously differentiable functions on their supports with $F_C(c) > F_P(c)$. Then there exists a strictly (-1)-concave demand function (unique up to a positive factor) that ratio-*

nalizes these distributions in the monopolistic competition model. Then the mark-up function $\mu(c)$ *(with* $\mu'(c) > 0$*) is found from (15); inverse marginal revenue is found from (12) and the demand function is given from (13), up to a positive multiplicative factor,* $h^*(\underline{c})$*. The output and profit distributions are determined, up to* $h^*(\underline{c})$*, by (7) and (8).*

Proof. Consider a distribution of costs, F_C and a distribution of prices, F_P satisfying $F_C(c) > F_P(c)$ (so that the price distribution is right of the cost one: note that $F_C(c) > F_P(c) = 0$ for c below the lower bound of the support of the price distribution). We wish to find a demand function satisfying A1. Define $p(c) = F_P^{-1}(F_C(c))$ which is an increasing function. Then Lemma 6 implies that there exists an $h(.)$ satisfying A1 (consistent with Corollary 1 that the price charged by a firm with cost c is a strictly increasing continuously differentiable function $p(c)$).

Hence we can write the price-cost margin, as a function of c, as

$$\mu(c) = F_P^{-1}(F_C(c)) - c, \tag{15}$$

with $\mu(c) > 0$ because $F_C(c) > F_P(c)$ and $\mu'(c) > -1$. Hence a unique such mark-up function $\mu(c)$ exists given the cost and price distributions. With the function $\mu(c)$ thus determined, we can invoke Lemma 6 to uncover the equilibrium demand function $h^*(\cdot)$ (unique up to a positive multiplicative factor) as given by (11) and (12), and the demand function is given from (13). By Lemma 6, this demand function satisfies A1, as postulated. ■

The idea behind the result is as follows. Given the first key property that prices rise with costs, we know that the $z\%$ of firms with cost below c are the $z\%$ of firms with an equilibrium price below p. This links the mark-up and the cost level, so we can use Lemma 6 to uncover the demand form and equilibrium output of the zth percentile firm, due to the second key property that equilibrium output is a decreasing function of cost. We hence uncover the output distribution. The profit distribution then follows immediately from knowing the output and mark-up distributions. The latter two distributions are only determined up to a positive factor because the mark-up function is consistent with any multiple of the demand (under the maintained hypothesis of constant returns to scale).

The construction of the demand function is illustrated in Figure 1 above. The only restriction we use here is that the cost distribution first-order stochastically dominates the price one. Given this property, any pair of (twice continuously differentiable) price and cost functions is consistent with the monopolistic competition model. We next show that the price and output distributions are restricted if they are to be consistent.

5.3 price and output

We now suppose that price and output distributions, F_P and F_Y, are known.

Theorem 5 *Let the price and output distributions, F_P and F_Y be two arbitrary strictly increasing and twice continuously differentiable functions on their supports. Then there exists a unique strictly (-1)-concave demand function $h(p) = F_Y^{-1}(1 - F_P(p))$ that rationalizes these distributions in the monopolistic competition model if and only if $F_Y^{-1}(1 - F_P(p))$ is (-1)-concave.*

Proof. Because from the two distributions $y = F_Y^{-1}(1 - F_P(p)) = h(p)$ this is the unique candidate demand function. While this is decreasing in p, as desired, we also require that the function $F_Y^{-1}(1 - F_P(p))$ is (-1)-concave to be consistent with the monopolistic competition model. ■

Therefore if the implied demand shape does not satisfy the given condition, the purported demand relation would not have a downward-sloping marginal revenue curve everywhere, and any price-output pair with an upward sloping marginal revenue could not be consistent with profit maximization by a firm.

5.4 cost and output

Although price and output distributions are jointly restricted, surprisingly, cost and output distributions are not. Suppose that F_C and F_Y are known.

Theorem 6 *Let the cost and output distributions, F_C and F_Y be two arbitrary strictly increasing and twice continuously differentiable functions on their supports. Then there exists a strictly (-1)-concave demand function (unique up to a positive constant) that rationalizes these distributions in the monopolistic competition model.*

Proof. From the two distributions $y = F_Y^{-1}(1 - F_C(c)) = h^*(c)$ is the candidate function for optimized demand. The only restriction is that it slope down, which is satisfied, and that it be continuous, which is also immediately satisfied. Hence it is rationalizable, and we can use earlier results (Lemma 5) to back up to the implied demand function, $h(p)$, which is therefore determined up to a positive constant. ■

5.5 cost and profit

This is another case where monopolistic competition restricts the distributions. Suppose that F_C and F_Π are known.

Theorem 7 *Let the cost and profit distributions, F_C and F_Π be two arbitrary strictly increasing and twice continuously differentiable functions on their supports. Then there exists a strictly (-1)-concave demand function (unique up to a positive constant) that rationalizes these distributions in the monopolistic competition model if and only if f_C/f_Π is increasing.*

Proof. From the two distributions, $\pi(c) = F_\Pi^{-1}(1 - F_C(c))$ is the candidate profit function. This is decreasing in c, as desired, but it also needs to be convex. The convexity condition is that f_C/f_Π is increasing in c in order to be consistent with the monopolistic competition model.

■

5.6 output and profit

The final case returns to no restrictions. Consider a distribution of output, F_Y and a distribution of profit, F_Π.

Theorem 8 *Let the output and profit distributions, F_Y and F_Π be two arbitrary strictly increasing and twice continuously differentiable functions on their supports. Then there exists a strictly (-1)-concave demand function (unique up to a positive constant) that rationalizes these distributions in the monopolistic competition model. This unique net demand function, and the other distributions, are determined explicitly in the proof.*

Proof. The assumption that F_Y and F_Π are continuously differentiable distributions means that we can invert them and write each of them as a function of the counter z. Both output and profit are increasing functions of cost, c. Therefore we can match the distributions: the firms with the highest $z\%$ of the costs are those with the lowest $z\%$ of the outputs and profits. Furthermore, because the distribution functions are differentiable, then we should have that z is a differentiable function of the underlying cost, and invert it. Call this inverted relation $c(z)$, with $c'(z) < 0$. Then we can write

$$\pi(c(z)) - \mu(c(z))\, h^*(c(z)) = 0.$$

Differentiating this identity,

$$\pi'(c) - \mu'(c)\, h^*(c) - \mu(c)\, h^{*\prime}(c) = 0.$$

But, by Lemma 1 $\pi'(c) = -h^*(c)$ so that

$$\mu'(c) = -\mu(c)\frac{h^{*\prime}(c)}{h^*(c)} - 1 > -1$$

which means that A1 holds.

We now determine the net inverse demand, the mark-up function, and the other distributions. We know that $h^*(c)$ is strictly decreasing in c, and so too is $\pi^*(c) = \mu(c)\, h^*(c)$ (by Lemma 1). We hence choose some arbitrary level $z \in (0,1)$ such that $1 - F_C(c) = F_Y(y) = F_\Pi(\pi) = z$. This means that all firm types with cost levels above $c(z) = F_C^{-1}(1-z)$ are the firms with outputs and profits below y and π. For this proof, we introduce z as an argument into the various outcome variables to track the dependence of the variables on the level of $z(c) = 1 - F_C(c)$. From (??) we can write $y(z) = F_Y^{-1}(z)$ and demand is

$$h^*(c) = y(z(c)) = F_Y^{-1}(1 - F_C(c)). \tag{16}$$

Because $\pi^*(z) = m(z)\, y(z) = F_\Pi^{-1}(z)$ then

$$m(z(c)) = \frac{F_\Pi^{-1}(z(c))}{F_Y^{-1}(z(c))} = \mu(c), \tag{17}$$

and equilibrium profit is $\pi^*(c) = \mu(c)\, h^*(c) = F_\Pi^{-1}(z(c))$.

It remains to find the relation $z(c)$. From (17), $\mu'(c) = m'(z(c))\, z'(c)$ and similarly $h^{*\prime}(c) = y'(z(c))\, z'(c)$ (where $m'(z(c)) = \frac{dm(z(c))}{dz(c)}$, etc.). The two functions $\mu(c)$ and $h^*(c)$, which are to be determined, satisfy condition (11), which implies

$$\frac{h^{*\prime}(c)}{h^*(c)} = \frac{y'(z(c))\, z'(c)}{y(z(c))} = -\frac{m'(z(c))\, z'(c) + 1}{m(z(c))}.$$

Rearranging the last equality to solve out for $z'(c)$ gives[18]

$$z'(c) = -\frac{y(z(c))}{\left[m(z(c))\, y(z(c))\right]'} = -\frac{F_Y^{-1}(z(c))}{\left[F_\Pi^{-1}(z(c))\right]'}. \tag{18}$$

Thus:

$$\int_0^z \frac{\left[F_\Pi^{-1}(r)\right]'}{F_Y^{-1}(r)} dr = -\int_{\bar{c}}^c dv = \bar{c} - c,$$

[18] An alternative derivation is to use Theorem 1 to write $\pi^{*\prime}(c) = -h^*(c)$, so the relation between the counter z and the cost level c is $dz/dc = -h^*(c) / [\pi^*(z(c))]'$, which is (18).

or $c(z) = \bar{c} - \Psi(z)$, where $\Psi(z)$ is the key transformation between z and c given by

$$\Psi(z) = \int_0^z \frac{\left[F_\Pi^{-1}(r)\right]'}{F_Y^{-1}(r)} dr. \tag{19}$$

Because $\Psi'(z) = \frac{\left[F_\Pi^{-1}(z)\right]'}{F_Y^{-1}(z)} > 0$, the required relation between z and c is $z(c) = \Psi^{-1}(\bar{c} - c)$.

Observe that $h(p) = h(\mu(c) + c)$ so that inverse demand is $p = \frac{F_\Pi^{-1}(z(c))}{F_Y^{-1}(z(c))} + c = \frac{F_\Pi^{-1}\left(\Psi^{-1}(\bar{c}-c)\right)}{F_Y^{-1}(\Psi^{-1}(\bar{c}-c))} + c$. This makes clear that a shift up in all costs by Δ and a corresponding shift up in the inverse demand by Δ (so the support of the cost distribution shifts up by Δ, i.e., $\bar{c}$ becomes $\bar{c} + \Delta$) keeps both the firm's output choice and mark-up constant so output and profit are not changed. This means that these two distributions can only pin down *net* (inverse) demand.

The distribution of cost is thus given by

$$F_C(c) = 1 - z(c) = 1 - \Psi^{-1}(\bar{c} - c). \tag{20}$$

The remaining unknowns can be backed out now knowing $z(c)$: equilibrium demand is $h^*(c) = F_Y^{-1}(\Psi^{-1}(\bar{c} - c))$ from (16), and the mark-up function is $\mu(c) = \frac{F_\Pi^{-1}\left(\Psi^{-1}(\bar{c}-c)\right)}{F_Y^{-1}(\Psi^{-1}(\bar{c}-c))}$ from (17). ∎

What the Theorem ties down is net demand (inverse demand minus cost): if both demand price and cost shift by the same amount then equilibrium quantity (output) and mark-up are unaffected, so profit is unchanged too. Thus output and profit distributions tie down the shape of the inverse demand and the shape of the other distributions, but not the inverse demand curve height. As we saw above, price and cost distributions alone do not tie down the demand scale, and nor do price and profit distributions. But the other pairs of distribution combinations fully determine the demand function and all distributions.

5.7 examples

We illustrate the theorem above with distributions that generate ρ-linear demand.

Example 1: *ρ-linear demands and uniform cost distribution.*

Suppose that $F_Y(y) = \frac{(1+\rho)y^\rho - 1}{\rho}$, $y \in \left[\frac{1}{(1+\rho)^{1/\rho}}, 1\right]$, *and* $F_\Pi(\pi) = \frac{(1+\rho)\pi^{\rho/(1+\rho)} - 1}{\rho}$, $\pi \in \left[\frac{1}{(1+\rho)^{(1+\rho)/\rho}}, 1\right]$, *with* $\rho > -1$.

Hence $F_Y^{-1}(z) = \left(\frac{\rho z+1}{1+\rho}\right)^{1/\rho}$ *and* $F_\Pi^{-1}(z) = \left(\frac{\rho z+1}{1+\rho}\right)^{(1+\rho)/\rho}$. *By (17), the ratio of these two yields the mark-up,* $m(z) = \frac{\rho z+1}{1+\rho} > 0$. *Because* $\left[F_\Pi^{-1}(z)\right]' = \left(\frac{\rho z+1}{1+\rho}\right)^{1/\rho}$, *we can write* $\Psi(z) = \int_0^z \frac{\left[F_\Pi^{-1}(r)\right]'}{F_Y^{-1}(r)} dr = z$, *and, because* $c(z) = F_C^{-1}(1-z)$, *then* $c(z) = \bar{c} - z$ *(with* $c'(z) = -1$*), so that* $\underline{c} = \bar{c} - 1$. *Now,* $F_C(c) = 1 - \Psi^{-1}(\bar{c} - c) = c - \underline{c}$. *Hence* $\mu(c) = \frac{\rho(\bar{c}-c)+1}{1+\rho}$. *Then* $y(c) = F_Y^{-1}(z(c)) = \left(\frac{\rho(\bar{c}-c)+1}{1+\rho}\right)^{1/\rho}$, *and* $h^*(c) = y(c)$. *We now want to find the associated demand,* $h(p)$. *We use the fact that* $p = \mu(c) + c = \frac{1+c+\rho\bar{c}}{1+\rho}$ *to write* $h(p) = (1 + \rho(\bar{c} - p))^{1/\rho}$, *which is therefore a* $\rho-$*linear demand function (see (6)) with the parameter* k *set at* $k = \bar{c}$, *and* $\rho > -1$ *implies* $h(.)$ *is* (-1)*-concave.*

Note that $y(\bar{c}) = \left(\frac{1}{1+\rho}\right)^{1/\rho}$, *as verified by the upper bound,* $\bar{c}$, *while the lower bound condition* $\underline{c} = \bar{c} - 1$ *implies that* $y(\underline{c}) = 1$, *so costs are uniformly distributed on* $[\underline{c}, \bar{c}]$. *Lastly,* $\lim_{\rho\to 0} y(c) = \exp(\bar{c} - c)$ *gives the logit equilibrium demand (see Section 6).*

The uniform cost example gives a useful benchmark for some important properties relating cost distribution to profit distribution. For the example above, we have $f_\Pi(\pi) = \pi^{-1/(1+\rho)}$, so that the density of the profit distribution is decreasing, despite the underlying cost distribution that generates it being flat. This property indicates how profit density "piles up" at the low end. The output density shape is also interesting. For linear demand ($\rho = 1$), it is clearly flat

– equilibrium quantity is a linear function of cost. For convex demand ($\rho < 1$), it is decreasing, but for concave demand it is *increasing*, despite the property just noted that the profit density is decreasing. This suggests that (for concave demand), a decreasing output density requires an increasing cost density, which *a fortiori* entails a decreasing profit density.

As per Theorem 8, the (output, profit) distribution pair does not tie down the value of the constant. Theorems 5, 6, and 7 show which distribution pairs do tie down the full model, and knowing the demand form plus any distribution ties down everything (Theorem 1.) We return to the above example to illustrate, with the same parameters as Example 1, that knowing the profit and cost distributions ties down the full model.

Example 2: *ρ-linear demands from uniform cost distribution.*

Suppose that it is known that $F_C(c) = c$ for $c \in [0,1]$ and (as above) $F_\Pi(\pi) = \frac{(1+\rho)\pi^{\rho/(1+\rho)}-1}{\rho}$, $\pi \in \left[\frac{1}{(1+\rho)^{(1+\rho)/\rho}}, 1\right]$. We first write $\pi^*(c)$ to find $h^*(c) = -\pi^{*\prime}(c)$. Matching the distribution levels, $1 - c = \frac{(1+\rho)\pi^{\rho/(1+\rho)}-1}{\rho}$, or $\pi^*(c) = \left(\frac{\rho(1-c)+1}{1+\rho}\right)^{(1+\rho)/\rho}$ and hence $y(c) = h^*(c) = \left(\frac{\rho(1-c)+1}{1+\rho}\right)^{1/\rho}$, so both output and profit are power functions. Then we use $c = 1 - F_Y(y)$ with $F_Y(y) = \frac{(1+\rho)y^\rho-1}{\rho}$ to get $\mu(c) = \frac{\pi^*(c)}{h^*(c)} = \left(\frac{\rho(1-c)+1}{1+\rho}\right) = [h^*(c)]^\rho$. Now use $p = \mu(c) + c$ to find $h(p) = (1+\rho(1-p))^{1/\rho}$ and hence the (ρ-linear) demand form is tied down, including the value of the constant ($k = 1$: see (6), and consistent with the specification $\overline{c} = 1$).

6 The Logit model of monopolistic competition

We here derive specific results for the Logit model with quality-cost heterogeneity and a continuum of active firms, using a log-linear demand form. Total demand is normalized to 1, so output for Firm i is a Logit function of active firms' qualities and prices:

$$y_i = \hat{h}\left(v_i - p_i\right) = \frac{\exp\left(\frac{v_i - p_i}{\mu}\right)}{\int_{\omega\in\Omega} \exp\left(\frac{v(\omega)-p(\omega)}{\mu}\right) d\omega + \exp\left(\frac{v_0}{\mu}\right)}, \quad i \in \Omega, \tag{21}$$

where $\mu > 0$ measures the degree of product heterogeneity and $v_0 \in (-\infty, \infty)$ measures the attractivity of the outside option (which could also be a competitive sector). We thus adapt the continuous Logit model (see Ben-Akiva and Watanada, 1981) to monopolistic competition.[19]

As before, the (gross) profit for Firm i is $\pi_i = (p_i - c_i)\, y_i$, $i \in \Omega$. Since there is a continuum of firms, the own-demand derivative is $\frac{dy_i}{dp_i} = \frac{-y_i}{\mu}$, $i \in \Omega$, so that $\frac{d\pi_i}{dp_i} = y_i \left[1 - \frac{(p_i - c_i)}{\mu}\right]$, $i \in \Omega$: the term inside the square brackets is strictly decreasing in p_i, so the profit function is strictly quasi-concave and the profit-maximizing price of Firm i is[20]

$$p_i = c_i + \mu, \quad i \in \Omega. \tag{22}$$

As we showed earlier for log-linear demand, the absolute mark-up is the same for all firms.[21] The corresponding equilibrium outputs are

$$y_i = \frac{\exp\left(\frac{x_i}{\mu}\right)}{\int_{\omega\in\Omega} \exp\left(\frac{x(\omega)}{\mu}\right) d\omega + \mathcal{V}_0}, \quad i \in \Omega, \tag{23}$$

where $\mathcal{V}_0 \equiv \exp\left(\frac{v_0}{\mu} + 1\right) \geq 0$, and recall that $x(\omega) = v(\omega) - c(\omega)$ is a one-dimensional parameterization of quality-cost. (23) verifies the output ranking over firms seen before: $y_i > y_j$

[19] Anderson et al. (1992) show that logit demands can be generated from an entropic representative consumer utility function as well as the traditional discrete choice theoretic root (see McFadden, 1978).

[20] For oligopoly with n firms, the equilibrium prices are (implicit) solutions to $p_i = c_i + \frac{\mu}{1-y_i}$, $i = 1...n$. Under symmetry, $p = c + \frac{\mu n}{n-1}$, which converges to $c + \mu$ as $n \to \infty$ (Anderson, de Palma, and Thisse, 1992, Ch.7).

[21] The CES model gives a constant *relative* mark-up property, $p_i^* = c_i(1+\mu)$, regardless of quality (see Section **??**). The similarity between the Logit and CES is not fortuitous: μ is related to ρ in CES models by $\mu = \frac{1-\rho}{\rho}$. Both models can be construed as sharing their individual discrete choice roots (Anderson et al. 1992).

if and only if $x_i > x_j$, for $i, j \in \Omega$. Equilibrium (gross) profit is μy_i, $i \in \Omega$, so outputs and profits are fully characterized by quality-cost levels, yielding the following special case of Theorem **??**:

Proposition 1 *In the Logit Monopolistic Competition model, all firms set the same absolute mark-up, μ. Higher quality-cost entails higher equilibrium output and profit.*

As seen in Section **??**, insofar as higher qualities also bear higher costs then they are also higher priced, but output and profit may well be highest for medium-quality products.

6.1 Quality-cost, output, and profit distributions

Recall that the distribution of quality-cost is $F_X(x) = \Pr(X < x)$, with density $f_X(\cdot)$ and support $[\underline{x}, \infty)$. The corresponding distribution of equilibrium output, $F_Y(y)$, and the relation between x and y is[22]

$$y = \frac{1}{D} \exp\left(\frac{x}{\mu}\right), \quad y \geq \underline{y} = \frac{1}{D} \exp\left(\frac{\underline{x}}{\mu}\right), \tag{24}$$

where we assume henceforth that $f_X(\cdot)$ ensures the output denominator D (which is the aggregate variable) is finite, as is true for any finite support and the examples below:

$$D = M \int_{u \geq \underline{x}} \exp(u/\mu) f_X(u)\, du + \mathcal{V}_0, \tag{25}$$

and $M = \|\Omega\|$ is the total mass of firms. Equilibrium (gross) profit is $\pi = \mu y \geq \underline{\pi} = \mu \underline{y}$. The following uses results from Section **??**.

Proposition 2 *For the Logit Monopolistic Competition model, the quality-cost distribution, $F_X(x)$, generates the equilibrium output distribution $F_Y(y) = F_X(\mu \ln(yD))$ and the equilib-*

[22]Here all firms are active. Section **??** introduces fixed costs to render endogenous the set of active firms.

rium profit distribution $F_\Pi(\pi) = F_X(\mu \ln(\pi D/\mu))$*, where* D *is given by (25). Conversely,* $F_X(x)$ *can be derived from the equilibrium output distribution as* $F_Y\left(\frac{1}{D_Y}\exp\left(\frac{x}{\mu}\right)\right)$ *with* $D_Y = \frac{\mathcal{V}_0}{(1-My_{av})}$*, or from the equilibrium profit distribution as* $F_\Pi\left(\frac{1}{D_\Pi}\exp\left(\frac{x}{\mu}\right)\right)$ *with* $D_\Pi = \frac{\mathcal{V}_0}{(\mu-M\pi_{av})}$*, where* y_{av} *and* π_{av} *denote average output and profit, respectively.*

(Proof in online Appendix 2).

Finally, the distribution of costs $F_C(c)$ and the distribution of prices $F_P(p)$ are related by the mark-up shift, so $F_C(c) = F_P(c+\mu)$, with $\underline{p} = \underline{c} + \mu$. Conversely, knowing the price distribution ties down the cost distribution when the mark-up level, μ, is known. Note some special case results. First, there is no price dispersion if and only if there is no cost dispersion. Second, there is no profit dispersion if and only if there is no quality-cost dispersion: then there is only cost dispersion, which price dispersion mirrors. The "classic" symmetry assumption often analyzed in the literature since Chamberlin (1933) has neither cost nor profit dispersion.

6.2 Specific distributions

We derive the equilibrium profit distributions for the Logit model.[23] Proofs are in Appendix 2.

The *normal* distribution is perhaps the most natural primitive assumption to take for quality-costs. Then profit $\Pi \in (0,\infty)$ is log-normally distributed. The log-normal has sometimes been fitted to firm size distribution (see Cabral and Mata, 2003, for a well-cited study of Portuguese firms). Note that a truncated normal begets a truncated log-normal (which is therefore important once we consider free entry equilibria below).

The simplest text-book case is the *uniform* distribution. Then the equilibrium profit Π has

[23] The reverse relations hold by Proposition 2. Equilibrium output distributions are analogous (as $\pi = \mu y$).

distribution $F_{\Pi}(\pi) = \mu \ln\left(\frac{\pi D}{\mu}\right)$ and its density is unit elastic. A *truncated Pareto* distribution leads to a truncated Log-Pareto for profit (or output).

At a simplistic level, Proposition 2 indicates that we just need to find the log-distribution of the seed distribution. However, we still need to match parameters, as done in Appendix 2 for the examples, and we also need to find the corresponding expression for D and ensure it is defined. Notice too that the methods described above work for more general demands under monopolistic competition (see Section 2).

The most successful function to fit the distribution of firm size has been the Pareto. We reverse-engineer using Proposition 2 to find the distribution of quality-cost. This gives:

Proposition 3 *Let quality-cost be exponentially distributed:* $F_X(x) = 1 - \exp(-\lambda(x - \underline{x}))$, $\lambda > 0$, $\underline{x} > 0$, $x \in [\underline{x}, \infty)$, *with* $\lambda\mu > 1$. *Then equilibrium output and profit are Pareto distributed:* $F_Y(y) = 1 - \left(\frac{\underline{y}}{y}\right)^{\alpha_y}$ *and* $F_{\Pi}(\pi) = 1 - \left(\frac{\underline{\pi}}{\pi}\right)^{\alpha_\pi}$, *where* $\alpha_y = \alpha_\pi = \lambda\mu > 1$. *Conversely, a Pareto distribution for equilibrium output or profit can only be generated by an exponential distribution of quality-costs.*

Thus the shape parameter, $\alpha_y = \alpha_\pi$, for the endogenous economic distributions depends just on the product of the taste heterogeneity and the technology shape parameter.[24]

If the price distribution follows the Pareto form (suggested as empirically viable) $F_P(p) = 1 - \left(\frac{\underline{p}}{p}\right)^{\alpha_p}$, with $p \in [\underline{p}, \infty)$ and $\alpha_p > 1$, the corresponding cost distribution is also Pareto:

$$F_C(c) = 1 - \left(\frac{\underline{c} + \mu}{c + \mu}\right)^{\alpha_p}, \quad c \in [\underline{c}, \infty), \ \alpha_p > 1. \tag{26}$$

[24] D is bounded if $\mu > 1/\lambda$: which requires that taste heterogeneity exceeds average quality-cost.

Suppose for illustration that prices are Pareto distributed so that $F_C(c)$ is given by (26). If there were no quality heterogeneity, then we would find a power distribution for quality-cost,[25] which would therefore be inconsistent with the required exponential function predicated in Proposition 3. It is the extra relation that we are afforded via $\beta(c)$ that decouples the allowable distributions, and therefore can enable us to fit (for example) both Pareto prices and profits. The next Proposition gives the underlying $\beta(c)$ function.

Proposition 4 *Let $F_P(p)$ be Pareto distributed with shape parameter α_p and let $F_\Pi(\pi)$ be Pareto distributed with shape parameter $\alpha_\pi > 1$ and $\underline{\pi} < \frac{\mu}{M}\frac{\alpha_\pi - 1}{\alpha_\pi}$, and suppose that $x = \beta(c)$ is an increasing function. Then $\beta(c) = \beta(\underline{c}) + \mu\frac{\alpha_p}{\alpha_\pi}\ln\left(\frac{c+\mu}{\underline{c}+\mu}\right)$, where $\beta(\underline{c}) = -\mu\ln\left[\frac{1}{\mathcal{V}_0}\left(\frac{\mu}{\underline{\pi}} - \frac{\alpha_\pi M}{\alpha_\pi - 1}\right)\right]$.*

Proof. Let $F_X(x)$ be the exponential function given in Proposition 3. Because $\beta(\cdot)$ is increasing, $F_C(c) = F_X(\beta(c)) = 1 - \exp[-\lambda(\beta(c) - \beta(\underline{c}))]$, $\lambda = \frac{\alpha_\pi}{\mu} > 0$, $\underline{c} > 0$, $c \in [\underline{c}, \infty)$. A Pareto price distribution with shape parameter α_p delivers the cost distribution (26). Equating these two expressions gives

$$\beta(c) = \beta(\underline{c}) + \mu\frac{\alpha_p}{\alpha_\pi}\ln\left(\frac{c+\mu}{\underline{c}+\mu}\right), \ \beta(c) \in [\beta(\underline{c}), \infty].$$

Thus $\beta'(c) > 0$ so that valuations rise faster than costs. The lower bound of the distribution $\beta(\underline{c}) = \underline{x}$ is given from Proposition 3 and is given in the Proposition statement. ■

Thus we can close the loop and deliver a model consistent with Pareto distributions (for example) for both profit and price, once we allow for a quality-cost relation. Figure 1 above is based on just such a logit example (with parameters $\alpha_p = \alpha_\pi = \mu = 2$, $\beta(\underline{c}) = 0$, $\underline{c} = 0$, and

[25] I.e., calling the common quality level $\bar{v}$, $F_X(x) = \left(\frac{\bar{v}-\underline{x}+\mu}{\bar{v}-x+\mu}\right)^{\alpha_p}$, $x \in (-\infty, \underline{x}]$.

$\underline{x} = 1$). So far, we have assumed the population of firm types is fixed: we next use the logit as a vehicle to endogenize the surviving form types when there are entry costs.

6.3 Comparative statics of distributions

So far in the paper we have considered cross-section properties of distributions and how they can be uncovered. Although it is not our main theme, we can also give some indication of comparative static properties across equilibria. How do distributions change with underlying preference and technology changes? This is a more tricky question because the values of the aggregate variables in demand, which we were able to suppress in the cross section analysis because they do not change, are now endogenously determined. While a full treatment is beyond our current scope, we can give some pointers (and deliver results) for the current Logit model, and determine how economic distributions change with fundamentals.

We here therefore briefly consider the comparative static properties for the Logit model. Because we are dealing with distributions, the natural way of doing so is to engage first order stochastic dominance (fosd). Proofs for this sub-section are in the CEPR Discussion Paper.

Proposition 5 *A fosd increase or a mean-preserving spread in quality-cost raises mean output and mean profit, and strictly so if the market is not fully covered (i.e., if $\mathcal{V}_0 > 0$).*

While moving up quality-cost mass will move up output mass ceteris paribus, it also increases competition for all the other firms (a D effect), which ceteris paribus reduces their output. Mean output does not necessarily rise if mean quality-cost rises.[26]

[26] Mean output rises with a mean-preserving spread but then if the mean of quality-cost is reduced slightly, mean output can still go up overall, so the two means can move in opposite directions.

Because the relation between output and profit distributions does not involve D, a fosd increase in output implies an increase in profit, and vice versa. However, a fosd increase in quality-cost does not necessarily lead to a fosd increase in output. Suppose for example that the increase in quality-cost is small for low quality-costs, but large for high ones. Then competition is intensified (an increase in D), and output at the bottom end goes down, while rising at the top end. So then there can be a rotation of $F_Y(.)$ (in the sense of Johnson and Myatt, 2006) without fosd (a similar rotation is delivered in Proposition 6 below). Nevertheless, specific examples do deliver stronger relations, as we show in Section 6.2.

We next determine how taste parameters feed through into the endogenous economic distributions.

Proposition 6 *A more attractive outside option ($\mathcal{V}_0$) fosd decreases outputs and profits. More product differentiation (μ) fosd increases outputs and profits for low quality-cost and fosd decreases them for high quality-cost; a lower profit implies a firm has a lower output.*

The first result is quite obvious, but the impact of higher product heterogeneity is more subtle.[27] When μ goes up, weak (low quality-cost) firms are helped and good ones are hurt. The intuition is as follows. With little product differentiation, consumers tend to buy the best quality-cost products. With more product differentiation (which increases the mark-up), consumers tend to buy more of the low quality-cost goods (which have lower outputs) and less of the high quality-cost goods (which have higher outputs). Hence, higher μ evens out demands across options. The fact that output may decrease and profit increase with μ follows because

[27] The proof proceeds by showing that the derivative of consumer surplus is given by the Shannon (1948) measure of information (entropy), which is positive.

$\pi_i = \mu y_i$. Thus it can happen that doubling μ does not double the profit of the top quality-cost firms, but it may more than double the profit of the lowest quality-cost firms. Whether high or low qualities are most profitable depends on whether quality-costs rise or fall with quality.

7 Conclusions

The basic ideas here are simple. Market performance depends on the economic fundamentals of tastes and technologies, and how these interact in the market-place.[28] The fundamental distribution of tastes and technologies feeds through the economic process to generate the endogenous distribution of economic variables, such as prices, outputs, and profits. Invoking the monopolistic competition assumption delivers a straight feed-through from fundamental distributions to performance distributions.

Conversely, the derived economic distributions can be reverse engineered to back out the model's primitives. Lemma 6 inverts the mark-ups to deliver both the equilibrium output choices and the form of the demand curve. This analysis constitutes an stand-alone contribution to the theory of monopoly pass-through, extending Weyl and Fabinger (2013) by working from pass-through *back* to implied demands. Theorem 4 shows how to use price and cost distributions to find the shape of the profit and output distributions and demand form (up to a positive factor). Theorem 8 shows how to invert the (potentially observable) output and profit distributions to find the underlying net inverse demand form (i.e., demand up to a cost shift), and the underlying primitive cost distribution, $F_C(c)$. Theorem 3 does likewise with price and profit distributions as the starting point (again up to a positive constant). Theorem 5, 6,

[28]Firm size distributions came to the fore in Chris Anderson's (2006) work on the Long Tail of internet sales.

and 7 shows that all other distribution pairs tie down all primitives and outcome distributions (including constants).

The Logit model gives some similar properties to the CES, while differing on others. For example, the simple CES has constant percentage mark-ups while the Logit has constant absolute mark-ups.[29] The Logit can be deployed for similar purposes as the CES, and has an established pedigree in its micro-economic underpinnings. It has a strong econometric backdrop which is at the heart of much of the structural empirical industrial organization revolution. For Logit, a normal quality-cost distribution leads to a log-normal distribution of firm size, and an exponential quality-cost distribution generates a Pareto distribution. But, our main cross-sectional analysis extends far beyond the restrictive IIA property of Logit and CES.

Indeed, our main results in the heart of the paper show how to back out the demand form from distributions. Surprisingly, this can be done just from profit and price distributions (for example) if firms differ only by production costs.

Another future research direction we only initiated here (in the context of Logit) is the comparative statics of distributions. Hopefully, we have given some pointers how to find how equilibrium distributions change with changes in underlying fundamentals, namely the quality-cost relation and consumer tastes. More can be contributed on understanding the inheritance properties of distributions.

[29] The latter property is perhaps quite descriptive for cinema movies, DVDs, and CDs.

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Appendix 1

Proof of Proposition 3

We first calculate the logit denominator, D, from (25), using the exponential CDF $F_X(x) = 1 - \exp(-\lambda(x - \underline{x}))$ with density $f_X(x) = \lambda \exp(-\lambda(x - \underline{x}))$ and $\lambda > 0$, $\underline{x} > 0$, and $x \in [\underline{x}, \infty)$. Integrating,

$$D = \frac{M\lambda\mu}{\lambda\mu - 1} \exp\left(\frac{\underline{x}}{\mu}\right) + \mathcal{V}_0,$$

which is positive and bounded for any $\mathcal{V}_0$ under the assumption that $\lambda\mu > 1$. Now, from Proposition 2,

$$F_\Pi(\pi) = 1 - \exp\left(-\lambda\left(\mu \ln\left(\frac{\pi D}{\mu}\right) - \underline{x}\right)\right) = 1 - \left(\frac{\pi D}{\mu}\right)^{-\lambda\mu} \exp(\lambda\underline{x}).$$

The profit, $\underline{\pi}$, of the lowest quality-cost firm solves $F_\Pi(\underline{\pi}) = 0$, and thus verifies the expected property $\underline{\pi} = \frac{\mu}{D} \exp\left(\frac{\underline{x}}{\mu}\right)$. Inserting this value back into $F_\Pi(\pi)$ gives the expression in Proposition 3. The output distribution follows from the profit distribution:

$$F_Y(y) = \Pr(Y < y) = \Pr\left(\frac{\Pi}{\mu} < y\right) = F_\Pi(\mu y) = 1 - \left(\frac{\underline{\pi}}{\mu y}\right)^{\lambda\mu} = 1 - \left(\frac{\underline{y}}{y}\right)^{\lambda\mu},$$

where the lowest output, $\underline{y}$, is associated to the lowest profit, $\underline{\pi} = \mu\underline{y}$.

The last statement follows from Theorem 2: starting with a Pareto distribution for output or profit implies an underlying exponential distribution for quality-cost. The lowest quality-cost is given by the condition $\underline{y} = \frac{1}{D} \exp\left(\frac{\underline{x}}{\mu}\right)$, so

$$\underline{\pi} = \mu\underline{y} = \frac{\mu}{\frac{M\lambda\mu}{\lambda\mu - 1} \exp\left(\frac{\underline{x}}{\mu}\right) + \mathcal{V}_0} \exp\left(\frac{\underline{x}}{\mu}\right) < \frac{1}{M}\left(\mu - \frac{1}{\lambda}\right). \tag{27}$$

Inverting (27) gives $\underline{x}$.

Proof of Theorem ??

Because $F_X(x)$ is continuous and increasing on support $[\underline{x}, \bar{x}]$ and because $\beta^{-1}(x)$ is continuous and increasing on support $[\underline{x}, \bar{x}]$ with $\beta^{-1}(\underline{x}) > 0$. Then $F_C(c) = \Pr(C < c)$ is uniquely defined and continuous and increasing on support $[\underline{c}, \bar{c}]$:

$$F_C(c) = \Pr\left(\beta^{-1}(X) < c\right) = \Pr(X < \beta(c)) = F_X(\beta(c)).$$

The last term is a continuous and increasing function of a continuous and increasing function, so $F_C(c)$ is recovered. Constructing $F_X(x)$ from $F_C(c)$ and $\beta(c)$ is completely analogous.

We now show how to construct a unique increasing $\beta(c)$ from the two distributions: let $F_X(x) = \Pr(X < x)$ and we postulate that there exists a continuous increasing function $\beta(C) = X$ and so $F_X(x) = \Pr(\beta(C) < x) = \Pr\left(C < \beta^{-1}(x)\right)$ which is then equal to $F_C\left(\beta^{-1}(x)\right)$. Now, since $F_X(x) = F_C\left(\beta^{-1}(x)\right)$, then $\beta^{-1}(x) = F_C^{-1}(F_X(x))$, so $\beta(x) = \left[F_C^{-1}(F_X(x))\right]^{-1}$ and $\beta(x) = F_X^{-1}(F_C(x))$. This is clearly increasing and continuous in x as desired.

The claim in the Theorem is shown because $F_X(x)$ can be used to construct the other distributions on its leg, and can be constructed from them; and likewise for $F_C(c)$.

Appendix 2 Distribution details (NOT FOR PUBLICATION)

Proof of Proposition 2

We first seek the distribution of outputs, $F_Y(y) = \Pr(Y < y)$, that is generated from the primitive distribution of quality-cost. First note from (24) that $Y = \frac{\exp\left(\frac{X}{\mu}\right)}{D}$, so:

$$F_Y(y) = \Pr\left(\frac{\exp\left(\frac{X}{\mu}\right)}{D} < y\right) = F_X(\mu \ln(yD)),$$

where D is given by (25). Because equilibrium profit is proportional to output ($\pi = \mu y$), we have a similar relation for the distribution of profit, $F_\Pi(\pi) = \Pr(\Pi < \pi)$:

$$F_\Pi(\pi) = \Pr\left(\mu\frac{\exp\left(\frac{X}{\mu}\right)}{D} < \pi\right) = F_X\left(\mu \ln\left(\frac{\pi D}{\mu}\right)\right),$$

where D is given by (25).

We next prove the converse result. We first determine the distribution of quality-costs consistent with a given observed distribution of output. Suppose that output has a distribution $F_Y(y)$. Applying the increasing transformation $y = \frac{1}{D}\exp\left(\frac{x}{\mu}\right)$, and $Y = \frac{1}{D}\exp\left(\frac{X}{\mu}\right)$, we get:

$$\begin{aligned} F_X(x) &= \Pr\left(\frac{1}{D}\exp\left(\frac{X}{\mu}\right) < \frac{1}{D}\exp\left(\frac{x}{\mu}\right)\right) \\ &= \Pr\left(Y < \frac{1}{D}\exp\left(\frac{x}{\mu}\right)\right) = F_Y\left(\frac{1}{D}\exp\left(\frac{x}{\mu}\right)\right). \end{aligned}$$

However, D is written in terms of $f_X(x)$, and we want to find the distribution solely in terms of $F_Y(y)$: this means writing D in terms of $f_Y(y)$. The corresponding expression, denoted D_y

is derived below as (28). Similar reasoning gives the profit expression:

$$F_X(x) = \Pr(X < x) = \Pr\left(\Pi < \frac{\mu}{D}\exp\left(\frac{x}{\mu}\right)\right) = F_\Pi\left(\frac{\mu}{D}\exp\left(\frac{x}{\mu}\right)\right),$$

where the expression for D in terms of $f_\Pi(\pi)$ (i.e., D_π) is given in the Theorem and derived below as (29).

We now show here how to write the function D as a function of $f_Y(.)$ or $f_\pi(.)$.

We first find the value of D in terms of the distribution of Y. Recall (25):

$$D = M\int_{u \geq \underline{x}} \exp\left(\frac{u}{\mu}\right) f_X(u)\, du + \mathcal{V}_0.$$

Now, $\Pr(x < X) = \Pr(y < Y)$, so $y(x) = \frac{\exp\left(\frac{x}{\mu}\right)}{M\int \exp\left(\frac{u}{\mu}\right) f_X(u) du + \mathcal{V}_0}$; hence $D\int y(u) f_Y(u)\, du = (D - \mathcal{V}_0)/M$, and thus:

$$D_y = \frac{\mathcal{V}_0}{1 - My_{av}}. \tag{28}$$

The denominator in the last expression is necessarily positive because My_{av} is total output, which is less than one when $\mathcal{V}_0 > 0$ because the market is not fully covered. Similarly, $F_X(x) = F_\Pi\left(\frac{\mu}{D}\exp\left(\frac{x}{\mu}\right)\right)$, so that

$$D_\pi = \frac{\mathcal{V}_0}{1 - \frac{M}{\mu}\pi_{av}}, \tag{29}$$

which is now expressed as a function of $f_\Pi(.)$, and where π_{av} is average firm profit. (29) is positive because total profit, $M\pi_{av}$, is less than μ because the market is not fully covered ($\mathcal{V}_0 > 0$).

Study of specific distributions

We now derive the distributions in Section 6.2: these involve parameter matching for the distribution examples.

Normal: For the normal, $F_X(x) = \frac{1}{\sigma\sqrt{2\tilde{\pi}}}\int_{-\infty}^{x}\exp\left(-\frac{(u-m)^2}{2\sigma^2}\right)du$, where $\tilde{\pi} = 3.1415...$. From Theorem 2, we have

$$F_\Pi(\pi) = F_X\left(\mu\ln\left(\frac{\pi D}{\mu}\right)\right),$$

where $\mu\ln\left(\frac{\pi D}{\mu}\right) \in (-\infty, \infty)$, so

$$F_\Pi(\pi) = F_X\left(\mu\ln\left(\frac{\pi D}{\mu}\right)\right) = \frac{1}{\sigma\sqrt{2\tilde{\pi}}}\int_{-\infty}^{\mu\ln\left(\frac{\pi D}{\mu}\right)}\exp\left(-\frac{(u-m)^2}{2\sigma^2}\right)du.$$

Using the change of variable $\Pi = \frac{\mu}{D}\exp\left(\frac{u}{\mu}\right)$ (so $u = \mu\ln\left(\frac{\Pi D}{\mu}\right)$ and $du = \frac{\mu}{\Pi}d\Pi$) we obtain

$$F_\Pi(\pi) = \frac{\mu}{\sigma\sqrt{2\tilde{\pi}}}\int_0^\pi\exp\left(-\frac{\left(\mu\ln\left(\frac{\Pi D}{\mu}\right)-m\right)^2}{2\sigma^2}\right)\frac{d\Pi}{\Pi},$$

which can be written in a standard form as:

$$F_\Pi(\pi) = \frac{1}{\left(\frac{\sigma}{\mu}\right)\sqrt{2\tilde{\pi}}}\int_0^\pi\frac{1}{\Pi}\exp\left(-\frac{\left(\ln\Pi-\left(\ln\left(\frac{D}{\mu}\right)-\frac{m}{\mu}\right)\right)^2}{2\left(\frac{\sigma}{\mu}\right)^2}\right)d\Pi.$$

Hence profits are log-normally distributed with parameters $\left[\ln\left(\frac{D}{\mu}\right)-\frac{m}{\mu}\right]$ and $\left(\frac{\sigma}{\mu}\right)$.

Recall: $D = M\int_{u\geq\underline{x}}\exp\left(\frac{u}{\mu}\right)f_X(u)\,du + \mathcal{V}_0$. Then:

$$D = \frac{M}{\sigma\sqrt{2\tilde{\pi}}}\int_{-\infty}^{\infty}\exp\left(\frac{x}{\mu}\right)\exp\left(\frac{-(x-m)^2}{2\sigma^2}\right)dx + \mathcal{V}_0.$$

Routine computation shows that:

$$D = M\exp\left(\frac{m}{\mu}+\frac{\sigma^2}{2\mu^2}\right)+\mathcal{V}_0.$$

Logistic.

A *logistic* distribution for quality-cost has a CDF given by $F_X(x) = \left(1 + \exp\left(-\frac{x-m}{s}\right)\right)^{-1}$, $x \in (\underline{x}, \infty)$, with mean m and variance $s^2\pi^2/3$. The PDF is similar in shape to the normal, but it has thicker tails (see the discussion in Fisk, 1961, and the comparison with the Weibull distribution). Hence, for $\mu > s$, profit $\Pi \in (0, \infty)$ is log-logistically distributed with parameters $\left(\frac{D}{\mu}\right) \exp\left(-\frac{m}{\mu}\right)$ and $\frac{\mu}{s}$:

$$F_\Pi(\pi) = \left(1 + \left(\frac{\pi}{\left(\frac{D}{\mu}\right) \exp\left(-\frac{m}{\mu}\right)}\right)^{-\frac{\mu}{s}}\right)^{-1}, \qquad \pi \in [0, \infty).$$

There is no closed form expression for D in this case. However, it can be shown that the condition $\mu > s$ guarantees that the output denominator D exists. The Log-logistic distribution (which provides a one parameter model for survival analysis) is very similar in shape to the log-normal distribution, but it has fatter tails. It has an explicit functional form, in contrast to the Log-normal distribution.

The logistic distribution (with mean m and standard deviation $s\tilde{\pi}/\sqrt{3}$) is given by:

$$F_X(x) = \frac{1}{1 + \exp\left(-\frac{x-m}{s}\right)}, x \in (\underline{x}, \infty).$$

From Theorem 2,

$$F_\Pi(\pi) = F_X\left(\mu \ln\left(\frac{\pi D}{\mu}\right)\right) = \frac{1}{1 + \exp\left(-\frac{\mu \ln\left(\frac{\pi D}{\mu}\right) - m}{s}\right)},$$

where $F_{\Pi}(0) = 0$ and $F_{\Pi}(\infty) = 1$. Thus

$$F_{\Pi}(\pi) = \frac{1}{1 + \exp\left(\frac{m}{s}\right) \exp\left(\ln\left(\frac{\pi D}{\mu}\right)^{-\frac{\mu}{s}}\right)} = \frac{1}{1 + \left(\frac{\pi}{\left(\frac{D}{\mu}\right)\exp\left(-\frac{m}{\mu}\right)}\right)^{-\frac{\mu}{s}}}.$$

Recall the log-logistic distribution is defined as:

$$F^{LL}(x; \alpha, \beta) = \frac{1}{1 + \left(\frac{x}{\alpha}\right)^{-\beta}}, \quad x > 0.$$

Thus, the parameter matching is:

$$F_{\Pi}(\pi) = F^{LL}\left(x; \left(\frac{D}{\mu}\right) \exp\left(-\frac{m}{\mu}\right), \frac{\mu}{s}\right).$$

We need to check when D converges, i.e., when $\int_{-\infty}^{\infty} \exp\left(\frac{x}{\mu}\right) f_X(x)\, dx$ converges. Because

$$f_X(x) = \frac{1}{s} \frac{\exp\left(-\frac{x-m}{s}\right)}{\left(1 + \exp\left(-\frac{x-m}{s}\right)\right)^2},$$

we need to ensure the convergence of the expression

$$\int_{-\infty}^{\infty} \frac{\exp\left(-x\left(\frac{1}{s} - \frac{1}{\mu}\right)\right)}{\left(1 + \exp\left(-\frac{x-m}{s}\right)\right)^2} dx.$$

Convergence is guaranteed if and only if $\mu > s$.

Pareto: The Pareto distribution is given by: $F_X(x) = \frac{1-\left(\frac{\underline{x}}{x}\right)^{\alpha}}{1-\left(\frac{\overline{x}}{\underline{x}}\right)^{-\alpha}}$. From Theorem 2

$$F_{\Pi}(\pi) = \frac{1 - \left(\frac{\underline{x}}{\mu \ln\left(\frac{\pi D}{\mu}\right)}\right)^{\alpha}}{1 - \left(\frac{\overline{x}}{\underline{x}}\right)^{-\alpha}}.$$

Recall that $\pi = \frac{\mu \exp\left(\frac{x}{\mu}\right)}{D}$ or $x = \mu \ln\left(\frac{\pi D}{\mu}\right)$, so that $\underline{x} = \mu \ln\left(\frac{\underline{\pi} D}{\mu}\right)$ and $\overline{x} = \mu \ln\left(\frac{\overline{\pi} D}{\mu}\right)$, so that $F_{\Pi}\left(\underline{\pi}\right) = 0$ and $F_{\Pi}\left(\overline{\pi}\right) = 1$. D is bounded because the quality-cost distribution is bounded.

Consider a log-Pareto distribution with scale parameter σ and shape parameters γ and β:

$$F^{LP}\left(\pi; \gamma, \beta, \sigma\right) = \frac{1 - \left(1 + \frac{1}{\beta} \ln\left(1 + \frac{\pi - \underline{\pi}}{\sigma}\right)\right)^{-\frac{1}{\gamma}}}{1 - \left(1 + \frac{1}{\beta} \ln\left(1 + \frac{\overline{\pi} - \underline{\pi}}{\sigma}\right)\right)^{-\frac{1}{\gamma}}}, \qquad \pi > \underline{\pi}.$$

In order to match parameters, of $F_{\Pi}\left(\pi\right)$ with $F^{LP}\left(\pi; \gamma, \beta, \sigma\right)$ observe that

$$\mu \ln\left(\frac{\pi D}{\mu}\right) = \mu \ln\left(\frac{\pi D}{\mu} \frac{\underline{\pi}}{\underline{\pi}}\right) = \underline{x} + \mu \ln\left(\frac{\pi}{\underline{\pi}}\right) = \underline{x} + \mu \ln\left(1 + \frac{\pi - \underline{\pi}}{\underline{\pi}}\right).$$

Therefore:

$$F_{\Pi}\left(\pi\right) = \frac{1 - \underline{x}^{\alpha}\left(\mu \ln\left(\frac{\pi D}{\mu}\right)\right)^{-\alpha}}{1 - \left(\frac{\overline{x}}{\underline{x}}\right)^{-\alpha}} = \frac{1 - \left(1 + \frac{\mu}{\underline{x}} \ln\left(1 + \frac{\pi - \underline{\pi}}{\underline{\pi}}\right)\right)^{-\alpha}}{1 - \left(\frac{\overline{x}}{\underline{x}}\right)^{-\alpha}}.$$

Thus π obeys a Log-Pareto distribution $F^{LP}\left(\pi; \frac{1}{\alpha}, \frac{\underline{x}}{\mu}, \underline{\pi}\right)$, i.e., $\gamma = \frac{1}{\alpha}, \beta = \frac{\underline{x}}{\mu}, \sigma = \underline{\pi}$.

It remains to check that the normalization factors are equal. Recall that $\overline{\pi}/\underline{\pi} = \exp \frac{\overline{x} - \underline{x}}{\mu}$. Using the specification $\gamma = \frac{1}{\alpha}, \beta = \frac{\underline{x}}{\mu}, \sigma = \underline{\pi}$, we get:

$$\begin{aligned}
\left(1 + \frac{\mu}{\overline{x}} \ln\left(1 + \frac{\overline{\pi} - \underline{\pi}}{\underline{\pi}}\right)\right)^{-\alpha} &= \left(1 + \frac{\mu}{\overline{x}} \ln\left(\frac{\overline{\pi}}{\underline{\pi}}\right)\right)^{-\alpha} \\
&= \left(1 + \left(\frac{\overline{x} - \underline{x}}{\overline{x}}\right)\right)^{-\alpha} = \left(\frac{\overline{x}}{\underline{x}}\right)^{-\alpha}.
\end{aligned}$$