Revenue sharing on hierarchies

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Abstract

We consider a model of joint venture where agents are organized on a hierarchical network and each agent produces her revenue through collaborating with her superiors. We study the problem of allocating the total revenue among these agents. A hierarchy is represented by a directed tree. We investigate allocation rules that are robust to reallocation of revenues within any coalition that includes all the superiors of its members. We obtain characterizations of such rules imposing standard axioms in the literature of fair allocation theory.

1 Introduction

Hierarchies are commonly observed in most organizations. Members in an organization take different levels of responsibilities, which give rise to a hierarchical decision structure. A hierarch may also describe a structure of command or permission that the organization relies on for her functionings. The organization can achieve stability when her members cooperate on a hierarchy as shown by Demange [1]. We consider a model of joint venture where agents are organized on a hierarchical network. Each agent in our model produces her revenue through collaborating with her superiors. For example, superiors supervise subordinates and offer advice or guidance, which plays a role in generating revenues. We investigate rules of allocating the total revenue among the agents in this model.

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The allocation rules of our interest are the family of transfer rules that are described by "upward transfer" of revenues (in a hierarchy) suggested by Hougaard et al. [2] and their extension. In transfer rules, each agent receives a fixed fraction of the sum of her revenue and the transferred revenues from her subordinates and transfers the rest to her immediate superiors equally. Hougaard et al. [2] provide an axiomatic characterization of these rules. Their conclusion relies on the feature of population variability in their model. By considering a suitable weakening of their axioms in our setup without population variability, we provide an axiomatic characterization of a larger family of rules that include all transfer rules as well as their extensions that allow different transfer rates at different positions in the hierarchy.

Hougaard et al. [2] investigate what rules are non-manipulable when a group of agents and all their superiors can merge into a single representative agent or when the representative agent can split into herself and dummies. They show that such a merging or a splitting cannot increase the payoff under any transfer rule and moreover, transfer rules are the only non-manipulable rules satisfying some other standard axioms. In our model, the set of agents is fixed and so a group of agents cannot merge or split. Nevertheless, they may reallocate their revenues by forming a coalition inclusive of all superiors. Our non-manipulability by such a reallocation is called reallocation-proofness. A well-behaved rule from a normative point of view may fail to achieve its normative goal, if it is not reallocation-proof, due to tactical reallocation of revenues. Ju [3] characterizes reallocation-proof rules in the setting where a coalition is made feasible by its connectivity in a network structure. In our investigation, a coalition can form only if they are connected, and in addition, it includes all the superiors of its members. Thus our reallocation-proofness is weaker than his. This stricter restriction arises from the hierarchical structure and its nature of directed permission. This restriction in coalition formation allows us to obtain much more diverse reallocation-proof rules than in Ju [3].

Types of hierarchies we consider in this paper are extensive enough to include those represented by a directed tree. Agents can have not only one immediate superior, but multiple immediate superiors; hierarchical networks here are any directed network not containing any undirected cycle. We characterize a family of rules called here the generalized transfer rules. They are asymmetric variants of transfer rules; two symmetry conditions of transfer rules are relaxed. First, when each agent gets a share of collected revenues from her subordinates including herself, she is allowed to take the share at her own rate, not necessarily at the same rate as other agents. Second, when each agent then transfers the rest of revenues to her immediate superiors, she is allowed to distribute it at different ratio; an immediate superior may receive more than another immediate superior.

2 Model

We consider a problem of allocating revenues generated by a set of agents through their hierarchical collaboration. Let $N = \{1, 2, \dots, n\}$ be the set of agents. These agents form a hierarchy given by a directed tree, a directed network not containing any undirected cycle. The hierarchy is represented by a correspondence $S : N \to 2^N$ that maps each agent $i \in N$ to her *immediate superiors* $S(i) \subset N$. When agent i has no immediate superior, that is, $S(i) = \emptyset$, agent i is referred to as a *top* agent. Note that there can be multiple top agents. Agent j is an *immediate subordinate* of agent i if $i \in S(j)$. For each pair $i, j \in N$, i is a *superior* of j, if $i = j$ or there is a finite sequence of agents $(a_1, a_2, \dots, a_m)$ such that $a_1 = i$, $a_m = j$, and $a_k \in S(a_{k+1})$ $\forall k = 1, \dots, m-1$. In this case, we also call j a *subordinate* of i. Let $sp : N \to 2^N$ be the correspondence that maps each agent $i \in N$ to all her superior agents (including herself), and $sp^0(i) \equiv sp(i) \setminus \{i\}$. Similarly let $sb : N \to 2^N$ be the correspondence that maps each agent $i \in N$ to all her subordinates and $sb^0(i) \equiv sb(i) \setminus \{i\}$. Refer to an agent who has no subordinate other than herself (i.e., $sb(i) = \{i\}$) as a *bottom agent*. A (undirected) *path* is a finite sequence of different agents $(a_1, a_2, \dots, a_m)$ such that either $a_k \in S(a_{k+1})$ or $a_{k+1} \in S(a_k)$ holds for all $k = 1, \dots, m-1$. This sequence is called a path from a_1 to a_m, and we say that agent a_k is on the path for every k. The path is unique as the network is a directed tree. A subset of agents $T \subset N$ is (weakly) *connected* if for every $i, j \in T$, we can find a path from i to j such that any agent on the path is a member of T.

In what follows, we fix hierarchy (N, S) and consider allocation problems associated with (N, S). Agents generate individual revenues over the hierarchy through cooperating with their superiors. More precisely, at each position $i \in N$ in the hierarchy, agent i together with her superiors generate revenue $r_i \in \mathbb{R}_+$. Denote the profile of revenues by $r = (r_i)_{i \in N}$. The problem is to allocate the total revenue $\sum_{i \in N} r_i$ among n agents. We denote this (*revenue sharing*) *problem* by r. A (feasible) *allocation* for r is a profile of non-negative payoffs $x \in \mathbb{R}^n_+$ satisfying $\sum_{i \in N} x_i \leq \sum_{i \in N} r_i$. A (budget) *bal-*

anced allocation is an allocation satisfying $\sum_{i\in N} x_i = \sum_{i\in N} r_i$. An *allocation rule* is a function $f : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ associating with each problem r an allocation $f(r)$. It satisfies *efficiency* if it always selects a balanced allocation.

The following notation will be used. For a vector $x \in \mathbb{R}^n_+$ and a subset of agents $T \subset N$, $x(T) \equiv \sum_{i\in T} x_i$ and $x_T \equiv (x_i)_{i\in T}$. For all $i \in N$, let (x'_i, x_{-i}) be the revenue profile obtained by replacing the i-th component x_i with x'_i.

We now present a family of allocation rules crucial in our investigation. Under these rules, each agent in the hierarchy keeps a fraction of her *gross revenue*, the sum of her own revenue and the transferred revenues from her immediate subordinates, and transfers the rest to her immediate superiors. A simple case of these rules called here as *transfer rules* is suggested by Hougaard et al. [2]. Sole parameter $\lambda \in [0, 1]$ is the rate at which each agent keeps her gross revenue. First, consider a bottom agent i. The gross revenue is just her revenue, $R_i(r; f^\lambda) = r_i$. She receives a fraction of her gross revenue, and allocates the rest $(1-\lambda)R_i(r; f^\lambda)$ equally to her immediate superiors;

$$f_i^\lambda(r) = \lambda R_i(r; f^\lambda) = \lambda r_i.$$

Next, consider agent j whose all immediate subordinates are bottom agents. She first collects her gross revenue, which is her revenue plus transferred amounts from immediate subordinates.

$$R_j(r; f^\lambda) = r_j + \sum_{h\colon j\in S(h)} \frac{1-\lambda}{|S(l)|} R_h(r; f^\lambda).$$

She then takes λ fraction of gross revenue,

$$f_j^\lambda(r) = \lambda R_j(r; f^\lambda) = \lambda \left(r_j + \sum_{h\colon j\in S(h)} \frac{1-\lambda}{|S(l)|} R_h(r; f^\lambda) \right),$$

and again transfers the rest of gross revenue to her immediate superiors equally. Payoffs of other agents are recursively computed. If t is a top agent, she receives all her gross revenue; $f_t^\lambda(r) = R_t(r; f^\lambda)$.

An alternative representation of transfer rules that follows does not impose recursive argument.

Definition. [Transfer rule]

An allocation rule is a transfer rule with λ if, for some $\lambda \in [0, 1]$, it holds that for all $r \in \mathbb{R}^n_+$,

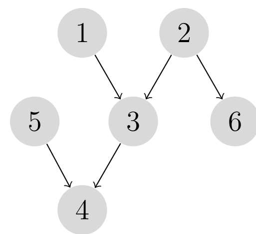


Figure 1: An example of hierarchy.

if $i \in N$ is not a top agent,

$$f_i(r) = \lambda \left[r_i + \sum_{j \in sb^0(i)} \left[\prod_{l \in sp(j) \cap sb^0(i)} \frac{(1-\lambda)}{|S(l)|} \right] r_j \right],$$

and if $t \in N$ is a top agent,

$$f_t(r) = r_t + \sum_{j \in sb^0(t)} \left[\prod_{l \in sp(j) \cap sb^0(t)} \frac{(1-\lambda)}{|S(l)|} \right] r_j.$$

We call the transfer rule with $\lambda = 0$ as the full transfer rule and the one with $\lambda = 1$ as the zero-transfer rule.

Example 1. Consider a set of agents $N = \{1,2,3,4,5,6\}$ with $r = (4,1,11,3,6,9)$, and $S(3) = \{1,2\}$, $S(5) = \{2\}$, $S(4) = \{3\}$, $S(6) = \{5\}$. The hierarchy is represented as in Figure 1.

The transfer rule with λ allocates $f_4(r) = 3\lambda$, $f_6(r) = 9\lambda$, $f_5(r) = 6 + \frac{3}{2}(1-\lambda)$, $f_3(r) = \lambda(11 + \frac{3}{2}(1-\lambda))$, $f_1(r) = 4 + \frac{1}{2}(1-\lambda)(11 + \frac{3}{2}(1-\lambda))$, $f_2(r) = 1 + \frac{1}{2}(1-\lambda)(11 + \frac{3}{2}(1-\lambda)) + 9(1-\lambda)$. When $\lambda = 1/3$, the allocation vector is $f(r) = (8, 11, 4, 1, 7, 3)$. The full transfer rule allocates $(\frac{41}{4}, \frac{65}{4}, 0, 0, \frac{15}{2}, 0)$ and the zero-transfer rule allocates $(4,1,11,3,6,9)$.

Another family of allocation rules is a generalized version of the family of transfer rules. The family of *generalized transfer rules* relaxes symmetry in two ways; first, upward transfers of gross revenue at differentiated rates across agents are allowed, and second, each agent may divide the upward transfer at any rate among her immediate superiors. For

Definition. [Generalized Transfer rule]

An allocation rule is a generalized transfer rule with λ and μ if, for some $\lambda \equiv (\lambda_i) \in [0,1]^n$ with $\lambda_t = 1 \; \forall t \in T$ where$T$ is a set of top agents, and $\mu \equiv (\mu_i)_{i \in N \setminus T}$ where $\mu_i = (\mu_{ip})_{p \in S(i)} \in \mathbb{R}_+^{|S(i)|}$ satisfies $\mu_{ip} \geq 0$ for all $p \in S(i)$ and $\sum_{p \in S(i)} \mu_{ip} = 1$, it holds that for all $r \in \mathbb{R}_+^n$, and for all $i \in N$,

$$f_i(r) = \lambda_i \left[r_i + \sum_{j \in sb^0(i)} \left[\prod_{k \in sp(j) \cap sb^0(i)} (1-\lambda_k)\mu_{ki} \right] r_j \right].$$

By abuse of notation, we write μ_{ki} instead of $\mu_{kp_{(k,i)}}$, where $p_{(k,i)}$ is defined as the immediate superior of k that lies between k and i.[1]

Let us revisit the hierarchy in Example 1. The set of top agents is $T = \{1,2,5\}$. Let $\lambda_3 = \frac{3}{5}$, $\lambda_4 = \frac{1}{3}$, $\lambda_6 = \frac{2}{3}$ so that $\lambda = (1, 1, \frac{3}{5}, \frac{1}{3}, 1, \frac{2}{3})$, and $\mu_{31} = \frac{3}{7}$, $\mu_{32} = \frac{4}{7}$, $\mu_{43} = \frac{1}{3}$, $\mu_{45} = \frac{2}{3}$, $\mu_{62} = 1$, so that $\mu = (\frac{3}{7}, \frac{4}{7}, \frac{1}{3}, \frac{2}{3}, 1)$. The generalized transfer rule with λ and μ allocates $f(r) = (6, \frac{20}{3}, 7, 1, \frac{22}{3}, 6)$.

Note that generalized transfer rules with λ and μ are transfer rules when for all $i \in N \setminus T$, $\lambda_i = \bar{\lambda}$ and $\mu_{ip} = \frac{1}{|S(i)|}$ for all $p \in S(i)$. When

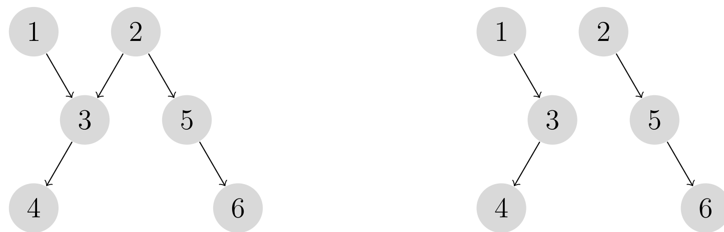
Definition. [Hierarchical Equal Sharing rule]
The hierarchical equal sharing rule f^{HES} allocates for all $r \in \mathbb{R}_+^n$, and all $i \in N$,

$$f_i^{HES}(r) = \sum_{j \in sb(i)} \frac{1}{|sp(j)|} r_j.$$

3 Axioms

In this section, we introduce axioms. The first one prevents advantageous coalitional manipulation. It requires that any feasible coalition should not gain from reallocating revenues within the coalition. To be concrete, the total payoff allocated to the coalition cannot change by any reallocation of revenues among its members. In the hierarchy, an agent generates revenue by collaborating with its superiors. Superiors supervise subordinates and take responsibility of their performances. Hence, a coalition is feasible if it is connected, and contains all superiors of each member. We denote the set of all these feasible coalitions by $\mathcal{F}(N,S)$. Formally, $F \in \mathcal{F}(N,S)$ if and only if $F \subset N$ is connected and $\forall j \in F,\ sup(j) \subset F$.

[1]This is possible because we are considering directed trees, in which any undirected path between two agents are unique.



(a) (b)

Figure 2: (a) An example of a hierarchy (N, S). (b) The edge between agent 2 and 3 is removed and N separates into two components.

Definition. [Superiors-Reallocation-Proofness, SRP]: For all $r, r' \in \mathbb{R}^n_+$, and all $F \in \mathcal{F}(N, S)$, if $r(F) = r'(F)$ and $r_{N\setminus F} = r'_{N\setminus F}$, then

$$\sum_{i\in F} f_i(r) = \sum_{i\in F} f_i(r').$$

The next axiom states that any top agent's revenue is irrelevant for the payoff of all other agents. It is exactly the same as the one in Section 4 of Hougaard et al. [2].

Definition. [Highest Rank Revenue Independence, HRRI] For all $r \in \mathbb{R}^n_+$, if $i \in N$ is a top agent, then for all $\hat{r}_i \in \mathbb{R}_+$,

$$f_{N\setminus\{i\}}(r) = f_{N\setminus\{i\}}(\hat{r}_i, r_{-i}).$$

The following property states that for each agent, only the revenues of her superiors and subordinates is relevant.

Definition. [Independence of Irrelevant Agents, IIA] For all $i \in N$, all $r \in \mathbb{R}^n_+$, and all $\hat{r} \in \mathbb{R}^{|N\setminus(sp(i)\cup sb(i))|}_+$,

$$f_i(r) = f_i(\hat{r}, r_{sp(i)\cup sb(i)}).$$

The family of generalized transfer rules, and thus the family of transfer rules, satisfy all three axioms above.

We observe that if agent i is an immediate superior of agent j, removing the edge between them leaves us with two components.[2] N separates into

[2]Let us say a set of agents $C \subset A$ is a (connected) component of a bigger set of agents A if C is connected, and $C \cup \{i\}$ is not connected for any $i \in A \setminus C$.

two connected components, one including j and the other one including i. Pick the set of agents in the connected component including j, and denote it by $C_{j,i}$. Similarly name the subset of agents linked to i as $C_{i,j}$. Surely $C_{j,i} \cup C_{i,j} = N$. For example, see the hierarchy in Figure 2a. If we remove the edge between agent 2 and 3, two connected components consisting N are $C_{2,3} = \{2, 5, 6\}$ and $C_{3,2} = \{1, 3, 4\}$ as we observe in Figure 2b .

Consider an agent i and her immediate subordinate j. The next axiom says that if agents linked to j (members of $C_{j,i}$) transfer any surplus from the component to i, and generate zero revenue instead, then the payoffs of agents linked to i (members of $C_{i,j}$) remain unchanged. This property is very similar to that in Section 4 of Hougaard et al. [2] . The difference is that in this paper the set of agents is fixed therefore the revenue of leaving agents are nulled instead.

Definition. [Component Null-Consistency, CNC] For all $r \in \mathbb{R}^n_+$, all $i \in N$, and all $p \in S(i)$, if r' is defined as

$$r'_j = \begin{cases} r_j + \sum_{k \in C_{i,p}} (r_k - f_k(r)), & \text{for } j = p. \\ 0, & \text{for } j \in C_{i,p}. \\ r_j, & \text{otherwise.} \end{cases}$$

then

$$f_{C_{p,i}}(r) = f_{C_{p,i}}(r').$$

The following is a standard axiom.

Definition. [Scale Invariance, SI] For all $r \in \mathbb{R}^n_+$, and all $\alpha > 0$,

$$f(\alpha r) = \alpha f(r).$$

Note that the two families of generalized transfer rules and transfer rules satisfy component null-consistency and scale invariance.

We next consider two symmetry axioms. First, for a given agent i, consider the ith unit vector, $(1, 0_{-i})$. The axiom requires that every agent $i \in N$ should get the same payoff when the revenue profile is ith unit vector.

Definition. [Unit Revenue Symmetry, URS] For all $r \in \mathbb{R}^n_+$, and all pair $i, j \in N$, if both are not top agents,

$$f_i(1, 0_{-i}) = f_j(1, 0_{-j}).$$

For the second one, consider an agent i, and her immediate superiors p and p' with identical revenues. Suppose that all agents in $C_{p,i}$ except p have zero revenue, and all agents in $C_{p',i}$ except p' have zero revenue too. Then the axiom says that the payoff of the two immediate superiors are the same.

Definition. [Superiors Symmetry, SS] For all $r \in \mathbb{R}^n_+$, all $i \in N$, and each pair $p, p' \in S(i)$, if $r_j = 0$ for all $j \in C_{p,i} \setminus \{p\}$ and $j \in C_{p',i} \setminus \{p'\}$, and $r_p = r_{p'}$, then

$$\sum_{l \in C_{p,i}} f_l(r) = \sum_{l \in C_{p',i}} f_l(r).$$

The two symmetries help us pin down the family of generalized transfer rules to the family of transfer rules. The last axiom states that agents with zero revenue gets zero payoff.

Definition. [No Award for Null, NA] For all $r \in \mathbb{R}^n_+$, and all $i \in N$, if $r_i = 0$, then

$$f_i(r) = 0.$$

The zero-transfer rule satisfies this property whereas the family of transfer rule does not. A top agent, for example, earns positive payoff if the transfer rate λ is positive and at least one of her subordinates creates positive revenue.

Definition. [Null Superior Symmetry, NSS] For all $i \in N$, and all $r_{sb(i)} \in \mathbb{R}^{|sb(i)|}_+$, if $j, k \in sp(i)$, then

$$f_j(0_{-sb(i)}, r_{sb(i)}) = f_k(0_{-sb(i)}, r_{sb(i)}).$$

The followings are two examples of allocation rules that satisfy superiors-reallocation-proofness.

Example 2. Equal division rule $f_i(r) = \frac{1}{N} r(N)$ for every $i \in N$ satisfies SRP, SI, URS. It does not satisfy HRRI, IIA, CNC, SS, NA.

Example 3. Any agent who is not a top agent gets $f_i = \lambda r_i$ and the top agents that are superiors of i shares the rest, $(1-\lambda) r_i$. Assume they share it equally. This rule satisfies SRP, HRRI, IIA, SI, URS, SS. It does not satisfy CNC, NA.

4 Results

We will now give a representation of rules satisfying superiors-reallocation-proofness, highest rank revenue independence, and independence of irrelevant agents. To that end, let us introduce new notations. For a set of agents S, let $sb^0(S) \equiv \{i \notin S : i \in sb(j) \text{ for some } j \in S\}$. An agent belongs to $sb^0(S)$ when the agent herself is not a member of S but is a subordinate of some member of S. Proposition **??** states that an allocation rule satisfies SRP, HRRI, and IIA if and only if every agent's payoff depends only on revenues of its subordinates.

Proposition 1. *Let f be an allocation rule. The following are equivalent.*

1. The rule f satisfies Superiors-Reallocation-Proofness, Highest Rank Revenue Independence, Independence of Irrelevant Agents, and Scale Invariance.

2. There exist $\alpha \in \mathbb{R}_+$, and nonnegative and degree-1-homogeneous functions $\{g^F : \mathbb{R}_+^{|sb^0(F)|} \to \mathbb{R}_+\}_{F \in \mathcal{F}(N,S) \setminus \{N\}}$ such that for all $r \in \mathbb{R}^n_+$,

$$\sum_{i \in N} f_i(r) = \alpha r(N),$$

and all $F \in \mathcal{F}(N,S) \setminus \{N\}$,

$$\sum_{i \in F} f_i(r) = \alpha r(F) + g^F(r_{sb^0(F)}).$$

3. There exist $\alpha \in \mathbb{R}_+$, and nonnegative and degree-one homogeneous functions $\{h_i : \mathbb{R}_+^{|sb(i)|} \to \mathbb{R}_+\}_{i \in N}$ such that for all $r \in \mathbb{R}^n_+$, and all $i \in N$,

$$f_i(r) = h_i(r_{sb(i)}), \tag{1}$$

and

$$\sum_{i \in N} h_i(r_{sb(i)}) = \alpha r(N). \tag{2}$$

Proof. ($1 \Rightarrow 2$) Fix $r \in R^n_+$. If $r = 0_N$, because of SI we must have $\sum_{i \in N} f_i(r) = 0 = r(N)$. Otherwise, define $\hat{r} = \frac{1}{r(N)} r$, then by SI, $\sum_{i \in N} f_i(r) = r(N) \sum_{i \in N} f_i(\hat{r})$. Note that by SRP, there exists some $\alpha \in \mathbb{R}_+$ such that for all $\hat{r}(N) = 1$, $\sum_{i \in N} f_i(\hat{r}) = \alpha$. For this α,

$$\sum_{i \in N} f_i(r) = \alpha r(N). \tag{3}$$

For $F \in \mathcal{F}(N,S) \setminus \{N\}$, pick a top player $t \in F$ from the coalition. Consider another problem r' where the total revenue of F is reallocated to t: $r'_t = r(F)$, $r'_{F\setminus\{t\}} = 0$, and $r'_{N\setminus F} = r_{N\setminus F}$. By SRP, this reallocation should not change total payoff given to F,

$$\sum_{i\in F} f_i(r) = \sum_{i\in F} f_i(r'). \tag{4}$$

For every $i \neq t$, by HRRI, $f_i(r') = f_i(0, r'_{-t}) = f_i(0_F, r_{N\setminus F})$. Since by (3) we have

$$\sum_{i\in N} f_i(r') = \alpha r(N)$$

and

$$\sum_{i\in N} f_i(0_F, r_{N\setminus F}) = \alpha[r(N) - r(F)],$$

we may write $f_t(r') = \alpha r(F) + f_t(0_F, r_{N\setminus F})$, hence

$$\sum_{i\in F} f_i(r) = \alpha r(F) + \sum_{i\in F} f_i(0_F, r_{N\setminus F}).$$

Next, IIA tells us that for all $i \in F$, $f_i(0_F, r_{N\setminus F}) = f_i(0_{N\setminus sb^0(F)}, r_{sb^0(F)})$. To see why, compare the revenue profile of i's superiors and subordinates in two problems $(0_F, r_{N\setminus F})$ and $(0_{N\setminus sb^0(F)}, r_{sb^0(F)})$. If $j \in sp(i) \subset F$ or $j \in sb(i) \cap F$, j's revenue is the same in two problems (as zero). If $j \in sb(i) \cap F^{\mathsf{c}}$, then $j \in sb^0(F)$, therefore j's revenue is r_j in both problems.

We then may write

$$\sum_{i\in F} f_i(r) = \alpha r(F) + \sum_{i\in F} f_i(0_{N\setminus sb^0(F)}, r_{sb^0(F)}), \tag{5}$$

and naturally define g^F as for $x \in \mathbb{R}_+^{|sb^0(F)|}$,

$$g^F(x) \equiv \sum_{i\in F} f_i(0_{N\setminus sb^0(F)}, x).$$

Clearly g^F is nonnegative, and is homogeneous of degree one: for $\beta > 0$, by SI,

$$g^F(\beta x) = \beta \sum_{i\in F} f_i(0_{N\setminus sb^0(F)}, x) = \beta g^F(x).$$

($2 \Rightarrow 3$) For α, we may just pick the same α from 2. Next, fix $r \in R^n_+$. If $i \in N$ is a top agent, $\{i\}$ is feasible. By 2, $f_i(r) = \alpha r_i + g^{\{i\}}(r_{sb^0(i)})$. Define h_i as for $x \in \mathbb{R}^{|sb(i)|}_+$, $h_i(x) = \alpha x_i + g^{\{i\}}(x_{-i})$. Showing that h_i is homogeneous of degree one is straightforward.

If $i \in N$ is not a top agent, let $P \equiv \{i\} \cup (\cup_{p\in S(i)} C_{p,i})$ be the union of components connected to her immediate superiors, and herself (note that $\{\{i\}\} \cup \{C_{p,i} : p \in S(i)\}$ is a partition of P). Then, we may write i's payoff as the following difference:

$$f_i(r) = \sum_{j\in P} f_j(r) - \sum_{p\in S(i)} \sum_{k\in C_{p,i}} f_k(r).$$

Because each $C_{p,i}$ and P are all feasible, we may use 2 to write

$$f_i(r) = \alpha r(P) + g^P(r_{sb^0(P)}) - \sum_{p\in S(i)} [\alpha r(C_{p,i}) + g^{C_{p,i}}(r_{sb^0(C_{p,i})})],$$

which reduces to

$$f_i(r) = \alpha r_i + g^P(r_{sb^0(P)}) - \sum_{p\in S(i)} g^{C_{p,i}}(r_{sb^0(C_{p,i})}).$$

Observing $sb^0(C_{p,i}) = \{i\} \cup sb^0(i) = sb(i)$ for all $p \in S(i)$, and $sb^0(P) = sb^0(i)$, this is rewritten as

$$f_i(r) = \alpha r_i + g^P(r_{sb^0(i)}) - \sum_{p\in S(i)} g^{C_{p,i}}(r_{sb(i)}).$$

Because the right hand side is fully decided once $r_{sb(i)}$ is known, we can define h_i as for $x \in \mathbb{R}^{|sb(i)|}_+$, $h_i(x) = \alpha x_i + g^P(x_{-i}) - \sum_{p\in S(i)} g^{C_{p,i}}(x)$. Also, h_i is homogeneous of degree one because g^P and $g^{C_{p,i}}$ are.

($3 \Rightarrow 1$) HRRI and IIA are immediately verified. For SRP, fix $F \in \mathcal{F}(N, S)$ and $r \in \mathbb{R}^n_+$, and let $r' \in \mathbb{R}^n_+$ be another problem that satisfies $r'(F) = r(F)$, and $r'_{N\setminus F} = r_{N\setminus F}$. First, define the immediate subordinate set for F as: $K(F) = \{k \in N : k \notin F,\ sp(p) \subset F \text{ for some } p \in S(k)\}$.[3] An agent k is a member of $K(F)$ when k is not a member of F but is an immediate subordinate of a member of F. For $i \in N \setminus F$, i is connected to a member of immediate subordinate set of F; $i \in C_{k,q}$ for some $k \in K(F)$ and

[3]This K set is similar to that in the proof of theorem 1 in Demange [1].

$q \in S(k) \cap F$. It follows that $sb(i) \subset N \setminus F$. From 3, this implies that for all $i \in N \setminus F$, $f_i(r) = f_i(r')$. Since $\sum_{i \in N} f_i(r) = \sum_{i \in N} f_i(r') = \alpha r(N)$, we have$\sum_{j \in F} f_j(r) = \sum_{j \in F} f_j(r')$. $\square$

Because a generalized transfer rule allocates each agent the value that only depends on revenues of her own and her subordinates, it is now clear that generalized transfer rules satisfy SRP, HRRI and IIA.

The following result is a characterization of the family of generalized transfer rules.

Proposition 2. *(Directed trees, not including rooted trees) An allocation rule f satisfies Superiors-Reallocation-Proofness, Highest Rank Revenue Independence, Independence of Irrelevant Agents, Scale Invariance, and Component Null-Consistency if and only if it is a generalized transfer rule, or for all $i \in N$ and $r \in \mathbb{R}^n_+$, $f_i(r) = 0$.*

Proof. It is easy to see the rules satisfy axioms. For the converse, suppose that f satisfies five axioms. When the hierarchy is a directed tree, we may pick an agent $i \in N$ who has two or more immediate superiors. With abuse of notation, for the proof we use the notation S instead of $S(i)$. Let $k = |S| \geq 2$ be the number of i's immediate superiors. Consider a problem where all other agents except i's immediate superiors make zero revenue: $\hat{r} = (r, 0_{N \setminus S})$ for some $r \in \mathbb{R}^k_+$, and let $x = f(\hat{r})$. If $j \notin \cup_{p \in S} C_{p,i}$, then in problem $\hat{r}$, revenue of all her subordinates including herself is zero, so by Proposition 1, $x_j = f(0_N) = 0$ (especially, $x_i = 0$). Therefore, the Proposition tells us that there exists $\alpha \geq 0$ such that

$$\sum_{p \in S} x(C_{p,i}) = \alpha \hat{r}(N) = \alpha r(S). \tag{6}$$

For $p \in S$, by SRP, $x(C_{p,i}) = y(C_{p,i})$ for $y = f(\hat{r}_p + \hat{r}(C_{i,p}) - x(C_{i,p}), 0_{-p})$. Because $\hat{r}_p + \hat{r}(C_{i,p}) = r(S)$ and $x(C_{i,p}) = \sum_{p' \in S, p' \neq p} x(C_{p',i})$, by SI,

$$x(C_{p,i}) = [r(S) - \sum_{p' \in S, p' \neq p} x(C_{p',i})] \sum_{j \in C_{p,i}} f_j(1, 0_{-p}0).$$

Note that by Proposition 1, for $j \in C_{p,i}$, $f_j(1, 0_{-p}) = 0$, so $\sum_{j \in C_{p,i}} f_j(1, 0_{-p}0) = \alpha$. The argument applies to any agent in S, therefore, for all $p \in S(i)$,

$$x(C_{p,i}) + \alpha[\sum_{p' \in S, p' \neq p} x(C_{p',i})] = \alpha r(S).$$

Sum up all k equations, then

$$[1+(k-1)\alpha][\sum_{p\in S} x(C_{p,i})] = k\alpha r(S).$$

If we multiply (6) by $1+(k-1)\alpha$ and subtract the previous equation from it, we get

$$\alpha(1+(k-1)\alpha-k)r(S) = \alpha(\alpha-1)(k-1)r(S) = 0.$$

Because $k \geq 2$, and the equation holds for arbitrary $r \in \mathbb{R}^k_+$, either $\alpha = 0$ or $\alpha = 1$. If $\alpha = 0$, equation (2) and non-negativity of payoff imply that for any $r' \in \mathbb{R}^n_+$, $f_i(r') = 0$. If $\alpha = 1$, f is efficient.

When f is efficient, we will show that f is a generalized transfer rule. Let T denote the set of top agents. Construct λ and μ as follows: $\lambda_i \equiv f_i(1, 0_{-i})$ for each $i \in N \setminus T$, $\lambda_t = 1$ for each $t \in T$, and

$$\mu_{ip} \equiv \begin{cases} \frac{\sum_{k\in C_{p,i}} f_k(1,0_{-i})}{1-\lambda_i}, & \text{if } \lambda_i \neq 1. \\ \frac{1}{|S(i)|}, & \text{if } \lambda_i = 1. \end{cases}$$

for each $i \in N \setminus T$ and each $p \in S(i)$. Clearly we must have $0 \leq \lambda_i \leq 1$ and $0 \leq \mu_{ip} \leq 1$ for such i and p. To see why for each i, $\sum_{p\in S(i)} \mu_{ip} = 1$, assume $\lambda_i \neq 1$ (it is trivial if $\lambda_i = 1$). By the way μ_{ip} are defined, we must have $\sum_{k\in C_{p,i}} f_k(1, 0_{-i}) = \mu_{ip}(1-\lambda_i)$.[4] As a consequence of proposition 1, $f_j(1, 0_{-i}) = 0$ for each $j \in C_{b,i}$ for every immediate subordinate b of i. Then it follows from balance $f_i(1, 0_{-i}) + \sum_{p\in S(i)} \sum_{k\in C_{p,i}} f_k(1, 0_{-i}) = 1$. By substituting, we may rewrite the equality as $\lambda_i + (1-\lambda_i)\sum_{p\in S(i)} \mu_{ip} = 1$ which reduces to $\sum_{p\in S(i)} \mu_{ip} = 1$.

We now argue that f is a generalized transfer rule with λ and μ.

Claim: For all $r \in \mathbb{R}^n_+$ and all $i \in N$,

$$f_i(r) = \lambda_i \left[r_i + \sum_{j\in sb^0(i)} \left[\prod_{k\in sp(j)\cap sb^0(i)} (1-\lambda_k)\mu_{ki} \right] r_j \right]. \tag{7}$$

Proof of Claim. If $i \in N$ is a bottom agent, by proposition 1 and SI, $f_i(r) = f_i(r_i, 0_{-i}) = r_i f_i(1, 0_{-i}) = \lambda_i r_i$. Define $\hat{r} = (r_i, 0_{-i})$ and $x = f(\hat{r})$,

[4]Note that this equality also holds even when $\lambda_i = 1$ because both sides equal zero.

then for i's immediate superior $p \in S(i)$, $\hat{r}(C_{i,p}) = r_i$. By balance, SI, and definition of μ_{ip}, we may write $r_i = x(C_{p,i}) + x(C_{i,p}) = \mu_{ip}(1-\lambda_i)r_i + x(C_{i,p})$. Hence, $\hat{r}(C_{i,p}) - x(C_{i,p}) = \mu_{ip}(1-\lambda_i)r_i$. Note that the last equality holds for every bottom agent i, and every immediate superior $p \in S(i)$.

Next, we argue that for an agent $i \in N$ who is neither a bottom agent nor a top agent, if the following two statements hold,

1. For each $j \in sb^0(i)$, $f_j(r)$ equals the payoff from the generalized transfer rule with λ and μ.

2. For each $j \in sb^0(i)$, and for each $q \in S(j)$, define $\tilde{r} = (r_{sb(j)}, 0_{N\setminus sb(j)})$ and $y = f(\tilde{r})$, then

$$\tilde{r}(C_{j,q}) - y(C_{j,q}) = \sum_{k \in sb(j)} \left[\prod_{l \in sp(k) \cap sb(j)} (1-\lambda_l)\mu_{lq} \right] r_k. \tag{8}$$

then the followings hold.

1. $f_i(r)$ equals the payoff from the generalized transfer rule with λ and μ.

2. Define $\hat{r} = (r_{sb(i)}, 0_{N\setminus sb(i)})$ and $x = f(\hat{r})$. For each $p \in S(i)$,

$$\hat{r}(C_{i,p}) - x(C_{i,p}) = \sum_{k \in sb(i)} \left[\prod_{l \in sp(k) \cap sb(i)} (1-\lambda_l)\mu_{lp} \right] r_k.$$

Assume that the first two statements hold, and let B denote the set of immediate subordinates of i. Let $\hat{r} = (r_{sb(i)}, 0_{N\setminus sb(i)})$ and $x = f(\hat{r})$. Then

$$f_i(r) = f_i(\hat{r}) = f_i(r_i + \sum_{b \in B} (\hat{r}(C_{b,i}) - x(C_{b,i})), 0_{-i}) \tag{9}$$

holds by CNC. For $b \in B$, let $\tilde{r} = (r_{sb(b)}, 0_{N\setminus sb(b)})$ and $y = f(\tilde{r})$. Then $\hat{r}(C_{b,i}) = \tilde{r}(C_{b,i})$, and since an agent's payoff depends only on revenues of her subordinates, $x(C_{b,i}) = y(C_{b,i})$.[5] By (8) and SI, (9) equals

$$f_i(r) = \left[r_i + \sum_{b \in B} \sum_{k \in sb(b)} \left[\prod_{l \in sp(k) \cap sb(b)} (1-\lambda_l)\mu_{li} \right] r_k \right] f_i(1, 0_{-i}). \tag{10}$$

[5]If $l \in C_{b,i}$ is not a superior of b, $x_l = f_l(0_N) = 0$ because $sb(i) \cap sb(l) = \emptyset$.

Because $\lambda_i = f_i(1, 0_{-i})$, this is rewritten as

$$f_i(r) = \lambda_i \left[r_i + \sum_{k \in sb^0(i)} \left[\prod_{l \in sp(k) \cap sb^0(i)} (1 - \lambda_l)\mu_{li} \right] r_k \right]$$

which proves the first part.

For the second part, let us compute $\hat{r}(C_{i,p}) - x(C_{i,p})$ for every $p \in S(i)$. Obviously $\hat{r}(C_{i,p}) = r(sb(i))$ and by balance, $r(sb(i)) = x(C_{p,i}) + x(C_{i,p})$ holds. By CNC and SI,

$$\begin{aligned} x(C_{p,i}) &= \textstyle\sum_{l \in C_{p,i}} f_l(r_i + \sum_{b \in B} (\hat{r}(C_{b,i}) - x(C_{b,i}), 0_{-i}) \\ &= \left[r_i + \textstyle\sum_{k \in sb^0(i)} \left[\prod_{l \in sp(k) \cap sb^0(i)} (1 - \lambda_l)\mu_{li} \right] r_k \right] \mu_{ip}(1 - \lambda_i). \end{aligned}$$

Therefore we have $\hat{r}(C_{i,p}) - x(C_{i,p}) = \sum_{k \in sb(i)} \left[\prod_{l \in sp(k) \cap sb(i)} (1 - \lambda_l)\mu_{li} \right] r_k$.

Using the above argument, we can sequentially show that for agents that are not top agents, f gives the payoff from the generalized transfer rule with λ and μ. What remains is to verify the statement also holds for top agents. Let $i \in N$ be a top agent. Following the argument above similarly, we can reach the equality (10). In addition, because i is a top agent, $f_i(0_{-i}, 1) = 1$ by HRRI. Then, $f_i(r) = r_i + \sum_{k \in sb^0(i)} \left[\prod_{l \in sp(k) \cap sb^0(i)} (1 - \lambda_l)\mu_{li} \right] r_k$, which completes the proof. $\diamondsuit$ $\square$

Proposition 3. *(Rooted trees) An allocation rule f satisfies Superiors-Reallocation-Proofness, Highest Rank Revenue Independence, Independence of Irrelevant Agents, Scale Invariance, and Component Null-Consistency if and only if it is a generalized transfer rule, or there is $\alpha \geq 0$ such that for all $r \in \mathbb{R}^n_+$, $f_t(r) = \alpha r(N)$ for the top agent t and $f_i(r) = 0$ for all $i \neq t$.*

Proof. Suppose that f satisfies five axioms, and fix $r \in \mathbb{R}^n_+$. Because the hierarchy is a rooted tree, there is only one top agent t, and each other agent has only one immediate superior. Let $\lambda_i \equiv f_i(1, 0_{-i})$ for each $i \in N$. For all $i \neq t$, $f_i(1, 0_{-t}) = f_i(0)$ by part 3 of Prop. 1. Also by (2), for all $i \in N$, $f_i(0) = 0$. Therefore, for all $i \neq t$, $f_i(1, 0_{-t}) = 0$ and $f_t(1, 0_{-t}) = \sum_{i \in N} f_i(1, 0_{-t}) = \alpha$, for $\alpha \geq 0$, given in part 3 of Prop. 1. Thus

$$\lambda_t = f_t(1, 0_{-t}) = \alpha. \tag{11}$$

Claim: For all $r \in \mathbb{R}^n_+$ and all $i \in N$,

$$f_i(r) = \lambda_i \left[r_i + \sum_{j \in sb^0(i)} \left[\prod_{k \in sp(j) \cap sb^0(i)} (1 - \lambda_k) \right] r_j \right]. \tag{12}$$

Proof of Claim. If $i \in N$ is a bottom agent, applying Proposition 1 and SI, $f_i(r) = f_i(r_i, 0_{-i}) = \lambda_i r_i$. Next, we show that for all $i \in N$, if i is not a bottom agent and for all $j \in sb^0(i)$,

$$f_j(r) = \lambda_j \left[r_j + \sum_{l \in sb^0(j)} \left[\prod_{k \in sp(l) \cap sb^0(j)} (1 - \lambda_k) \right] r_l \right], \tag{13}$$

we have

$$f_i(r) = \lambda_i \left[r_i + \sum_{j \in sb^0(i)} \left[\prod_{k \in sp(j) \cap sb^0(i)} (1 - \lambda_k) \right] r_j \right].$$

Let j be i's immediate subordinate. Then by CNC, $f_i(r) = f_i(\hat{r})$, when in problem $\hat{r}$, agents linked to j transfer any surplus from the component to i, and generate zero revenue instead: $\hat{r} = (r_i + x, 0_{sb(j)}, r_{N \setminus sb(i)})$, where $x = \sum_{k \in sb(j)} r_k - \sum_{k \in sb(j)} f_k(r)$. (Note that $C_{j,i} = sb(j)$ in rooted trees.) Substituting (13), $x = \sum_{l \in sb(j)} \left[\prod_{k \in sp(l) \cap sb^0(j)} (1 - \lambda_k) \right] r_l$. Now applying similar argument for other immediate subordinates of i, we get $f_i(r) = f_i(r_i + y, 0_{sb^0(i)}, r_{N \setminus sb(i)})$, where

$$y = \sum_{j : S(j) = \{i\}} \sum_{l \in sb(j)} \left[\prod_{k \in sp(l) \cap sb^0(j)} (1 - \lambda_k) \right] r_l = \sum_{j \in sb^0(i)} \left[\prod_{k \in sp(j) \cap sb^0(i)} (1 - \lambda_k) \right] r_j.$$

The last equality holds because $sb^0(i) = \cup_{j : S(j) = \{i\}} sb(j)$ in rooted trees. Proposition 1 then tells us $f_i(r) = f_i(r_i + y, 0_{-i}) = \lambda_i \left[r_i + \sum_{j \in sb^0(i)} \left[\prod_{k \in sp(j) \cap sb^0(i)} (1 - \lambda_k) \right] r_j \right]$. $\diamondsuit$

Moreover, by Proposition 1, for all $r' \in \mathbb{R}^n_+$,

$$\sum_{i \in N} f_i(r') = \alpha r'(N). \tag{14}$$

When $r' = (1, 0_{-t})$, the left-hand side of (14) equals $f_t(1, 0_{-t}) = \lambda_t = \alpha$ by (11).

When $r' = (1, 0_{-i})$ for $i \neq t$, using Claim, the left-hand side of (14) is given by

$$\sum_{j \in sp(i)} f_j(r') = \lambda_i + \sum_{j \in sp^0(i)} \lambda_j \prod_{k \in sp(i) \cap sb^0(j)} (1 - \lambda_k) = \sum_{j \in sp(i)} \lambda_j [\prod_{k \in sp(i) \cap sb^0(j)} (1 - \lambda_k)], \tag{15}$$

where $\prod_{k\in sp(i)\cap sb^0(j)}(1-\lambda_k)=1$ if $sp(i)\cap sb^0(j)=\emptyset$.

Using (11), we obtain:

$$\lambda_t[\prod_{k\in sp(i)\setminus\{t\}}(1-\lambda_k)]=\alpha[1-\sum_{l\in sp(i)\setminus\{t\}}\lambda_l\prod_{k\in sp(i)\cap sb^0(l)}(1-\lambda_k)].$$

Therefore we may rewrite (15) as

$$\alpha+(1-\alpha)[\sum_{j\in sp(i)\setminus\{t\}}\lambda_j[\prod_{k\in sp(i)\cap sb^0(j)}(1-\lambda_k)],$$

which equals α, the right-hand side of (14). Then we obtain: for all $i\neq t$,

$$(1-\alpha)[\sum_{j\in sp(i)\setminus\{t\}}\left[\prod_{k\in sp(i)\cap sb^0(j)}(1-\lambda_k)\right]\lambda_j]=0. \tag{16}$$

Suppose $\alpha=1$. Then it follows from non-negativity and $\sum_{j\in N}f_j(1,0_{-i})=\alpha$ that $0\leq\lambda_i=f_i(1,0_{-i})\leq 1$ for each $i\neq t$, and $\lambda_t=\alpha=1$. This means that f is a generalized transfer rule. Now suppose $\alpha\neq 1$. We show that $\lambda_i=0$ for each $i\in N\setminus\{t\}$. To see this, start from agent i who has only one superior (the top agent) other than herself, then (16) becomes $(1-\alpha)\lambda_i=0$, which implies $\lambda_i=0$. It is easy to see that if $\lambda_j=0$ holds for all $j\in sp^0(i)\setminus\{t\}$, then $\lambda_i=0$ as well. □

If we in addition add the two symmetries, the family of transfer rules is singled out from the family of generalized transfer rules.

Proposition 4. *An efficient allocation rule f satisfies Superiors-Reallocation-Proofness, Highest Rank Revenue Independence, Independence of Irrelevant Agents, Component Null-Consistency, Scale Invariance, Superior Symmetry, and Unit Revenue Symmetry if and only if it is a transfer rule.*

Proof. Verifying that a transfer rule satisfies SS and URS are straightforward. For the inverse, first we recall that f is a generalized transfer rule with λ and μ as defined in the proof of Proposition **??**. For $i\in N$ that is not a top agent, since $\lambda_i=f_i(0_{-i},1)$, by URS $\lambda_i=\lambda$ for all such i. Next, given two immediate superiors $p,p'\in S(i)$, suppose r is a problem such that $r_j=0$ for all $j\in C_{p,i}\setminus\{p\}$, $j\in C_{p',i}\setminus\{p'\}$, and $r_p=r_{p'}$. Because μ_{ip} is defined as to satisfy $\sum_{l\in C_{p,i}}f_l(0_{-i},1)=\mu_{i,p}(1-\lambda)$, by SS, $\mu_{ip}=\mu_{ip'}$. Since this holds for any i and any pair $p,p'\in S(i)$, $\mu_{ip}=\frac{1}{|S(i)|}$ for all i and $p\in S(i)$. □

The last result offers a characterization of zero-transfer rules. The no award for null axiom together with superiors-reallocation-proofness and highest revenue independence is strong enough to allow us drop three axioms, independence of irrelevant agents, component null consistency, and scale invariance.

Proposition 5. *An efficient allocation rule f satisfies Superiors-Reallocation-Proofness, Highest Rank Revenue Independence, and No award for Null if and only if it is the zero-transfer rule.*

Proof. Apparently the zero-transfer rule satisfies all three axioms. To show the inverse, we start by showing that if $i \in N$ is a top agent, then $f_i(r) = r_i$ for every $r \in \mathbb{R}^n_+$. For all $j \neq i$, by HRRI, $f_j(r) = f_j(0, r_{-i})$. By balance, we have two equalities $\sum_{k \in N} f_k(r) = r(N)$ and $\sum_{k \in N} f_k(0, r_{-i}) = r(N) - r_i$. Then, $f_i(r) = r_i + f_i(0, r_{-i}) = r_i$ holds by NA.

When i is not a top agent, if $f_j(r) = r_j$ for every $j \in sp^0(i)$ then $f_i(r) = r_i$. To see this, pick a top agent among superiors of i, $t \in sp(i)$. Let r' be a new problem where the total revenue of i's superiors is reallocated as t generating $r(sp(i))$, and other agents making zero revenue: $r'_t = r(sp(i))$, $r'_{sp(i)\setminus\{t\}} = 0$, and $r'_{N \setminus sp(i)} = r_{N \setminus sp(i)}$. Because $sp(i) \in \mathcal{F}(N, S)$, by SRP, $r(sp^0(i)) + f_i(r) = r'(sp^0(i)) + f_i(r')$. NA then implies $r(sp^0(i)) + f_i(r) = r(sp(i))$. Therefore, $f_i(r) = r_i$.

Now we are ready to show that $f_i(r) = r_i$ holds for every i that is not a top agent. For given i, if all superiors of i other than herself are top agents, i must be in the second position from the top, so that $f_i(r) = r_i$. If every superior of i other than herself is either a top agent or in the second position from the top, again we see $f_i(r) = r_i$. Repeatedly, we can show that $f_i(r) = r_i$ holds wherever i lies on hierarchy. □

Proposition 6. *An efficient allocation rule f satisfies Reallocation-Proofness, Highest Rank Revenue Independence, Independence of Irrelevant Agents and Null Superior Symmetry if and only if it is the Hierarchical Equal Sharing rule.*

Proof. If part is easy to see. Conversely, suppose that f satisfy the four axioms. If $i \in N$ is a bottom agent, by proposition **??**, $f_i(r) = f_i(0_{-i}, r_i)$. By NSS, for each $j \in sp(i)$, $f_j(0_{-i}, r_i) = f_i(0_{-i}, r_i)$. For $k \notin sp(i)$, $f_k(0_{-i}, r_i) = f_k(0_N) = 0$ by IIA and balance. Then by balance, it follows that $f_i(r) = \frac{1}{|sp(i)|} r_i = f_i^{HES}(r)$.

Next we claim that for each $i \in N$ who is not a bottom agent, if $f_j(r) = f_j^{HES}(r)\ \forall j \in sb^0(i)$ then $f_i(r) = f_i^{HES}(r)$. First, define $x = f(0_{-sb(i)}, r_{sb(i)})$, then proposition **??** tells us $f_i(r) = x_i$. For i's superior $k \in sp(i)$, by NSS, $x_k = x_i$. For an irrelevant agent $l \notin sp(i) \cup sb(i)$, by IIA $x_l = y_l$ if we denote $y = f(0_{-sb^0(i)}, r_{sb^0(i)})$. There are two cases; first assume that (i) l is a superior to a subordinate of i. Since an agent's payoff only depends on revenue profile of her subordinates by proposition **??**, we must have $y_l = z_l$ for $z = (0_{-(sb^0(i)\cap sb(l))}, r_{sb^0(i)\cap sb(l)})$. By NSS this equals $\sum_{j\in sb^0(i)\cap sb(l)} \frac{1}{|sp(j)|} r_j$. If it is the case that (ii) l is not a superior to any of i's subordinates, so that $l \notin sp(j)\backslash sp(i)$ for any $j \in sb^0(i)$, then by IIA and balance we have $y_l = f_l(0_N) = 0$.

Taking sum of payoff given to all agents other than superiors of i, we see $\sum_{j\in N\backslash sp(i)} x_j = \sum_{j\in sb^0(i)} \frac{|sp(j)|-|sp(i)|}{|sp(j)|} r_j$. By balance, $x_i = \frac{1}{|sp(i)|}(\sum_{j\in sb(i)} r_j - \sum_{j\in sb^0(i)} \frac{|sp(j)|-|sp(i)|}{|sp(j)|} r_j) = \sum_{j\in sb(i)} \frac{1}{|sp(j)|} r_j$. That is, $f_i(r) = f_i^{HES}(r)$. The proof becomes complete by using the claim in order starting from bottom agents. $\square$

5 Conclusion

In a revenue sharing model with a hierarchy, transfer rules and their asymmetric multiple-parameter extensions comprise the family of generalized transfer rules. We offer a characterization of this family with superiors-reallocation-proofness and some other axioms. As a corollary, we obtain an alternative characterization of transfer rules suggested by Hougaard et al. [2].

Similar results for generalized transfer rules establishes for hierarchies where each agent has a single immediate superior. For those simpler hierarchies, parameters of generalized transfer rules are only the rates at which an agent transfers to her immediate superior, and we may drop independence of irrelevant agents.

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