Optimal Index versus Simple Index for Monetary Policy

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Abstract

If the degree of nominal rigidities is heterogeneous across sectors (or regions), relative price distortions and resulting welfare losses are more pronounced in a more rigid sector/region. In this environment, a well-established economic principle suggests that it is welfare improving to stabilize an *optimally* constructed price index that places a disproportionately larger weight on 'stickier' sectors/regions. In practice, however, policy discussions are often centered around (standard) *simple* indices – such as CPI or PCEPI – that overlook sectoral heterogeneity in rigidities, thereby treating all sectors symmetrically. In this paper, we explore two potential reasons for the disconnection between the theory and the practice in a stylized currency union setting where a single utilitarian central bank governs multiple member countries that produce differentiated goods and thus are naturally characterized by a different degree of nominal rigidities. First, the welfare gain from adopting an optimal index over a simple index may be small depending on the level of real and financial integration within the union. Second, adopting optimal index may be politically infeasible as some member countries would be better off exiting the union when the relative weight on country inflation are too imbalanced.

Keywords: Inflation targeting, Multi-sector New-Keynesian, Heterogeneity in price stickiness, Currency union, Financial frictions, Trade frictions, Monetary cooperation.

JEL Code: E52, E58, F41, F42, F45

1 Introduction

Inflation has been one of the most researched subject in economics. It is associated with relative price distortions (and resource misallocations) when not all agents respond instantaneously and simultaneously to disturbances – due to various types of nominal rigidities. Because the welfare costs can potentially be significant, the maintenance of price stability has been a primary goal of central banks around the world.

The question of which price index is appropriate for stabilization policy is nontrivial.[1] If not all regions (or sectors) are equally rigid, the resource misallocations and the resulting welfare losses

[1] Another important question is what the appropriate numerical target is for a *given* index. Coibion et al. (2012) is a recent important contribution.

would be more pronounced in more rigid regions/sectors. Two regions in an economy can be heterogeneous in their overall level of nominal rigidity for various reasons: they may be subject to different regulations and institutional frictions; they produce different goods, and thus a region may have more competitive industries than other regions. In this environment, it is welfare improving to stabilize an *optimally* constructed price index that places a disproportionately larger weight on 'stickier' regions. Aoki (2001), Benigno (2004), Mankiw and Reis (2003), and Eusepi, Hobijin, and Tambalotti (2012) propose to target such an optimal price index.[2]

In practice, however, policy discussions are often centered around *simple* and standard indices – such as CPI – that overlook the regional/sectoral heterogeneity in rigidities, thereby treating all regions/sectors symmetrically. For example, the U.S. Federal Open Market Committee (FOMC) stated that the personal consumption expenditure price index (PCEPI) of 2 percent is the long-run goal. The European Central Bank (ECB) on the other hand uses the Harmonised Index of Consumer Prices (HICP) as the primary target.[3]

In this paper, we attempt to reconcile the disconnection between the theory and the practice. Why do inflation targeting central banks not adopt a simple index over an optimal index? We explore two usual suspects: 1) *they don't want to*; 2) *they can't*. Specifically, central banks may have little incentive to adopt an optimal index over a simple index because the welfare gain is insignificant.[4] Furthermore, switching from a simple to optimal index may be politically infeasible – even if central banks were eager for the switch – because changing the relative weights across regions has distributional implications.

We use as our laboratory a stylized currency union that involves a single (utilitarian) central bank and countries with heterogeneous degree of nominal rigidity. Since the member countries share the same currency while being subject to idiosyncratic shocks, the union central bank faces a policy trade-off: inflation stability of member countries vs. efficient movements of the relative price between countries (i.e. the terms of trade.) Completely stabilizing national inflation would eliminate resource misallocations *within* countries, but at the cost of amplifying resource misallocations *across* countries. In other words, the relative production *between* countries (hit by idiosyncratic shocks) would deviate greatly from the first best allocation when the price levels of all member countries are fixed simultaneously. The welfare-optimizing central bank therefore chooses to focus on stabilizing inflation of the sticky country (where inflation creates relatively larger welfare costs), while allowing the flexible country's price level to float in order to achieve more efficient level of relative production. Optimal index– placing a larger weight on the stickier country – is, thus, built to mimic such optimal policy behavior. We indeed find that the optimal inflation targeting (OIT) scheme that stabilizes optimal index not only outperforms the simple inflation targeting (SIT) but

[2]Moreover, Benigno (2004) convincingly shows that stabilizing optimal index is not only welfare improving but also quite close to optimal monetary policy which is hard to implement in reality. We reconfirm the important result in our setting.

[3]Those indices are a simple expenditure weighted average of prices.

[4]Evidently, there must be myriad types of costs in constructing an optimal index (which are absent in our analysis) – given its complexity. While (sector-specific) nominal rigidity is probably the most important and robust factor that decides optimal weights across sectors, other factors such as the labor share in each sector may also be relevant. The central bank will stick to a simple (standard) index if the costs exceed the benefits.

comes quite close to the optimal monetary policy, which does not have to follow any parametric rule, yet is difficult to implement in practice.

The extent of welfare gain from adopting OIT over SIT, however, depends on economic environments. In this paper, we focus on the degree of union integration. One of the original arguments for a currency union was that it would lower trade and financial barriers, thereby promoting the cross-country flows of goods and funds. The integration process however takes time, and at a given point in time a union may be in a certain stage of integration, which may be some steps away from the frictionless utopia. It is then important for the union central bank to understand how SIT would perform relative to the alternative for varying degrees of trade and financial frictions. Are the central bank making a big mistake not adopting OIT or not?

The answer depends on the types of frictions under consideration. We find that the gap between SIT and OIT is less pronounced under financial frictions – the opposite is the case under trade frictions. Besides the degree of union integration, we also find that the nature of the driving forces of inflation are relevant. When shocks are more correlated across countries and when *inefficient* shocks (that would not affect the first best allocation) are relatively more important, the gap between SIT and OIT is narrower.

As mentioned above, the main advantage of OIT over SIT is clear. The central bank under OIT can move the relative price (i.e. the terms of trade) more easily (so that it can promote efficient cross-country relative production) while better stabilizing the sticky country's inflation. Financial frictions diminish the relative benefit of OIT over SIT through two channels. First, the central bank has less incentive to move the relative price due to a new policy tradeoff and thus views OIT less attractive. When one country enjoys a productivity growth, policy should aim to expand output of that country relative to other members by adjusting the terms of trade. But, that expansion, under financial frictions, raises the high-productivity country's consumption relative to other countries, which creates relative consumption distortions. The central bank therefore does not move the relative price as much. Second, such increase in relative consumption stabilizes the marginal cost, which in turn stabilizes inflation – in particular, that of sticky country, which further diminishes the relative merit of OIT. For example, a high (idiosyncratic) productivity shock lowers a country's marginal cost, which increases its output and consumption. Consumption increase in turn raises the real wage through income effects, which increases the marginal cost. The later increase in the marginal cost offsets the initial decrease, hence there will be smaller marginal cost fluctuations.

In contrast, trade frictions de-stabilize the marginal cost and thus inflation (of sticky country, in particular), which pronounces the relative merit of OIT. When a country with a positive productivity shock decreases its price due to a lower marginal cost, the demand for the country's output will be higher. The increased production pushes back the marginal cost, which offsets the initial marginal cost decrease. This offsetting effect however is less under trade frictions because the extent of the demand increase for the country's product will be smaller as other countries will import less.

Finally, when shocks are more correlated or *inefficient* shocks are the main driving forces, the efficient level of cross-country relative production does not change much over time. In this case, the central bank has no need to move the relative price, which again diminishes the relative merit of OIT.

Overall, our analysis shows a general point that the extent of the benefit of moving away from SIT to OIT depends on the situation in which the central bank finds itself. The central bank then may choose to adhere to SIT when the benefit is lower relative to the cost of conducting OIT – which is absent in our model.

We then turn to a different (yet related) question. Would the central bank be able to adopt OIT even if it desired to? It may not. In general, the weights (in the price index) optimal for the union do not necessarily coincide with the optimal weights for individual countries. Some countries under OIT may be better off exiting the union and conducting a self-oriented monetary policy individually – especially when the index shows too much disparity. In that case, it is evident that the optimal weights that satisfy "participation constraints" are different from the solution of "unconstrained" problem, and furthermore they can be close to simple weights.

Interestingly (and somewhat counter-intuitively), we find that a country does not necessarily prefer a larger weight on its own inflation in the standard model because of relative demand shift effects. Forward looking firms have a precautionary motive and set a higher price when they expect future inflations will be higher and more volatile. This makes a country to prefer stable inflation in *other* countries, ceteris paribus. With low and stable inflation, foreign firms set prices low, which raises the relative demand for the foreign goods. Domestic households then work less, which improves welfare as long as they value leisure. How strong this effect is depends on the cross-country elasticity of demand. When the elasticity is sufficiently large, then the relative demand shift effects is strong enough to create a situation in which a country attaches a smaller relative weight on its own inflation.

The rest of the paper is as follows. After discussing the related literatures, we present the model in section 2. Section 3 provides the characteristics of an efficient and decentralized equilibrium. Section 4 details the analytical and numerical analysis results and the economic mechanism at work. Section 5 concludes. For interested readers, we provide all derivations and technical details in the appendix.

Related literatures

This paper is closely related to and built from a broad set of researches on what price index a central bank should target. Among others, an important stream of studies focus on the construction of target inflation index when multiple sectors and a single central bank constitute a model economy. Such multi-sector models (or equivalently, monetary union models) emphasize the implication of the sectoral heterogeneities when nominal rigidity is present in the form either of sticky price or sticky information. Aoki (2001) and Benigno (2004) conclude from their two-sector model with heterogeneous price-stickiness that strongly targeting the sticky sector's inflation improves the social welfare. Eusepi, Hobijin, and Tambalotti (2011) builds a fifteen-sector economy with heterogeneous labor shares, through which they confirm the "stickiness principle" holds in a more realistic setting.

In a similar spirit, Mankiw and Reis (2003), assuming nominal rigidities from information lags, analytically derives a target price index that minimizes the output volatility when a number of sectoral heterogeneities are taken into account. We set these prior literatures as our benchmark, and assess the performance of the second-best target inflation index compared to the first-best when financial and trade frictions exist.

A number of works test the the extent of financial integration in a monetary union (Among others, see Jappelli and Pistaferri (2010)). Although the source of the financial frictions is a frequent subject of recent theoretical and empirical researches, based on Villa (2013) who tests the empirical relevance of the financial frictions in the DSGE model for the euro area, we take as given that the financial market in our monetary union may be imperfect. In this regard, this paper is closely related to Anand, Prasad, and Zhang (2015) and Catão and Chang (2015), which search for an optimal inflation targeting when access to financial assets is restricted. Considering a small open economy, both explicitly model a food-producing sector that is flexible in price-setting, and exploit the effect of a low price elasticity of food consumption on the monetary transmission mechanism. In Anand, Prasad, and Zhang (2015), households in a food-producing sector live hand-to-mouth while those in the other can buy and sell risk-free government bonds. In this economy, targeting headline inflation index may yield higher social welfare than targeting core index, which reverses the former optimal monetary policy recipe to target the inflation of the stickier sector. Meanwhile, Catão and Chang (2015) assumes risk-sharing is imperfect only in a food-importing sector, and shows targeting production price index may be welfare-improving than targeting consumption price index. (See Huand and Liu (2005) for a welfare implication of targeting the production and consumption price index in a one-sector model.) In our paper, we take a more general approach in modelling. The cost of relocating funds across sectors can range from zero to infinity, goods produced in each sector is highly substitutable, and the level of nominal rigidity in not restricted.

As a strand of literature that introduces financial frictions in a monetary union economy, also related to our paper are Bhattarai, Lee, and Park (2015), Auray and Eyquem (2014), and Corsetti et al. (2014). They all contain a source of financial imperfections, yet focus on various aspects of the business cycle. Bhattarai, Lee, and Park (2015) highlights the wealth redistributing role of monetary policy when net asset position is imbalanced, risk-sharing is imperfect, and interest rate spreads increase endogenously. Auray and Eyquem (2014) suggests that under strict inflation targeting the welfare may be lower under perfect financial market than under financial autarky due to a less volatile movement of the terms of trade in the latter case. Finally, Corsetti et al. (2014) illustrates the widened equilibrium indeterminacy regions when a higher level of sovereign risk renders the private borrowing of the indebted countries more costly.

This paper also is in line with literatures on trade frictions in two directions. The first lies on empirical and theoretical researches on testing the real integration in a monetary union. Important related pieces include Engel and Rogers (1996), Engel and Rose (2000) and Rogers (2002), which measure the trade costs within a monetary union. The other stream of researches related to ours focus on the effects of trade frictions on monetary policy. The wide convention in the literatures has been to model the trade integration as a decrease in the home bias in consumer preferences or as an increase in the import share (see Faia and Monacelli (2008) and Lombardo and Ravenna (2010) for an example). On the other hand, Caccierto and Ghironi (2014) explicitly models the trade frictions and relates to an optimal monetary policy. But, to the best of our knowledge, our paper is the first to study the implication of the trade frictions in the central bank's inflation targeting.

The last stream of researches this paper is related to is on central bank cooperation (for a recent survey, see Engel (2015)). Fuchs and Lippi (2006) is closest to our paper in that member countries are modelled to voluntarily participate in the currency union and exit if necessary. In Fuchs and Lippi (2006), member countries have incentives to stay in the union when the union central bank cannot change the policy unexpectedly. In contrast, we assume the formation of the currency union promotes real integration by inducing zero trade costs (see Rose(2000), Rose and Wincoop (2001), Rose and Engel (2002), Baldwin, Skudelny, and Taglioni (2005), and Lane (2006) for an empirical evidence for enhanced trade links in a currency union.). Outside the union, we impose import technology is inefficient and volatile nominal exchange rate additionally increases the import price. Though the debate on whether the exchange rate stability increases the trade flows does not seem to have reached consensus yet (see Chowdhury (1993) for a positive empirical effect of exchange rate volatility on trade volume and see Bacchetta and Wincoop (200) for a theoretical model showing no impact of exchange rate system on trade.), we take for granted that the member states do have incentives to stay within the union and improved trading conditions can be one of the reasons.

2 The model

This section presents a parsimonious currency union model with sticky prices in line with Woodford (2003) and Benigno (2004). Our monetary union is composed of two countries (A and B) that produce differentiated goods and thus are naturally characterized by a different degree of nominal rigidities. The countries trade their products and state-contingent assets. Cross-country flows of goods and funds however are subject to certain frictions, which is referred respectively as (i) trade frictions and (ii) financial frictions. The frictions are parameterized in a simple way, which enables us to analyze frictionless economy in a single framework and derive an analytic expression of the central bank loss function.

Various types of market imperfections impinging on the economy create deadweight losses both in the short and the long run. Monetary and fiscal authorities reduce the losses via different policy measures. The fiscal authority in each country eliminates long-run (steady state) distortions by providing subsidies to firms and intermediaries, yet has no ability to address inefficient fluctuations over the business cycle. Such short-run inefficiencies are then managed by the utilitarian central bank through (optimal) inflation targeting.

2.1 Households

The union lies on the interval $[0, 1]$, which is divided into two subintervals for member countries, $\mathcal{I}_A = [0, n_a]$ and $\mathcal{I}_B = (n_a, 1]$. For simplicity, we assume that there is a representative agent in each country.

The representative household in country j ($j = A$ and B), who gains utility from consumption and disutility from working, maximizes the following expected lifetime utility:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[U(C_{j,t}) - \frac{1}{n_j} \int_{\mathcal{I}_j} V(N_{j,t}(i))\, di \right]$$

where β is a time discount factor, $C_{j,t}$ is country j household's consumption on final goods and $N_{j,t}(i)$ indicates the representative agent's labor supply to firm i at period t. The period (dis)utility functions have the following functional form:

$$U(C_{j,t}) = \frac{C_{j,t}^{1-\sigma} - 1}{1-\sigma} \quad \text{and} \quad V(N_{j,t}(i)) = \frac{N_{j,t}^{1+\varphi}(i)}{1+\varphi}$$

where σ refers to the the coefficient of relative risk aversion, and $\frac{1}{\varphi}$ to the Frisch elasticity.

The household faces the flow budget constraint:

$$P_{j,t}^C C_{j,t} + P_{j,t}^C \mathbb{E}_t[Q_{t,t+1} B_{j,t+1}] + P_{j,t}^C \Phi(C_{j,t}, \xi_{j,t}) = P_{j,t}^C B_{j,t} + P_{j,t}^C \xi_{j,t},$$

where $P_{j,t}^C$ is a price index of country j's final consumption goods (CPI.) As in standard models, a complete set of Arrow-Debrew securities is available: $B_{j,t}$ is the real payoffs of the portfolio that is delivered to household at period t where $Q_{t,t+1}$ is the stochastic discount factor for the one-period-ahead real assets' payoffs. Our model however departs from perfect risk-sharing because transferring resources across countries is costly. Household j pays an extra cost of $\Phi(C_{j,t}, \xi_{j,t})$ units of consumption to financial agency when it consumes a different amount from its real non-financial income $\xi_{j,t}$ (to be detailed below) via borrowing and lending. Specifically, we follow Schulhofer-Wohl (2011), Lee (2012, 2014), and Catao and Chang (2015) and assume a convex cost function of the form:

$$\Phi(C_{j,t}, \xi_{j,t}) = \frac{\phi}{2} C_{j,t} \left(\log \frac{C_{j,t}}{\xi_{j,t}} \right)^2,$$

where $\phi \geq 0$ reflects the level of financial frictions. When $\phi = 0$, cross-country risk sharing is perfect. On the other hand, if $\phi > 0$, a country's consumption would track the non-financial income $\xi_{j,t}$ which in equilibrium corresponds to the country's domestic product. Financial autarky arises when ϕ goes to infinity.

The non-financial income $\xi_{j,t}$ is given as the sum of labor and profit income net of taxes:

$$P_{j,t}^C \xi_{j,t} \equiv \underbrace{\frac{1}{n_j} \int_{\mathcal{I}_j} W_{j,t}(i) N_{j,t}(i) di + W^T N_{j,t}^T}_{\text{labor income}} + \underbrace{\frac{1}{n_j} \int_{\mathcal{I}_j} \Pi_{j,t}(i) di + \Pi_{j,t}^F + \Pi_{j,t}^T}_{\text{profit income}} - \underbrace{P_{j,t}^C \tau_{j,t}}_{\text{taxes}}. \tag{1}$$

Aside from providing labor services to the production firm i and receiving nominal compensation $W_{j,t}(i) N_{j,t}(i)$, the household inelastically supply labor hours $N_{j,t}^T$ to import agency (to be detailed) at a given wage W^T which is exogenously fixed for simplicity. The household owns firms and intermediary agencies in country j. Therefore, the profit of firms ($\frac{1}{n_j}\int_{\mathcal{I}_j}\Pi_{j,t}(i)di$) and that of the financial and import agency ($\Pi^F_{j,t}$ and $\Pi^T_{j,t}$) are given to the household as a dividend.

The household combines domestic and foreign composite country goods to make final consumption goods:

$$C_{j,t} = \left[n_a{}^{\frac{1}{\eta}} C_{j,a,t}^{\frac{\eta-1}{\eta}} + n_b{}^{\frac{1}{\eta}} C_{j,b,t}^{\frac{\eta-1}{\eta}} \right]^{\frac{\eta}{\eta-1}},$$

where $C_{j,a,t}$ and $C_{j,b,t}$ are country j household's consumption of country A goods and of country B goods respectively. The households buy domestic goods directly from producers. However, foreign goods can be purchased only via import agencies that transport goods from abroad, which incurrs costs and results in a higher price. The consumption price index (CPI) of the two countries is constructed as:

$$P^C_{a,t} = \left[n_a P_{a,t}^{1-\eta} + n_b \tilde{P}_{b,t}^{1-\eta} \right]^{\frac{1}{1-\eta}} \quad \text{and} \quad P^C_{b,t} = \left[n_a \tilde{P}_{a,t}^{1-\eta} + n_b P_{b,t}^{1-\eta} \right]^{\frac{1}{1-\eta}},$$

where $\tilde{P}_{k,t}$ is the import price of a composite country k goods. Note that the Law of One Price does not generally hold ($\tilde{P}_{k,t} \neq P_{k,t}$) due to transportation costs.

The intra- and inter-temporal optimality conditions are

$$\frac{C_{a,a,t}}{C_{a,b,t}} = \left(\frac{P_{a,t}}{\tilde{P}_{b,t}} \right)^{-\eta}; \quad \frac{C_{b,a,t}}{C_{b,b,t}} = \left(\frac{\tilde{P}_{a,t}}{P_{b,t}} \right)^{-\eta},$$

$$N^{\varphi}_{j,t}(i) C^{\sigma}_{j,t} \frac{\{1 + \Phi_c(C_{j,t}, \xi_{j,t})\}}{\{1 - \Phi_{\xi}(C_{j,t}, \xi_{j,t})\}} = \frac{W_{j,t}(i)}{P^C_{j,t}},$$

$$\frac{1}{R_t} = \beta \mathbb{E}_t \left[\frac{C^{\sigma}_{j,t} \left(1 + \Phi_c\left(C_{j,t}, \xi_{j,t}\right)\right)}{C^{\sigma}_{j,t+1} \left(1 + \Phi_c\left(C_{j,t+1}, \xi_{j,t+1}\right)\right)} \right],$$

where $R_t^{-1} \equiv \mathbb{E}_t Q_{t,t+1}$. The three equations characterize a consumption bundle (domestic-foreign goods) decision, a consumption-leisure decision, and a consumption-saving decision respectively. The relative consumption of country A to country B is obtained as:

$$\left(\frac{C_{a,t}}{C_{b,t}} \right)^{\sigma} = \frac{\{1 + \Phi_c(C_{b,t}, \xi_{b,t})\}}{\{1 + \Phi_c(C_{a,t}, \xi_{a,t})\}} \tag{2}$$

2.2 Firms

2.2.1 Composite country goods producer

In each country *j*, perfectly competitive firms produce composite country goods $Y_{j,t}$ by assembling intermediate goods $Y_{j,t}(i)$ with a Dixit-Stiglitz technology:

$$Y_{j,t} = \left[\left(\frac{1}{n_j} \right)^{\frac{1}{\theta}} \int_{\mathcal{I}_j} (Y_{j,t}(i))^{\frac{\theta-1}{\theta}} \right]^{\frac{\theta}{\theta-1}},$$

The firms sell the composite goods to the domestic household and the foreign trade agency at $P_{j,t}$. The profit maximization of the firms yields the demand function:

$$Y_{j,t}(i) = \left(\frac{P_{j,t}(i)}{P_{j,t}} \right)^{-\theta} Y_{j,t},$$

while the producer price index (PPI) of country j is given by:

$$P_{j,t} = \left[\frac{1}{n_j} \int_{\mathcal{I}_j} (P_{j,t}(i))^{1-\theta} di \right]^{\frac{1}{1-\theta}}.$$

2.2.2 Intermediate goods producer

Monopolistically competitive firms produce intermediate goods using a linear production technology, $Y_{j,t}(i) = Z_{j,t} N_{j,t}(i)$. Country j productivity shock $Z_{j,t}$ follows AR(1) in logs

$$\begin{pmatrix} \log Z_{a,t} \\ \log Z_{b,t} \end{pmatrix} = \begin{pmatrix} \rho_a & 0 \\ 0 & \rho_b \end{pmatrix} \begin{pmatrix} \log Z_{a,t-1} \\ \log Z_{b,t-1} \end{pmatrix} + \begin{pmatrix} \sigma_{aa} & \sigma_{ab} \\ \sigma_{ba} & \sigma_{bb} \end{pmatrix} \begin{pmatrix} \varepsilon_{a,t} \\ \varepsilon_{b,t} \end{pmatrix},$$

where $\begin{pmatrix} \varepsilon_{a,t} \\ \varepsilon_{b,t} \end{pmatrix} \sim \mathcal{N}(0, I)$. Exogenous variations in the productivity are the only source of disturbances in the union.

As in Calvo (1983) and Yun (1996), firms in country j sets its price with a probability $1 - \alpha_j$ each period to maximize the expected discounted profits:

$$\max_{P_{j,t}(i)} \mathbb{E}_t \sum_{k=0}^{\infty} (\alpha_j \beta)^k Q_{t,t+k} \underbrace{[P_{j,t}(i) Y_{j,t+k}(i) - (1-s) W_{j,t+k}(i) N_{j,t+k}(i)]}_{\Pi_{j,t+k}(i)}, \tag{3}$$

where $Q_{t,t+k}$ is the stochastic discount factor between period *t* and *t+k*. The first term in the bracket is the expected revenue, and the second is the production costs net of employment subsidy. We assume a time-invariant subsidy rate *s* to induce an efficient steady state. The parameter *s* is set to offset the gross mark-up charged by firms.[5]

[5] Note also that a stochastic mark-up is obtained if we consider a time-varying subsidy rate u_t. By assuming that the

As firms that optimize choose a common price $P^*_{j,t}$, country *j*'s PPI evolves as:

$$P_{j,t} = \left[(1-\alpha_j)\left(P^*_{j,t}\right)^{1-\theta} + \alpha_j\left(P_{j,t-1}\right)^{1-\theta}\right]^{\frac{1}{1-\theta}} \tag{4}$$

2.3 Intermediary agency

Financial and trade intermediaries relocate wealth and goods across border on behalf of the households. The service fees the intermediary agencies charge increase in the level of financial and trade frictions.

2.3.1 Financial intermediation

We consider the simplest form of financial intermediation. Unless the household chooses not to insure against any idiosyncratic income risks (*i.e.* holding no securities), it has to pay the financial agency for its intermediation service: $\Phi(C_{j,t}, \xi_{j,t}) > 0$ unless $C_{j,t} = \xi_{j,t}$. We assume that the agency moves resources at zero cost and simply returns the profit as a lump-sum dividend to the household. This assumption is not consequential to our results.

2.3.2 Trade intermediation

The trade intermediaries buy foreign country goods and sell them to the household in country j. Providing the import services is costly: when importing $M_{j,t}$ amounts, the trade intermediaries buy foreign country goods from foreign producers at the producer price and relocate them to country j. The import technology $M_{j,t} = f(N^T_{j,t})$ requires hiring $N^T_{j,t}$ of labor services to provide the import services. Our modelling of the transport technology departs from the conventional iceberg-type trade costs, which are assumed to be 'melt down'. Instead, trade services in our model create a value-added. The trade intermediaries hire labor and compensate for the services. After transporting the goods, the intermediaries then sell the products to the household at a given import price. Lastly, to induce an efficient steady state, we assume the government provides a fixed subsidy for each unit of imports. Again, this assumption is only for convenience and unimportant for the result.

The import agency in country A chooses the amount of import $M_{a,t}$ to maximize its profits given the producer price and import price of foreign country goods, the fixed wage of the labor services, the transport technology, and the subsidy.

$$\max_{M_{a,t}} \Pi^T_{a,t} = \left(\tilde{P}_{b,t} - P_{b,t} + s^T_a\right) M_{b,t} - W^T_{a,t} N^T_{a,t}$$

subject to $M = f(N)$ where $f' > 0$ and $f'' < 0$ and $s^T = \frac{W^T}{f'(N^T)}$. According to the transport technology, the demand for the labor services is a convex increasing function of the amount of imports.

steady-state value of u_t equals $\frac{1}{\theta}$, an efficient steady state equilibrium is achieved as well. For an inefficient steady state, see Benigno and Woodford (2005).

The first order condition is

$$\tilde{P}_{b,t} = P_{b,t} + \frac{W^T_{a,t}}{f'\left(N^T_{a,t}\right)} - s^T_a = P_{b,t} + W^T_{a,t} g'(M_{a,t}) - s^T_a \tag{5}$$

where $g = f^{-1}$. Hence, $g' > 0$, $g'' > 0$, and $g'(M) = \frac{1}{f'(N)}$. Note that trade frictions introduce an endogenous wedge, $W^T_{a,t} g'(M_{a,t}) - s^T_a$, between the import price $\tilde{P}_{b,t}$ and the producer price $P_{b,t}$. The wedge however disappears in the steady state thanks to the subsidy. Likewise, for the trade intermediaries in country B, we get

$$\tilde{P}_{a,t} = P_{a,t} + W^T_{b,t} g'(M_{b,t}) - s^T_b \tag{6}$$

2.4 Governments

We suppose a simple form of fiscal policy. The government in each country maintains a balanced budget and collects lumpsum taxes to finance its expenditures and subsidies to trade intermediaries as well as production firms. The subsidy rates (s and s^T) are not state-contingent. The fiscal policy therefore eliminates only the steady-state distortions, but is unable to address inefficient fluctuations over the business cycles. The government budget constraint is given as:

$$P^C_{j,t} G_{j,t} + \frac{s}{n_j} \int_{\mathcal{I}_j} W_{j,t}(i) N_{j,t}(i) di + s^T M_{j,t} = P^C_{j,t} \tau_{j,t}$$

We further assume zero government expenditure for simplicity.

2.5 Market clearing conditions

All contingent claims are in zero net supply for every state and every t. The financial market clearing condition is given by:

$$n_a P^C_{a,t} B_{a,t} + n_b P^C_{b,t} B_{b,t} = 0$$

The imported goods market clears when import demand equals import supply:

$$M_{a,t} = C_{a,b,t}; \quad M_{b,t} = C_{b,a,t}$$

where $C_{a,b,t}$ is country A household's demand for country B goods, and $C_{b,a,t}$ is country B household's demand for country A goods.

Combining the households and the government budget constraints, the resource constraint is derived:

$$Y_{a,t} = n_a C_{a,a,t} + n_b C_{b,a,t} \tag{7a}$$

$$Y_{b,t} = n_a C_{a,b,t} + n_b C_{b,b,t} \tag{7b}$$

No resources are lost despite the presence of financial and trade frictions since the related costs are collected and redistributed to the household.

2.6 Monetary policy

We confine the monetary policy to a class of inflation targeting where the central bank (i) decides *what inflation (or equivalently, price) index to target* and (ii) sets an explicit quantitative inflation target. [6] Following Benigno (2004), we assume that the central bank constructs and perfectly stabilizes a target inflation index π_{target}.

$$\pi_{target,t} = \delta\pi_{a,t} + (1-\delta)\pi_{b,t} = 0$$

The central bank decides how to weight national inflations when composing a target index. The weighting scheme determines the relative volatility of $\pi_{a,t}$ and $\pi_{b,t}$. When δ is large, $\pi_{a,t}$ strongly contributes to $\pi_{target,t}$ and thus is strongly stabilized – while $\pi_{b,t}$ is let to float.

We compare two target indices. Simple inflation targeting (SIT) stabilizes a simple expenditure-weighted average of prices and thus sets $\delta = n_a$. Optimal inflation targeting (OIT), on the other hand, sets $\delta = \delta^*$ such that δ^* maximizes the *ex-ante* expected currency union welfare, $\mathcal{V}$:

$$\mathcal{V} \equiv \mathbb{E}\sum_{t=0}^{\infty}\beta^t W_t^{CU}$$

where the per period union utility

$$W_t^{CU} = n_a U(C_{a,t}) + n_b U(C_{b,t}) - \int_0^1 V(N_{j,t}(i))di.$$

In this paper, we consider up to a second-order approximation of the union welfare. This approach has an obvious disadvantage that it overlooks third or higher order effects. We nevertheless take such the approach because it is beneficial to develop an intuition (as will be evident below) and we believe the benefit outweighs the cost for our purpose.

Following the standard approach (Benigno and Woodford, 2005), the per period utility can be approximated as:

$$\frac{W_t^{CU} - \overline{W}^{CU}}{\overline{W}_C^{CU}\bar{C}} = -\frac{1}{2}L_t + t.i.p + O(\|\xi\|^3),$$

where L_t contains the second order terms. Therefore, the currency union welfare, $\mathcal{V}$, can be approximated as the *minus* of the welfare loss $\mathcal{L}$:

$$\mathcal{L} = \mathbb{E}\,\frac{1}{2}\sum_{t=0}^{\infty}\beta^t L_t$$

which indicates an ex-ante permanent welfare loss as a fraction of steady-state consumption. Fi-

[6] See Bernanke and Mishkin (1997) for a survey of the diverse inflation targeting schemes each central bank adopts. More specifically, Mankiw and Reis (2003) focus on what inflation measure the central bank should target.

nally, we define the welfare gap between OIT and SIT as

$$(\mathcal{L}_{SIT} - \mathcal{L}_{OIT}) * 100 \geq 0,$$

the percentage additional welfare loss of SIT against OIT expressed in terms of the steady-state consumption. A 1% of welfare gap, for example, implies that an additional permanent welfare loss from adopting SIT instead of OIT equals 1% of the steady-state consumption.

As a final note, we find that the optimal inflation targeting performs almost as good as the (unconstrained) optimal monetary policy, and thus the later is not presented for brevity in the paper. We instead discuss below how OIT mimics the optimal policy.

3 The equilibrium

3.1 An efficient allocation

With nominal, financial and trade frictions, the market outcome departs from an efficient allocation. The goal of the monetary policy is to move market equilibrium as close as possible to the efficient allocation given the constraints (this will be clear in the following section). We therefore characterize the efficient benchmark first before analyzing the monetary policy.

The social planner maximizes the following objective function with Pareto weights $\{\omega_a, \omega_b\}$.

$$\sum_{j\in\{a,b\}} \omega_j n_j \left[U(C_j) - \frac{1}{n_j} \int_{\mathcal{I}_j} V\left(\frac{Y_j(i)}{Z_j}\right) di \right]$$

subject to the resource and technology constraints: for $j \in \{a, b\}$,

$$\left[\left(\frac{1}{n_j}\right)^{\frac{1}{\theta}} \int_{\mathcal{I}_j} Y_j(i)^{\frac{\theta-1}{\theta}} \right]^{\frac{\theta}{\theta-1}} = n_a C_{a,j} + n_b C_{b,j}$$

The first order conditions with respect to $\{C_{j,a}, C_{j,b,}, Y_j(i)\}$ are:

$$\omega_j n_j C_j^{-\sigma} \left(\frac{n_a C_j}{C_{j,a}}\right)^{\frac{1}{\eta}} = \mu_a n_j$$
$$\omega_j n_j C_j^{-\sigma} \left(\frac{n_b C_j}{C_{j,b}}\right)^{\frac{1}{\eta}} = \mu_b n_j$$
$$\omega_j \left(\frac{Y_j(i)}{Z_j}\right)^{\varphi} \frac{1}{Z_j} = \mu_j \left(\frac{Y_j}{n_j Y_j(i)}\right)^{\frac{1}{\theta}}$$

where μ_j are Lagrangian multipliers. Substituting $Y_j(i) = \frac{Y_j}{n_j}$, the third conditions reduce to $\mu_j = \omega_j n_j^{-\varphi} Y_j^{\varphi} Z_j^{-(1+\varphi)}$. We consider a utilitarian social planner who equally weights country A and B (i.e. $\omega_a = \omega_b$). In this case, we get: $C^{R,E} \equiv \frac{C_a^E}{C_b^E} = 1$, $Y^{R,E} \equiv \frac{Y_a^E}{Y_b^E} = \frac{n_a}{n_b}\left(\frac{Z_a}{Z_b}\right)^{\frac{\eta(1+\varphi)}{1+\varphi\eta}}$, $\frac{C_{a,a}^E}{C_{a,b}^E} = \frac{C_{b,a}^E}{C_{b,b}^E} = Y^{R,E}$, and $C^{U,E} \equiv n_a C_a^E + n_b C_b^E = \left[\sum_j n_j Z_j^{\frac{(\eta-1)(1+\varphi)}{1+\varphi\eta}}\right]^{\frac{1+\varphi\eta}{(\eta-1)(\sigma+\varphi)}}$. This particular allocation coincides with the decentralized equilibrium that would arise in the absence of the aforementioned frictions, in which $P_t^{R,E} \left(\equiv \frac{P_{a,t}^E}{P_{b,t}^E}\right)$ and $Y_t^{R,E}$ have a one-to-one relationship: $Y_t^{R,E} = \frac{n_a}{n_b}\left(P^{R,E}\right)^{-\eta}$. [7]

3.2 A decentralized Equilibrium

Here we characterize the equilibrium of the decentralized economy. To solve the model, we linearize our model around a symmetric, efficient steady state equilibrium. We assume zero initial wealth. For any variable X_t, $\hat{X}_t$ refers to a log deviation of X_t from its steady state $\bar{X}$, and we define the relative and union-wide variable to be $\hat{X}_t^R = \hat{X}_{a,t} - \hat{X}_{b,t}$ and $\hat{X}_t^U = n_a\hat{X}_{a,t} + n_b\hat{X}_{b,t}$.

The decentralized equilibrium deviates from the efficient one due to the presence of nominal, financial, and trade frictions. To start with, we consider when acquiring perfect consumption insurance becomes costly. Such financial frictions hamper perfect risk sharing in consumption as reflected in the relative consumption Euler equation Eq. (2).

$$(\sigma + \phi)\hat{C}_t^R = \phi\hat{\xi}_t^R$$

In a complete financial market (i.e. $\phi = 0$), consumption is synchronized across countries ($\hat{C}_t^R = 0$). Otherwise, relative consumption depends on relative real non-financial income $\hat{\xi}_t^R$ (to be detailed in the following section).

On the other hand, when trade intermediaries use real resources (labor) in providing the import services, import goods become more expensive. As the prices offered to domestic and foreign customers differ, the relative price of two country goods that prevails in country A and B diverges. We delve into the inefficiency arising from the depressed import demand, for which a formal representation of the endogenous import price wedge is derived from Eqs. (5) and (6).

$$\hat{\tilde{P}}_{b,t} = \hat{P}_{b,t} + \nu n_b \hat{C}_{a,b,t}; \quad \hat{\tilde{P}}_{a,t} = \hat{P}_{a,t} + \nu n_a \hat{C}_{b,a,t} \tag{8}$$

where $\nu = W^T(\bar{P}_a^C)^{-1} g''(\bar{C}_{a,b}) = W^T(\bar{P}_b^C)^{-1} g''(\bar{C}_{b,a})$ captures the level of trade frictions: the more convex the trading costs, the higher ν. As Eqs. (8) imply, higher import demands raise the import prices. For further investigation, we define the relative price between two country goods in each country, $\hat{P}_{a,t}^R \equiv \hat{P}_{a,t} - \hat{\tilde{P}}_{b,t}$ and $\hat{P}_{b,t}^R \equiv \hat{\tilde{P}}_{a,t} - \hat{P}_{b,t}$. Then, by definition,

$$\hat{P}_{a,t}^R = \hat{P}_t^R - \nu n_b \hat{C}_{a,b,t} \tag{9a}$$

$$\hat{P}_{b,t}^R = \hat{P}_t^R + \nu n_a \hat{C}_{b,a,t} \tag{9b}$$

Under non-zero trade frictions, country-specific relative price deviates from the relative PPI ($\hat{P}_t^R \equiv \hat{P}_{a,t} - \hat{P}_{b,t}$) off the steady state, diverging further from each other when import levels are high. We are interested in the dynamics of the relative price as they play a crucial role in allocating resources

[7] The superscript E is used to indicate efficient allocation.

both at a within-country level and a cross-country level. The consumption demand functions and the resource constraints associate with the country-specific relative prices in the following way.

$$\hat{C}_{a,a,t} - \hat{C}_{a,b,t} = -\eta \hat{P}^R_{a,t}; \quad \hat{C}_{b,a,t} - \hat{C}_{b,b,t} = -\eta \hat{P}^R_{b,t} \tag{10}$$

$$\hat{Y}^R_t = -\eta \left(n_a \hat{P}^R_{a,t} + n_b \hat{P}^R_{b,t} \right) \tag{11}$$

The consumption ratio of country A to B goods depends upon the country-specific relative price, and the resulting relative production across countries relies on the average relative price. Without trade frictions (i.e. $\nu = 0$), we have $\hat{P}^R_{a,t} = \hat{P}^R_{b,t} = \hat{P}^R_t$. This is precisely when $\hat{Y}^R_t = -\eta \hat{P}^R_t$ and $\left(\hat{C}_{a,a,t} - \hat{C}_{a,b,t}\right) = \left(\hat{C}_{b,a,t} - \hat{C}_{b,b,t}\right)$ are satisfied as in the efficient allocation. In contrast, when trade frictions exist, country A and B have different relative prices which distort the allocation of relative output across countries and the consumption portfolio within each country.

Lastly, nominal frictions indicate infrequent price changes. Derived from Eqs. (3) and (4), the New Keynesian Phillips Curve (NKPC) of country A is:

$$\pi_{a,t} = \beta \mathbb{E}_t \pi_{a,t+1} + \kappa_a \widehat{mc}_{a,t}$$

where

$$\widehat{mc}_{a,t} = (\sigma + \varphi) \hat{Y}^U_t + n_b \left(\varphi \hat{Y}^R_t + \sigma \hat{C}^R_t - \hat{P}^R_{a,t} \right) - (1 + \varphi) \hat{Z}_{a,t}$$

and $\kappa_j = \frac{(1-\alpha_j)(1-\alpha_j \beta)}{\alpha_j (1+\theta\varphi)}$. The real marginal costs correspond to the real marginal employment costs. The first four terms of $\widehat{mc}_{a,t}$ refer to labor market equilibrium condition. On the demand-side, firms hire more labor services when the demand for their products ($\hat{Y}^U_t + n_b Y^R_t$) and the product price ($\hat{P}_{a,t}$) increase. On the supply-side, higher consumption level ($\hat{Y}^U_t + n_b C^R_t$) and higher CPI ($\hat{P}^C_{a,t}$) discourage labor supply. Note that $\hat{P}^C_{a,t} - \hat{P}_{a,t} = -n_b \hat{P}^R_{a,t}$ indicates the difference between CPI and PPI. The last term of $\widehat{mc}_{a,t}$ mirrors the technology of the production, particularly the marginal product of labor. With a higher level of nominal rigidity, hence lower κ_j, price level is less likely to respond to a change in the employment costs. The NKPC of country B is derived in the same way.

$$\pi_{b,t} = \beta \mathbb{E}_t \pi_{b,t+1} + \kappa_b \widehat{mc}_{b,t}$$

where

$$\widehat{mc}_{b,t} = (\sigma + \varphi) \hat{Y}^U_t - n_a \left(\varphi \hat{Y}^R_t + \sigma \hat{C}^R_t - \hat{P}^R_{b,t} \right) - (1 + \varphi) \hat{Z}_{b,t}$$

4 The analysis

Most parameter values adopted for numerical analysis are standard (Table 1). The time discount factor β is set to yield steady-state annual real interest rate of about 4 percent. An inverse Frisch elasticity φ between 0 and 2 (see Eusepi, Hobijin, and Tambalotti (2011) for the related discussions) and the risk aversion parameter σ from 1 to 3 are standard. The within-country and cross-country elasticity of substitution among goods, θ and η, are adopted from Bhattarai, Lee, and Park (2015) to be 6 and 3.[8] The heterogeneity in nominal rigidity is taken into account. We aim to target the price stickiness of Germany and France from Dhyne et al. (2006) that the frequency of monthly price changes in Germany is 13.5% and that of France is 20.9%. Matching these values, we assume country A has higher nominal rigidity ($\alpha_a = 0.65$) than country B ($\alpha_b = 0.5$). Therefore, we often call country A sticky and country B flexible. For the shock process, we let $\rho_a = \rho_b = 0.96$, $\sigma_{aa} = \sigma_{bb} = 0.007$, and $\sigma_{ab} = 0.001$. The details of the productivity shock estimation are in the appendix.

α_a	0.65	α_b	0.5	n_a	0.5	n_b	0.5
β	0.99	σ	2	φ	1	θ	6
η	3	$\rho_a = \rho_b$	0.96	$\sigma_{aa} = \sigma_{bb}$	0.007	σ_{ab}	0.001

Table 1: Parameter values

4.1 Baseline: the stickiness principle

Our baseline economy suffers from nominal frictions but is free from financial and trade frictions, which then reduces to a model economy in Benigno (2004). When price is sticky in both country A and B, an important departure from the efficient allocation is $\hat{P}_t^R \neq \hat{P}_t^{R,E}$. Suppose country A has a positive productivity shock, then the marginal costs imply:

$$\widehat{mc}_{a,t} = \underbrace{(\sigma + \varphi)\hat{Y}_t^U}_{(+)} \underbrace{-n_b(1+\varphi\eta)\hat{P}_t^R}_{(+)} \underbrace{-(1+\varphi)\hat{Z}_{a,t}}_{(-)}$$

Relative price decreases and union-wide output increases. The decrease in relative price, however, is not enough and the overall level of marginal costs decrease, resulting in the deflation in country A and inflation in country B. That is, $\hat{P}_t^R \neq \hat{P}_t^{R,E}$ drives the fluctuation of the marginal costs and of the inflation rate.

More importantly, the price level in each member state determines the relative price of country A goods compared to country b goods. As an identity, $\pi_{a,t} - \pi_{b,t} = \hat{P}_t^R - \hat{P}_{t-1}^R$ holds for all t. That is, under a fixed the nominal exchange rate, current inflation rates of each country directly set $\hat{P}_t^R$. Since resources are allocated across border following the relative price, the shared single currency imposes a trade off between inflation stabilization and efficient cross-country resource allocation which we call *a single currency trade off*. Due to this trade off, the dynamics of $\hat{P}_t^R$ depends upon the choice of target inflation index – or the choice of δ.

According to the analytical derivation of the law of motion of $\hat{P}_t^R$, it responds both to the its one-period-ahead value and to the exogenous shock, the extent of which relies on the monetary policy weight δ:

$$\hat{P}_t^R = \gamma_1 \hat{P}_{t-1}^R + \gamma_2 \hat{P}_t^{R,E}$$

[8]Note that Benigno (2009) takes $\theta = 7.66$ from Rotemberg and Woodford (1998) and considers a wide range of η between 0.8 and 6. He confirms that different values of η bigger than or equal to 1 do not change the direction of the numerical results. Our parameter choice of $\eta = 3$ is thus a compromise between 1 and 6.

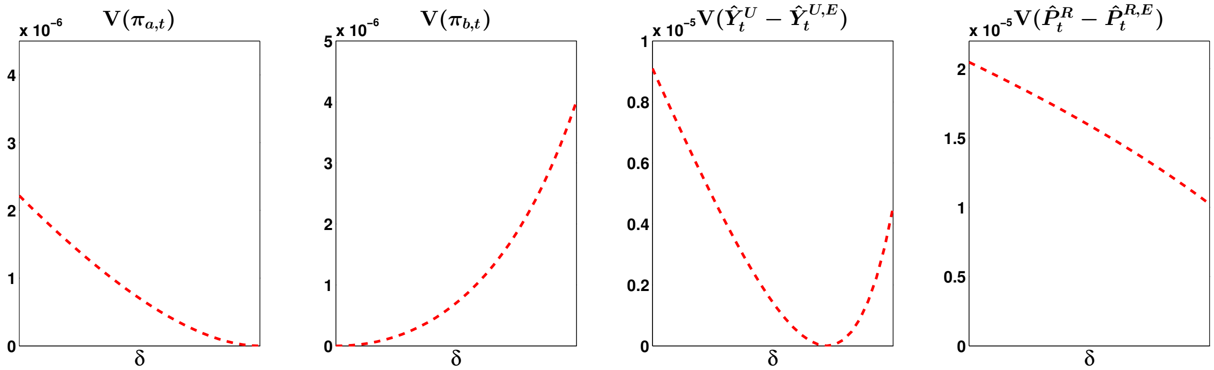

Figure 1: Model variance in the baseline ($\phi = \nu = 0$)

where γ_1 and γ_2 are functions of the model variables, and satisfy $\partial\gamma_1/\partial\delta < 0$ and $\partial\gamma_2/\partial\delta > 0$.[9] When the central bank stabilizes $\pi_{a,t}$ more strongly than $\pi_{b,t}$ (i.e. higher δ), $\hat{P}^R_t$ more closely follows $\hat{P}^{R,E}_t$ than $\hat{P}^R_{t-1}$. Intuitively, stabilizing the sticky country's inflation and destabilizing the flexible country's inflation make $\hat{P}^R_t$ more responsive to the productivity shock. We confirm this dynamics in Figure 1. When δ increases, $\pi_{a,t}$ and $\left(\hat{P}^R_t - \hat{P}^{R,E}_t\right)$ are stabilized while $\pi_{b,t}$ fluctuates further. Back to our previous example, following a positive productivity shock, $\hat{P}_{a,t}$ does not decrease by much but $\hat{P}_{b,t}$ increases sufficiently as δ increases. As $\hat{P}^R_t$ drops, higher production is induced union-wide so that $\left(\hat{Y}^U_t - \hat{Y}^{U,E}_t\right)$ becomes less volatile. However, when δ is too high, an excessive demand for country A goods leads to a higher than efficient level of union-wide output production. The third plot of Figure 1 shows this finding.

The optimal level of δ is derived from the following loss function.

$$L_t = \theta \sum_{j=a,b} \frac{n_j}{\kappa_j}(\pi_j)^2 + (\sigma+\varphi)\left(\hat{Y}^U_t - \hat{Y}^{U,E}_t\right)^2 + n_a n_b \eta(1+\varphi\eta)\left(\hat{P}^R_t - \hat{P}^{R,E}_t\right)^2$$

where the efficient equilibrium is:

$$\hat{Y}^{U,E}_t = \frac{1+\varphi}{\sigma+\varphi}\hat{Z}^U_t; \quad \hat{P}^{R,E}_t = -\frac{1+\varphi}{1+\varphi\eta}\hat{Z}^R_t$$

The interpretation of the source of welfare loss is direct. National inflation captures the price dispersion within country. And for an equal level of national inflation, higher rigidity (lower κ_j) leads to bigger real distortions since more firms are left with old prices while a few are dragging up the inflation. Also, non-zero output gap and the relative price gap respectively imply a union-wide and cross-country inefficient allocation of resources.

OIT weight relies heavily on the following features. First of all, perfectly stabilizing all sources of welfare loss is not feasible due to the single currency trade off. Second, *a utilitarian central bank* strongly targets national inflations, especially that from the sticky country, since they create an asymmetrically large amount of welfare loss to the entire union. Therefore, the central bank

[9]This result does not change when financial and trade frictions are present, the details of which are in the appendix.

makes balance between the volatility of the national inflations and the relative price gap. As a result, strongly stabilizing the sticky country's inflation becomes the monetary policy scheme of OIT, which is often referred to as *the stickiness principle*. (see Aoki (2001), Benigno (2004), and Eusepi, Hobijin, and Tambalotti (2011) for the related literature.) OIT achieves this result by setting the weight as high as $\delta = 0.7$. This is in contrast to SIT letting $\delta = 0.5$.

Acknowledging the fact that SIT has an obviously poor performance than OIT, can there be justifications for why SIT is adopted over OIT in practice? Since the stickiness principle is a product of the single currency trade off and the utilitarian central bank, we start from the bottom line. Our first conjecture is that the existence of additional frictions or the assumptions of external disturbances may diminish the welfare cost of the single currency trade off. If so, not following the stickiness principle becomes less welfare costly and the welfare gap between SIT and OIT decreases. Our second conjecture is that the central bank cannot be a perfect utilitarian because it conducts a monetary policy in a way to keep all the member countries from leaving the currency union. In such case, the participation constraints of the member states may restrict the range of monetary policy the central bank can employ.

4.2 Frictions and external disturbances in the currency union

4.2.1 The role of financial frictions

When trading state-contingent bonds becomes costly, consumption is not synchronized across countries – neither party wishes to save or borrow as much as in the efficient equilibrium. Hence, relative consumption depends positively upon relative real income $\hat{\xi}_t^R$, which in equilibrium satisfies:

$$\hat{\xi}_t^R = (1-\eta)\hat{P}_t^R$$

as derived from Eq. (1). Assuming country goods is highly substitutable ($\eta > 1$), a drop in relative price, for instance, attracts demand for country A goods and yields higher production revenue, therefore higher income, in country A. Consumption level of country A then exceeds that of country B.

This non-zero relative consumption becomes an additional source of inflation stabilization, which we explain with the real marginal costs.

$$\widehat{mc}_{a,t} = \underbrace{(\sigma+\varphi)\hat{Y}_t^U}_{(+)} \underbrace{-n_b(1+\varphi\eta)\hat{P}_t^R}_{(+)} \underbrace{+n_b\sigma\hat{C}_t^R}_{(+)} \underbrace{-(1+\varphi)\hat{Z}_{a,t}}_{(-)}$$

Having a productivity increase, $\hat{P}_t^R$ drops and additionally $\hat{C}_t^R$ increases. With households reducing the labor supply, equilibrium wage increases. Accordingly, the marginal costs are more stabilized, and so are the inflation rates. Since inflation is the major source of welfare loss, this stabilization of inflation for any given monetary policy weakens the welfare cost of the single currency trade off. The model dynamics under financial frictions are shown in Figure 2. For any δ, the inflation and the union-wide output gap are more stabilized. On the other hand, the relative price gap is more volatile as the relative price becomes stabilized. Note that $\hat{C}_t^R$ fluctuates more for higher δ because $\hat{P}_t^R$ closely follows $\hat{P}_t^{R,E}$.

From the loss function, we study how financial frictions and the resulting the model dynamics shown in Figure 2 affect the welfare.

$$L_t = \theta \sum_{j=a,b} \frac{n_j}{\kappa_j}(\pi_j)^2 + (\sigma+\varphi)\left(\hat{Y}_t^U - \hat{Y}_t^{U,E}\right)^2 + n_a n_b \eta(1+\varphi\eta)\left(\hat{P}_t^R - \hat{P}_t^{R,E}\right)^2 + n_a n_b \sigma\left(\hat{C}_t^R\right)^2$$

As the last term shows, unequalized consumption becomes a new source of welfare loss. Since $\hat{C}_t^R$ depends on $\hat{P}_t^R$, a new policy trade-off arises in maximizing social welfare: when $\hat{P}_t^R$ closely follows $\hat{P}_t^{R,E}$, $\hat{C}_t^R$ is destabilized. Therefore, the central bank does not want to move the relative price much if other things are equal. We illustrate the numerical result in Figure 3a. Note first that welfare deteriorates when financial frictions exist (i.e. $\phi > 0$). Second, the slope of the curve becomes flatter as ϕ increases, indicating a smaller welfare gap between SIT and OIT. Third, the optimal δ that minimizes welfare loss does not change much. SIT shows better performance than the baseline model, which comes directly from both the new policy trade off and the inflation-stabilizing effect of financial frictions. Figure 3b shows that welfare gap decreases continuously as ϕ increases, confirming the reduced merit of OIT over SIT.

4.2.2 The role of trade frictions

When relocating goods across border is not free, country goods becomes more expensive in foreign country and hence the relative price of country goods differs in country A and B ($\hat{P}_{a,t}^R \neq \hat{P}_{b,t}^R \neq \hat{P}_t^R$). To show the effect of the depressed import demand, we define new variables $D_t^U = n_a \hat{P}_{a,t}^R + n_b \hat{P}_{b,t}^R$ and $D_t^R = \hat{P}_{a,t}^R - \hat{P}_{b,t}^R$ – from Eqn. (11) the average movement of the relative prices determines the allocation of output across countries and the gap between the relative prices reflects the divergent consumption portfolio within each country:

$$\hat{Y}_t^R = -\eta \hat{D}_t^U$$
$$(\hat{C}_{a,a,t} - \hat{C}_{a,b,t}) - (\hat{C}_{b,a,t} - \hat{C}_{b,b,t}) = -\eta \hat{D}_t^R$$

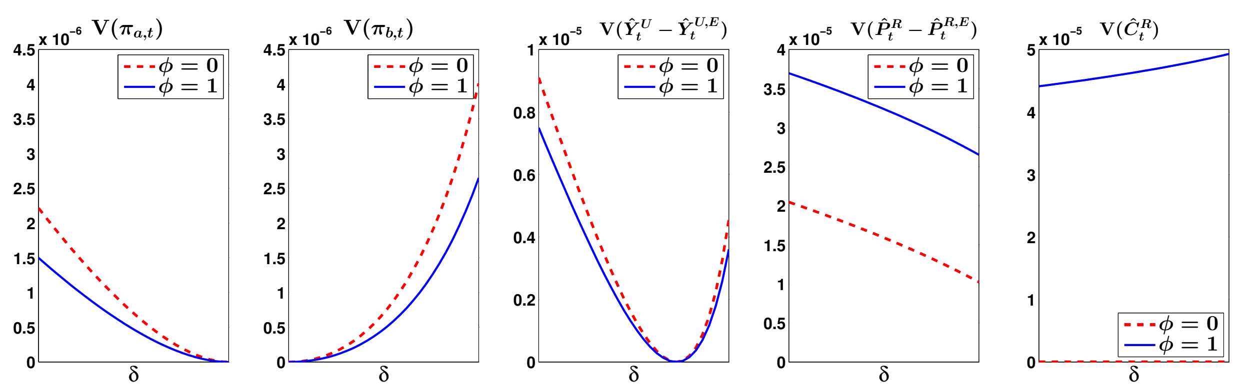

Figure 2: Financial frictions ($\phi > 0$) and model variance

(a) Welfare loss (%) **(b)** Welfare gap (%)

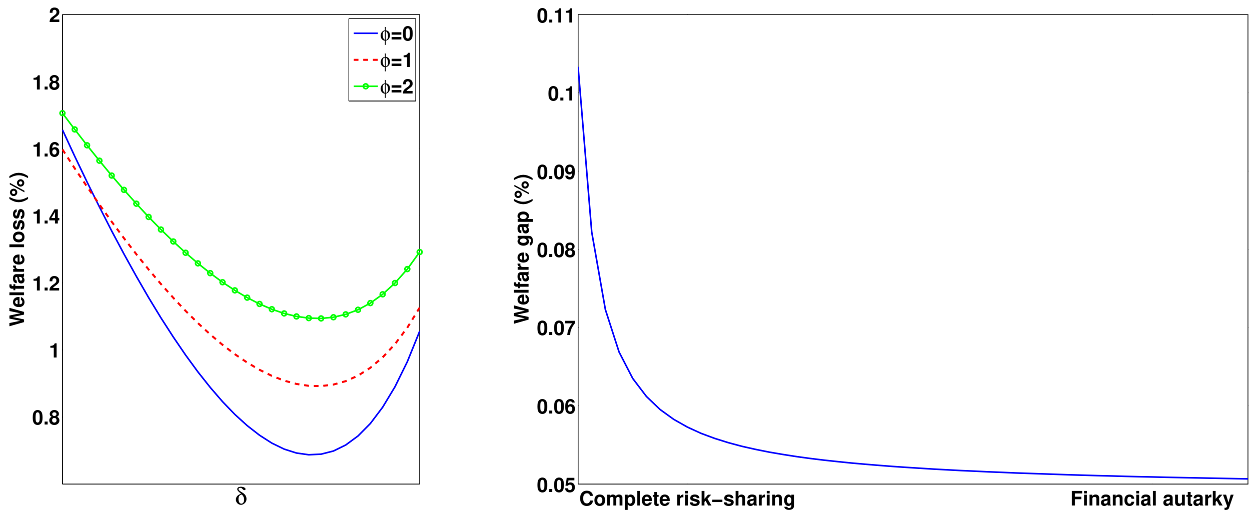

Figure 3: Financial frictions and the welfare implication

It is only without any trade frictions when $\hat{D}_t^U = \hat{P}_t^R$ and $\hat{D}_t^R = 0$ hold. Furthermore, using the consumption demand function we can show that $\hat{D}_t^U$ and $\hat{D}_t^R$ satisfy:

$$\hat{D}_t^U = \Lambda_D \hat{P}_t^R; \quad \hat{D}_t^R = -\nu \Lambda_D \hat{Y}_t^U \tag{12a}$$

where $\Lambda_D \equiv \frac{1}{1+n_a n_b \nu\eta} \in (0,1]$. That is, the average relative price follows relative PPI but in a muted way, while the gap between the relative price depends on the union-wide output. This implication has to do with the endogenous import price premium. First of all, shifts in import demand moderates the movement of the average relative price. When $\hat{P}_t^R$ drops (from any exogenous disturbances), from the substitution effects country A households decrease their import demand while country B households increase theirs. Therefore, import premium decreases in country A ($\widehat{\widetilde{P}}^{a}_{b,t} \downarrow$ while increases in country B ($\widehat{\widetilde{P}}^{b}_{a,t} \uparrow$). Accordingly, the country-specific relative prices in both countries decrease less than they would have without the premiums. Second, when consumers are expanding their consumption (*i.e.* higher union-wide production), the import demand from foreign country exceeds the domestic import demand. When country A goods becomes relatively cheaper from a positive productivity shock, country B's import outweighs country A's import, increasing the import premium further in country B. On the other hand, when country B goods becomes relatively cheaper, country A pays higher import premiums due to larger imports. In both cases, we get $\hat{P}_{a,t}^R \leq \hat{P}_{b,t}^R$.

From the real marginal costs, we learn that trade frictions destabilize the inflations as opposed to the financial frictions case.

$$\widehat{mc}_{a,t} = \underbrace{(\sigma+\varphi)\hat{Y}_t^U}_{(+)} \underbrace{-n_b\left(1+\varphi\eta\right)\hat{P}_t^R}_{(+)} + \underbrace{n_b(1+\varphi\eta)(1-\Lambda_D)\hat{P}_t^R}_{(-)} \underbrace{-n_b^2 \hat{D}_t^R}_{(+)} \underbrace{-(1+\varphi)\hat{Z}_{a,t}}_{(-)}$$

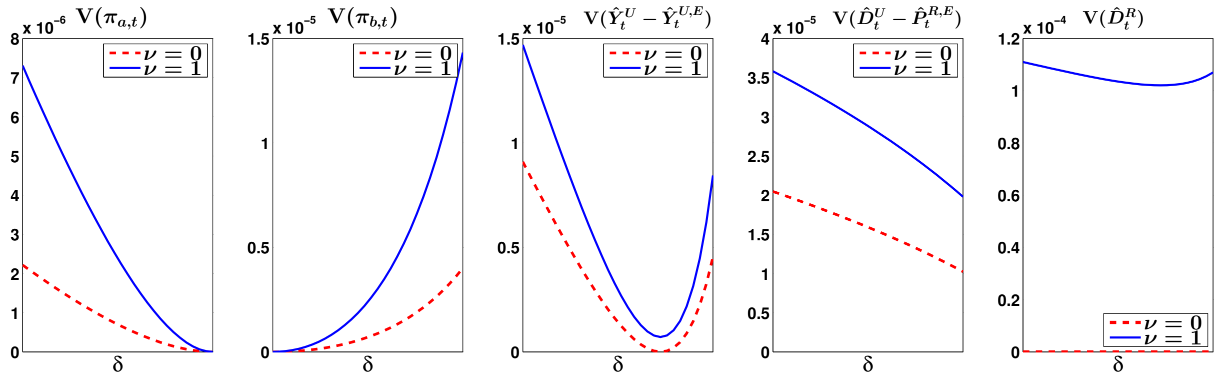

Figure 4: Trade frictions ($\nu > 0$) and model variance

When country A is hit by a positive shock, relative price drops and union-wide output increases. Here additional terms enter the marginal costs due to trade frictions. First, as the negative $\hat{D}_t^R$ term suggests, country B's import demand outweighs country A's which further stabilizes the marginal costs. However, a muted response of the average relative price captured by $(1 - \Lambda_D)\hat{P}_t^R$ dominates the former effects and destabilize the marginal costs. Since the marginal costs are less stabilized, so are the inflations as we verify in Figure 4. All variables of our interests become more volatile when trade frictions exist. Therefore, the welfare cost of the single currency trade off increases. The last plot roughly follows the third since $\hat{D}_t^R$ is a function of the union-wide output level.

The welfare implications of trade frictions are again derived from the loss function:

$$L_t = \theta \sum_{j=a,b} \frac{n_j}{\kappa_j}(\pi_j)^2 + (\sigma + \varphi)\left(\hat{Y}_t^U - \hat{Y}_t^{U,E}\right)^2 + n_a n_b \eta\,(1 + \varphi\eta)\left(\hat{D}_t^U - \hat{P}_t^{R,E}\right)^2 + (n_a n_b)^2 \eta \left(\hat{D}_t^R\right)^2$$

The inefficiency arising from different relative price in country A and B is reflected in the allocation of relative output and in the consumption portfolio difference across countries. We inspect the numerical result in Figure 5a. Expectedly, the welfare loss increases for higher trade frictions, and the curve becomes steeper as ϕ increases. Therefore, when trade frictions exist, the welfare gap between SIT and OIT increases. And the higher the frictions, the larger the gap as in Figure 5b.

4.2.3 The role of external disturbances

In this section, we focus on how external disturbances affect the welfare cost of the single currency trade off. As a reminder, the central bank in the currency union has trouble stabilizing all sources of welfare loss because national inflations and relative price are tightly connected: when inflations are stabilized, relative price does not respond flexibly to productivity shocks and the relative price gap is not closed. Hence, we consider two examples when the volatility of the efficient relative price changes.

First off, we extend our baseline model to have mark-up shocks in addition to the productivity

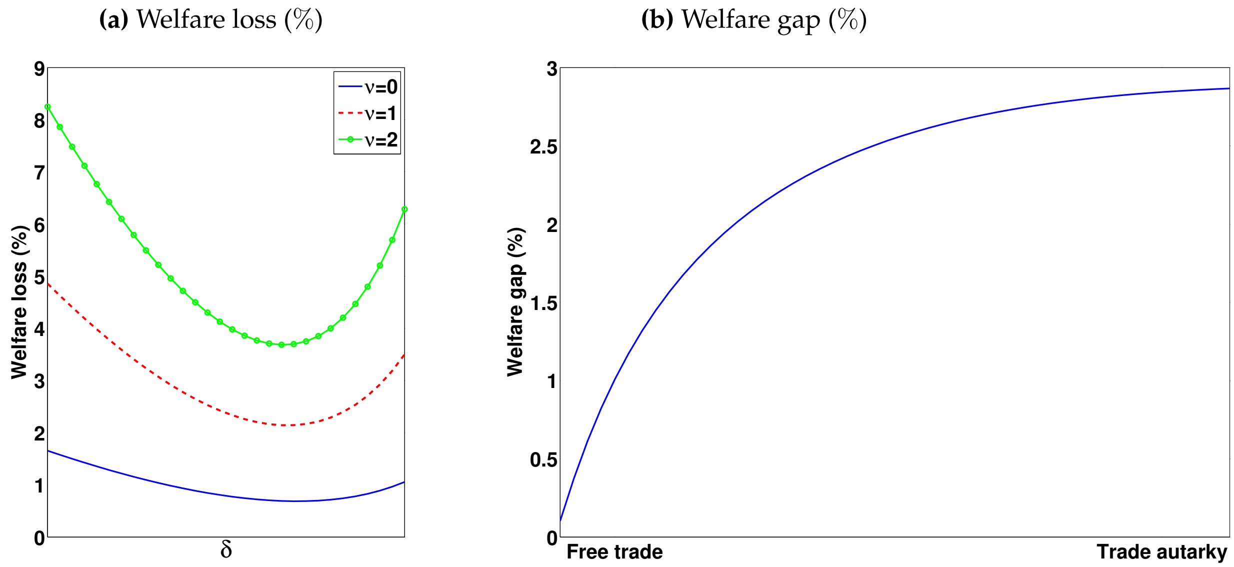

Figure 5: Trade frictions and the welfare implication

shocks. Then, the marginal costs become:

$$\widehat{mc}_{a,t} = (\sigma + \varphi)\hat{Y}_t^U - n_b(1 + \varphi\eta)\hat{P}_t^R - \underbrace{\left((1+\varphi)\hat{Z}_{a,t} + \frac{1}{\theta - 1}\hat{u}_{a,t}\right)}_{\equiv \mu_{a,t}}$$

We define the entire driving force of the real marginal costs as $\mu_{a,t}$. Our experiment is straightforward. We fix the volatility of $\mu_{a,t}$ as the baseline and assume the combination of productivity and mark-up shocks constitute the fixed volatility. We decompose the variance of $\mu_{a,t}$ under three scenarios: productivity is the sole driver, each shock is equally responsible, or mark-up shock is the sole driver. This experiment aims to reveal that when external disturbances that do not affect the efficient relative price are a main driver of the model dynamics, the welfare cost of the single currency trade off decreases. Then, the welfare gap between SIT and OIT diminishes. As in Figure 6a, the slope of the welfare loss becomes flatter as mark-up shocks become a major disturbance.

Our second experiment is similar in spirit with the first. This time, we show that the welfare cost of the single currency trade off depends on the comovement of the productivity shocks in two countries. For simplicity, we assume as in the baseline that the productivity shocks are the only external disturbances. When shocks positively comove, the efficient relative price is not as volatile as it would be when shocks negatively comove. We confirm the implication of the less volatile efficient relative price in Figure 6b. Under positive comovement, the slope of the welfare loss flattens and hence the welfare gap is reduced.

(a) Shock decomposition

(b) Comovement

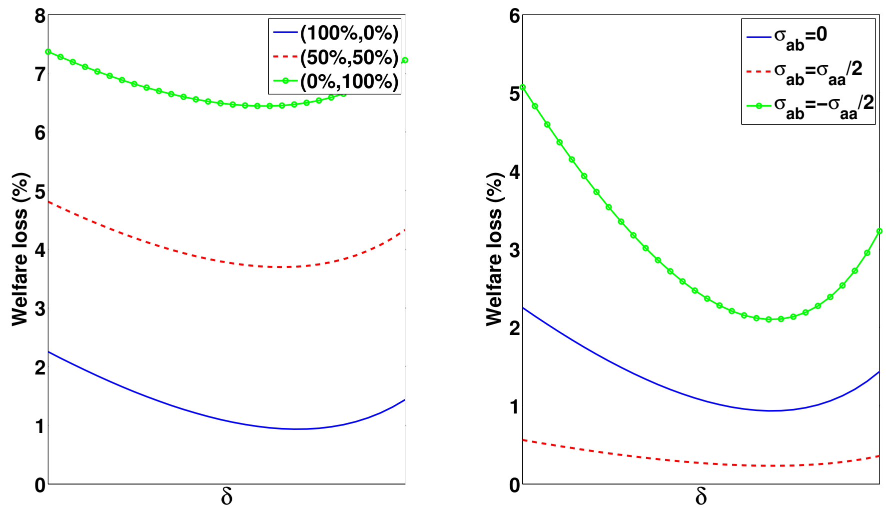

Figure 6: External disturbances and the welfare loss (%)

4.3 Participation constraints in the currency union

So far we take for granted that member countries of the currency union have no incentive to leave the union. However, in this section, we presume countries stay only if their welfare loss is smaller within the currency union than outside the union. So there are likely to be costs and benefits of joining a currency union.

Regarding the cost side, each country gives up national currency and an independent monetary policy. In the earlier section, we mention how a single currency trade off and a utilitarian central bank meet to result the stickiness principle. Some may gain under this monetary policy, while others may not. As to the benefits of staying in the currency union, we consider trade integration through efficient import technology and nominal exchange rate stabilization. For this purpose, we extend our model by imposing that trade intermediaries take a change in nominal exchange rate as part of their costs of supplying import services. Therefore, in the currency union, we presume trade costs are zero since goods are freely relocated across border and nominal exchange rate is fixed. On the other hand, trade costs are non-zero outside the currency union due to inefficient import technology and volatile nominal exchange rate.

To keep member countries from leaving, the central bank in the currency union gives up being a perfect utilitarian and instead provides all countries welfare loss that is smaller than what they would get outside the union. Formally writing, the range of δ that is politically feasible satisfies both $\mathcal{L}_a^{CU} \leq \mathcal{L}_a^{Exit}$ and $\mathcal{L}_b^{CU} \leq \mathcal{L}_b^{Exit}$. Under this participation constraint, the central bank constructs a target inflation index within the range of monetary policy weight δ that is politically feasible. In getting this range, we first derive the country-specific loss function and compute the welfare loss outside the union. We follow Benigno and Woodford (2005) and Benigno and Benigno (2006) in deriving the following loss function.

$$L_{a,t} = \sum_j \lambda_{\pi j}(\pi_{j,t})^2 + \sum_j \lambda_{yj}\left(\hat{Y}_{j,t} - \widetilde{Y}_{j,t}\right)^2 + \lambda_{du}\left(\hat{D}_t^U - \widetilde{D}_t^U\right)^2 + \lambda_{dr}\left(\hat{D}_t^R\right)^2 + \lambda_s\left(\Delta\hat{S}_t\right)^2$$

where the loss function coefficients and the welfare-related targets are:

$$\lambda_{\pi a} = \theta\frac{\widetilde{n}}{\kappa_a}; \quad \lambda_{\pi b} = \theta\frac{1-\widetilde{n}}{\kappa_b}; \quad \lambda_{ya} = \widetilde{n}(\sigma+\varphi); \quad \lambda_{yb} = (1-\widetilde{n})(\sigma+\varphi)$$

$$\lambda_{du} = n_a n_b \eta(1-\sigma\eta); \quad \lambda_{dr} = (n_a n_b)^2\eta; \quad \lambda_s = \omega$$

$$\widetilde{Y}_{a,t} = \frac{1+\varphi}{\sigma+\varphi}\hat{Z}_t^U + n_b\left(\frac{\widetilde{\eta}}{\widetilde{n}}\right)\frac{1+\varphi}{1+\varphi\eta}\hat{Z}_t^R; \quad \widetilde{Y}_{b,t} = \frac{1+\varphi}{\sigma+\varphi}\hat{Z}_t^U - n_a\left(\frac{\eta-\widetilde{\eta}}{1-\widetilde{n}}\right)\frac{1+\varphi}{1+\varphi\eta}\hat{Z}_t^R$$

$$\widetilde{D}_t^U = -\frac{1+\varphi}{1+\varphi\eta}\hat{Z}_t^R$$

and

$$\widetilde{n} = n_a + n_b\frac{1-\eta}{1+\varphi\eta} < n_a; \quad \widetilde{\eta} = n_b\left(\eta + \frac{1-\eta}{\sigma+\varphi}\right) < \eta$$

The first thing to notice is the interdependence between member countries. When country goods are substitutable ($\eta \neq 0$), country A gets welfare loss not only from domestic variables, but from foreign variables. When $\eta = 0$, on the contrary, we get $\tilde{n} = 1$ – only domestic inflation and output gap remain in the loss function. An even closer inspection of the coefficients informs us that the elasticity of substitution among country goods determines the size of such interdependency. When country goods are highly substitutable ($\eta > 1$), country A attaches relatively higher weights on foreign inflation rate and output gap ($\widetilde{n} < n_a$), and vice versa. Our parameterization considers the former case.

The heightened interdependence is a result of the relative demand shift channel that is active through the price setting behaviour of firms when $\eta > 1$. The second order approximation of the optimal prise-setting condition reveals that each firm raises its price when the volatility of the domestic inflation and output gap are expected to be higher. It is optimal for the price-changing firms to do so because the present value of future marginal costs convexly increase in expected future price and output levels. When expected price level is high in the future, the demand for the firm's product and its labor demand will increase convexly. Also, when expected wealth level increases, employment costs rise in a convex way due to lower labor supply. Therefore, firms anticipate higher marginal costs and raise current prices when volatile inflations and output gap are expected. This pricing behaviour is rooted on a precautionary motive as the Calvo pricing does not allow all firms to change their prices as they wish.

If firms in country B raise their price level following this motive, the relative demand for country A goods convexly increases as country A goods becomes cheaper. Then, country A household's disutility from working increases. Since both countries share this incentive for precautionary price-setting, the welfare loss from domestic inflation and output gap become relatively smaller whereas

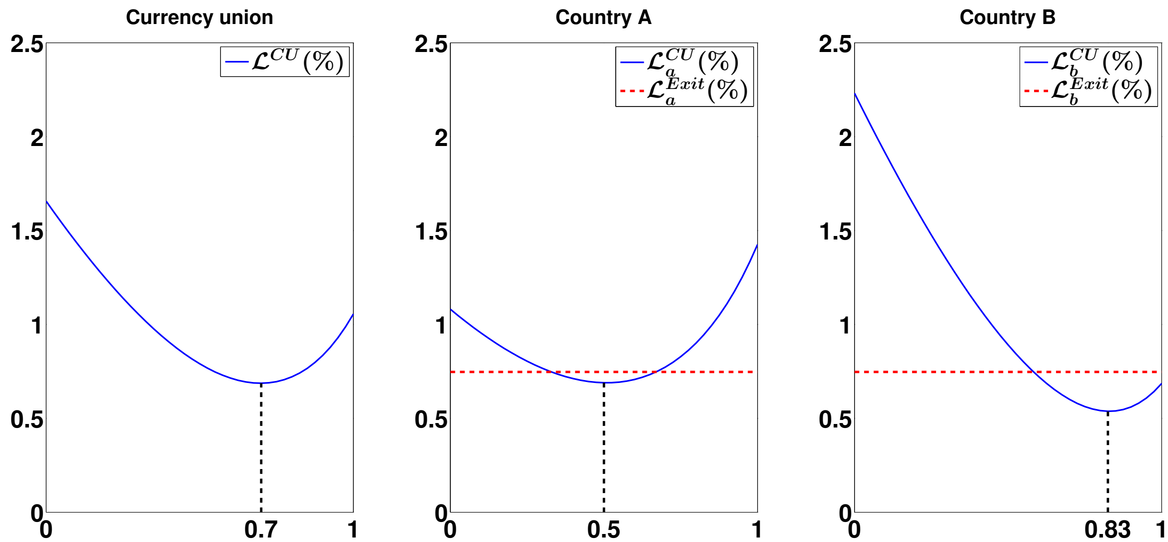

Figure 7: Welfare loss within and outside the union

that from foreign inflation and output gap grow larger. Therefore, while inflation, especially from sticky country, is still a dominant source of welfare loss, it is country B who is more exposed to the welfare loss from unstabilized $\pi_{a,t}$ – not country A itself. Accordingly, country B prefers a stronger stabilization of $\pi_{a,t}$ through a high level of δ.

Finally, the last term of the loss function $\left(\Delta\hat{S}_t\right)^2$ indicates the welfare loss from lost resources due to volatile nominal exchange rate. The extent to which the change of nominal exchange rate affects the welfare loss depends on how convexly trade costs increase in unstable nominal exchange rates. The parameter ω captures this inefficiency.

Now we compute the politically feasible range of δ in Figure 7. When leaving the union, we assume that each country conducts an optimal monetary policy under commitment and that the level of trade frictions are $\nu = 1.5$ and $\omega = 2$. The welfare loss outside the union is drawn as the straight line for country A and B. As the participation constraint implies, the central bank should set δ so that the welfare loss within the union is smaller than the straight line. We can find the intersection of the two politically feasible ranges as in Figure 8. The range of δ the central bank can choose from is $[0.58, 0.67]$. This range does not include the optimal level of $\delta^* = 0.7$ we derived in the baseline model with no trade frictions. Also, we report the point where the excess welfare gains from joining the currency union – the excess matching surplus in a Nash bargaining sense – coincides between country A and B. The level δ indicating an equal bargaining power among member countries equals to $\delta = 0.61$, which comes closer to the simple inflation target $\delta = 0.5$.

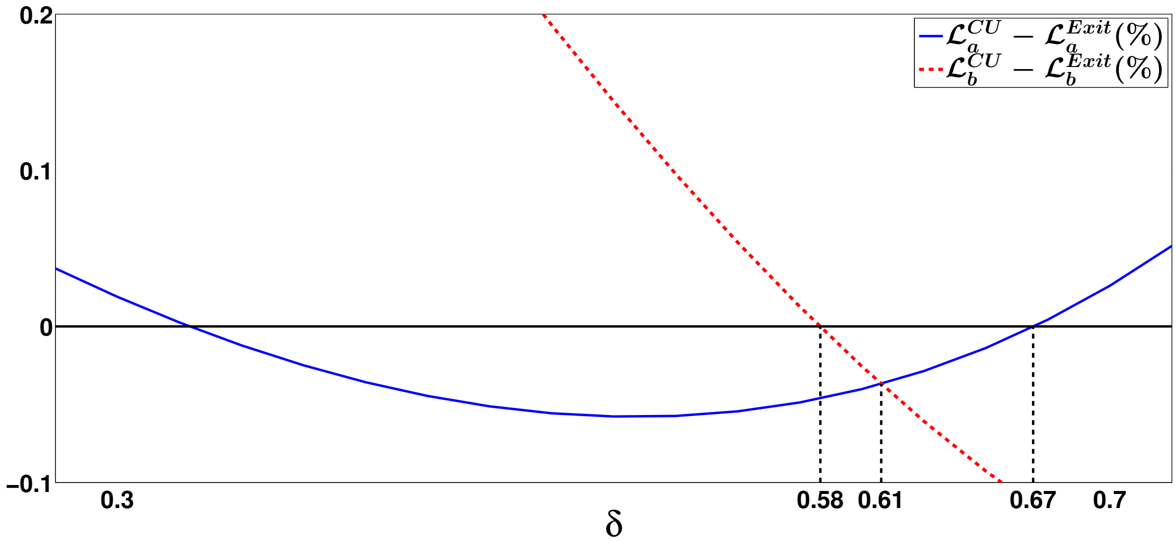

Figure 8: Politically feasible range of δ

5 Conclusion

With regard to the optimal monetary policy in a currency union, a line of researches stress the stickiness principle. That is, for the single central bank to maximize the union-wide welfare, the target inflation index should give high weights on the inflation from the sticky country. The optimal index, however, is not very popular among practitioners and is often replaced by a simple index. In this paper, we find possible justifications for this practice.

We take two approaches in doing so. First of all, we delve into the role of financial and trade frictions and that of external disturbances in altering the welfare cost of sharing a single currency. Second, we impose participation constraints so that the central bank cannot actively increase the welfare of some at the expense of the remaining countries. We conclude that not following the stickiness principle can be justified under certain scenarios.

For tractability, our model admittedly omits numerous potentially important factors in determining the robustness of our conclusion. In this sense, the stylized model should be complemented to yield numerical policy descriptions. We perfectly simplify the fiscal side of the currency union. But in reality, the coexistence of fiscal surplus and fiscal deficit countries can have important implications in a monetary union setting. There may also be interesting interaction between the monetary and fiscal policy when considering participation constraints.

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6 Appendix

6.1 Steady state

We consider a symmetric, efficient steady state equilibrium. Let $Z_{a,t} = Z_{b,t} = 1$ for all t and assume both countries have zero initial wealth for simplicity. Then, after normalization, we thereby confirm that each labor market structure yields equivalent steady state values. $\bar{Y}^U = \bar{C_a} = \bar{C_b} = \bar{\xi_a} = \bar{\xi_b} = 1$, $\bar{P_a^C} = \bar{P_b^C} = \bar{P}_a = \bar{P}_b = \bar{\tilde{P}}_a = \bar{\tilde{P}}_b$ and $\bar{Y}_j(i) = \bar{N}_j(i) = \frac{Y_j}{n_j}$. Also, $\bar{Y}_j = n_j$, $\bar{C}_{a,a} = \bar{C}_{b,a} = n_a\bar{C}$, $\bar{C}_{a,b} = \bar{C}_{b,b} = n_b\bar{C}$ while $\bar{R}^{-1} = Q = \beta$ by goods index and Euler Equation. Therefore, households do not pay transaction costs at steady state: $\Phi(\bar{C}_j, \bar{\xi}_j) = \Phi_c(\bar{C}_j, \bar{\xi}_j) = \Phi_\xi(\bar{C}_j, \bar{\xi}_j) = 0$ and $\Phi_{cc}(\bar{C}_j, \bar{\xi}_j) = \phi$. Lastly, $\Phi_{c\xi}(\bar{C}_j, \bar{\xi}_j) = -\phi$

6.2 Log-linearized System

- CPI

$$\hat{P}_{a,t}^C = n_a \widehat{P}_{a,t} + n_b \hat{\tilde{P}}_{b,t}$$
$$\hat{P}_{b,t}^C = n_a \hat{\tilde{P}}_{a,t} + n_b \hat{P}_{b,t}$$

- Country-specific relative price:

$$\hat{P}_{a,t}^R = \hat{P}_{a,t} - \hat{\tilde{P}}_{b,t}$$
$$\hat{P}_{b,t}^R = \hat{\tilde{P}}_{a,t} - \hat{P}_{b,t}$$

- Identities from the price indices:

$$\hat{P}_{a,t} - \hat{P}_{a,t}^C = n_b \hat{P}_{a,t}^R$$
$$\hat{\tilde{P}}_{b,t} - \hat{P}_{a,t}^C = -n_a \hat{P}_{a,t}^R$$
$$\hat{\tilde{P}}_{a,t} - \hat{P}_{b,t}^C = n_b \hat{P}_{b,t}^R$$
$$\hat{P}_{b,t} - \hat{P}_{b,t}^C = -n_a \hat{P}_{b,t}^R$$

- Identities from the consumption demand

$$\begin{aligned}
\hat{C}_{a,a,t} &= -\eta(\hat{P}_{a,t} - \hat{P}_{a,t}^C) + \hat{C}_{a,t} \\
&= -n_b \eta \hat{P}_{a,t}^R + \hat{C}_{a,t} \\
\hat{C}_{a,b,t} &= n_a \eta \hat{P}_{a,t}^R + \hat{C}_{a,t} \\
\hat{C}_{b,a,t} &= -n_b \eta \hat{P}_{b,t}^R + \hat{C}_{b,t} \\
\hat{C}_{b,b,t} &= n_a \eta \hat{P}_{b,t}^R + \hat{C}_{b,t}
\end{aligned}$$

- Goods market clearing:

$$
\begin{aligned}
& \hat{Y}_{a,t} = n_a \hat{C}_{a,a,t} + n_b \hat{C}_{b,a,t} = -n_b \eta \left(n_a \hat{P}^R_{a,t} + n_b \hat{P}^R_{b,t} \right) + \hat{C}^U_t \\
& \hat{Y}_{b,t} = n_a \hat{C}_{a,b,t} + n_b \hat{C}_{b,b,t} = n_a \eta \left(n_a \hat{P}^R_{a,t} + n_b \hat{P}^R_{b,t} \right) + \hat{C}^U_t \\
\Rightarrow \quad & \hat{Y}^R_t = -\eta \left(n_a \hat{P}^R_{a,t} + n_b \hat{P}^R_{b,t} \right) \\
& Y^U_t = \hat{C}^U_t
\end{aligned}
$$

- Households' FOC:

$$
\begin{aligned}
(\sigma + \phi)\hat{C}^R_t &= \phi \hat{\xi}^R_t \\
\hat{C}^U_t &= \mathbb{E}_t \hat{C}^U_{t+1} - \frac{1}{\sigma}\{\hat{R}_t - \mathbb{E}_t \pi^U_{t+1}\} \\
\varphi \hat{N}_{a,t}(i) + \sigma \hat{C}_{a,t} &= \hat{W}_{a,t}(i) - \hat{P}^C_{a,t} \\
\varphi \hat{N}_{b,t}(i) + \sigma \hat{C}_{b,t} &= \hat{W}_{b,t}(i) - \hat{P}^C_{b,t}
\end{aligned}
$$

  where $\pi^U_t = n_a \pi_{a,t} + n_b \pi_{b,t}$.

- Derivation of $\hat{\xi}^R_t$
  In equilibrium, the real non-financial income of each representative consumer in country A and B is:

$$
\begin{aligned}
P^C_{a,t}\xi_{a,t} &= P_{a,t}Y_{a,t} + (\tilde{P}_{b,t} - P_{b,t})C_{a,b,t} + P^C_{a,t}\Phi(C_{a,t}, \xi_{a,t}) \\
P^C_{b,t}\xi_{b,t} &= P_{b,t}Y_{b,t} + (\tilde{P}_{a,t} - P_{a,t})C_{b,a,t} + P^C_{b,t}\Phi(C_{b,t}, \xi_{b,t})
\end{aligned}
$$

  Therefore,

$$
\begin{aligned}
& \hat{P}^C_{a,t} + \hat{\xi}_{a,t} = \hat{P}_{a,t} + \hat{Y}_{a,t} + n_b \left(\hat{\tilde{P}}_{b,t} - \hat{P}_{b,t} \right) \\
& \hat{P}^C_{b,t} + \hat{\xi}_{b,t} = \hat{P}_{b,t} + \hat{Y}_{b,t} + n_a \left(\hat{\tilde{P}}_{a,t} - \hat{P}_{a,t} \right) \\
\Rightarrow \quad & \hat{E}_t + \hat{\xi}^R_t = \hat{P}^R_t + \hat{Y}^R_t + \hat{E}_t \\
& \hat{\xi}^R_t = \hat{Y}^R_t + \hat{P}^R_t
\end{aligned}
$$

  where $\hat{E}_t$ is the real exchange rate $\hat{E}_t \equiv \hat{P}^C_{a,t} - \hat{P}^C_{b,t}$.

- Import prices:

$$
\begin{aligned}
\hat{\tilde{P}}_{b,t} &= \hat{P}_{b,t} + \nu \hat{C}_{a,b,t} \\
\hat{\tilde{P}}_{a,t} &= \hat{P}_{a,t} + \nu \hat{C}_{b,a,t}
\end{aligned}
$$

  where $\nu = W^T (\bar{P}^C_a)^{-1} g''(\bar{C}_{a,b})\bar{C}_{a,b} = W^T (\bar{P}^C_b)^{-1} g''(\bar{C}_{b,a})\bar{C}_{b,a}$.

- From above, we get

$$
\begin{aligned}
\hat{D}_t^U &\equiv n_a \hat{P}_{a,t}^R + n_b \hat{P}_{b,t}^R \\
&= n_a(\hat{P}_{a,t} - \hat{\tilde{P}}_{b,t}) + n_b(\hat{\tilde{P}}_{a,t} - \hat{P}_{b,t}) \\
&= \hat{P}_t^R - n_a \nu \hat{C}_{a,b,t} + n_b \nu \hat{C}_{b,a,t} \\
&= \hat{P}_t^R - \nu \left(n_a \hat{C}_{a,b,t} - n_b \hat{C}_{b,a,t} \right)
\end{aligned}
$$

  When $\nu = 0$, $\hat{P}_t^R = \hat{D}_t^U$.

- The relative price gap between country A and B:

$$
\begin{aligned}
\hat{D}_t^R \equiv& \hat{P}_{a,t}^R - \hat{P}_{b,t}^R \\
&= (\hat{P}_{a,t} - \hat{\tilde{P}}_{b,t}) - (\hat{\tilde{P}}_{a,t} - \hat{P}_{b,t}) \\
&= \left(\hat{P}_{b,t} - \hat{\tilde{P}}_{b,t} \right) + \left(\hat{P}_{a,t} - \hat{\tilde{P}}_{a,t} \right) \\
&= -\nu (\hat{C}_{a,b,t} + \hat{C}_{b,a,t})
\end{aligned}
$$

  Therefore, when assuming $n_a = n_b$,

$$
\begin{aligned}
\hat{P}_{a,t}^R &= \hat{D}_t - \frac{2\nu}{2 + \nu\eta} \hat{C}_t^U \\
\hat{P}_{b,t}^R &= \hat{D}_t + \frac{2\nu}{2 + \nu\eta} \hat{C}_t^U
\end{aligned}
$$

- Firms' price setting (flexible case)

$$
\begin{aligned}
\hat{P}_{a,t}(i) - \hat{P}_{a,t} &= \hat{W}_{a,t}(i) - \hat{P}_{a,t} - \hat{Z}_{a,t} \\
&= \varphi \hat{N}_{A,t}(i) + \sigma \hat{C}_{a,t} + \hat{P}_{a,t}^C - \hat{P}_{a,t} - \hat{Z}_{a,t} \\
&= \varphi \left(\hat{Y}_{a,t}(i) - \hat{Z}_{a,t} \right) + \sigma \hat{C}_{a,t} + \hat{P}_{a,t}^C - \hat{P}_{a,t} - \hat{Z}_{a,t} \\
&= -\varphi\theta \left(\hat{P}_{a,t}(i) - \hat{P}_{a,t} \right) + \varphi \hat{Y}_{a,t} + \sigma \hat{C}_{a,t} + \hat{P}_{a,t}^C - \hat{P}_{a,t} - (1 + \varphi) \hat{Z}_{a,t} \\
&= -\varphi\theta \left(\hat{P}_{a,t}(i) - \hat{P}_{a,t} \right) \\
&+ (\sigma + \varphi) \hat{Y}_t^U + n_b \left(\varphi \hat{Y}_t^R + \sigma \hat{C}_t^R - \hat{P}_{a,t}^R \right) - (1 + \varphi) \hat{Z}_{a,t}
\end{aligned}
$$

  The last equality is derived by utilizing $\hat{P}_{a,t}^C - \hat{P}_{a,t} = -n_b \hat{P}_{a,t}^R$, $\hat{Y}_{a,t} = \hat{Y}_t^U + n_b \hat{Y}_t^R$, $\hat{C}_t = \hat{C}_t^U + n_b \hat{C}_t^R$ and $\hat{C}_t^U = \hat{Y}_t^U$.

  Likewise,

$$
\begin{aligned}
\hat{P}_{b,t}(i) - \hat{P}_{b,t} &= \hat{W}_{b,t}(i) - \hat{P}_{b,t} - \hat{Z}_{b,t} \\
&= \varphi \hat{N}_{B,t}(i) + \sigma \hat{C}_{b,t} + \hat{P}_{b,t}^C - \hat{P}_{b,t} - \hat{Z}_{b,t}
\end{aligned}
$$

$$
\begin{aligned}
&= \varphi\left(\hat{Y}_{b,t}(i) - \hat{Z}_{b,t}\right) + \sigma\hat{C}_{b,t} + \hat{P}^C_{b,t} - \hat{P}_{b,t} - \hat{Z}_{b,t} \\
&= -\varphi\theta\left(\hat{P}_{b,t}(i) - \hat{P}_{b,t}\right) + \varphi\hat{Y}_{b,t} + \sigma\hat{C}_{b,t} + \hat{P}^C_{b,t} - \hat{P}_{b,t} - (1+\varphi)\hat{Z}_{b,t} \\
&= -\varphi\theta\left(\hat{P}_{b,t}(i) - \hat{P}_{b,t}\right) \\
&+ (\sigma+\varphi)\hat{Y}^U_t - n_a\left(\varphi\hat{Y}^R_t + \sigma\hat{C}^R_t - \hat{P}^R_{b,t}\right) - (1+\varphi)\hat{Z}_{b,t}
\end{aligned}
$$

Similarly used were $\hat{P}^C_{b,t} - \hat{P}_{b,t} = n_a\hat{P}^R_{b,t}$, $\hat{Y}_{b,t} = \hat{Y}^U_t - n_a\hat{Y}^R_t$, and $\hat{C}_{b,t} = \hat{C}^U_t - n_a\hat{C}^R_t$.

- Flexible equilibrium: Since the price setting is symmetric, we get

$$
\begin{aligned}
(1+\varphi)\hat{Z}_{a,t} &= (\sigma+\varphi)\hat{Y}^{U,F}_t + n_b\left(\varphi\hat{Y}^{R,F}_t + \sigma\hat{C}^{R,F}_t - \hat{P}^{R,F}_{a,t}\right) \\
(1+\varphi)\hat{Z}_{b,t} &= (\sigma+\varphi)\hat{Y}^{U,F}_t - n_a\left(\varphi\hat{Y}^{R,F}_t + \sigma\hat{C}^{R,F}_t - \hat{P}^{R,F}_{b,t}\right)
\end{aligned}
$$

By aggregating and subtracting the above two,

$$
\begin{aligned}
(1+\varphi)\hat{Z}^U_t &= (\sigma+\varphi)\hat{Y}^{U,F}_t - \frac{1}{4}\left(\hat{P}^R_{a,t} - \hat{P}^R_{b,t}\right) \\
&= \left(\sigma + \varphi + \frac{\nu}{2+\nu\eta}\right)\hat{Y}^{U,F}_t \\
(1+\varphi)\hat{Z}^R_t &= \varphi\hat{Y}^{R,F}_t + \sigma\hat{C}^{R,F}_t - \hat{D}^f_t \\
&= \left(1 + \varphi\eta + \sigma\Lambda_C\right)\hat{D}^f_t
\end{aligned}
$$

- The efficient equilibrium ($\nu = 0$, $\phi = 0$):

$$
\begin{aligned}
(1+\varphi)\hat{Z}^U_t &= (\sigma+\varphi)\hat{Y}^{U,E}_t \\
(1+\varphi)\hat{Z}^R_t &= \frac{(1+\varphi\eta)}{\eta}\hat{Y}^{R,E}_t = -(1+\varphi\eta)\hat{P}^{R,E}_{a,t} = -(1+\varphi\eta)\hat{P}^{R,E}_{b,t} = -(1+\varphi)\hat{D}^e_t
\end{aligned}
$$

Therefore, the flexible equilibrium deviates from the efficient one as follows when the financial frictions and trade frictions present.

$$
\begin{aligned}
\hat{Y}^{U,F}_t &= \frac{\sigma+\varphi}{\sigma+\varphi+\frac{\nu}{2+\nu\eta}}\hat{Y}^{U,E}_t \le \hat{Y}^{U,E}_t \\
\hat{D}^f_t &= \frac{1+\varphi\eta}{1+\varphi\eta+\sigma\Lambda_C}\hat{P}^{R,E}_t \le \hat{P}^{R,E}_t
\end{aligned}
$$

- NK Phillips Curves:

$$
\begin{aligned}
\pi_{a,t} &= \beta\mathbb{E}_t\pi_{a,t+1} + \kappa_a\left[(\sigma+\varphi)\hat{Y}^U_t + n_b\left(\varphi\hat{Y}^R_t + \sigma\hat{C}^R_t - \hat{P}^R_{a,t}\right) - (1+\varphi)\hat{Z}_{a,t}\right] \\
\pi_{b,t} &= \beta\mathbb{E}_t\pi_{b,t+1} + \kappa_b\left[(\sigma+\varphi)\hat{Y}^U_t - n_a\left(\varphi\hat{Y}^R_t + \sigma\hat{C}^R_t - \hat{P}^R_{a,t}\right) - (1+\varphi)\hat{Z}_{a,t}\right]
\end{aligned}
$$

where $\kappa_j = \frac{(1-\alpha_j)(1-\alpha_j\beta)}{\alpha_j(1+\varphi\theta)}$.

System of equations

The equilibrium path of $\{\hat{Y}_t^U, \hat{Y}_t^R, \hat{C}_{a,t}, \hat{C}_{b,t}, \hat{C}_t^R, \hat{P}_{a,t}^R, \hat{P}_{b,t}^R, \hat{P}_t^R, \hat{D}_t^U, \hat{D}_t^R, \pi_{a,t}, \pi_{b,t}, \}_{t=0}^{\infty}$ are determined by the following equations. That last corresponds to the monetary policy specification.

$$\begin{aligned}
\pi_{a,t} &= \beta \mathbb{E}_t \pi_{a,t+1} + \kappa_a \left[(\sigma+\varphi)\hat{Y}_t^U + n_b \left(\varphi \hat{Y}_t^R + \sigma \hat{C}_t^R - \hat{P}_{a,t}^R \right) - (1+\varphi)\hat{Z}_{a,t} \right] \\
\pi_{b,t} &= \beta \mathbb{E}_t \pi_{b,t+1} + \kappa_b \left[(\sigma+\varphi)\hat{Y}_t^U - n_a \left(\varphi \hat{Y}_t^R + \sigma \hat{C}_t^R - \hat{P}_{b,t}^R \right) - (1+\varphi)\hat{Z}_{b,t} \right] \\
\hat{Y}_t^R &= -\eta \hat{D}_t^U \\
(\sigma+\varphi)\hat{C}_t^R &= \phi \left(\hat{Y}_t^R + \hat{P}_t^R \right) \\
\hat{D}_t^U &= \hat{P}_t^R - \nu \left(n_a \hat{C}_{a,b,t} - n_b \hat{C}_{b,a,t} \right) \\
\hat{D}_t^R &= -\nu (\hat{C}_{a,b,t} + \hat{C}_{b,a,t}) \\
\hat{C}_{a,b,t} &= n_a \eta \hat{P}_{a,t}^R + \hat{C}_{a,t} \\
\hat{C}_{b,a,t} &= -n_b \eta \hat{P}_{b,t}^R + \hat{C}_{b,t} \\
\hat{D}_t^U &= n_a \hat{P}_{a,t}^R + n_b \hat{P}_{b,t}^R \\
\hat{D}_t^R &= \hat{P}_{a,t}^R - \hat{P}_{b,t}^R \\
\hat{P}_t^R &= \hat{P}_{t-1}^R + \pi_{a,t} - \pi_{b,t} \\
0 &= \delta \pi_{a,t} + (1-\delta)\pi_{b,t}
\end{aligned}$$

6.3 Estimation of the shock processes

We assume productivity shocks of the form

$$\begin{pmatrix} \hat{Z}_{a,t} \\ \hat{Z}_{b,t} \end{pmatrix} = \begin{pmatrix} \rho_a & 0 \\ 0 & \rho_b \end{pmatrix} \begin{pmatrix} \hat{Z}_{a,t-1} \\ \hat{Z}_{b,t-1} \end{pmatrix} + \begin{pmatrix} \sigma_{aa} & \sigma_{ab} \\ \sigma_{ab} & \sigma_{bb} \end{pmatrix} \begin{pmatrix} \varepsilon_{a,t} \\ \varepsilon_{b,t} \end{pmatrix}$$

where $\begin{pmatrix} \varepsilon_{a,t} \\ \varepsilon_{b,t} \end{pmatrix} \sim \mathcal{N}(0, I)$. Since our model implies $\hat{Z}_{j,t} = \hat{Y}_{j,t} - \hat{N}_{j,t}$, we use the quarterly labour productivity per hour worked data of Germany and France. The labour productivity is defined as real output per unit of labour input. The data range from 1999Q1 to 2014Q1 and are from the eurostat. We remove the linear trend before the estimation process.

Restricting the shock process to be symmetric, we further assume $\rho_a = \rho_b$ and $\sigma_{aa} = \sigma_{bb}$. A loose prior distribution is adopted for the Bayesian estimation: we use a uniform distribution $(0, 1)$ for ρ_j and another uniform distribution $[0, 0.2]$ for both σ_{jj} and σ_{ab}. Accordingly, we get the posterior mode $\rho_j = 0.96$, $\sigma_{jj} = 0.007$, and $\sigma_{ab} = 0.001$.

6.4 The law of motion of $\hat{P}_t^R$

When we focus on the symmetric country size case ($n_a = n_b$), we can express the relative consumption and output as a scalar multiplication of the average terms of trade.

$$
\begin{aligned}
\hat{Y}_t^R &= -\eta \hat{D}_t^U \\
\hat{C}_t^R &= \Lambda_C \hat{D}_t^U = -\frac{\phi(\eta - 1 - n_a n_b \nu \eta)}{\sigma + \phi} \hat{D}_t^U \\
\hat{D}_t^U &= \Lambda_D \hat{P}_t^R
\end{aligned}
$$

Now define $\tilde{y}_t^U$, $\tilde{y}_t^R$, $\tilde{c}_t^R$, $\tilde{d}_t$, $\tilde{p}_{a,t}^R$, and $\tilde{p}_{b,t}^R$ as the gap of aggregate output, relative output, relative consumption, average relative price, and country-specific terms of trade gap from their efficient counterparts.

The monetary policy stabilizes the target inflation index:

$$\delta \pi_{a,t} + (1 - \delta)\pi_{b,t} = 0$$

Letting $s \equiv \frac{1-\delta}{\delta}$, $\pi_{a,t} = -s\pi_{b,t}$ and s is decreasing towards zero in δ. Expressing the NKPCs in the welfare-relevant gap terms,

$$
\begin{aligned}
\pi_{a,t} &= \beta \mathbb{E}_t \pi_{a,t+1} + \kappa_a \left\{ (\sigma + \varphi)\tilde{y}_t^U + \frac{1}{2}\left(\varphi \tilde{y}_t^R + \sigma \tilde{c}_t^R - \tilde{p}_{a,t}^R\right) \right\} \\
\pi_{b,t} &= \beta \mathbb{E}_t \pi_{b,t+1} + \kappa_b \left\{ (\sigma + \varphi)\tilde{y}_t^U - \frac{1}{2}\left(\varphi \tilde{y}_t^R + \sigma \tilde{c}_t^R - \tilde{p}_{b,t}^R\right) \right\}
\end{aligned}
$$

the given monetary policy then stabilizes the weighted sum of the current real marginal cost of production in each country.

$$\delta \kappa_a \widehat{mc}_{a,t} + (1 - \delta)\kappa_b \widehat{mc}_{b,t} = 0$$

where the real marginal cost of production in each country is:

$$
\begin{aligned}
\widehat{mc}_{a,t} &= (\sigma + \varphi)\tilde{y}_t^U + \frac{1}{2}\left(\varphi \tilde{y}_t^R + \sigma \tilde{c}_t^R - \tilde{p}_{a,t}^R\right) \\
\widehat{mc}_{b,t} &= (\sigma + \varphi)\tilde{y}_t^U - \frac{1}{2}\left(\varphi \tilde{y}_t^R + \sigma \tilde{c}_t^R - \tilde{p}_{b,t}^R\right)
\end{aligned}
$$

By definition, following relation holds between terms of trade and national inflations, which allows us to solve for the system of linear rational expectations for a terms of trade in a Blanchard-Kahn method.

$$
\begin{aligned}
\hat{P}_t^R &= \hat{P}_{t-1}^R + \pi_{a,t} - \pi_{b,t} \\
\Rightarrow \quad \left(\hat{P}_t^R - \hat{P}_{t-1}^R\right) &= -(1 + s)\pi_{b,t} = \frac{1 + s}{s}\pi_{a,t}
\end{aligned}
$$

We consider two cases: (i) $0 < \kappa_a < \infty$ and $\kappa_b \to \infty$, (ii) $0 < \kappa_a \le \kappa_b < \infty$.

(i) $0 < \kappa_a < \infty$ and $\kappa_b \to \infty$.

Since price is flexible in country B, $\widehat{mc}_{b,t} = 0$. Then, by perfectly stabilizing $\pi_{a,t}$, or by letting $\delta = 1$, we have $\widehat{mc}_{a,t} = 0$. Therefore, the equilibrium is equivalent to the flexible equilibrium except that we specify the dynamics of $\pi_{b,t}$ to follow $\pi_{b,t} = -\left(\hat{P}_t^R - \hat{P}_{t-1}^R\right)$. There is, however, no loss from the unstabilized $\pi_{b,t}$ because $\kappa_b \to \infty$. We confirm that $\delta = 1$ is the optimal weight as any δ less than unity yields higher welfare losses.

Now we consider a case where $\delta < 1$. Since $\widehat{mc}_{b,t} = 0$,

$$(\sigma + \varphi)\tilde{y}_t^U = \frac{1}{2}\left(\varphi\tilde{y}_t^R + \sigma\tilde{c}_t^R - \tilde{p}_{b,t}^R\right)$$

Substituting this relation in to $\widehat{mc}_{a,t}$,

$$\begin{aligned}
\widehat{mc}_{a,t} &= \left(\varphi\tilde{y}_t^R + \sigma\tilde{c}_t^R\right) - \frac{1}{2}\left(\tilde{p}_{a,t}^R + \tilde{p}_{b,t}^R\right) \\
&= \left(-\varphi\eta + \sigma\Lambda_C - 1\right)\hat{D}_t + \left(1 - \varphi\eta\right)\hat{P}_t^{R,E} \\
&= \frac{1-\varphi\eta}{\Lambda}\left(\Lambda_D\hat{P}_t^R + \Lambda\hat{P}_t^{R,E}\right)
\end{aligned}$$

For the second equality, I use $\tilde{y}_t^R = -\eta\tilde{d}_t$ and $\tilde{c}_t^R = \Lambda_C\hat{D}_t$, and for the last equality, I define $\Lambda \equiv \frac{1-\varphi\eta}{-\varphi\eta+\sigma\Lambda_C-1}$.

By substituting $\pi_{a,t} = \frac{s}{1+s}\left(\hat{P}_t^R - \hat{P}_{t-1}^R\right)$ into the NKPC of country A, we get the linear system of $\hat{P}_t^R$.

(ii) $0 < \kappa_a \le \kappa_b < \infty$.

Let $\kappa \equiv \frac{\kappa_b}{\kappa_a} \ge 1$. Then, applying the monetary policy is equivalent to having

$$\widehat{mc}_{a,t} + (s\kappa)\widehat{mc}_{b,t} = 0$$

Therefore, aggregating the NKPCs yields

$$(\sigma + \varphi)\tilde{y}_t^U = -\frac{1}{2}\left(\frac{1-s\kappa}{1+s\kappa}\right)\left(\varphi\tilde{y}_t^R + \sigma\tilde{c}_t^R\right) + \frac{1}{2}\left(\frac{\tilde{p}_{a,t}^R - (s\kappa)\tilde{p}_{b,t}^R}{1+s\kappa}\right)$$

Substituting this result into the NKPC of country B,

$$\widehat{mc}_{b,t} = -\frac{1}{1+s\kappa}\left(\left(-\varphi\eta + \sigma\Lambda_C - 1\right)\hat{D}_t + \left(1-\varphi\eta\right)\hat{P}_t^{R,E}\right)$$

$$= \frac{1-\varphi\eta}{(1+s\kappa)\,\Lambda}\left(\Lambda_D \hat{P}_t^R + \Lambda \hat{P}_t^{R,E}\right)$$

Using the relation $\pi_{b,t} = -\frac{1}{1+s}\left(\hat{P}_t^R - \hat{P}_{t-1}^R\right)$,

$$\pi_{b,t} = \beta \mathbb{E}_t \pi_{b,t+1} + \kappa_b \widehat{mc}_{b,t}$$

$$\Rightarrow \quad -\frac{1}{1+s}\left(\hat{P}_t^R - \hat{P}_{t-1}^R\right) = -\frac{\beta}{1+s}\mathbb{E}_t\left(\hat{P}_{t+1}^R - \hat{P}_t^R\right) + \frac{\kappa_b(1-\varphi\eta)}{(1+s\kappa)\,\Lambda}\left(\Lambda_D \hat{P}_t^R + \Lambda \hat{P}_t^{R,E}\right)$$

Let $A \equiv \frac{(1+s)\kappa_a\kappa_b(1+\varphi\eta)}{\beta(\kappa_a+s\kappa_b)\Lambda} > 0$ and rearrange the equation to yield:

$$\mathbb{E}_t \hat{P}_{t+1}^R = \left(1 + \frac{1}{\beta} + A\Lambda_D\right)\hat{P}_t^R - \frac{1}{\beta}\hat{P}_{t-1}^R - A\Lambda \hat{P}_t^{R,E}$$

Then,

$$\mathbb{E}_t \begin{bmatrix} \hat{P}_{t+1}^R \\ \hat{P}_t^R \end{bmatrix} = \begin{bmatrix} 1+\frac{1}{\beta}+A\Lambda_D & -\frac{1}{\beta} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{P}_t^R \\ \hat{P}_{t-1}^R \end{bmatrix} + \begin{bmatrix} -A\Lambda \\ 0 \end{bmatrix} \hat{P}_t^{R,E}$$

The characteristic polynomial of the system is:

$$P(\lambda) = \lambda^2 - \left(1 + \frac{1}{\beta} + A\Lambda_D\right)\lambda + \frac{1}{\beta}$$

Since $P(0) > 0$ and $P(1) = -A\Lambda_D < 0$, we have a unique and bounded solution where $0 < \lambda_1 < 1$ and $\lambda_2 > 1$. Letting $B \equiv 1 + \frac{1}{\beta} + A\Lambda_D$, we have

$$\lambda = \frac{1}{2}\left(B \pm \sqrt{B^2 - \frac{4}{\beta}}\right)$$

Denoting the eigenvector corresponding to λ_2 as $\begin{pmatrix} 1 & d_2 \end{pmatrix}$, we have

$$\begin{pmatrix} 1 & d_2 \end{pmatrix} \begin{bmatrix} B-\lambda_2 & -\frac{1}{\beta} \\ 1 & -\lambda_2 \end{bmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix}$$

Therefore, $d_2 = -\frac{1}{\beta\lambda_2} = -\lambda_1$ since $\lambda_1\lambda_2 = \frac{1}{\beta}$.

Letting $w_t \equiv \begin{pmatrix} 1 & d_2 \end{pmatrix} \begin{bmatrix} \hat{P}_t^R \\ \hat{P}_{t-1}^R \end{bmatrix}$, we can express the system as

$$\mathbb{E}_t w_{t+1} = \lambda_2 w_t - A\Lambda \hat{P}_t^{R,E}$$

Through forward iterations, we arrive at

$$w_t = \frac{A\Lambda}{\lambda_2}\hat{P}_t^{R,E}\sum_{k=0}^{\infty}\left(\frac{\rho}{\lambda_2}\right)^k = \frac{A\Lambda}{\lambda_2-\rho}\hat{P}_t^{R,E}$$

Therefore,

$$\begin{aligned}\hat{P}_t^R &= -d_2\hat{P}_{t-1}^R + \frac{A\Lambda}{\lambda_2-\rho}\hat{P}_t^{R,E} \\ &= \lambda_1\hat{P}_{t-1}^R + \frac{A\Lambda}{\lambda_2-\rho}\hat{P}_t^{R,E}\end{aligned}$$

In case of Benigno (2004), $\phi = \nu = 0$ implies $\Lambda_D = 1$ and $\Lambda = 1$, which results in $A = \frac{(1+s)\kappa_a\kappa_b}{\kappa_a+s\kappa_b}\frac{1+\varphi\eta}{\beta(1+\theta\varphi)}$ and $B = 1 + \frac{1}{\beta} + A$. We then can express the output gap as a function of the terms of trade gap.

$$\tilde{y}_t^U = \frac{s\kappa-1}{s\kappa+1}\frac{1+\varphi\eta}{2(\sigma+\varphi)}\tilde{p}_t^R$$

Note that $\frac{s\kappa-1}{s\kappa+1}$ is a decreasing function of δ and $s = \frac{1}{\kappa}$ always leads to $\tilde{y}_t^U = 0$. That is, the bigger κ (i.e. the bigger the heterogeneity in nominal rigidity), the bigger δ such that $\frac{s\kappa-1}{s\kappa+1} = 0$.

6.5 Outside-the-union model

6.5.1 Model

Outside the union, each country adopts a national currency. We let subscript j on $P_{j,t}^j$ denote the currency units. (*e.g.* $P_{a,t}^a$ is a price of country A goods denominated in country A currency.) Without a single currency, the nominal rate is no more fixed and we extend our model that the intermediaries have to make extra payments for a volatile nominal exchange rate S_t: $\Omega\left(\Delta_t\right)$ where S_t describes how much of country A currency is required to pay off one unit of country B currency. We assume $\Omega\left(\Delta S_t\right)$ is weakly convex. Hence, $\Omega_\Delta \geq 0$ and $\Omega_{\Delta\Delta} \geq 0$. We also suppose that there is no first order effect of the nominal exchange rate related trade frictions at the steady state. That is, $\Omega_\Delta\left(0\right) = 0$.

Country A

$$\max_{M_{a,t}} \Pi_{a,t}^T = \tilde{P}_{b,t}^a M_{a,t} - P_{b,t}^b S_t\left(M_{a,t} + \Omega\left(\Delta S_t\right)\right) + s_a^T M_{a,t} - W_{a,t}^T N_{a,t}^T$$

The first order condition is:

$$\tilde{P}_{b,t}^a = P_{b,t}^b S_t + W_{a,t}^T g'\left(M_{a,t}\right) - s_a^T$$

Country B

$$\max_{M_{b,t}} \Pi^T_{b,t} = \tilde{P}^b_{b,t} M_{b,t} - P^a_{a,t} S_t^{-1} \left(M_{b,t} + \Omega\left(\Delta S_t\right)\right) + s^T_b M_{b,t} - W^T_{b,t} N^T_{b,t}$$

The first order condition is:

$$\tilde{P}^b_{a,t} = P^a_{a,t} S_t^{-1} + W^T_{b,t} g'\left(M_{b,t}\right) - s^T_a$$

In equilibrium, the profit each trade intermediary yields reduces to:

$$\begin{aligned}\Pi^T_{a,t} &= W^T_{a,t} g'\left(M_{a,t}\right) M_{a,t} - W^T_{a,t} N^T_{a,t} - P^b_{b,t} S_t \Omega\left(\Delta S_t\right) \\ \Pi^T_{b,t} &= W^T_{b,t} g'\left(M_{b,t}\right) M_{b,t} - W^T_{b,t} N^T_{b,t} - P^a_{a,t} S_t^{-1} \Omega\left(\Delta S_t\right)\end{aligned}$$

Aggregating the household budget constraints of country A and B and applying the government budget constraints, the definition of firms' profits, and the bond market clearing condition, we get:

$$n_a P^C_{a,t} C_{a,t} + n_b P^C_{b,t} S_t C_{b,t} = P^a_{a,t} Y_{a,t} + P^b_{b,t} S_t Y_{b,t} + n_a \Pi^T_{a,t} + n_b \Pi^T_{b,t} S_t$$

Sufficient conditions for the above constraint to be satisfied are:

$$\begin{aligned}n_a P^a_{a,t} C_{a,a,t} + n_b \tilde{P}^b_{a,t} S_t C_{b,a,t} &= P^a_{a,t} Y_{a,t} + n_a \Pi^T_{b,t} S_t \\ n_a \tilde{P}^a_{b,t} C_{a,b,t} + n_b P^b_{b,t} S_t C_{b,b,t} &= P^b_{b,t} S_t Y_{b,t} + n_a \Pi^T_{a,t}\end{aligned}$$

By substituting the derived import price and import intermediaries' profit, we get to the goods market clearing conditions:

$$\begin{aligned}Y_{a,t} &= n_a C_{a,a,t} + n_b C_{b,a,t} + n_b \Omega\left(\Delta S_t\right) \\ Y_{b,t} &= n_a C_{a,b,t} + n_b C_{b,b,t} + n_a \Omega\left(\Delta S_t\right)\end{aligned}$$

6.5.2 Log-linearization

- The import prices

$$\begin{aligned}\widehat{\tilde{P}}^a_{b,t} &= \hat{P}^b_{b,t} + \hat{S}_t + \nu \hat{C}_{a,b,t} \\ \widehat{\tilde{P}}^b_{a,t} &= \hat{P}^a_{a,t} - \hat{S}_t + \nu \hat{C}_{b,a,t}\end{aligned}$$

- The relative price gap

$$\begin{aligned}\hat{D}^R_t = \left(\hat{P}^R_{a,t} - \hat{P}^R_{b,t}\right) &= \left(\hat{P}^a_{a,t} - \widehat{\tilde{P}}^a_{b,t}\right) - \left(\widehat{\tilde{P}}^b_{a,t} - \hat{P}^b_{b,t}\right) \\ &= \left(\hat{P}^a_{a,t} - \widehat{\tilde{P}}^b_{a,t}\right) + \left(\hat{P}^b_{b,t} - \widehat{\tilde{P}}^a_{b,t}\right)\end{aligned}$$

$$= \left(\hat{S}_t - \nu \hat{C}_{a,b,t}\right) + \left(-\hat{S}_t - \nu \hat{C}_{b,a,t}\right)$$
$$= -\nu \left(\hat{C}_{a,b,t} + \hat{C}_{b,a,t}\right)$$

- The path of the nominal exchange rate $\hat{S}_t$ are derived residually as follows. Since $\frac{\pi_{a,t}}{\pi_{b,t}} = \frac{P^a_{a,t}}{P^a_{a,t-1}} \frac{P^b_{b,t-1}}{P^b_{b,t}}$, in a log linear form:

$$\begin{aligned}
\pi_{a,t} - \pi_{b,t} &= \Delta\left(\hat{P}^a_{a,t} - \hat{P}^b_{b,t}\right) \\
&= \Delta\left(\widehat{\tilde{P}}^b_{a,t} + \hat{S}_t - \nu \hat{C}_{b,a,t} - \hat{P}^b_{b,t}\right) \\
&= \Delta\left(\hat{P}^R_{b,t} + \hat{S}_t - \nu \hat{C}_{b,a,t}\right) \\
&= \Delta\left(\hat{D}^U_t - n_a \hat{D}^R_t + \hat{S}_t - \nu \hat{C}_{b,a,t}\right)
\end{aligned}$$

where $\hat{D}^U_t = n_a \hat{P}^R_{a,t} + n_b \hat{P}^R_{b,t}$ and $\hat{D}^R_t = \hat{P}^R_{a,t} - \hat{P}^R_{b,t}$. Equivalently, $\pi_{a,t} - \pi_{b,t} = \Delta\left(\hat{P}^R_{a,t} - \hat{S}_t - \nu \hat{C}_{a,b,t}\right)$.

To derive the second order approximation of $\Omega(\Delta S_t)$, we make use of the following property:

$$\frac{\partial \Omega_t}{\partial S_t} = \Omega_\Delta; \quad \frac{\partial \Omega_t}{\partial S_{t-1}} = -\Omega_\Delta$$
$$\frac{\partial^2 \Omega_t}{\partial S_t^2} = \Omega_{\Delta\Delta}; \quad \frac{\partial^2 \Omega_t}{\partial S_{t-1}^2} = \Omega_{\Delta\Delta}; \quad \frac{\partial^2 \Omega}{\partial S_t \partial S_{t-1}} = -\Omega_{\Delta\Delta}$$

Then, we get:

$$\Omega(\Delta S_t) \simeq \frac{\Phi_{\Delta\Delta}}{2} \bar{S}^2 \left(\hat{S}_t^2 - 2\hat{S}_t \hat{S}_{t-1} + \hat{S}_{t-1}^2\right) = \frac{\Omega_{\Delta\Delta}}{2} \bar{S}^2 \left(\hat{S}_t - \hat{S}_{t-1}\right)^2$$

6.6 Loss functions

We follow Woodford (2003) in deriving the utility-based loss function. Taking a second order Taylor expansion of the utility function around the steady state, we obtain

$$U(C_t) = U(\bar{C}) + U_C(C_t - \bar{C}) + \frac{1}{2} U_{CC}(C_t - \bar{C})^2 + t.i.p + O\left(\|\xi\|^3\right) \tag{13}$$

where $O\left(\|\xi\|^3\right)$ represents all relevant terms that are of third or higher order, and $t.i.p$ denotes all the terms independent of monetary policy. We also take a second order Taylor expansion of C_t. Then we have

$$C_{j,t} = \bar{C}\left(1 + \hat{C}_{j,t} + \frac{1}{2}\hat{C}^2_{j,t}\right) + O\left(\|\xi\|^3\right) \tag{14}$$

where $\hat{C}_{j,t} \equiv \log C_{j,t} - \log \bar{C}_j$. This implies

$$C_{j,t} - \bar{C}_j = \bar{C}_j \hat{C}_{j,t} + \frac{1}{2}\bar{C}_j \hat{C}^2_{j,t} + O\left(\|\xi\|^3\right) \tag{15}$$

Substituting (15) into (13) gives

$$U(C_{j,t}) = U(\bar{C}_j) + U_C\bar{C}_j\hat{C}_{j,t} + \frac{1}{2}U_C\bar{C}_j\hat{C}_{j,t}^2 + \frac{1}{2}U_{CC}\bar{C}_j^2\hat{C}_{j,t}^2 + t.i.p + O\left(\|\xi\|^3\right) \tag{16}$$

Note that $U(\bar{C})$ is independent of monetary policy. We rewrite (16) as

$$U(C_{j,t}) = U_C\bar{C}_j\left\{\hat{C}_{j,t} + \frac{1}{2}\hat{C}_{j,t}^2 + \frac{1}{2}\frac{U_{CC}\bar{C}_j}{U_C}\hat{C}_{j,t}^2\right\} + t.i.p + O\left(\|\xi\|^3\right) \tag{17}$$

where $t.i.p$ denotes all the terms independent of monetary policy. From the utility function we assume in the text, we have $\frac{U_{CC}\bar{C}_j}{U_C} = -\sigma$. Thus we obtain

$$U(C_{j,t}) = U_C\bar{C}_j\left\{\hat{C}_{j,t} + \frac{1}{2}(1-\sigma)\hat{C}_{j,t}^2\right\} + t.i.p + O\left(\|\xi\|^3\right)$$

Now we also take a second order Taylor expansion of $V\left(N_{j,t}(i)\right)$.

$$V\left(N_{j,t}(i)\right) = V(\bar{N}) + V_N(N_{j,t}(i) - \bar{N}) + V_{NN}(N_{j,t}(i) - \bar{N})^2 + t.i.p + O\left(\|\xi\|^3\right) \tag{18}$$

The second order approximation of $N_{j,t}(i)$ is:

$$\frac{N_{j,t}(i)}{\bar{N}} = 1 + \hat{N}_{j,t}(i) + \frac{1}{2}\hat{N}_{j,t}(i)^2 + O\left(\|\xi\|^3\right) \tag{19}$$

Substituting (19) into (18) gives

$$\begin{aligned} V\left(N_{j,t}(i)\right) = {} & V_N\bar{N}\left\{\hat{N}_{j,t}(i) + \frac{1}{2}\hat{N}_{j,t}(i)^2 + \frac{1}{2}\frac{V_{NN}\bar{N}}{V_N}\hat{N}_{j,t}(i)^2\right\} \\ & + t.i.p. + O\left(\|\xi\|^3\right) \end{aligned} \tag{20}$$

Since $\frac{V_{NN}\bar{N}}{V_N} = \varphi$, we rewrite (20) as

$$V\left(N_{j,t}(i)\right) = V_N\bar{N}\left\{\hat{N}_{j,t}(i) + \frac{1}{2}(1+\varphi)\hat{N}_{j,t}(i)^2\right\} + t.i.p. + O\left(\|\xi\|^3\right) \tag{21}$$

From the production function, we have

$$\hat{Y}_{j,t}(i) = \hat{Z}_{j,t} + \hat{N}_{j,t}(i) \Longrightarrow \hat{N}_{j,t}(i) = \hat{Y}_{j,t}(i) - \hat{Z}_{j,t} \tag{22}$$

Substituting (22) into (21), we obtain

$$V\left(N_{j,t}(i)\right) = V_N\bar{N}\left\{\begin{array}{c} \hat{Y}_{j,t}(i) - \hat{Z}_{j,t} \\ +\frac{1}{2}(1+\varphi)\left[\begin{array}{c} \hat{Y}_{j,t}(i)^2 + \hat{Z}_{j,t}^2 \\ -2\hat{Z}_{j,t}\hat{Y}_{j,t}(i)\hat{Z}_{j,t} \end{array}\right] \end{array}\right\} + t.i.p. + O\left(\|\xi\|^3\right)$$

$$= V_N \bar{N} \left\{ \begin{array}{c} \hat{Y}_{j,t}(i) + \frac{1}{2}(1+\varphi)\hat{Y}_{j,t}(i)^2 \\ -(1+\varphi)\left[\hat{Z}_{j,t}\hat{Y}_{j,t}(i)\right] \end{array} \right\} + t.i.p. + O\left(\|\xi\|^3\right) \tag{23}$$

By integrating (23), we obtain

$$\frac{1}{n_j}\int_{\mathcal{I}_j} V\left(N_{j,t}(i)\right) di = V_N \bar{N} \left\{ \begin{array}{c} E_i^j\left[\hat{Y}_{j,t}(i)\right] + \frac{1}{2}(1+\varphi)Var_i^j\left[\hat{Y}_{j,t}(i)\right] + \\ \frac{1}{2}(1+\varphi)E_i^j\left[\hat{Y}_{j,t}(i)\right]^2 - (1+\varphi)\left(\hat{Z}_{j,t}\right)E_i^j\left[\hat{Y}_{j,t}(i)\right] \end{array} \right\} \tag{24}$$
$$+ t.i.p. + O\left(\|\xi\|^3\right)$$

Taking a second order approximation of the aggregators gives

$$\hat{Y}_{j,t}(i) = E_i^j\left[\hat{Y}_{j,t}(i)\right] + \frac{1}{2}\left(\frac{\theta-1}{\theta}\right)Var_i^j\left[\hat{Y}_{j,t}(i)\right] + O\left(\|\xi\|^3\right),$$

which implies

$$E_i^j\left[\hat{Y}_{j,t}(i)\right] = \hat{Y}_{j,t} - \frac{1}{2}\left(\frac{\theta-1}{\theta}\right)Var_i^j\left[\hat{Y}_{j,t}(i)\right] + O\left(\|\xi\|^3\right) \tag{25}$$
$$E_i^j\left[\hat{Y}_{j,t}(i)\right]^2 = \hat{Y}_{j,t}^2 + O\left(\|\xi\|^3\right) \tag{26}$$

We substitute (25) and (26) into (24) obtaining

$$\frac{1}{n_j}\int_{\mathcal{I}_j} V\left(N_{j,t}(i)\right) di = V_N \bar{N} \left\{ \begin{array}{c} \hat{Y}_{j,t} + \frac{1}{2}(1+\varphi)\hat{Y}_{j,t}^2 - (1+\varphi)\left(\hat{Z}_{j,t}\right)\hat{Y}_{j,t} \\ +\frac{1}{2}\left(\varphi+\theta^{-1}\right)Var_i^j\left[\hat{Y}_{j,t}(i)\right] \end{array} \right\}$$
$$+ t.i.p. + O\left(\|\xi\|^3\right)$$

Recall that $\bar{N} = \frac{\bar{Y}}{\bar{A}_j} = \bar{Y}$. From the household's labor supply relation, we have

$$-\frac{V_N}{U_C} = \frac{\bar{W}}{\bar{P}} = \bar{A}_j = 1 \Longrightarrow -V_N\bar{Y} = U_C\bar{Y} = U_C\bar{C}$$

Country A household's period utility is then given by

$$\begin{aligned} W_{a,t} &\equiv \left\{U(C_{a,t}) - \frac{1}{n_a}\int_{\mathcal{I}_a} V\left(N_{a,t}(i)\right) di\right\} \\ &= U_C\bar{C}\left\{\hat{C}_{a,t} + \frac{1}{2}(1-\sigma)\hat{C}_{a,t}^2\right\} - U_C\bar{Y}\left\{ \begin{array}{c} \hat{Y}_{a,t} + \frac{1}{2}(1+\varphi)\hat{Y}_{a,t}^2 - (1+\varphi)\hat{Z}_{a,t}\hat{Y}_{a,t} \\ +\frac{1}{2}\left(\varphi+\theta^{-1}\right)Var_i^a\left[\hat{Y}_{a,t}(i)\right] \end{array} \right\} + t.i.p. + O\left(\|\xi\|^3\right) \\ &= U_C\bar{C}\left\{ \begin{array}{c} \hat{C}_{a,t} + \left(\frac{1-\sigma}{2}\right)\hat{C}_{a,t}^2 - \hat{Y}_{a,t} - \left(\frac{1+\varphi}{2}\right)\hat{Y}_{a,t}^2 \\ +(1+\varphi)\hat{Z}_{a,t}\hat{Y}_{a,t} - \left(\frac{\varphi+\theta^{-1}}{2}\right)Var_i^a\left[\hat{Y}_{a,t}(i)\right] \end{array} \right\} + t.i.p. + O\left(\|\xi\|^3\right) \end{aligned}$$

The second order approximation of the resource constraint

Beginning with the second order approximation of the price indices,

$$\hat{P}^C_{a,t} + \frac{1}{2}(1-\eta)(\hat{P}^C_{a,t})^2 = n_a\hat{P}^a_{a,t} + n_b\hat{\widetilde{P}}^a_{b,t} + \frac{1-\eta}{2}\left(n_a\left(\hat{P}^a_{a,t}\right)^2 + n_b\left(\hat{\widetilde{P}}^a_{b,t}\right)^2\right) + +O(\|\xi\|^3)$$

$$\hat{P}^C_{b,t} + \frac{1}{2}(1-\eta)(\hat{P}^C_{b,t})^2 = n_a\hat{\widetilde{P}}^b_{a,t} + n_b\hat{P}^b_{b,t} + \frac{1-\eta}{2}\left(n_a\left(\hat{\widetilde{P}}^b_{a,t}\right)^2 + n_b\left(\hat{P}^b_{b,t}\right)^2\right) + +O(\|\xi\|^3)$$

Note that $\hat{P}^a_{j,t} = \hat{P}^b_{j,t}$ within the union. Above expression is rearranged to yield

$$\hat{P}^C_{a,t} - \hat{P}^a_{a,t} = -n_b\left(\hat{P}^a_{a,t} - \hat{\widetilde{P}}^a_{b,t}\right) + \left(\frac{1-\eta}{2}\right)\left(n_a\left(\hat{P}^a_{a,t}\right)^2 + n_b\left(\hat{\widetilde{P}}^a_{b,t}\right)^2 - (\hat{P}^C_{a,t})^2\right)$$

$$\hat{P}^C_{a,t} - \hat{\widetilde{P}}^a_{b,t} = n_a\left(\hat{P}^a_{a,t} - \hat{\widetilde{P}}^a_{b,t}\right) + \left(\frac{1-\eta}{2}\right)\left(n_a\left(\hat{P}^a_{a,t}\right)^2 + n_b\left(\hat{\widetilde{P}}^a_{b,t}\right)^2 - (\hat{P}^C_{a,t})^2\right)$$

$$\hat{P}^C_{b,t} - \hat{\widetilde{P}}^b_{a,t} = -n_b\left(\hat{\widetilde{P}}^b_{a,t} - \hat{P}^b_{b,t}\right) + \left(\frac{1-\eta}{2}\right)\left(n_a\left(\hat{\widetilde{P}}^b_{a,t}\right)^2 + n_b\left(\hat{P}^b_{b,t}\right)^2 - (\hat{P}^C_{b,t})^2\right)$$

$$\hat{P}^C_{b,t} - \hat{P}^b_{b,t} = n_a\left(\hat{\widetilde{P}}^b_{a,t} - \hat{P}^b_{b,t}\right) + \left(\frac{1-\eta}{2}\right)\left(n_a\left(\hat{\widetilde{P}}^b_{a,t}\right)^2 + n_b\left(\hat{P}^b_{b,t}\right)^2 - (\hat{P}^C_{b,t})^2\right)$$

From the first order Taylor expansion of the CPI index,

$$\hat{P}^C_{a,t} = n_a\hat{P}^a_{a,t} + n_b\hat{\widetilde{P}}^a_{b,t} + O(\|\xi\|^2)$$

$$\Rightarrow \quad (\hat{P}^C_{a,t})^2 = n_a^2\left(\hat{P}^a_{a,t}\right)^2 + n_b^2\left(\hat{\widetilde{P}}^a_{b,t}\right)^2 + 2n_an_b\hat{P}^a_{a,t}\hat{\widetilde{P}}^a_{b,t} + O(\|\xi\|^3)$$

Likewise,

$$(\hat{P}^C_{b,t})^2 = n_a^2\left(\hat{\widetilde{P}}^b_{a,t}\right)^2 + n_b^2\left(\hat{P}^b_{b,t}\right)^2 + 2n_an_b\hat{\widetilde{P}}^b_{a,t}\hat{P}^b_{b,t} + O(\|\xi\|^3)$$

As an exact relation, $\hat{P}^R_{a,t} = \hat{P}^a_{a,t} - \hat{\widetilde{P}}^a_{b,t}$ and $\hat{P}^R_{b,t} = \hat{\widetilde{P}}^b_{a,t} - \hat{P}^b_{b,t}$. Therefore,

$$\hat{P}^C_{a,t} - \hat{P}^a_{a,t} = -n_b\hat{P}^R_{a,t} + \frac{1-\eta}{2}n_an_b(\hat{P}^R_{a,t})^2 + O(\|\xi\|^3)$$

$$\hat{P}^C_{a,t} - \hat{\widetilde{P}}^a_{b,t} = n_a\hat{P}^R_{a,t} + \frac{1-\eta}{2}n_an_b(\hat{P}^R_{a,t})^2 + O(\|\xi\|^3)$$

$$\hat{P}^C_{b,t} - \hat{\widetilde{P}}^b_{a,t} = -n_b\hat{P}^R_{b,t} + \frac{1-\eta}{2}n_an_b(\hat{P}^R_{b,t})^2 + O(\|\xi\|^3)$$

$$\hat{P}^C_{b,t} - \hat{P}^b_{b,t} = n_a\hat{P}^R_{b,t} + \frac{1-\eta}{2}n_an_b(\hat{P}^R_{b,t})^2 + O(\|\xi\|^3)$$

We have the following demand functions from each country for each composite country goods.

$$
\begin{aligned}
\hat{C}_{a,a,t} &= -\eta(\hat{P}_{a,t} - \hat{P}^C_{a,t}) + \hat{C}_{a,t} \\
\hat{C}_{a,b,t} &= -\eta(\hat{\tilde{P}}_{b,t} - \hat{P}^C_{a,t}) + \hat{C}_{a,t} \\
\hat{C}_{b,a,t} &= -\eta(\hat{\tilde{P}}_{a,t} - \hat{P}^C_{b,t}) + \hat{C}_{b,t} \\
\hat{C}_{b,b,t} &= -\eta(\hat{P}_{b,t} - \hat{P}^C_{b,t}) + \hat{C}_{b,t}
\end{aligned}
$$

Then, by plugging in the above price indices,

$$
\begin{aligned}
\hat{C}_{a,a,t} &= -n_b\eta\hat{P}^R_{a,t} + \hat{C}_{a,t} + \frac{\eta(1-\eta)}{2}n_a n_b(\hat{P}^R_{a,t})^2 + O(\|\xi\|^3) \\
\hat{C}_{a,b,t} &= n_a\eta\hat{P}^R_{a,t} + \hat{C}_{a,t} + \frac{\eta(1-\eta)}{2}n_a n_b(\hat{P}^R_{a,t})^2 + O(\|\xi\|^3) \\
\hat{C}_{b,a,t} &= -n_b\eta\hat{P}^R_{b,t} + \hat{C}_{b,t} + \frac{\eta(1-\eta)}{2}n_a n_b(\hat{P}^R_{b,t})^2 + O(\|\xi\|^3) \\
\hat{C}_{b,b,t} &= n_a\eta\hat{P}^R_{b,t} + \hat{C}_{b,t} + \frac{\eta(1-\eta)}{2}n_a n_b(\hat{P}^R_{b,t})^2 + O(\|\xi\|^3)
\end{aligned}
$$

Finally, the resource constraint is second order approximated to be:

$$
\begin{aligned}
\hat{Y}_{a,t} + \frac{1}{2}\left(\hat{Y}_{a,t}\right)^2 &= n_a\left(\hat{C}_{a,a,t} + \frac{1}{2}\left(\hat{C}_{a,a,t}\right)^2\right) + n_b\left(\hat{C}_{b,a,t} + \frac{1}{2}\left(\hat{C}_{b,a,t}\right)^2\right) + \frac{\omega}{2}\left(\Delta\hat{S}_t\right)^2 + O(\|\xi\|^3) \\
\hat{Y}_{b,t} + \frac{1}{2}(\hat{Y}_{b,t})^2 &= n_a\left(\hat{C}_{a,b,t} + \frac{1}{2}(\hat{C}_{a,b,t})^2\right) + n_b\left(\hat{C}_{b,b,t} + \frac{1}{2}\left(\hat{C}_{b,b,t}\right)^2\right) + \frac{\omega}{2}\left(\Delta\hat{S}_t\right)^2 + O(\|\xi\|^3)
\end{aligned}
$$

$\omega = n_b\Omega_{\Delta\Delta}\bar{S}^2\bar{C}_{ba}^{-1} = n_a\Omega_{\Delta\Delta}\bar{S}^2\bar{C}_{ab}^{-1}$. Plugging in the above demand function,

$$
\begin{aligned}
\hat{Y}_{a,t} + \frac{1}{2}(\hat{Y}_{a,t})^2 = &\sum_j n_j\hat{C}_{j,t} - n_b\eta\sum_j n_j\hat{P}^R_{j,t} + \left(\frac{\eta(1-\eta)}{2}n_a n_b + \frac{n_b\eta^2}{2}\right)\sum_j n_j\left(\hat{P}^R_{j,t}\right)^2 \\
&+ \frac{1}{2}\sum_j n_j\left(\hat{C}_{j,t}\right)^2 - n_b\eta\hat{C}^U_t\hat{D}^U_t + \frac{\omega}{2}\left(\Delta\hat{S}_t\right)^2 + O(\|\xi\|^3) \\
\hat{Y}_{b,t} + \frac{1}{2}(\hat{Y}_{b,t})^2 = &\sum_j n_j\hat{C}_{j,t} + n_b\eta\sum_j n_j\hat{P}^R_{j,t} + \left(\frac{\eta(1-\eta)}{2}n_a n_b + \frac{n_a\eta^2}{2}\right)\sum_j n_j\left(\hat{P}^R_{j,t}\right)^2 \\
&+ \frac{1}{2}\sum_j n_j\left(\hat{C}_{j,t}\right)^2 + n_a\eta\hat{C}^U_t\hat{D}^U_t + \frac{\omega}{2}\left(\Delta\hat{S}_t\right)^2 + O(\|\xi\|^3)
\end{aligned}
$$

Since $\hat{Y}_{a,t} = \hat{C}^U_t - n_b\eta\hat{D}^U_t + O(\|\xi\|^2)$ and $\hat{Y}_{b,t} = \hat{C}^U_t + n_a\eta\hat{D}^U_t + O(\|\xi\|^2)$ hold in the first order,

$$
\begin{aligned}
\left(\hat{Y}_{a,t}\right)^2 &= \left(\hat{C}^U_t\right)^2 - 2n_b\eta\hat{C}^U_t\hat{D}^U_t + (n_b\eta)^2\left(\hat{D}^U_t\right)^2 + O(\|\xi\|^3) \\
\left(\hat{Y}_{b,t}\right)^2 &= \left(\hat{C}^U_t\right)^2 + 2n_a\eta\hat{C}^U_t\hat{D}^U_t + (n_a\eta)^2\left(\hat{D}^U_t\right)^2 + O(\|\xi\|^3
\end{aligned}
$$

By definition,

$$\hat{C}_{a,t} = \hat{C}_t^U + n_b\hat{C}_t^R + O(\|\xi\|^2); \quad \hat{C}_{b,t} = \hat{C}_t^U - n_a\hat{C}_t^R + O(\|\xi\|^2)$$
$$\hat{Y}_{a,t} = \hat{Y}_t^U + n_b\hat{Y}_t^R + O(\|\xi\|^2); \quad \hat{Y}_{b,t} = \hat{Y}_t^U - n_a\hat{Y}_t^R + O(\|\xi\|^2)$$
$$\hat{P}_{a,t}^R = \hat{D}_t^U + n_b\hat{D}_t^R + O(\|\xi\|^2); \quad \hat{P}_{b,t}^R = \hat{D}_t^U - n_a\hat{D}_t^R + O(\|\xi\|^2)$$

Then, we get:

$$\sum_j n_j(\hat{C}_{j,t})^2 = \left(\hat{C}_t^U\right)^2 + n_a n_b(\hat{C}_t^R)^2$$
$$\sum_j n_j(\hat{Y}_{j,t})^2 = \left(\hat{Y}_t^U\right)^2 + n_a n_b(\hat{Y}_t^R)^2$$
$$\sum_j n_j(\hat{P}_{j,t}^R)^2 = \left(\hat{D}_t^U\right)^2 + n_a n_b(\hat{D}_t^R)^2$$

Rearrange the second order approximation of $\hat{Y}_{a,t}$ and $\hat{Y}_{b,t}$ as

$$\begin{aligned}
\hat{Y}_{a,t} = {} & \sum_j n_j\hat{C}_{j,t} - n_b\eta\sum_j n_j\hat{P}_{j,t}^R \\
& + \frac{n_a n_b}{2}(\hat{C}_t^R)^2 + n_a n_b\frac{\eta(1-\eta)}{2}\left(\hat{D}_t^U\right)^2 + \frac{\tilde{\lambda}_{dr}}{2}\left(\hat{D}_t^R\right)^2 + \frac{\omega}{2}\left(\Delta\hat{S}_t\right)^2 + O(\|\xi\|^3) \\
\hat{Y}_{b,t} = {} & \sum_j n_j\hat{C}_{j,t} + n_a\eta\sum_j n_j\hat{P}_{j,t}^R \\
& + \frac{n_a n_b}{2}(\hat{C}_t^R)^2 + n_a n_b\frac{\eta(1-\eta)}{2}\left(\hat{D}_t^U\right)^2 + \frac{\tilde{\lambda}_{dr}^*}{2}\left(\hat{D}_t^R\right)^2 + \frac{\omega}{2}\left(\Delta\hat{S}_t\right)^2 + O(\|\xi\|^3)
\end{aligned}$$

where $\tilde{\lambda}_{dr} \equiv (n_a n_b)^2\eta\left(1-\eta+\frac{n_b}{n_b}\eta\right)$ and $\tilde{\lambda}_{dr}^* \equiv (n_a n_b)^2\eta\left(1-\eta+\frac{n_b}{n_a}\eta\right)$. Substituting the second order expression of the resource constraint, country A household's period utility is then given by

$$\begin{aligned}
W_{a,t} = U_C\bar{C} & \left\{ \begin{array}{c} n_b\hat{C}_t^R + n_b\eta\hat{D}_t^U + n_b\eta(1+\varphi)\hat{C}_t^U\hat{D}_t^U + n_b(1-\sigma)\hat{C}_t^U\hat{C}_t^R \\ -\left(\frac{\sigma+\varphi}{2}\right)\left(\hat{C}_t^U\right)^2 - \frac{n_a n_b}{2}\left(1-\frac{n_b}{n_a}(1-\sigma)\right)\left(\hat{C}_t^R\right)^2 - \frac{\tilde{\lambda}_{du}}{2}\left(\hat{D}_t^U\right)^2 - \frac{\tilde{\lambda}_{dr}}{2}\left(\hat{D}_t^R\right)^2 - \frac{\omega}{2}\left(\Delta\hat{S}_t\right)^2 \\ +(1+\varphi)\hat{Z}_{a,t}\hat{C}_t^U - n_b\eta(1+\varphi)\hat{Z}_{a,t}\hat{D}_t^U - \left(\frac{\varphi+\theta^{-1}}{2}\right)Var_i^a\left[\hat{Y}_{a,t}(i)\right] \end{array} \right\} \\
& + t.i.p. + O\left(\|\xi\|^3\right)
\end{aligned}$$

where $\tilde{\lambda}_{du} \equiv n_a n_b\eta\left(1-\eta+\frac{n_b}{n_a}\eta(1+\varphi)\right)$. Likewise, country B household's period utility is given as

$$W_{b,t} = U_C\bar{C} \left\{ \begin{array}{c} -n_a\hat{C}_t^R - n_a\eta\hat{D}_t^U - n_a\eta(1+\varphi)\hat{C}_t^U\hat{D}_t^U - n_a(1-\sigma)\hat{C}_t^U\hat{C}_t^R \\ -\left(\frac{\sigma+\varphi}{2}\right)\left(\hat{C}_t^U\right)^2 - \frac{n_a n_b}{2}\left(1-\frac{n_a}{n_b}(1-\sigma)\right)\left(\hat{C}_t^R\right)^2 - \frac{\tilde{\lambda}_{du}^*}{2}\left(\hat{D}_t^U\right)^2 - \frac{\tilde{\lambda}_{dr}^*}{2}\left(\hat{D}_t^R\right)^2 - \frac{\omega}{2}\left(\Delta\hat{S}_t\right)^2 \\ +(1+\varphi)\hat{Z}_{b,t}\hat{C}_t^U + n_a\eta(1+\varphi)\hat{Z}_{b,t}\hat{D}_t^U - \left(\frac{\varphi+\theta^{-1}}{2}\right)Var_i^b\left[\hat{Y}_{b,t}(i)\right] \end{array} \right\}$$

$$+ t.i.p. + O\left(\|\xi\|^3\right)$$

where $\tilde{\lambda}^*_{du} \equiv n_a n_b \eta \left(1 - \eta + \frac{n_a}{n_b}\eta(1+\varphi)\right)$. We now solve for $\sum_j n_j Var_i^j \left[\hat{Y}_{j,t}(i)\right]$. The demand for $Y_{j,t}(i)$ is given by

$$Y_{j,t}(i) = \left(\frac{P_{j,t}(i)}{P_{j,t}}\right)^{-\theta}\left(\frac{P_{j,t}}{P_t}\right)^{-\eta} Y_t$$

Then

$$\hat{Y}_{j,t}(i) = -\theta\left(\hat{P}_{j,t}(i) - \hat{P}_{j,t}\right) - \eta\left(\hat{P}_{j,t} - \hat{P}_t\right) + \hat{Y}_t$$

This implies that

$$Var_i^j\left[\hat{Y}_{j,t}(i)\right] = \theta^2 Var_i^j\left[\hat{P}_{j,t}(i)\right]$$

where $\Delta_t^j \equiv Var_i^j\left[\hat{P}_{j,t}(i)\right]$ is a measure of price dispersion within a country. When prices are staggered as in the discrete time Calvo fashion, Woodford (2003) has shown that

$$\begin{aligned}
\Delta_t^j &= \alpha_j \Delta_{t-1}^j + \frac{\alpha_j}{1-\alpha_j}\pi_{j,t}^2 + O\left(\|\xi\|^3\right) \Longrightarrow \\
&= \alpha_j^{t+1}\Delta_{-1}^j + \sum_{k=0}^{t} \alpha_j^{t-s}\left(\frac{\alpha_j}{1-\alpha_j}\right)\pi_{j,k}^2 + O\left(\|\xi\|^3\right)
\end{aligned}$$

If a new policy is conducted from $t \geqslant 0$, the first term, $\alpha_j^{t+1}\Delta_{-1}^j$, is independent of policy. If we take the discounted sum over time, we obtain

$$\sum_{t=0}^{\infty}\beta^t\Delta_t^j = \frac{\alpha_j}{(1-\alpha_j)(1-\alpha_j\beta)}\sum_{t=0}^{\infty}\beta^t\pi_{j,t}^2 + t.i.p. + O\left(\|\xi\|^3\right)$$

Accordingly, the lifetime utility loss of country A is defined as

$$\begin{aligned}
\mathcal{L}_a &\equiv -\mathbb{E}\sum \beta^t \frac{W_{a,t} - \bar{W}_a}{U_C \bar{C}} \\
&= \frac{1}{2}\mathbb{E}\sum\beta^t \left\{\begin{array}{c}
\frac{\theta}{\kappa_a}(\pi_{a,t})^2 + (\sigma+\varphi)\left(\hat{C}_t^U\right)^2 + \tilde{\lambda}_{du}\left(\hat{D}_t^U\right)^2 \\
+n_a n_b\left(1 - \frac{n_b}{n_a}(1-\sigma)\right)\left(\hat{C}_t^R\right)^2 + \tilde{\lambda}_{dr}\left(\hat{D}_t^R\right)^2 + \omega\left(\Delta\hat{S}_t\right)^2 \\
-2n_b\hat{C}_t^R - 2n_b\eta\hat{D}_t^U - 2n_b\eta(1+\varphi)\hat{C}_t^U\hat{D}_t^U - 2n_b(1-\sigma)\hat{C}_t^U\hat{C}_t^R \\
-2(1+\varphi)\hat{Z}_{a,t}\hat{C}_t^U + 2n_b\eta(1+\varphi)\hat{Z}_{a,t}\hat{D}_t^U
\end{array}\right\} + t.i.p. + O\left(\|\xi\|^3\right)
\end{aligned}$$

where $\kappa_j = \frac{(1-\alpha_j)(1-\alpha_j\beta)}{\alpha_j(1+\varphi\theta)}$ for $j = \{a, b\}$. Similarly, the lifetime utility loss of country B is defined as

$$\mathcal{L}_b \equiv -\mathbb{E}\sum\beta^t\frac{W_{b,t} - \bar{W}_b}{U_C\bar{C}}$$

$$
= \frac{1}{2}\mathbb{E}\sum \beta^t \left\{ \begin{array}{c} \frac{\theta}{\kappa_b}(\pi_{b,t})^2 + (\sigma+\varphi)\left(\hat{C}_t^U\right)^2 + \tilde{\lambda}^*_{du}\left(\hat{D}_t^U\right)^2 \\ +n_a n_b\left(1-\frac{n_a}{n_b}(1-\sigma)\right)\left(\hat{C}_t^R\right)^2 + \tilde{\lambda}^*_{dr}\left(\hat{D}_t^R\right)^2 + \omega\left(\Delta\hat{S}_t\right)^2 \\ +2n_a\hat{C}_t^R + 2n_a\eta\hat{D}_t^U + 2n_a\eta(1+\varphi)\hat{C}_t^U\hat{D}_t^U + 2n_a(1-\sigma)\hat{C}_t^U\hat{C}_t^R \\ -2(1+\varphi)\hat{Z}_{b,t}\hat{C}_t^U + 2n_a\eta(1+\varphi)\hat{Z}_{b,t}\hat{D}_t^U \end{array} \right\} + t.i.p. + O\left(\|\xi\|^3\right)
$$

6.6.1 The union-wide loss function

A lifetime union-wide welfare loss is derived as the weighted sum of the countries' welfare loss. Substituting $\hat{C}_t^U = \hat{Y}_t^U + O\left(\|\xi\|^2\right)$ and $\hat{S}_t = 0$, we get:

$$
\begin{aligned}
\mathcal{L}^{CU} &\equiv \sum_{j=a,b} n_j \mathcal{L}_j \\
&= \frac{1}{2}\mathbb{E}\sum \beta^t \left\{ \begin{array}{c} \theta\sum_j \frac{n_j}{\kappa_j}(\pi_{j,t})^2 + (\sigma+\varphi)(\hat{Y}_t^U)^2 + (1+\varphi)\hat{Z}_t^U\hat{Y}_t^U + n_a n_b\eta(1+\varphi\eta)(\hat{D}_t^U)^2 \\ -n_a n_b\eta(1+\varphi)\hat{Z}_t^R\hat{D}_t^U + n_a n_b\sigma(\hat{C}_t^R)^2 + (n_a n_b)^2\eta\left(\hat{D}_t^R\right)^2 \end{array} \right\} + t.i.p. + O\left(\|\xi\|^3\right) \\
&= \frac{1}{2}\mathbb{E}\sum \beta^t \left\{ \begin{array}{c} \theta\sum_j \frac{n_j}{\kappa_j}(\pi_{j,t})^2 + (\sigma+\varphi)\left(\hat{Y}_t^U - \hat{Y}_t^{U,E}\right)^2 \\ +n_a n_b\eta(1+\varphi\eta)\left(\hat{D}_t^U - \hat{P}_t^{R,E}\right)^2 + n_a n_b\sigma\left(\hat{C}_t^R\right)^2 + (n_a n_b)^2\eta\left(\hat{D}_t^R\right)^2 \end{array} \right\} + t.i.p. + O\left(\|\xi\|^3\right)
\end{aligned}
$$

The last equality is derived from the first order approximation of the efficient equilibrium:

$$
\hat{Y}_t^{U,E} = \frac{1+\varphi}{\sigma+\varphi}\hat{Z}_t^U; \quad \hat{P}_t^{R,E} = -\frac{1+\varphi}{1+\varphi\eta}\hat{Z}_t^R
$$

Therefore, the loss function of the currency union equals:

$$
\begin{aligned}
L_t^{CU} = \theta &\sum_{j=a,b}\frac{n_j}{\kappa_j}(\pi_{j,t})^2 + (\sigma+\varphi)\left(\hat{Y}_t^U - \hat{Y}_t^{U,E}\right)^2 \\
&+ n_a n_b\eta(1+\varphi\eta)\left(\hat{D}_t^U - \hat{P}_t^{R,E}\right)^2 + n_a n_b\sigma\left(\hat{C}_t^R\right)^2 + (n_a n_b)^2\eta\left(\hat{D}_t^R\right)^2
\end{aligned}
$$

6.6.2 Country-specific loss function

In deriving the loss function of country A and B, we suppose an equal country size ($n_a = n_b$) and perfect consumption risk-sharing ($\phi = 0$).[10] In this case, $\tilde{\lambda}_{du} = \tilde{\lambda}^*_{du} = n_a n_b\eta(1+\varphi\eta)$, $\tilde{\lambda}_{dr} = \tilde{\lambda}^*_{dr} = (n_a n_b)^2\eta$, and $\hat{C}_t^R = 0$ hold. For consistency, we write $\hat{C}_t^U = \hat{C}_{a,t} = \hat{C}_{b,t}$. Then, the lifetime utility loss of country A is:

$$
\mathcal{L}_a = \frac{1}{2}\mathbb{E}\sum \beta^t \left\{ \begin{array}{c} \frac{\theta}{\kappa_a}(\pi_{a,t})^2 + (\sigma+\varphi)\left(\hat{C}_t^U\right)^2 + \tilde{\lambda}_{du}\left(\hat{D}_t^U\right)^2 + \tilde{\lambda}_{dr}\left(\hat{D}_t^R\right)^2 + \omega\left(\Delta\hat{S}_t\right)^2 \\ -2n_b\eta\hat{D}_t^U - 2n_b\eta(1+\varphi)\hat{C}_t^U\hat{D}_t^U - 2(1+\varphi)\hat{Z}_{a,t}\hat{C}_t^U + 2n_b\eta(1+\varphi)\hat{Z}_{a,t}\hat{D}_t^U \end{array} \right\} + t.i.p. + O\left(\|\xi\|^3\right)
$$

[10] We make these assumptions for simplicity's sake and they do not make a significant contribution in either quantitative or qualitative aspect of the results.

For country B,

$$\mathcal{L}_b = \frac{1}{2}\mathbb{E}\sum \beta^t \left\{ \begin{array}{c} \frac{\theta}{\kappa_b}(\pi_{b,t})^2 + (\sigma+\varphi)\left(\hat{C}_t^U\right)^2 + \tilde{\lambda}_{du}\left(\hat{D}_t^U\right)^2 + \tilde{\lambda}_{dr}\left(\hat{D}_t^R\right)^2 + \omega\left(\Delta\hat{S}_t\right)^2 \\ +2n_a\eta\hat{D}_t^U + 2n_a\eta(1+\varphi)\hat{C}_t^U\hat{D}_t^U - 2(1+\varphi)\hat{Z}_{b,t}\hat{C}_t^U + 2n_a\eta(1+\varphi)\hat{Z}_{b,t}\hat{D}_t^U \end{array} \right\} + t.i.p. + O\left(\|\xi\|^3\right)$$

When each country adopts national currency ($\hat{S} \neq 0$), the weighted sum of the welfare loss becomes:

$$\mathcal{L}^{CU}|_{\hat{S}\neq 0} = \frac{1}{2}\mathbb{E}\sum \beta^t \left\{ \begin{array}{c} \theta\sum_j \frac{n_j}{\kappa_j}(\pi_{j,t})^2 + (\sigma+\varphi)\left(\hat{C}_t^U - \hat{C}_t^{U,E}\right)^2 \\ +n_a n_b\eta(1+\varphi\eta)\left(\hat{D}_t^U - \hat{P}_t^{R,E}\right)^2 + (n_a n_b)^2\eta\left(\hat{D}_t^R\right)^2 + \omega\left(\Delta\hat{S}_t\right)^2 \end{array} \right\} + t.i.p. + O\left(\|\xi\|^3\right)$$

We express our derived loss function in terms of country-specific output gaps, for which we make use of the first order approximation of the demand function for the country goods:

$$\hat{Y}_{a,t} = \hat{C}_t^U - n_b\eta\hat{D}_t^U + O\left(\|\xi\|^2\right); \quad \hat{Y}_{b,t} = \hat{C}_t^U + n_a\eta\hat{D}_t^U + O\left(\|\xi\|^2\right)$$

Above relationship holds in the efficient equilibrium:

$$\begin{aligned} \hat{Y}_{a,t}^E &= \hat{C}_t^{U,E} - n_b\eta\hat{P}_t^{R,e} + O\left(\|\xi\|^2\right) = \frac{1+\varphi}{\sigma+\varphi}\hat{Z}_t^U + n_b\frac{\eta(1+\varphi)}{1+\varphi\eta}\hat{Z}_t^R \\ \hat{Y}_{b,t}^E &= \hat{C}_t^{U,E} + n_a\eta\hat{P}_t^{R,e} + O\left(\|\xi\|^2\right) = \frac{1+\varphi}{\sigma+\varphi}\hat{Z}_t^U - n_a\frac{\eta(1+\varphi)}{1+\varphi\eta}\hat{Z}_t^R \end{aligned}$$

We can then show:

$$\begin{aligned} \left(\hat{C}_t^U - \hat{C}_t^{U,E}\right)^2 &= n_a\left(\hat{C}_t^U - \hat{C}_t^{U,E}\right)^2 + n_b\left(\hat{C}_t^U - \hat{C}_t^{U,E}\right)^2 \\ &= n_a\left\{\left(\hat{Y}_{a,t} - \hat{Y}_{a,t}^E\right) + n_b\eta\left(\hat{D}_t^U - \hat{P}_t^{R,E}\right)\right\}^2 + n_b\left\{\left(\hat{Y}_{b,t} - \hat{Y}_{b,t}^E\right) - n_a\eta\left(\hat{D}_t^U - \hat{P}_t^{R,E}\right)\right\}^2 \\ &= n_a\left(\hat{Y}_{a,t} - \hat{Y}_{a,t}^E\right)^2 + n_b\left(\hat{Y}_{b,t} - \hat{Y}_{b,t}^E\right)^2 + n_a n_b\eta^2\left(\hat{D}_t^U - \hat{P}_t^{R,E}\right)^2 \\ &\quad + 2n_a n_b\eta\left(\hat{D}_t^U - \hat{P}_t^{R,E}\right)\underbrace{\left\{\left(\hat{Y}_{a,t} - \hat{Y}_{a,t}^E\right) - \left(\hat{Y}_{b,t} - \hat{Y}_{b,t}^E\right)\right\}}_{= -\eta\left(\hat{D}_t^U - \hat{P}_t^{R,E}\right)} \\ &= n_a\left(\hat{Y}_{a,t} - \hat{Y}_{a,t}^E\right)^2 + n_b\left(\hat{Y}_{b,t} - \hat{Y}_{b,t}^E\right)^2 - n_a n_b\eta^2\left(\hat{P}_t^R - \hat{P}_t^{R,E}\right)^2 \end{aligned}$$

Substituting this relationship back into $\mathcal{L}^{CU}|_{\hat{S}\neq 0}$,

$$\mathcal{L}^{CU}|_{\hat{S}\neq 0} = \frac{1}{2}\mathbb{E}\sum \beta^t \left\{ \begin{array}{c} \theta\sum_j \frac{n_j}{\kappa_j}(\pi_{j,t})^2 + n_a(\sigma+\varphi)\left(\hat{Y}_{a,t} - \hat{Y}_{a,t}^E\right)^2 + n_b(\sigma+\varphi)\left(\hat{Y}_{b,t} - \hat{Y}_{b,t}^E\right)^2 \\ +n_a n_b(1-\sigma\eta)\left(\hat{D}_t^U - \hat{P}_t^{R,E}\right)^2 + (n_a n_b)^2\eta\left(\hat{D}_t^R\right)^2 + \omega\left(\Delta\hat{S}_t\right)^2 \end{array} \right\} + t.i.p. + O\left(\|\xi\|^3\right)$$

Second order approximation of the AS relation

A crucial step in deriving the country-specific loss function is to replace the linear term ($\hat{D}_t^U$) so that our loss function is accurate in the second order. Following Woodford and Benigno (2005), we take a second order approximation of the AS relation. The firm's optimal prise-setting conditions are:

$$\sum_{k=0}^{\infty} \alpha_j^k \mathbb{E}_t \left\{ Q_{t,t+k} Y_{j,t+k|t}(i) \left(P_{j,t}^*(i) - \frac{W_{j,t+k}(i)}{Z_{j,t+k}} \right) \right\} = 0 \tag{28}$$

where $W_{j,t+k}(i)$ is the nominal wage at time $t+k$ for the firm i. Note that following are satisfied:

$$\begin{aligned}
W_{j,t+k}(i) &= C_{j,t+k}^{\sigma} N_{j,t+k}^{\varphi}(i) P_{t+k}^C \\
&= C_{j,t+k}^{\sigma} \left(\frac{Y_{j,t+k|t}(i)}{Z_{j,t+k}} \right)^{\varphi} P_{t+k}^C \\
Y_{j,t+k|t}(i) &= \left(\frac{P_{j,t}^*(i)}{P_{j,t+k}} \right)^{-\theta} Y_{j,t+k} = \left(\frac{P_{j,t}^*(i)}{P_{j,t}} \right)^{-\theta} \left(\frac{P_{j,t}}{P_{j,t+k}} \right)^{-\theta} Y_{j,t+k} \\
Q_{t,t+k} &\equiv \beta^k \frac{C_{j,t+k}^{-\sigma}}{C_{e,t}^{-\sigma}} \frac{P_t^C}{P_{t+k}^C}
\end{aligned}$$

After substituting $Q_{t,t+k}$ into Eqn. (28), we get:

$$\sum_{k=0}^{\infty} (\alpha_j \beta)^k \mathbb{E}_t Y_{j,t+k}(i) \left[\left(\frac{C_{j,t+k}^{-\sigma}}{P_{t+k}^C} P_{j,t}^*(i) \right) - Y_{j,t+k|t}^{\varphi}(i) Z_{j,t+k}^{-(1+\varphi)} \right] = 0$$

Since firms that change price choose an equivalent price, let $P_t^*(i) = P_t^*$ hereafter. Rearranging and substituting $Y_{j,t+k|t}(i)$,

$$\sum_{k=0}^{\infty} (\alpha_j \beta)^k \mathbb{E}_t \left[\left(\frac{P_{j,t}^*}{P_{j,t+k}} \right)^{1-\theta} C_{j,t+k}^{-\sigma} Y_{j,t+k} \frac{P_{j,t+k}}{P_{t+k}^C} - \left(\frac{P_{j,t}^*}{P_{j,t+k}} \right)^{-\theta(1+\varphi)} Y_{j,t+k}^{(1+\varphi)} Z_{j,t+k}^{-(1+\varphi)} \right] = 0 \tag{29}$$

After rearrangements, we get:

$$\begin{aligned}
&\left(\frac{P_t^*}{P_t} \right)^{1+\varphi\theta} \underbrace{\sum_{k=0}^{\infty} (\alpha_j \beta)^k \mathbb{E}_t \left\{ \left(\frac{P_{j,t}}{P_{j,t+k}} \right)^{1-\theta} C_{j,t+k}^{-\sigma} Y_{j,t+k} \frac{P_{j,t+k}}{P_{t+k}^C} \right\}}_{\equiv F_{j,t}} \\
&= \underbrace{\sum_{k=0}^{\infty} (\alpha_j \beta)^k \mathbb{E}_t \left\{ \left(\frac{P_{j,t}}{P_{j,t+k}} \right)^{-\theta(1+\varphi)} Y_{j,t+k}^{(1+\varphi)} Z_{j,t+k}^{-(1+\varphi)} \right\}}_{\equiv K_{j,t}}
\end{aligned}$$

Therefore,

$$\frac{P_{j,t}^*}{P_{j,t}} = \left(\frac{K_{j,t}}{F_{j,t}} \right)^{\frac{1}{1+\varphi\theta}} \tag{30}$$

From the aggregate price dynamics,

$$1=\alpha_j\left(\frac{P_{j,t-1}}{P_{j,t}}\right)^{1-\theta}+(1-\alpha_j)\left(\frac{P^*_{j,t}}{P_{j,t}}\right)^{1-\theta} \tag{31}$$

Combining Eqns.(30) and (31),

$$\left(\frac{1-\alpha_j{\Pi_{j,t}}^{\theta-1}}{1-\alpha_j}\right)=\left(\frac{K_{j,t}}{F_{j,t}}\right)^{\frac{1-\theta}{1+\varphi\theta}} \tag{32}$$

Taking log,

$$\log\left(\frac{1-\alpha_j{\Pi_{j,t}}^{\theta-1}}{1-\alpha_j}\right)=\left(\frac{1-\theta}{1+\varphi\theta}\right)(\log K_{j,t}-\log F_{j,t}) \tag{33}$$

Take a second order approximation of the LHS of Eqn.(33):

$$-\frac{\alpha_j(\theta-1)}{1-\alpha_j}\left(\pi_{j,t}+\frac{1}{2}\frac{\theta-1}{1-\alpha_j}{\pi_{j,t}}^2\right)+O\left(\|\xi\|^3\right) \tag{34}$$

Accurate to the first order, Eqn.(33) reduces to

$$\pi_{j,t}=\frac{1-\alpha_j}{\alpha_j(1+\varphi\theta)}\left(\hat{K}_{j,t}-\hat{F}_{j,t}\right) \tag{35}$$

Turning to the RHS of Eqn.(30), the second order expansion of $K_{j,t}$ and $F_{j,t}$ are:

$$\hat{K}_{j,t}+\frac{1}{2}\hat{K}^2_{j,t}=(1-\alpha_j\beta)\sum_{k=0}^{\infty}(\alpha_j\beta)^k\mathbb{E}_t\left\{\hat{k}_{j,t,t+k}+\frac{1}{2}\hat{k}^2_{j,t,t+k}\right\}+O\left(\|\xi\|^3\right) \tag{36}$$

$$\hat{F}_{j,t}+\frac{1}{2}\hat{F}^2_{j,t}=(1-\alpha_j\beta)\sum_{k=0}^{\infty}(\alpha_j\beta)^k\mathbb{E}_t\left\{\hat{f}_{j,t,t+k}+\frac{1}{2}\hat{f}^2_{j,t,t+k}\right\}+O\left(\|\xi\|^3\right) \tag{37}$$

where $\hat{k}_{j,t,t+k}$ and $\hat{f}_{j,t,t+k}$ are defined as:

$$\hat{k}_{j,t,t+k}\equiv\hat{k}_{j,t+k}+\theta(1+\varphi)\sum_{s=1}^{k}\pi_{j,t+s}$$

$$\hat{f}_{j,t,t+k}\equiv\hat{f}_{j,t+k}+(\theta-1)\sum_{s=1}^{k}\pi_{j,t+s}$$

where we applied $\sum_{s=1}^{k}\pi_{j,t+s}=\hat{p}_{j,t+k}-\hat{p}_t$. And from the definition of $K_{j,t}$ and $F_{j,t}$,

$$\hat{k}_{j,t+k}=(1+\varphi)\hat{Y}_{j,t+k}-(1+\varphi)\hat{Z}_{j,t+k}$$

$$\hat{f}_{j,t+k}=-\sigma\hat{C}_{j,t+k}+\hat{Y}_{j,t+k}+\hat{P}_{j,t+k}-\hat{P}^C_{t+k}$$

From Eqn.(37),

$$
\hat{K}_{j,t} - \hat{F_{j,t}} + \underbrace{\frac{1}{2}\left(\hat{K}^2_{j,t} - \hat{F}^2_{j,t}\right)}_{\text{(A)}}
$$

$$
= \underbrace{(1-\alpha_j\beta)\sum_{k=0}^{\infty}(\alpha_j\beta)^k\mathbb{E}_t\left(\hat{k}_{j,t,t+k} - \hat{f}_{j,t,t+k}\right)}_{\text{(B)}} + \underbrace{(1-\alpha_j\beta)\sum_{k=0}^{\infty}(\alpha_j\beta)^k\mathbb{E}_t\frac{1}{2}\left(\hat{k}^2_{j,t,t+k} - \hat{f}^2_{j,t,t+k}\right)}_{\text{(C)}} + O\left(\|\xi\|^3\right)
$$

Solve (A):

$$
\begin{aligned}
&\frac{1}{2}\left(\hat{K}^2_t - \hat{F}^2_t\right) \\
&= \frac{1}{2}\left(\hat{K}_t - \hat{F}_t\right)\left(\hat{K}_t + \hat{F}_t\right) \\
&= \frac{1}{2}\left(\frac{\alpha_j(1+\varphi\theta)}{1-\alpha_j}\pi_{j,t}\right)\left(\hat{K}_t + \hat{F}_t\right) \quad \text{by Eqn.(35)} \\
&= \frac{1}{2}\left(\frac{\alpha_j(1+\varphi\theta)}{1-\alpha_j}\pi_{j,t}\right)(1-\alpha_j\beta)\underbrace{\sum_{k=0}^{\infty}(\alpha_j\beta)^k\mathbb{E}_t\left(\hat{k}_{j,t,t+k} + \hat{f}_{j,t,t+k}\right)}_{\equiv Z_{j,t}} \\
&= \frac{1}{2}(1-\alpha_j\beta)\frac{\alpha_j(1+\varphi\theta)}{1-\alpha_j}\pi_{j,t}Z_{j,t}
\end{aligned}
$$

Solve (B):

$$
\begin{aligned}
&(1-\alpha_j\beta)\sum_{k=0}^{\infty}(\alpha_j\beta)^k\mathbb{E}_t\left(\hat{k}_{j,t,t+k} - \hat{f}_{j,t,t+k}\right) \\
&= (1-\alpha_j\beta)\sum_{k=0}^{\infty}(\alpha_j\beta)^k\mathbb{E}_t\left\{\left(\hat{k}_{j,t+k} - \hat{f}_{j,t+k}\right) + (1+\varphi\theta)\sum_{s=1}^{k}\pi_{j,t+s}\right\} \\
&= (1-\alpha_j\beta)\sum_{k=0}^{\infty}(\alpha_j\beta)^k\mathbb{E}_t\left(\hat{k}_{j,t+k} - \hat{f}_{j,t+k}\right) + (1+\varphi\theta)\mathcal{P}_{j,t}
\end{aligned}
$$

For the last equality, we apply the following:

$$
\begin{aligned}
&\sum_{k=0}^{\infty}(\alpha_j\beta)^k\mathbb{E}_t\sum_{s=1}^{k}\pi_{j,t+s} \quad \text{where} \quad \sum_{s=1}^{k=0}\pi_{j,t+s} \equiv 0 \\
&= \frac{1}{1-\alpha_j\beta}\mathbb{E}_t\left[(\alpha_j\beta)\pi_{j,t+1} + (\alpha_j\beta)^2\pi_{j,t+2} + \cdots\right] \\
&= \frac{1}{1-\alpha_j\beta}\underbrace{\sum_{k=1}^{\infty}(\alpha_j\beta)^k\mathbb{E}_t\pi_{j,t+k}}_{\equiv \mathcal{P}_{j,t}}
\end{aligned}
$$

Solve (C):

$$
\begin{aligned}
&(1-\alpha_j\beta)\sum_{k=0}^{\infty}(\alpha_j\beta)^k\mathbb{E}_t\frac{1}{2}\left(\hat{k}^2_{j,t,t+k}-\hat{f}^2_{j,t,t+k}\right)\\
&=(1-\alpha_j\beta)\sum_{k=0}^{\infty}(\alpha_j\beta)^k\mathbb{E}_t\frac{1}{2}\left(\hat{k}^2_{j,t+k}-\hat{f}^2_{j,t+k}\right)\\
&+\underbrace{(1-\alpha_j\beta)\sum_{k=0}^{\infty}(\alpha_j\beta)^k\mathbb{E}_t\left(\theta(1+\varphi)\hat{k}_{j,t+k}-(\theta-1)\hat{f}_{j,t+k}\right)\sum_{s=1}^{k}\pi_{j,t+s}}_{\text{(C-1)}}\\
&+\underbrace{(1-\alpha_j\beta)\sum_{k=0}^{\infty}(\alpha_j\beta)^k\mathbb{E}_t\frac{1}{2}\left((\theta(1+\varphi))^2-(\theta-1)^2\right)\left(\sum_{s=1}^{k}\pi_{j,t+s}\right)^2}_{\text{(C-2)}}
\end{aligned}
$$

To simplify (C-1), define $\mathcal{N}_{j,t}$ as

$$
\mathcal{N}_{j,t}\equiv\sum_{k=0}^{\infty}(\alpha_j\beta)^k\mathbb{E}_t\left(\theta(1+\varphi)\hat{k}_{j,t+k}-(\theta-1)\hat{f}_{j,t+k}\right)
$$

Then, we can verity that (C-1) reduces to

$$
(1-\alpha_j\beta)\sum_{k=1}^{\infty}(\alpha_j\beta)^k\mathbb{E}_t\pi_{j,t+k}\mathcal{N}_{j,t+k}
$$

Lastly, in solving (C-2), we apply the following:

$$
\sum_{k=0}^{\infty}(\alpha_j\beta)^k\mathbb{E}_t\left(\sum_{s=0}^{k}\pi_{j,t+s}\right)^2=\frac{1}{1-\alpha_j\beta}\sum_{k=1}^{\infty}(\alpha_j\beta)^k\mathbb{E}_t\pi_{j,t+k}\left(\pi_{j,t+k}+2\mathcal{P}_{j,t+k}\right)
$$

Therefore, (C-2) becomes

$$
\frac{1}{2}(\varphi\theta+2\theta-1)\sum_{k=1}^{\infty}(\alpha_j\beta)^k\mathbb{E}_t\pi_{j,t+k}\left(\pi_{j,t+k}+2\mathcal{P}_{j,t+k}\right)
$$

Summing over (A)-(C), we get:

$$
\begin{aligned}
&\hat{K}_{j,t}-\hat{F_{j,t}}\\
&=-\frac{1}{2}(1-\alpha_j\beta)\frac{\alpha_j(1+\varphi\theta)}{1-\alpha_j}\pi_{j,t}Z_{j,t}+(1-\alpha_j\beta)\sum_{k=0}^{\infty}(\alpha_j\beta)^k\mathbb{E}_t\left\{\left(\hat{k}_{j,t+k}-\hat{f}_{j,t+k}\right)+\frac{1}{2}\left(\hat{k}^2_{j,t+k}-\hat{f}^2_{j,t+k}\right)\right\}\\
&+\sum_{k=1}^{\infty}(\alpha_j\beta)^k\mathbb{E}_t\left\{(1+\varphi\theta)\pi_{j,t+k}+(1-\alpha_j\beta)\pi_{j,t+k}\mathcal{N}_{j,t+k}+\frac{1}{2}(\varphi\theta+2\theta-1)\pi_{j,t+k}\left(\pi_{j,t+k}+2\mathcal{P}_{j,t+k}\right)\right\}
\end{aligned}
$$

Substituting Eqn.(34) and recursively writing,

$$
\begin{aligned}
&\frac{\alpha_j(1+\varphi\theta)}{1-\alpha_j}\left(\pi_{j,t}+\frac{1}{2}\frac{\theta-1}{1-\alpha_j}\pi_t^2+\frac{1}{2}(1-\alpha_j\beta)\pi_{j,t}Z_{j,t}\right)\\
&=(1-\alpha_j\beta)\left(\hat{k}_{j,t}-\hat{f}_{j,t}+\frac{1}{2}\left(\hat{k}_{j,t}^2-\hat{f}_{j,t}^2\right)\right)\\
&+\alpha_j\beta\mathbb{E}_t\left((1+\varphi\theta)\pi_{j,t+1}+(1-\alpha_j\beta)\pi_{j,t+1}\mathcal{N}_{j,t+1}+\frac{1}{2}(\varphi\theta+2\theta-1)\pi_{j,t+1}\left(\pi_{j,t+1}+2\mathcal{P}_{j,t+1}\right)\right)\\
&+\alpha_j\beta\frac{\alpha_j(1+\varphi\theta)}{1-\alpha_j}\mathbb{E}_t\left(\pi_{j,t+1}+\frac{1}{2}\frac{\theta-1}{1-\alpha_j}\pi_{j,t+1}^2+\frac{1}{2}(1-\alpha_j\beta)\pi_{j,t+1}Z_{j,t+1}\right)
\end{aligned}
$$

Dividing $\frac{\alpha_j(1+\varphi\theta)}{1-\alpha_j}$ by both sides,

$$
\begin{aligned}
&\pi_{j,t}+\frac{1}{2}\frac{\theta-1}{1-\alpha_j}\pi_{j,t}^2+\frac{1}{2}(1-\alpha_j\beta)\pi_{j,t}Z_{j,t}\\
&=\kappa_j\left(\hat{k}_{j,t}-\hat{f}_{j,t}+\frac{1}{2}\left(\hat{k}_{j,t}^2-\hat{f}_{j,t}^2\right)\right)\\
&+\mathbb{E}_t\left(\beta(1-\alpha_j)\pi_{j,t+1}+\frac{\beta(1-\alpha_j\beta)(1-\alpha_j)}{1+\varphi\theta}\pi_{j,t+1}\mathcal{N}_{j,t+1}+\frac{1}{2}\beta(1-\alpha_j)\frac{\varphi\theta+2\theta-1}{1+\varphi\theta}\pi_{j,t+1}\left(\pi_{j,t+1}+2\mathcal{P}_{j,t+1}\right)\right)\\
&+\alpha_j\beta\mathbb{E}_t\left(\pi_{j,t+1}+\frac{1}{2}\frac{\theta-1}{1-\alpha_j}\pi_{j,t+1}^2+\frac{1}{2}(1-\alpha_j\beta)\pi_{j,t+1}Z_{j,t+1}\right)
\end{aligned}
$$

where $\kappa_j\equiv\frac{(1-\alpha_j\beta)(1-\alpha_j)}{\alpha_j(1+\varphi\theta)}$. We can substitute out $\mathcal{N}_{j,t}$ as follows:

$$
\begin{aligned}
\mathcal{N}_{j,t}&=\sum_{k=0}^{\infty}(\alpha_j\beta)^k\mathbb{E}_t\left(\left(\theta(1+\varphi)\right)\hat{k}_{j,t+k}-(\theta-1)\,\hat{f}_{j,t+k}\right)\\
&=\frac{1}{2}\sum_{k=0}^{\infty}(\alpha_j\beta)^k\mathbb{E}_t\left\{(1+\varphi\theta)\left(\hat{k}_{j,t,t+k}+\hat{f}_{j,t,t+k}\right)+(\varphi\theta+2\theta-1)\left(\hat{k}_{j,t,t+k}-\hat{f}_{j,t,t+k}\right)\right\}\\
&-(1+\varphi\theta)(\varphi\theta+2\theta-1)\frac{1}{1-\alpha_j\beta}\mathcal{P}_{j,t}\\
&=\frac{1+\varphi\theta}{2}Z_{j,t}+\frac{1}{2}\frac{\varphi\theta+2\theta-1}{\kappa_j}\pi_{j,t}-\frac{(1+\varphi\theta)(\varphi\theta+2\theta-1)}{1-\alpha_j\beta}\mathcal{P}_{j,t}
\end{aligned}
$$

Substituting $\pi_{j,t+1}\mathcal{N}_{j,t+1}$ and rearranging,

$$
\begin{aligned}
&\pi_{j,t}+\frac{1}{2}\frac{\theta-1}{1-\alpha_j}\pi_{j,t}^2+\frac{1}{2}(1-\alpha_j\beta)\pi_{j,t}Z_{j,t}\\
&=\kappa_j\left(\hat{k}_{j,t}-\hat{f}_{j,t}+\frac{1}{2}\left(\hat{k}_{j,t}^2-\hat{f}_{j,t}^2\right)\right)+\frac{\beta(\varphi\theta+2\theta-1)}{2}\left(\frac{(1-\alpha_j\beta)(1-\alpha_j)}{\kappa_j(1+\varphi\theta)}+(1-\alpha_j)\right)\mathbb{E}_t\pi_{j,t+1}^2\\
&+\beta\mathbb{E}_t\left(\pi_{t+1}+\frac{1}{2}\frac{\alpha_j(\theta-1)}{1-\alpha_j}\pi_{t+1}^2+\frac{1}{2}(1-\alpha_j\beta)\pi_{j,t+1}Z_{j,t+1}\right)-\frac{1}{2}\beta(\theta-1)\mathbb{E}_t\pi_{t+1}^2
\end{aligned}
$$

which can be rewritten as

$$\begin{aligned}
&\pi_{j,t} + \frac{1}{2}\frac{\theta-1}{1-\alpha_j}\pi_t^2 + \frac{1}{2}(1-\alpha_j\beta)\pi_{j,t}Z_{j,t} \\
&= \kappa_j\left(\hat{k}_{j,t} - \hat{f}_{j,t} + \frac{1}{2}\left(\hat{k}_{j,t}^2 - \hat{f}_{j,t}^2\right)\right) + \frac{\beta}{2}\theta(1+\varphi)\mathbb{E}_t\pi_{j,t+1}^2 \\
&+ \beta\mathbb{E}_t\left(\pi_{j,t+1} + \frac{1}{2}\frac{\theta-1}{1-\alpha_j}\pi_{j,t+1}^2 + \frac{1}{2}(1-\alpha_j\beta)\pi_{j,t+1}Z_{j,t+1}\right)
\end{aligned}$$

Then, we can describe the AS relation in a recursive formula:

$$V_{j,t} = \hat{k}_{j,t} - \hat{f}_{j,t} + \frac{1}{2}\left(\hat{k}_{j,t}^2 - \hat{f}_{j,t}^2\right) + \frac{1}{2}\frac{\theta(1+\varphi)}{\kappa_j}\pi_{j,t}^2 + \beta\mathbb{E}_tV_{j,t+1} \tag{38}$$

where

$$V_{j,t} \equiv \frac{1}{\kappa_j}\left[\pi_{j,t} + \frac{1}{2}\frac{\theta-1}{1-\alpha_j}\pi_{j,t}^2 + \frac{1}{2}(1-\alpha_j\beta)\pi_{j,t}Z_{j,t} + \frac{1}{2}\theta(1+\varphi)\pi_{j,t}^2\right]$$

Forward iteration of Eqn. (38) yields:

$$V_{j,0} = \mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\left[\hat{k}_{j,t} - \hat{f}_{j,t} + \frac{1}{2}\left(\hat{k}_{j,t}^2 - \hat{f}_{j,t}^2\right) + \frac{1}{2}\frac{\theta(1+\varphi)}{\kappa_j}\pi_{j,t}^2\right] + t.i.p \tag{39}$$

As defined earlier,

$$\begin{aligned}
\hat{k}_{j,t} &= (1+\varphi)\hat{Y}_{j,t} - (1+\varphi)\hat{Z}_{j,t} \\
\hat{f}_{j,t} &= -\sigma\hat{C}_{j,t} + \hat{Y}_{j,t} + \hat{P}_{j,t} - \hat{P}_{j,t}^C
\end{aligned}$$

Then,

$$\begin{aligned}
\hat{k}_{a,t} - \hat{f}_{a,t} &= \varphi\hat{Y}_{a,t} + \sigma\hat{C}_t^U + \hat{P}_{a,t}^C - \hat{P}_{a,t} - (1+\varphi)\hat{Z}_{a,t} \\
\hat{k}_{b,t} - \hat{f}_{b,t} &= \varphi\hat{Y}_{b,t} + \sigma\hat{C}_t^U + \hat{P}_{b,t}^C - \hat{P}_{b,t} - (1+\varphi)\hat{Z}_{b,t}
\end{aligned}$$

Subtracting the two and applying the second order approximation of the price indices,

$$\left(\hat{k}_{a,t} - \hat{f}_{a,t}\right) - \left(\hat{k}_{b,t} - \hat{f}_{b,t}\right) = -\left(1+\varphi\eta\right)\hat{D}_t^U + n_an_b\left(1-\eta\right)\hat{D}_t^U\hat{D}_t^R + t.i.p$$

Also,

$$\begin{aligned}
&\left(\hat{k}_{a,t}^2 - \hat{f}_{a,t}^2\right) = \left(\hat{k}_{a,t} - \hat{f}_{a,t}\right)\left(\hat{k}_{a,t} + \hat{f}_{a,t}\right) \\
&= \left\{(\sigma+\varphi)\hat{C}_t^U - n_b\varphi\eta\hat{D}_t^U - n_b\hat{P}_{a,t}^R - (1+\varphi)\hat{Z}_{a,t}\right\}\left\{(2+\varphi-\sigma)\hat{C}_t^U - n_b\eta(2+\varphi)\hat{D}_t^U + n_a\hat{P}_{a,t}^R - (1+\varphi)\hat{Z}_{a,t}\right\} \\
&= \left\{(\sigma+\varphi)\hat{C}_t^U - n_b(1+\varphi\eta)\hat{D}_t^U - n_b^2\hat{D}_t^R - (1+\varphi)\hat{Z}_{a,t}\right\} \\
&\times\left\{(2+\varphi-\sigma)\hat{C}_t^U - n_b(2\eta+\varphi\eta-1)\hat{D}_t^U + n_b^2\hat{D}_t^R - (1+\varphi)\hat{Z}_{a,t}\right\}
\end{aligned}$$

$$= (\sigma+\varphi)(2+\varphi-\sigma)\left(\hat{C}_t^U\right)^2 - n_b\left[(\sigma+\varphi)(2\eta+\varphi\eta-1)+(1+\varphi\eta)(2+\varphi-\sigma)\right]\hat{C}_t^U\hat{D}_t^U$$
$$+ 2n_b^2(\sigma-1)\hat{C}_t^U\hat{D}_t^R - 2n_b^3(1-\eta)\hat{D}_t^U\hat{D}_t^R - 2(1+\varphi)^2\hat{Z}_{a,t}\hat{C}_t^U + 2n_b\eta(1+\varphi)^2\hat{Z}_{a,t}\hat{D}_t^U$$

Likewise,

$$\begin{aligned}
&\left(\hat{k}_{b,t}^2 - \hat{f}_{b,t}^2\right) = \left(\hat{k}_{b,t} - \hat{f}_{b,t}\right)\left(\hat{k}_{b,t} + \hat{f}_{b,t}\right) \\
&= \left\{(\sigma+\varphi)\hat{C}_t^U + n_a(1+\varphi\eta)\hat{D}_t^U - n_a^2\hat{D}_t^R - (1+\varphi)\hat{Z}_{b,t}\right\} \\
&\times \left\{(2+\varphi-\sigma)\hat{C}_t^U + n_a(2\eta+\varphi\eta-1)\hat{D}_t^U + n_a^2\hat{D}_t^R - (1+\varphi)\hat{Z}_{b,t}\right\} \\
&= (\sigma+\varphi)(2+\varphi-\sigma)\left(\hat{C}_t^U\right)^2 + n_a\left[(\sigma+\varphi)(2\eta+\varphi\eta-1)+(1+\varphi\eta)(2+\varphi-\sigma)\right]\hat{C}_t^U\hat{D}_t^U \\
&+ 2n_a^2(\sigma-1)\hat{C}_t^U\hat{D}_t^R + 2n_a^3(1-\eta)\hat{D}_t^U\hat{D}_t^R - 2(1+\varphi)^2\hat{Z}_{b,t}\hat{C}_t^U - 2n_a\eta(1+\varphi)^2\hat{Z}_{b,t}\hat{D}_t^U
\end{aligned}$$

Then,

$$\begin{aligned}
&\left(\hat{k}_{a,t}^2 - \hat{f}_{a,t}^2\right) - \left(\hat{k}_{b,t}^2 - \hat{f}_{b,t}^2\right) \\
&= -2\left[(1+\varphi\eta)(1+\varphi) - (\sigma+\varphi)(1-\eta)\right]\hat{C}_t^U\hat{D}_t^U \\
&- 2(n_a^3+n_b^3)(1-\eta)\hat{D}_t^U\hat{D}_t^R - 2(1+\varphi)^2\hat{Z}_t^R\hat{C}_t^U + 2\eta(1+\varphi)^2\hat{Z}_t^U\hat{D}_t^U
\end{aligned}$$

Accordingly,

$$\begin{aligned}
&\left(\hat{k}_{a,t} - \hat{f}_{a,t}\right) + \frac{1}{2}\left(\hat{k}_{a,t}^2 - \hat{f}_{a,t}^2\right) - \left(\hat{k}_{b,t} - \hat{f}_{b,t}\right) - \frac{1}{2}\left(\hat{k}_{b,t}^2 - \hat{f}_{b,t}^2\right) \\
&= -(1+\varphi\eta)\hat{D}_t^U - \left[(1+\varphi\eta)(1+\varphi) - (\sigma+\varphi)(1-\eta)\right]\hat{C}_t^U\hat{D}_t^U - (1+\varphi)^2\hat{Z}_t^R\hat{C}_t^U + \eta(1+\varphi)^2\hat{Z}_t^U\hat{D}_t^U
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{1}{1+\varphi\eta}\left(V_{a,0} - V_{b,0}\right) \\
&= \frac{1}{2}\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\left[\begin{array}{c} \frac{\theta(1+\varphi)}{1+\varphi\eta}\left(\frac{\pi_{a,t}^2}{\kappa_a} - \frac{\pi_{b,t}^2}{\kappa_b}\right) - 2\hat{D}_t^U - 2\left[(1+\varphi) - \frac{(\sigma+\varphi)(1-\eta)}{1+\varphi\eta}\right]\hat{C}_t\hat{D}_t^U \\ -\frac{2(1+\varphi)^2}{1+\varphi\eta}\hat{Z}_t^R\hat{C}_t^U + \frac{2\eta(1+\varphi)^2}{1+\varphi\eta}\hat{Z}_t^U\hat{D}_t^U \end{array}\right] + t.i.p
\end{aligned}$$

Relative welfare

A relative welfare is defined to be the gap between the welfare loss of country A and B:

$$\begin{aligned}
\mathcal{L}^R &\equiv \mathcal{L}_a - \mathcal{L}_b \\
&= \frac{1}{2}\mathbb{E}\sum_{t=0}^{\infty}\beta^t\left\{\begin{array}{c} \theta\left(\frac{(\pi_{a,t})^2}{\kappa_a} - \frac{(\pi_{b,t})^2}{\kappa_b}\right) - 2\eta\hat{D}_t^U - 2\eta(1+\varphi)\hat{C}_t^U\hat{D}_t^U \\ -2(1+\varphi)\hat{Z}_t^R\hat{C}_t^U + \eta(1+\varphi)\hat{Z}_t^U\hat{D}_t^U \end{array}\right\}
\end{aligned}$$

We first substitute out the linear term in $\mathcal{L}^R$ using the second order approximated AS relations.

$$\begin{aligned}V_0 &\equiv \frac{\eta}{1+\varphi\eta}\left(V_{a,0}-V_{b,0}\right)\\ &= \frac{1}{2}\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\left[\begin{array}{c}\frac{\eta\theta(1+\varphi)}{1+\varphi\eta}\left(\frac{\pi_{a,t}^2}{\kappa_a}-\frac{\pi_{b,t}^2}{\kappa_b}\right)-2\eta\hat{D}_t^U-2\left[\eta(1+\varphi)-\frac{\eta(\sigma+\varphi)(1-\eta)}{1+\varphi\eta}\right]\hat{C}_t^U\hat{D}_t^U\\ -\frac{2\eta(1+\varphi)^2}{1+\varphi\eta}\hat{Z}_t^R\hat{C}_t^U+\frac{2\eta^2(1+\varphi)^2}{1+\varphi\eta}\hat{Z}_t^U\hat{D}_t^U\end{array}\right]+t.i.p\end{aligned}$$

Then,

$$\begin{aligned}W^R-V_0 &= \frac{1}{2}\mathbb{E}\sum_{t=0}^{\infty}\beta^t\left[\begin{array}{c}\frac{\theta(1-\eta)}{1+\varphi\eta}\left(\frac{(\pi_{a,t})^2}{\kappa_a}-\frac{\left(\pi_{b,t}\right)^2}{\kappa_b}\right)-\frac{2\eta(\sigma+\varphi)(1-\eta)}{(1+\varphi\eta)}\hat{C}_t^U\hat{D}_t^U\\ -\frac{2(1+\varphi)(1-\eta)}{(1+\varphi\eta)}\hat{Z}_t^R\hat{C}_t^U+\frac{2\eta(1+\varphi)(1-\eta)}{(1+\varphi\eta)}\hat{Z}_t^U\hat{D}_t^U\end{array}\right]+t.i.p\\ &= \frac{1}{2}\mathbb{E}\sum_{t=0}^{\infty}\beta^t\left[+\frac{\theta(1-\eta)}{(1+\varphi\eta)}\left(\frac{(\pi_{a,t})^2}{\kappa_a}-\frac{\left(\pi_{b,t}\right)^2}{\kappa_b}\right)-\frac{2\eta(\sigma+\varphi)(1-\eta)}{(1+\varphi\eta)}\left(\hat{C}_t^U-\hat{C}_t^{U,E}\right)\left(\hat{D}_t^U-\widetilde{D}_t^{U,1}\right)\right]+t.i.p\end{aligned}$$

where $\widetilde{D}_t^{U,1}=-\frac{1+\varphi}{\eta(\sigma+\varphi)}\hat{Z}_t^R$. We now replace the cross terms, for which we define the output target in minimizing the relative welfare losses:

$$\widetilde{Y}_{a,t}^R\equiv\hat{C}_t^{U,E}-n_b\eta\widetilde{D}_t^{U,1};\quad \widetilde{Y}_{b,t}^R\equiv\hat{C}_t^{U,E}+n_a\eta\widetilde{D}_t^{U,1}$$

Utilizing relationship the demand function as before, we show the following.

$$\begin{aligned}&\eta\left(\hat{C}_t^U-\hat{C}_t^E\right)\left(\hat{D}_t^U-\widetilde{D}_t^{U,1}\right)=\eta\left[n_a\left(\hat{C}_t^U-\hat{C}_t^E\right)+n_b\left(\hat{C}_t^U-\hat{C}_t^E\right)\right]\left(\hat{D}_t^U-\widetilde{D}_t^{U,1}\right)\\ &=n_a\eta\left[\left(\hat{Y}_{a,t}-\widetilde{Y}_{a,t]}^R\right)+n_b\eta\left(\hat{D}_t^U-\widetilde{D}_t^{U,1}\right)\right]\left(\hat{D}_t^U-\widetilde{D}_t^{U,1}\right)\\ &+n_b\eta\left[\left(\hat{Y}_{b,t}-\widetilde{Y}_{b,t}^R\right)-n_a\eta\left(\hat{D}_t^U-\widetilde{D}_t^{U,1}\right)\right]\left(\hat{D}_t^U-\widetilde{D}_t^{U,1}\right)\\ &=n_a\eta\left(\hat{Y}_{a,t}-\widetilde{Y}_{a,t}^R\right)\left(\hat{D}_t^U-\widetilde{D}_t^{U,1}\right)+n_b\eta\left(\hat{Y}_{b,t}-\widetilde{Y}_{b,t}^R\right)\left(\hat{D}_t^U-\widetilde{D}_t^{U,1}\right)\\ &=-\left(\hat{Y}_{a,t}-\widetilde{Y}_{a,t}^R\right)^2+\left(\hat{Y}_{b,t}-\widetilde{Y}_{b,t}^R\right)^2-\eta\left(\hat{C}_t^U-\widetilde{C}_t^W\right)\left(\hat{D}_t^U-\widetilde{D}_t^{U,1}\right)\end{aligned}$$

Hence, we just derive

$$2\eta\left(\hat{C}_t^U-\hat{C}_t^{U,E}\right)\left(\hat{D}_t^U-\widetilde{D}_t^{U,1}\right)=-\left(\hat{Y}_{a,t}-\widetilde{Y}_{a,t}^R\right)^2+\left(\hat{Y}_{b,t}-\widetilde{Y}_{b,t}^R\right)^2 \tag{40}$$

Substituting (40) back into the relative welfare loss function and rearranging the terms,

$$\mathcal{L}^R=\frac{1}{2}\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\left[\frac{\theta(1-\eta)}{1+\varphi\eta}\left(\frac{(\pi_{a,t})^2}{\kappa_a}-\frac{\left(\pi_{b,t}\right)^2}{\kappa_b}\right)+\frac{(\sigma+\varphi)(1-\eta)}{(1+\varphi\eta)}\left(\hat{Y}_{a,t}-\widetilde{Y}_{a,t}^R\right)^2-\frac{(\sigma+\varphi)(1-\eta)}{(1+\varphi\eta)}\left(\hat{Y}_{b,t}-\widetilde{Y}_{b,t}^R\right)^2\right]+t.i.p$$

where $\widetilde{Y}_{a,t}^R=\frac{1+\varphi}{\sigma+\varphi}\hat{Z}_t^U+n_b\frac{1+\varphi}{\sigma+\varphi}\hat{Z}_t^R$ and $\widetilde{Y}_{b,t}^R=\frac{1+\varphi}{\sigma+\varphi}\hat{Z}_t^U-n_a\frac{1+\varphi}{\sigma+\varphi}\hat{Z}_t^R$.

Country-wide welfare

Finally, the lifetime welfare loss of each country is derived:

$$\mathcal{L}_a = \mathcal{L}^{CU} + n_b \mathcal{L}^R; \quad \mathcal{L}_b = \mathcal{L}^{CU} - n_a \mathcal{L}^R$$

And we derive the corresponding the loss function of country A and B (i.e. $\mathcal{L}_a = \frac{1}{2}\mathbb{E}\sum_{t=0}^{\infty}\beta^t L_{a,t}$ and $\mathcal{L}_b = \frac{1}{2}\mathbb{E}\sum_{t=0}^{\infty}\beta^t L_{b,t}$). For country A,

$$L_{a,t} = \sum_j \lambda_{\pi j}(\pi_{j,t})^2 + \sum_j \lambda_{yj}\left(\hat{Y}_{j,t} - \widetilde{Y}_{j,t}\right)^2 + \lambda_{du}\left(\hat{D}_t^U - \widetilde{D}_t^U\right)^2 + \lambda_{dr}\left(\hat{D}_t^R\right)^2 + \lambda_s\left(\Delta\hat{S}_t\right)^2$$

where the loss function coefficients and the welfare-related targets are:

$$\lambda_{\pi a} = \theta\frac{\widetilde{n}}{\kappa_a}; \quad \lambda_{\pi b} = \theta\frac{1-\widetilde{n}}{\kappa_b}; \quad \lambda_{ya} = \widetilde{n}(\sigma+\varphi); \quad \lambda_{yb} = (1-\widetilde{n})(\sigma+\varphi)$$
$$\lambda_{du} = n_a n_b \eta(1-\sigma\eta); \quad \lambda_{dr} = (n_a n_b)^2\eta; \quad \lambda_s = \omega$$
$$\widetilde{Y}_{a,t} = \frac{1+\varphi}{\sigma+\varphi}\hat{Z}_t^U + n_b\left(\frac{\widetilde{\eta}}{\widetilde{n}}\right)\frac{1+\varphi}{1+\varphi\eta}\hat{Z}_t^R; \quad \widetilde{Y}_{b,t} = \frac{1+\varphi}{\sigma+\varphi}\hat{Z}_t^U - n_a\left(\frac{\eta-\widetilde{\eta}}{1-\widetilde{n}}\right)\frac{1+\varphi}{1+\varphi\eta}\hat{Z}_t^R$$
$$\widetilde{D}_t^U = -\frac{1+\varphi}{1+\varphi\eta}\hat{Z}_t^R$$

and

$$\widetilde{n} = n_a + n_b\frac{1-\eta}{1+\varphi\eta} < n_a; \quad \widetilde{\eta} = n_b\left(\eta + \frac{1-\eta}{\sigma+\varphi}\right) < \eta$$

For country B,

$$L_{b,t} = \sum_j \lambda^*_{\pi j}(\pi_{j,t})^2 + \sum_j \lambda^*_{yj}\left(\hat{Y}_{j,t} - \widetilde{Y}^*_{j,t}\right)^2 + \lambda^*_{du}\left(\hat{D}_t^U - \widetilde{D}_t^U\right)^2 + \lambda^*_{dr}\left(\hat{D}_t^R\right)^2 + \lambda^*_s\left(\Delta\hat{S}_t\right)^2$$

where

$$\lambda^*_{\pi a} = \theta\frac{1-\widetilde{n}}{\kappa_a}; \quad \lambda^*_{\pi b} = \theta\frac{\widetilde{n}}{\kappa_b}; \quad \lambda^*_{ya} = (1-\widetilde{n})(\sigma+\varphi); \quad \lambda^*_{yb} = \widetilde{n}(\sigma+\varphi)$$
$$\lambda^*_{dw} = \lambda_{dw}; \quad \lambda^*_{dr} = \lambda_{dr}; \quad \lambda^*_s = \lambda_s$$
$$\widetilde{Y}^*_{a,t} = \frac{1+\varphi}{\sigma+\varphi}\hat{Z}_t^U + n_b\left(\frac{\eta-\widetilde{\eta}}{1-\widetilde{n}}\right)\frac{1+\varphi}{1+\varphi\eta}\hat{Z}_t^R; \quad \widetilde{Y}^*_{b,t} = \frac{1+\varphi}{\sigma+\varphi}\hat{Z}_t^U - n_a\left(\frac{\widetilde{\eta}}{\widetilde{n}}\right)\frac{1+\varphi}{1+\varphi\eta}\hat{Z}_t^R$$

6.7 Optimal monetary policy outside the union

When each country adopts national currency and employs an optimal monetary policy, the central bank chooses domestic inflation path given the foreign inflation paths. The common constraints faced by the central banks of country A and B are as follows:

$$\begin{aligned}
&\pi_{a,t} = \beta\mathbb{E}_t\pi_{a,t+1} + \kappa_a\left\{(\sigma+\varphi)\hat{Y}_{a,t} - n_b(1-\sigma\eta)\hat{D}_t^U - n_b^2\hat{D}_t^R - (1+\varphi)\hat{Z}_{a,t}\right\}\\
&\pi_{b,t} = \beta\mathbb{E}_t\pi_{b,t+1} + \kappa_b\left\{(\sigma+\varphi)\hat{Y}_{b,t} + n_a(1-\sigma\eta)\hat{D}_t^U - n_a^2\hat{D}_t^R - (1+\varphi)\hat{Z}_{b,t}\right\}\\
&\hat{Y}_{a,t} - \hat{Y}_{b,t} = -\eta\hat{D}_t^U\\
&\hat{D}_t^R = -\nu\left(\hat{C}_{a,b,t} + \hat{C}_{b,a,t}\right)\\
&\pi_{a,t} - \pi_{b,t} = \Delta\left(\hat{D}_t^U - n_a\hat{D}_t^R + \hat{S}_t - \nu\hat{C}_{b,a,t}\right)\\
&\hat{C}_{a,b,t} = \hat{Y}_{b,t} + n_a n_b\eta\hat{D}_t^R\\
&\hat{C}_{b,a,t} = \hat{Y}_{a,t} + n_a n_b\hat{D}_t^R
\end{aligned}$$

We get the last two relationship from the demand function for both domestic and foreign consumption of country goods. Therefore the Lagrangian problem of the country A central bank is as follows.

$$\begin{aligned}
\mathcal{L} =&\frac{1}{2}\mathbb{E}_0\sum\beta^t\left[\begin{array}{c}\lambda_{\pi a}(\pi_{a,t})^2 + \lambda_{\pi b}(\pi_{b,t})^2 + \lambda_{ya}\Big(\hat{Y}_{a,t} - \widetilde{Y}_{a,t}\Big)^2 + \lambda_{yb}\Big(\hat{Y}_{b,t} - \widetilde{Y}_{b,t}\Big)^2\\ +\lambda_{du}\Big(\hat{D}_t^U - \widetilde{D}_t^U\Big)^2 + \lambda_{dr}\Big(\hat{D}_t^R\Big)^2 + \lambda_s\Big(\Delta\hat{S}_t\Big)^2\end{array}\right]\\
&+\phi_{1,t}\left[\frac{\pi_{a,t}}{\kappa_a} - \frac{\beta\pi_{a,t+1}}{\kappa_a} - (\sigma+\varphi)\hat{Y}_{a,t} + n_b(1-\sigma\eta)\hat{D}_t^U + n_b^2\hat{D}_t^R + (1+\varphi)\hat{Z}_{a,t}\right]\\
&+\phi_{2,t}\left[\frac{\pi_{b,t}}{\kappa_b} - \frac{\beta\pi_{b,t+1}}{\kappa_b} - (\sigma+\varphi)\hat{Y}_{b,t} - n_a(1-\sigma\eta)\hat{D}_t^U + n_a^2\hat{D}_t^R + (1+\varphi)\hat{Z}_{b,t}\right]\\
&+\phi_{3,t}\left[\hat{Y}_{a,t} - \hat{Y}_{b,t} + \eta\hat{D}_t^U\right]\\
&+\phi_{4,t}\left[\hat{D}_t^R + \nu\left(\hat{C}_{a,b,t} + \hat{C}_{b,a,t}\right)\right]\\
&+\phi_{5,t}\left[\pi_{a,t} - \pi_{b,t} - \left(\hat{D}_t^U - \frac{1}{2}\hat{D}_t^R + \hat{S}_t - \nu\hat{C}_{b,a,t}\right) + \left(\hat{D}_{t-1}^U - \frac{1}{2}\hat{D}_{t-1}^R + \hat{S}_{t-1} - \nu\hat{C}_{b,a,t-1}\right)\right]\\
&+\phi_{6,t}\left[\hat{C}_{a,b,t} - \hat{Y}_{b,t} - n_a n_b\eta\hat{D}_t^R\right]\\
&+\phi_{7,t}\left[\hat{C}_{b,a,t} - \hat{Y}_{a,t} - n_a n_b\eta\hat{D}_t^R\right]
\end{aligned}$$

Substituting $n_a = n_b = \frac{1}{2}$, the first order conditions are:

$$\lambda_{\pi a}\pi_{a,t} + \frac{1}{\kappa_a}\left(\phi_{1,t} - \phi_{1,t-1}\right) = 0 \tag{41a}$$

$$\lambda_{ya}\left(\hat{Y}_{a,t} - \widetilde{Y}_{a,t}\right) - (\sigma+\varphi)\phi_{1,t} + \phi_{3,t} - \phi_{7,t} = 0 \tag{41b}$$

$$\lambda_{yb}\left(\hat{Y}_{b,t} - \widetilde{Y}_{b,t}\right) - (\sigma+\varphi)\phi_{2,t} - \phi_{3,t} - \phi_{6,t} = 0 \tag{41c}$$

$$\lambda_{du}\left(\hat{D}_t^U - \widetilde{D}_t^U\right) + \frac{1-\sigma\eta}{2}\left(\phi_{1,t} - \phi_{2,t}\right) + \eta\phi_{3,t} - \phi_{5,t} + \beta\mathbb{E}_t\phi_{5,t+1} = 0 \tag{41d}$$

$$\lambda_{dr}\hat{D}_t^R + \frac{1}{4}\left(\phi_{1,t} + \phi_{2,t}\right) + \phi_{4,t} + \frac{1}{2}\phi_{5,t} - \frac{\beta}{2}\mathbb{E}_t\phi_{5,t+1} - \frac{\eta}{4}\left(\phi_{6,t} + \phi_{7,t}\right) = 0 \tag{41e}$$

$$\lambda_s\left(\Delta\hat{S}_t\right) - \phi_{5,t} + \beta\mathbb{E}_t\phi_{5,t+1} = 0 \tag{41f}$$

$$\nu\phi_{4,t} + \nu\phi_{5,t} - \nu\beta\mathbb{E}_t\phi_{5,t+1} + \phi_{7,t} = 0 \tag{41g}$$

$$\nu\phi_{4,t} + \phi_{6,t} = 0 \tag{41h}$$

From Eq. (41f), we get $\phi_{5,t} - \beta\mathbb{E}_t\phi_{5,t+1} = \lambda_s\Delta\hat{S}_t$, and from Eq. (41h), $\nu\phi_{4,t} = -\phi_{6,t}$. Substituting this into Eq. (41g), we get:

$$\phi_{6,t} - \phi_{7,t} = \nu\lambda_s\Delta\hat{S}_t \tag{42a}$$

$$\phi_{6,t} + \phi_{7,t} = -2\nu\phi_{4,t} - \nu\lambda_s\Delta\hat{S}_t \tag{42b}$$

Now by adding and subtracting Eqs. (41b) and (41c),

$$\lambda_{ya}\left(\hat{Y}_{a,t} - \widetilde{Y}_{a,t}\right) + \lambda_{yb}\left(\hat{Y}_{b,t} - \widetilde{Y}_{b,t}\right) - (\sigma+\varphi)\left(\phi_{1,t} + \phi_{2,t}\right) - \phi_{6,t} - \phi_{7,t} = 0 \tag{43a}$$

$$\lambda_{ya}\left(\hat{Y}_{a,t} - \widetilde{Y}_{a,t}\right) - \lambda_{yb}\left(\hat{Y}_{b,t} - \widetilde{Y}_{b,t}\right) - (\sigma+\varphi)\left(\phi_{1,t} - \phi_{2,t}\right) + 2\phi_{3,t} + \phi_{6,t} - \phi_{7,t} = 0 \tag{43b}$$

Substituting Eqs. (42a) and (43b) into Eq. (41d) and rearranging yields:

$$\begin{aligned} &\phi_{1,t} - \phi_{2,t} \\ &= \frac{1}{1+\varphi\eta}\left[\eta\lambda_{ya}\left(\hat{Y}_{a,t} - \widetilde{Y}_{a,t}\right) - \eta\lambda_{yb}\left(\hat{Y}_{b,t} - \widetilde{Y}_{b,t}\right) - 2\lambda_{du}\left(\hat{D}^U_t - \widetilde{D}^U_t\right) + \lambda_s\left(2+\nu\eta\right)\Delta\hat{S}_t\right] \end{aligned} \tag{44}$$

Combining Eqs. (41d) and (42b), we can solve for $\phi_{4,t}$:

$$\phi_{4,t} = \frac{-1}{2(\nu\eta+2)}\left[\phi_{1,t} + \phi_{2,t} + 4\lambda_{dr}\hat{D}^R_t + \lambda_s\left(2+\nu\eta\right)\Delta\hat{S}_t\right] \tag{45}$$

Then, Eq. (42b) becomes:

$$\phi_{6,t} + \phi_{7,t} = \frac{\nu}{\nu\eta+2}\left[\phi_{1,t} + \phi_{2,t} + 4\lambda_{dr}\hat{D}^R_t\right] \tag{46}$$

Substituting Eq. (46) into Eq. (43a) and rearranging,

$$\phi_{1,t} + \phi_{2,t} = (\nu\eta+2)\,\mu\left[\lambda_{ya}\left(\hat{Y}_{a,t} - \widetilde{Y}_{a,t}\right) + \lambda_{yb}\left(\hat{Y}_{b,t} - \widetilde{Y}_{b,t}\right)\right] - 4\nu\mu\lambda_{dr}\hat{D}^R_t \tag{47}$$

where $\mu^{-1} = (\sigma+\varphi)(\nu\eta+2) + \nu$. Finally, combining Eqs. (44) and (47),

$$\begin{aligned} \phi_{1,t} = &\frac{1}{2}\left((\nu\eta+2)\,\mu + \frac{\eta}{1+\varphi\eta}\right)\lambda_{ya}\left(\hat{Y}_{a,t} - \widetilde{Y}_{a,t}\right) + \frac{1}{2}\left((\nu\eta+2)\,\mu - \frac{\eta}{1+\varphi\eta}\right)\lambda_{yb}\left(\hat{Y}_{b,t} - \widetilde{Y}_{b,t}\right) \\ &- \frac{1}{1+\varphi\eta}\lambda_{du}\left(\hat{D}^U_t - \widetilde{D}^U_t\right) - 2\nu\mu\lambda_{dr}\hat{D}^R_t + \frac{\lambda_s\left(2+\nu\eta\right)}{2(1+\varphi\eta)}\Delta\hat{S}_t \end{aligned}$$

The targeting rule of country A is:

$$\pi_{a,t} = -\frac{1}{\theta\widetilde{n}}\Delta\phi_{1,t}$$

Equivalently, we can derive the targeting rule of country B:

$$\pi_{b,t} = -\frac{1}{\theta\widetilde{n}}\Delta\phi^*_{2,t}$$

where $\phi^*_{2,t}$ is composed of target variables:

$$\begin{aligned}\phi^*_{2,t} = &\frac{1}{2}\left(\left(\nu\eta+2\right)\mu - \frac{\eta}{1+\varphi\eta}\right)\lambda^*_{ya}\left(\hat{Y}_{a,t} - \widetilde{Y}^*_{a,t}\right) + \frac{1}{2}\left(\left(\nu\eta+2\right)\mu + \frac{\eta}{1+\varphi\eta}\right)\lambda^*_{yb}\left(\hat{Y}_{b,t} - \widetilde{Y}^*_{b,t}\right)\\ &+ \frac{1}{1+\varphi\eta}\lambda^*_{du}\left(\hat{D}^U_t - \widetilde{D}^U_t\right) - 2\nu\mu\lambda^*_{dr}\hat{D}^R_t - \frac{\lambda_s\left(2+\nu\eta\right)}{2(1+\varphi\eta)}\Delta\hat{S}_t\end{aligned}$$