

# Prices and Inflation when Government Bonds are Net Wealth\*

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## Abstract

In this paper I show that models in which government bonds are net wealth - that is, their value exceeds that of tax liabilities (Barro, 1974) - offer a new perspective on several issues in monetary economics.

First and foremost, prices and inflation are jointly and uniquely determined by fiscal and monetary policy. In contrast to the conventional view, the long-run inflation rate here is, in the absence of output growth, and even when monetary policy operates an interest rate rule with a different inflation target, equal to the growth rate of nominal fiscal variables, which are controlled by fiscal policy.

This novel theory also offers a different perspective on the fiscal and monetary transmission mechanism, policies at the zero-lower bound, U.S. inflation history, recent attempts to stimulate inflation in the Euro area and several puzzles which arise in New Keynesian models during a liquidity trap. To derive my findings, I first use a reduced form approach in which households derive utility from holding bonds. I prove how and for which policy rules the price level is globally determinate, then showing that the reduced form results carry over to a Bewley-Imrohorglu-Huggett-Aiyagari heterogeneous agent incomplete markets model.

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# 1 Introduction

The prevailing view on prices and inflation and the conduct of monetary policy (Woodford, 2003; Galí, 2015) is that the latter works through setting nominal interest rates, that monetary policy controls the inflation rate and that prices are determinate if monetary policy responds sufficiently strongly to inflation - the Taylor principle. Fiscal policy is largely irrelevant in this view.

In this paper, I propose that adding nominal bonds that are net wealth - i.e. the real value exceeds the value of tax liabilities (Barro, 1974) - to standard models offers a new and different perspective on these topics. While monetary policy continues to control the nominal interest rate, fiscal policy is now assigned a significant role. Two results stand out.

First, in contrast to the conventional view, the long-run inflation rate is, in the absence of output growth and even when monetary policy operates an interest rate rule with a different inflation target, equal to the growth rate of nominal fiscal variables, which are controlled by fiscal policy. A tough, independent central bank is not only insufficient to ensure price stability in the long-run, it also has no direct control over long-run inflation even if it follows an interest rate rule which satisfies the Taylor principle.<sup>1</sup> The fiscal determination of long-run inflation leads to a reinterpretation of one of Friedman's (1963; 1968; 1970) key propositions that "inflation is always and everywhere a monetary phenomenon in the sense that it is and can be produced only by a more rapid increase in the quantity of money than in output." Friedman assumed that money supply, purposefully controlled by central banks, serves as a nominal anchor to control inflation. 50 years after Friedman (1963, 1968, 1970) however, central bank practice and academic research has shifted to using a short-term nominal interest rate as the policy instrument. Money supply is not controlled by monetary policy anymore, but is determined endogenously as clearing the money market. I incorporate these features into the model, which strips money of its anchoring role envisaged by Friedman. However, the nominal fiscal variables, controlled by fiscal policy, such as nominal government debt and spending, replace money in this role.

Second, models in which government bonds are net wealth provide a new theory of price-level determination. Monetary policy works through setting an arbitrary sequence of nominal interest rates, for example through an interest rate rule. Fiscal policy sets sequences of nominal government spending, taxes, and government debt, for example through a fiscal rule, and these sequences satisfy the present value government budget constraint at all times and for all prices. In this environment, I show that the steady-state price level is determinate, even if nominal interest rates are constant, and I derive conditions for policy rules to ensure global determinacy. It is this

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<sup>1</sup>Central bank independence ensures, however, that the treasury cannot impose fiscal policies on the central bank, e.g. monetizing its debt to finance the government.

global determinacy result which rules out sunspot-driven inflation movements and establishes that the price level and the inflation rate are uniquely determined by policy. As I will show, the price level jointly by monetary and fiscal policy and the long-run inflation rate by fiscal policy.

To understand these results, it is sufficient to combine a few simple insights and it is instructive to start with a steady state. First, when government bonds are net wealth, shifts in the stock of real public debt affect real aggregate demand (Barro, 1974). When in addition, government bonds are nominal, shifts in the price level shift the real value of debt and thus affect real aggregate demand. The price level is then determined such that aggregate demand equals aggregate supply. Second, there is no unique steady-state real interest rate determined by a household discount factor, but instead, it depends on the amount of real bonds available. When government bonds are net wealth, households are willing to accept a lower return than suggested by their discount factor, as bonds provide some extra service. Bonds have a “liquidity premium”, which under standard assumptions is the lower the more bonds households own. To absorb more government bonds, households thus require a higher real interest rate. This indicates that depending on the amount of bonds available, a continuum of steady-state real interest rates is feasible. Monetary and fiscal policy now jointly choose one of this continuum of potential steady-state real interest rates. Monetary policy sets the steady-state nominal interest rate, whereas fiscal policy sets the growth rate of nominal government debt. In a steady state, the value of real government debt is constant, such that the steady-state condition for fiscal policy is that the growth rate of nominal debt equals the inflation rate (in the absence of economic growth). The real interest rate is then determined by the Fisher equation, as the ratio of the nominal interest rate to the inflation rate. Clearly, this logic for determining the long-run inflation rate does not apply if the steady-state real interest rate is pinned down by the discount factor, as is the case in standard New Keynesian models.

Outside of steady states, I demonstrate determinacy for all monetary policy rules, that is those not responding, weakly responding or strongly responding to prices. Interestingly, by responding too strongly to price increases, fiscal policy may induce, not remove, indeterminacy. In such an expansionary fiscal policy scenario, reestablishing determinacy requires monetary policy to increase nominal interest rates in response to price increases. I provide a characterization of these determinacy conditions, including how responsive monetary policy has to be if fiscal policy is too expansionary. I also provide several proofs to rule out both hyperinflations and hyperdeflations. For example, while price stickiness is largely irrelevant for steady-state and local determinacy it rules out hyperinflations. An equilibrium hyperinflation requires the price level to be ultimately infinite. Setting the price to infinity is, however, not an optimal decision for a firm subject to price adjustment costs and frictions.

This novel theory of price and inflation determination also calls for a rethinking of various other issues in monetary economics. Applied to recent attempts by the ECB to increase inflation in the Euro area, the findings in this paper suggest that these efforts are unlikely to succeed. Instead, higher inflation would require an expansion of nominal fiscal spending by Euro area member states, in order to stimulate nominal demand, assigning an important role to large countries such as Germany. A fiscal stimulus by a small country would have very little impact on inflation, as it has only a negligible effect on area-wide demand, but would lead to a real exchange rate appreciation (with likely adverse economic consequences) for this small country.

Applied to growing concerns that the US or the world economy may be stuck in a liquidity trap with zero nominal and real interest rates for an extended period, the findings in this paper suggest an easy solution. Although the ZLB prevents further cuts of the nominal interest rate, fiscal policy can increase the growth rate of nominal spending and therefore the inflation rate, leading to lower real interest rates, provided that this policy is sufficiently persistent and credible to households and firms. If instead fiscal policy continues its current austerity plan of bringing low inflation rates to around zero, then the real interest rate will hover around zero too, even in the long run.

The theory set out in this paper also offers a different perspective on US inflation history. After experiencing high rates in the 1970s, the 1980s witnessed success in keeping inflation low. The standard interpretation is that central banks eventually recognized that keeping inflation low was their primary objective and as a consequence, were successful in doing so. The framework proposed in this paper suggests that it was not the change in monetary policy that kept inflation in check, but rather a shift to a less expansionary fiscal policy during the Reagan administration, perhaps forced on by the prolonged high nominal interest rates set by central banks under chairman Paul Volcker and the resulting high deficits.

Hagedorn (2016) and Hagedorn et al. (2018b,a, 2017) show that when government bonds constitute net wealth several puzzles disappear, which are otherwise observed during a liquidity trap in New Keynesian models. There is no forward guidance puzzle, as commitment to future monetary policy has only negligible effects in this context. Improvements in technology increase output when the price level is determinate, in contrast to New Keynesian models. The size of the fiscal multiplier becomes smaller if prices are less sticky, whereas it becomes arbitrarily large in New Keynesian models.

It is important to emphasize that these results do not hold in all models in which Ricardian equivalence fails. For example, the price level is not determinate in an economy where a fraction of households simply consume their current income, “hand-to-mouth”, while the remaining households act according to the permanent income hypothesis (PIH). The reason for the indeterminacy is that

government bonds are not net wealth, since only permanent-income households hold bonds, and shifts in the value of public debt have no aggregate demand effects, but only shift consumption from one group to the other. Similar arguments apply to the perpetual youth model and its variants (Yaari, 1965; Blanchard, 1985; Bénassy, 2005, 2008), in which Ricardian equivalence does not hold, but the price level is indeterminate, as explained in the Appendix.

To derive my results formally, I start with a reduced form model in Section 2, in which government bonds are net wealth, because they are an argument of the utility function. This simplification allows for a clear and accessible exposition, which explains how the steady-state price level is determined, when the economy is locally determinate, how to rule out hyperinflations and hyperdeflations and how the steady-state inflation rate is determined. I also explain the mechanism behind the determinacy result; why models in which government bonds are net wealth deliver determinacy and why those in which they are not net wealth lead to indeterminacy when nominal interest rates are constant; why fiscal policy has to be partially nominal; why this is not the Fiscal Theory of the Price Level (FTPL); and why adding capital or cash to the model does not alter these conclusions.

In Section 3, I move to a Bewley-Imrohoroglu-Huggett-Aiyagari heterogeneous agents incomplete markets economy, and show that the results from the reduced form analysis carry over to this model class. One reason to study incomplete markets models is that it is desirable to have a model in which government bonds have net wealth, not because bonds enter the utility function, but because net wealth arises endogenously as a result of household optimization. In incomplete markets models, government bonds are net wealth since they allow households to smooth consumption more effectively in response to uninsurable idiosyncratic income shocks.

Another reason is that the net wealth is large in these models. Indeed, a large body of empirical research (Campbell and Deaton, 1989; Attanasio and Davis, 1996; Johnson et al., 2006; Blundell et al., 2008; Attanasio and Pavoni, 2011, among many others) has documented significant deviations in households' consumption behavior from the complete markets benchmark. There is, for example, substantial heterogeneity in the marginal propensity to consume (MPC) across households, with some households behaving according to the permanent income hypothesis and having a small MPC, while others have an MPC of a higher-order magnitude. As incomplete markets models match these micro-consumption facts, the value of government bonds is large in these models. While it is sufficient for the theoretical results in this paper that government bonds are some possibly arbitrarily small net wealth, quantitative work is motivated by the observation that the deviation from complete markets models, in which bonds are not net wealth, is large. This brings me to the last reason to consider incomplete markets models.

A growing literature has recently emerged which incorporates price rigidities into incomplete

markets models.<sup>2</sup> One motivation to do so is that, while able to generate a realistic distribution of marginal propensities to consume, the textbook incomplete markets model does not allow output to be demand-determined, as prices are fully flexible, potentially limiting its applicability to many questions raised by the Great Recession. Adding a nominal side to the model and allowing for price rigidities on the other hand, forces us to address the same questions we confront in complete markets models: How is the price level determined? What type of monetary and fiscal policies ensure determinacy? How does taking the zero lower bound (ZLB) into account affect the conclusions derived from the model? This paper provides these answers and shows that they are quite different from the standard analysis based on complete markets. Without answers to these questions, for example, one of the most important policy questions, the magnitude of the fiscal multiplier, cannot be addressed satisfactorily. Section 4 concludes and discusses several contributions, Hagedorn (2016) and Hagedorn et al. (2017, 2018a,b), who use incomplete markets models in which government bonds are net wealth and which build on these answers and the insights provided in this paper to consider forward guidance, the fiscal multiplier and several other puzzles which arise in New Keynesian models during a liquidity trap.

## 2 Price Level and Inflation Determinacy when Government Bonds are Net Wealth

In this Section, I use a reduced form approach to explain why and how the price level and inflation are determinate if government bonds are both net wealth and are nominal. For pedagogical reasons, I start with an infinite horizon representative agent economy in which households derive utility from consumption, leisure and money holdings. Importantly, monetary policy works through setting the nominal interest rate and I assume first that the nominal rate is constant, the scenario considered by Sargent and Wallace (1975). Money supply is not set through monetary policy, but instead adjusts endogenously to satisfy households' money demand at the nominal interest rate

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<sup>2</sup>Gornemann et al. (2012), Kaplan et al. (2016), Auclert (2016) and Lütticke (2015) study monetary policy in a model with incomplete markets and pricing frictions, but with a different focus, emphasizing and quantifying several redistributive channels of the transmission mechanism of monetary policy which are absent in standard complete markets models. Earlier contributions are Oh and Reis (2012) and Guerrieri and Lorenzoni (2015), who were among the first to add nominal rigidities to a Bewley-Imrohoroglu-Huggett-Aiyagari model and Gornemann et al. (2012), who were the first to study monetary policy in the same environment. More recent contributions include McKay and Reis (2016) (impact of automatic stabilizers), McKay et al. (2015) (forward guidance), Bayer et al. (2015) (impact of time-varying income risk), Ravn and Sterk (2013) (increase in uncertainty causes a recession) and Den Haan et al. (2015) (increase in precautionary savings magnifies deflationary recessions). While all these papers address issues that are complimentary to my paper, the price level is shown to be endogenously determined in equilibrium only in my paper. In terms of assumptions, the main reason for this difference is that government bonds are nominal here, whereas they are assumed to be real in the cited work.

set by the central bank. In this world, I show that the steady-state price level is indeterminate as in Sargent and Wallace (1975), and that price stickiness does not change this conclusion, echoing findings in Nakajima and Polemarchakis (2005). I then assume that households additionally also derive utility from holding real bonds, a reduced form of modeling that government bonds have value (which I use interchangeably with constituting net wealth in what follows). In this extended model, I show three layers of determinacy: The steady-state price level is determinate, the economy is locally determinate and I rule out both speculative hyperinflations and hyperdeflations. Whereas price stickiness is irrelevant for the first two layers, it becomes important for the latter one. I also characterize the properties of fiscal and monetary policy rules which imply local determinacy. In Section 3, I go beyond the reduced form analysis and present a heterogeneous agent incomplete markets model of the Bewley-Imrohorglu-Huggett-Aiyagari type, and show that the same arguments and the same results apply, derived using the reduced form analysis.

## 2.1 Indeterminacy when Government Bonds are not Net Wealth

The economy is populated by a large number of identical households with preferences over consumption  $c_t$ , hours worked  $h_t$ , and real money balances  $m_t$

$$\sum_{t=0}^{\infty} \beta^t (u(c_t) - v(h_t) + \mu(m_t)), \quad (1)$$

where  $u$  and  $\chi$  are increasing and concave, and  $v$  is increasing and convex. Households carry nominal money  $M_{t-1}$  and nominal bonds  $B_{t-1}$  into period  $t$  from the previous period and acquire money  $M_t$  and nominal bonds  $B_t$  in that period. The period  $t$  price level is  $P_t$ , so that nominal consumption expenditures are  $P_t c_t$  and real balances  $m_t = M_t/P_t$ . The real wage is  $w_t$ , the time endowment is normalized to one and labor income is  $w_t h_t$ . Households then maximize utility for a budget constraint

$$M_t + B_t = R_t B_{t-1} + M_{t-1} + P_t(1 - \tau)w_t h_t - P_t c_t - T_t + P_t d_t, \quad (2)$$

where  $R_t = 1 + i_t$  is the nominal interest rate on bonds,  $T_t$  are lump-sum taxes,  $\tau$  is a linear tax on labor income and  $d_t$  are real dividend payments. The real interest rate  $1 + r_t = R_t \frac{P_{t-1}}{P_t}$  and the inflation rate is  $1 + \pi_t = \frac{P_t}{P_{t-1}}$ .<sup>3</sup> I now follow the standard approach in the New Keynesian

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<sup>3</sup>It is important to bear in mind that here, as in all of the recent literature in monetary economics, the central bank sets the nominal interest rate on short-term bonds and not money supply, which gives rise to price level indeterminacy in the first place (Sargent and Wallace (1975)). In addition to setting the nominal return on bonds, the central bank can also pay interest rates on reserves. But this does not overcome the indeterminacy issue and only changes the opportunity costs of holding money. I thus omit this complication. For determinacy, setting

literature to add price stickiness to the model.

**Final Good Producer** A competitive representative final goods producer aggregates a continuum of intermediate goods indexed by  $j \in [0, 1]$  and with prices  $p_{jt}$ :

$$Y_t = \left( \int_0^1 y_{jt}^{\frac{\epsilon-1}{\epsilon}} dj \right)^{\frac{\epsilon}{\epsilon-1}},$$

where  $\epsilon > 1$ . Given a level of aggregate demand  $Y_t$ , cost minimization for the final goods producer implies that the demand for the intermediate good  $j$  is given by

$$y_{jt} = y(p_{jt}; P_t, Y_t) = \left( \frac{p_{jt}}{P_t} \right)^{-\epsilon} Y_t, \quad (3)$$

where  $P_t$  is the (equilibrium) price of the final good and can be expressed as

$$P_t = \left( \int_0^1 p_{jt}^{1-\epsilon} dj \right)^{\frac{1}{1-\epsilon}}.$$

**Intermediate good producer** Each intermediate good  $j$  is produced by a monopolistically competitive producer using labor input  $n_{jt}$ . The production technology is linear,

$$y_{jt} = n_{jt}.$$

Intermediate producers hire labor at the nominal wage  $P_t w_t$  in a competitive labor market. With this technology, the real marginal cost of a unit of intermediate good is

$$mc_{jt} = w_t.$$

Each firm chooses its price to maximize profits subject to real price adjustment costs as in Rotemberg (1982),

$$\Phi(p_{jt}; p_{jt-1}, P_{t-1}) Y_t, \quad (4)$$

which depend on the set price  $p_{jt}$ , on the previous period's price  $p_{jt-1}$  and/or the previous period's price level  $P_{t-1}$ . Costs  $\Phi$  are increasing and convex in its first argument and zero in a steady state with price level  $P_t^*$ ,  $\Phi(P_t^*, P_t^*, P_t^*) = 0$  and  $\lim_{p_{jt} \rightarrow \infty} \Phi(p_{jt}; p_{jt-1}, P_{t-1}) = \infty$ . In what follows, I

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money supply and paying interest rates on reserves is equivalent to not setting the nominal return on bonds, and setting money supply instead.



consider two specifications. For tractability, I always start with a simple specification where deviations of the price  $p_{jt}$  from previous period's price level  $P_{t-1}$  are costly,

$$\Phi(p_{jt}; p_{jt-1}, P_{t-1}) = \Phi\left(\frac{p_{jt}}{P_{t-1}} - \pi^{ss}\right), \quad (5)$$

where the steady-state inflation rate is  $\pi^{ss}$ . I also consider a more standard specification,

$$\Phi(p_{jt}, p_{jt-1}; P_t^*) = \Phi\left(\frac{p_{jt}}{p_{jt-1}} - \pi^{ss}\right), \quad (6)$$

which leads to a standard forward-looking New-Keynesian Phillips curve.

Given previous period's individual price  $p_{jt-1}$  and the aggregate state  $(P_t, Y_t, w_t, r_t)$ , the firm chooses this period's price  $p_{jt}$  to maximize the present discounted value of future profits. The firm satisfies all demand  $y(p_{jt}; P_t, Y_t)$  by hiring the necessary amount of labor,

$$n_{jt} = y(p_{jt}; P_t, Y_t) = \left(\frac{p_{jt}}{P_t}\right)^{-\epsilon} Y_t. \quad (7)$$

The firm's pricing problem is

$$V_t(p_{jt-1}) \equiv \max_{p_{jt}} \frac{p_{jt}}{P_t} y(p_{jt}; P_t, Y_t) - w_t y(p_{jt}; P_t, Y_t) - \Phi(p_{jt}; p_{jt-1}, P_{t-1}) Y_t + \beta V_{t+1}(p_{jt}).$$

In equilibrium all firms choose the same price, and thus, aggregate consistency implies  $p_{jt} = P_t$  for all  $j$  and  $t$ . Thus,  $\frac{p_{jt}}{p_{jt-1}} = \frac{P_t}{P_{t-1}} = 1 + \pi_t$  and  $\frac{p_{jt+1}}{p_{jt}} = \frac{P_{t+1}}{P_t} = 1 + \pi_{t+1}$ . The equilibrium real profit of each intermediate goods firm is then

$$d_t = Y_t(1 - w_t).$$

To avoid ruling out some price sequence merely because the associated adjustment costs exhaust the available resources, I assume that the costs are "as-if". They affect firms' pricing decisions, but neither lower firms' profits nor enter the aggregate resource constraint. None of my conclusions are affected by this assumption. The steady-state and the local determinacy analysis are unaffected, since steady-state adjustment costs are zero. Exploding adjustment costs, which eventually exceed available resources, would be an easy way to rule out hyperinflations. As I will explain, there are alternatives for ruling out hyperinflations, and exploding costs are not the reason why price rigidities are helpful in this respect.

Nominal lump-sum taxes  $T_t$  are set to satisfy the government budget constraint at all times

$$T_t := R_t B_{t-1} + P_t(g_t - \tau w_t h_t) + G_t - B_t + M_{t-1} - M_t, \quad (8)$$

where  $g_t$  is real and  $G_t$  is nominal government expenditure. Since  $T_t$  is adjusted to balance the government budget at all times, this shows that the theory presented here is not the Fiscal Theory of the Price Level (FTPL). A more detailed discussion of the differences between my theory and the FTPL using Ljungqvist and Sargent's (2012) "Ten Monetary Doctrines" environment is in Appendix A.I. The transversality condition for bonds,

$$\lim_{T \rightarrow \infty} \beta^T u'(c_T) R_T \frac{B_{T-1}}{P_T} = 0, \quad (9)$$

holds, which is written such that it also applies in the model in Section 2.2 in which government bonds are net wealth.<sup>4</sup> The transversality condition for money is

$$\lim_{T \rightarrow \infty} \beta^T u'(c_T) \frac{M_{T-1}}{P_T} = 0. \quad (10)$$

### 2.1.1 Equilibrium and Steady State

Given sequences of nominal interest rates  $\{R_t\}_{t=0}^{\infty}$ , nominal and real government spending  $\{G_t, g_t\}_{t=0}^{\infty}$ , nominal taxes  $\{T_t\}_{t=0}^{\infty}$  and nominal bonds  $\{B_t\}_{t=0}^{\infty}$ , a *competitive equilibrium* is a sequence of prices  $\{P_t\}_{t=0}^{\infty}$ , money  $\{M_t\}_{t=0}^{\infty}$ , consumption  $\{c_t\}_{t=0}^{\infty}$ , hours  $\{h_t\}_{t=0}^{\infty}$ , real interest rates  $\{r_t\}_{t=1}^{\infty}$  and real wages  $\{w_t\}_{t=0}^{\infty}$  such that:

1. Households choose  $c_t$ ,  $h_t$  and  $M_t$  to maximize utility, given prices  $P_t$ ,  $R_t$ ,  $r_t$  and  $w_t$ .
2. Firm choose prices to maximize profits.
3. The government budget constraint is satisfied.
4. The resource constraint is satisfied:  $Y_t = h_t = c_t + g_t + \frac{G_t}{P_t}$ .

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<sup>4</sup>Kocherlakota and Phelan (1999) (KP) impose an additional limiting condition,  $\lim_{t \rightarrow \infty} (M_t + R_{t+1} B_t) \prod_{s=1}^{t+1} \frac{1}{R_s} = 0$ , on the government budget (equation (25) in KP) to rule out the FTPL. In their complete markets model in which government bonds are not net wealth, this condition is equivalent to the household transversality condition which has to be satisfied in any equilibrium. But the KP condition is not equivalent to the transversality condition when government bonds are net wealth, which, for example, and in contrast to KP, is consistent with a real interest rate below the growth rate of the economy. For example, the KP limiting condition would not be satisfied in a steady state where  $R \equiv 1$  (ZLB), constant  $M, B > 0$ ,  $\tau = g = 0$  and  $G = T$  although the government budget constraint and the TVC (9) are clearly satisfied. Therefore (25) in KP does not apply in the model in which government bonds are net wealth (Section 2.2) and does not indicate whether the FTPL is operating or not.

5. The transversality condition (9) holds.

6. All markets clear.

A *steady state* is a competitive equilibrium where  $c, h, Y, r$  and  $R$  are constant and

$$\frac{B_t}{B_{t-1}} = \frac{M_t}{M_{t-1}} = \frac{T_t}{T_{t-1}} = \frac{G_t}{G_{t-1}} = \frac{P_t}{P_{t-1}} = 1 + \pi^{ss}. \quad (11)$$

I impose here the condition that the real value of bonds,  $B_t/P_t$ , is constant in a steady-state, which I show below in Section 2.2 to be a result. Adding the optimality conditions,

$$1 + r_t = \frac{R_t}{1 + \pi^{ss}} = \frac{R_t}{\frac{P_t}{P_{t-1}}} = \frac{R_t}{\frac{B_t}{B_{t-1}}} = 1/\beta \quad [\text{Bond market}] \quad (12)$$

$$\frac{R_t - 1}{R_t} = \frac{\mu'(M_t/P_t)}{u'(c_t)} \quad [\text{Money market}] \quad (13)$$

$$(1 - \tau)w_t = \frac{v'(h_t)}{u'(c_t)} \quad [\text{Labor Supply}] \quad (14)$$

$$w_t = \frac{\epsilon - 1}{\epsilon} \quad [\text{Pricing}] \quad (15)$$

then characterizes a steady-state.

### 2.1.2 Steady-state price level indeterminacy

The indeterminacy of the steady-state price level is quite evident. Start with a steady-state price sequence  $\{P_t\}_{t=0}^{\infty}$ . Then,  $\{\lambda P_t\}_{t=0}^{\infty}$  is also a steady-state price sequence for every  $\lambda > 0$ . All real variables  $c_t, h_t, w_t, r_t$  and  $Y_t$ , as well as the inflation rate  $\frac{P_t}{P_{t-1}} = \frac{\lambda P_t}{\lambda P_{t-1}}$  are unchanged. Since the nominal interest rate  $R_t$  set by the central bank is unchanged, households demand the same amount of real balances and thus  $\lambda M_t$  nominal money. All firms setting a price  $\lambda P_t$  is profit-maximizing. The government budget constraint is satisfied and the adjustments of lump sum taxes do not affect consumption and hours worked. By construction, these allocations and prices form a new equilibrium for every  $\lambda$ . Predictably (see Nakajima and Polemarchakis (2005)), this conclusion holds if prices are flexible or sticky, simply because a steady-state requires no price-adjustment other than an increase by the steady-state inflation rate  $\pi_{ss}$ .

**Result 1.** *There is a continuum of steady-state price levels. The steady-state price level is indeterminate.*

This confirms the well-known result of price level indeterminacy if the nominal interest is fixed. There is no point in discussing local determinacy in this environment if the steady-state price level

is already indeterminate. The standard approach is to derive properties of the interest rate rule to ensure determinacy. In this paper, I take a different route and argue that a slight modification of the model, such that government bonds are net wealth, implies global determinacy even if the nominal interest rate is fixed.

## 2.2 Steady-State Price Level Determinacy when Government Bonds are Net Wealth

I make just one (reduced-form) change to the previous model, by adding a preference  $\chi(B_t/P_t)$  for holding real bonds so that the utility function is

$$\sum_{t=0}^{\infty} \beta^t \left[ u(c_t) - v(h_t) + \mu \left( \frac{M_t}{P_t} \right) + \chi \left( \frac{B_t}{P_t} \right) \right], \quad (16)$$

where  $\chi(\cdot)$  is increasing and strictly concave. The same definition of equilibrium, households are just maximizing a different objective function, and the same transversality condition apply.<sup>5</sup> I now show that this modification leads to price level determinacy, even if the preference for holding bonds is quite small. The separability between  $\mu$  and  $\chi$  is just for convenience and all results are unchanged if I were to add a non-separable function  $\chi(\frac{M_t}{P_t}, \frac{B_t}{P_t})$  instead. The starting point for establishing price-level determinacy is the first-order condition for bonds which is now

$$u'(c_t) = \chi' \left( \frac{B_t}{P_t} \right) + \beta \frac{R_{t+1}}{1 + \pi_{t+1}} u'(c_{t+1}). \quad (17)$$

In a steady-state

$$u'(c^*) = \chi' \left( \frac{B}{P} \right) + \beta \frac{R}{1 + \pi_{ss}} u'(c^*), \quad (18)$$

where steady-state consumption  $c^* = h^*$  and hours  $h^*$  solve

$$\frac{v'(h^*)}{u'(h^*)} = (1 - \tau) \frac{\epsilon - 1}{\epsilon}, \quad (19)$$

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<sup>5</sup>The transversality condition for bonds (9) can be left unchanged since the FOC for bonds implies that it is equivalent to the transversality condition (see (Kamihigashi, 2006, 2016) for an intuitive treatment)

$$\lim_{T \rightarrow \infty} \beta^T [u'(c_T) - \chi \left( \frac{B_T}{P_T} \right)] \frac{B_T}{P_T} = 0.$$

$\frac{B_t}{B_{t-1}} = 1 + \pi^{ss}$ ,  $B_t = B(1 + \pi^{ss})^t$  and  $P_t = P(1 + \pi^{ss})^t$ , where for simplicity but inconsequentially  $g = G = 0$  (see Appendix A.I). Since consumption,  $c^*$ , and the real interest rate,  $\frac{R}{1+\pi^{ss}}$ , are constant in a steady state, the first-order condition shows that the real value of bonds has to be constant as well, implying that nominal bonds and prices grow at the same rate.

Households are willing to accept a real return  $\frac{R_{ss}}{1+\pi^{ss}}$  lower than  $1/\beta$  even in a steady state, since holding bonds provides some extra utility other than just intertemporal consumption substitution. The higher is this (marginal) utility  $\chi'(\frac{B}{P})$ , the lower is the real interest rate. The magnitude of this extra utility depends on preferences - the function  $\chi$  - and fiscal policy, such that the steady-state real interest rate also depends on fiscal policy, as this policy determines the amount of outstanding government debt.

**Result 2.** *The steady-state real interest rate satisfies  $\frac{R_{ss}}{1+\pi^{ss}} < 1/\beta$ .*

This is the key difference to the previous model in which government bonds are not net wealth. Since bonds are nominal, the first-order condition now depends on the price level, but did not do so above. Indeed, given a steady-state nominal interest rate  $R$  set by monetary policy and a sequence of nominal government bonds growing at a constant rate  $\pi^{ss}$ , the first-order condition can be used to solve for the steady-state price level. Clearly, the same conclusion is reached if  $g$  and  $G$  are not zero.

**Result 3.** *The steady-state inflation rate is equal to the growth rate of nominal government bonds,*

$$1 + \pi^{ss} = \frac{B_t}{B_{t-1}}$$

*The first-order condition for bonds allows solving for the steady-state price level,*

$$P_t^* = P^*(1 + \pi^{ss})^t.$$

The infinite horizon assumption ensures that the first-order condition for bonds can be used in each period to determine the price level. To understand this better, assume a finite horizon such that in the last period, households derive no utility from bonds and the demand for them is therefore zero in this period. There is then no first-order condition for bonds that can be used to determine the price level, so that the logic from Section 2.1 applies which implies price-level indeterminacy. This last-period indeterminacy carries over to previous periods, such that the price level is indeterminate in all periods, as for example in Geanakoplos and Mas-Colell (1989) and Balasko and Cass (1989).

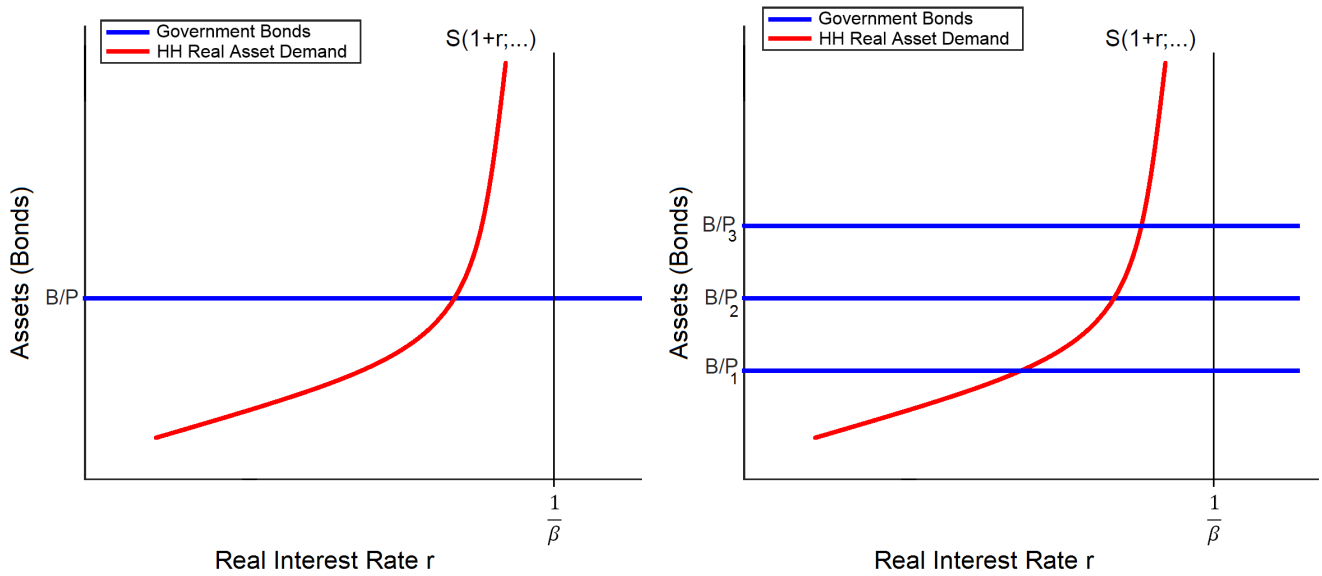


Figure 1: Asset Market when Bonds are Net Wealth

The arguments allow for a graphical analysis which also serves to illustrate the property needed for price-level determinacy, a well-defined demand for bonds. Solving the first order condition (18),

$$u'(c^*) = \chi'(b) + \beta(1+r)u'(c^*), \quad (20)$$

for real bonds  $b$  yields the demand for bonds  $S(1+r, \dots)$  as a function of the real interest rate. The bond market clears if the demand for real bonds  $S(1+r, \dots)$  is equal to the supply of real bonds  $B/P$ , as represented in Figure 1 (left panel). Households' asset demand  $S(1+r, \dots)$  is an upward sloping function of the real interest rate  $1+r$ . Preferences, productivity and other exogenous variables shift the asset demand function. Real asset supply by the government equals  $B/P$ , where  $B$  is nominal bonds and  $P$  is the price level.<sup>6</sup> The equilibrium condition is

$$S(1+r, \dots) = \frac{B}{P}. \quad (21)$$

This is one equation with two unknowns, the real interest rate  $1+r$  and the price level  $P$ , suggesting that a continuum of price levels (associated with a continuum of real interest rates), e.g.  $P_1, P_2, P_3$ , clears the asset market as illustrated in the right panel of Figure 1. But this argument would overlook the fact that  $(1+r_{ss}) = \frac{1+i_{ss}}{1+\pi_{ss}} = \frac{R_t}{\frac{B_t}{B_{t-1}}}$  is set by policy so as to eliminate the real interest rate from the list of unknowns; equation (21) now has just one unknown, the price level  $P^*$ :

$$S\left(\frac{1+i_{ss}}{1+\pi_{ss}}, \dots\right) = \frac{B}{P^*}, \quad (22)$$

<sup>6</sup>With positive inflation rate  $\pi$ , bonds in a steady state at time  $t$  equal  $B(1+\pi)^t$  and the price level equals  $P(1+\pi)^t$  for initial values  $B$  and  $P$ , so that the term  $(1+\pi)^t$  term cancels itself out when computing the real value of bonds  $B/P$ .

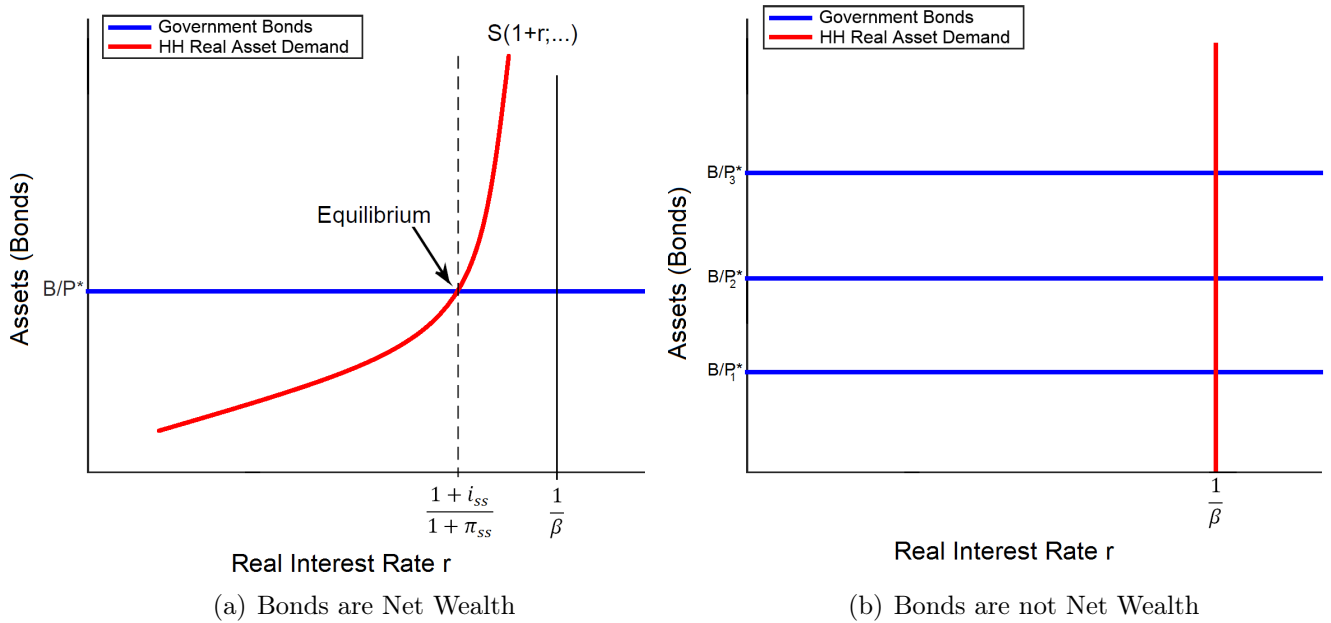


Figure 2: Asset Market Equilibrium: a) Determinacy b) Indeterminacy.

which serves to determine the unique price level, as illustrated in the left panel of Figure 2.

The above reasoning does not extend to the the first model in Section 2.1 in which government bonds are not net wealth. The key implication of this model is that the steady-state real interest rate is determined by the discount factor only,  $(1 + r_{ss})\beta = 1$ , whereas in models in which government bonds are net wealth, the real interest rate depends on virtually all model primitives. Ricardian equivalence implies that the steady-state demand for real bonds is not a well-defined function, but a correspondence and thus, the above arguments cannot be used to determine the price level. The right panel of Figure 2 illustrates the indeterminacy, depicting supply and demand in the asset market as before, but with the difference that now, the steady-state savings curve is a vertical line at the steady-state interest rate  $1/\beta$ . When government bonds are net wealth, it is an upward sloping curve. The vertical asset demand curve when government bonds are not net wealth, reflects the result that the real interest rate is independent of the quantity of real bonds, such that a continuum of price levels, e.g.  $P_1^*, P_2^*, P_3^*$ , satisfies all equilibrium conditions.

The graphical representation also suggests another simple way to understand why the price level is indeterminate in the first model, but is determinate in the second, where the demand for bonds is well defined. The number of endogenous variables exceeds the number of equilibrium conditions by one in the first model, but in the second model in which bonds are net wealth, provides an extra non-redundant equation, the first-order condition (18) for bond-market clearing. At first glance, the argument might seem wrong, since both models feature a consumption Euler equation. That is correct, but in the first model, this equation is redundant, as it merely determines the steady-state real interest rate to be equal to  $1/\beta$  with no role for bonds or the price level whatsoever. In contrast, the bond-FOC in the second model defines a trade-off between the real interest rate

and the value of bonds. This extra and non-redundant FOC, together with  $1 + r_{ss} = \frac{R_t}{\frac{B_t}{B_{t-1}}}$ , then determines the price level as illustrated analytically and graphically above.<sup>7</sup>

### How Monetary and Fiscal Policy Determine the Steady-State Real Interest Rate

A key step in the argument is how monetary and fiscal policy determine the steady-state real interest rate. In both models, a Fisher relationship between the steady-state nominal interest  $i_{ss}$ , real interest rate  $r_{ss}$  and inflation  $\pi_{ss}$  holds:

$$1 + r_{ss} = \frac{1 + i_{ss}}{1 + \pi_{ss}}. \quad (23)$$

Monetary policy sets the steady-state nominal interest rate  $i_{ss}$ . Fiscal policy sets the growth rate of nominal debt ( $B$ ) and adjusts nominal tax revenue ( $T$ ) to balance the government budget. In a steady state, real tax revenue and real government debt are constant, such that the steady-state condition for fiscal policy is that the growth rates of nominal tax revenue, nominal government expenditure and nominal debt all equal the inflation rate (in the absence of economic growth),<sup>8</sup>

$$1 + \pi_{ss} = \frac{B' - B}{B} = \frac{T' - T}{T} = \frac{G' - G}{G}. \quad (24)$$

To be clear about the interpretation of these steady-state conditions: If fiscal policy decides on a 2% nominal growth rate in nominal debt,  $\frac{B' - B}{B}$ , then the steady-state condition for steady-state real government debt to be constant requires that the steady-state inflation rate equals 2% as well. The steady-state further requires that nominal tax revenue  $T$  also grows at 2%. It is important to note that these considerations do not determine the levels of real taxes and real debt except in the sense that these are unchanging over time in a steady state. In particular, the price level has not yet been determined.

Equation (24) means that the inflation rate is set by fiscal policy and is equal to the growth rate of nominal government debt, implying that the equilibrium real interest rate is determined jointly by monetary and fiscal policy.<sup>9</sup> These conclusions about the steady-state inflation rate are

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<sup>7</sup>The Fiscal Theory of the Price Level (FTPL) provides an additional equation as it assumes that the government budget constraint is satisfied by only one price level. In this paper, I pursue a different approach. Instead of using the government budget constraint as an additional binding equation to pin down the price level, I assume that the government balances the budget for all price levels, and show that the asset-market-clearing condition is the necessary additional equation in models in which government bonds have real value.

<sup>8</sup>With real economic growth of rate  $\gamma$ ,  $(1 + \pi_{ss})(1 + \gamma) = \frac{B' - B}{B} = \frac{T' - T}{T} = \frac{G' - G}{G}$ .

<sup>9</sup>Monetary and fiscal policy cannot implement any arbitrary steady-state real interest rate, but only one that is consistent with a steady state. In particular  $\beta(1 + r_{ss}) < 1$ , since otherwise, asset demand would become infinite, also a well-known result in incomplete markets models.



valid, even if monetary policy implements an interest rate rule such as

$$i_{t+1} = \max(\bar{i} + \phi(\pi_t - \pi^*), 0), \quad (25)$$

for an inflation target  $\pi^*$ , an intercept  $\bar{i}$  and  $\phi > 0$ . In this case inflation is still determined by equation (24) and the steady-state nominal interest rate equals<sup>10</sup>

$$i^{ss} = \max(\bar{i} + \phi(\frac{B' - B}{B} - \pi^*), 0). \quad (26)$$

Note that this line of reasoning requires a continuum of potential steady-state real interest rates, and not just one equal to  $1/\beta$  as in models where government bonds are not net wealth. Therefore, this logic for determining the long-run inflation rate does not apply if government bonds are not net wealth.

#### Money Demand, Endogenous Money and Open-Market Operations

Note that for these monetary and fiscal policy choices to be consistent with equilibrium, the central bank has to be ready to satisfy the resulting nominal money demand (13) and the fiscal authority has to set taxes as in (8) to balance the steady-state government budget constraint. After the price level is obtained as solving (18), the first-order condition for money is used only to solve for  $M_t$ , also implying that the real value of money is constant in a steady state,

$$1 + \pi_{ss} = \frac{B' - B}{B} = \frac{M' - M}{M}. \quad (27)$$

This is a steady-state condition which has to hold in any model with government bonds and money. In simple textbook models, the central bank sets money supply and according to the standard interpretation determines the steady-state inflation rate equal to the growth rate of money. If the central bank sets  $M'/M$  to 2 percent then the inflation rate equals 2 percent in a steady state. If in another steady state  $M'/M$  equals 4 percent then the inflation rate equals 4 percent. Fiscal policy then has no choice but to adjust the growth rate of bond supply to be equal to the inflation rate determined by monetary policy. Causality runs from  $M'/M$  to  $\pi_{ss}$  to  $B'/B$ . What is different here is that money is not a policy instrument but adjusts to money demand and price movements, so that causality runs from  $\pi_{ss}$  to  $M'/M$ . Fiscal policy sets  $B'/B$  which is then equal to the steady-state inflation rate, so that causality runs from fiscal policy to inflation. Again that is a comparison of steady states where fiscal policy follows a simple constant debt-growth-rate

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<sup>10</sup>For example if  $\bar{i} = 0.02$ ,  $\phi = 1.5$ , debt grows at  $\frac{B' - B}{B} = 0.02$  and the inflation target  $\pi^* = 0$  then the steady-state inflation is 2% and the nominal interest rate equals  $i^{ss} = 0.02 + 1.5 * 0.02 = 0.05$ . In the (less realistic) case that the inflation target of monetary policy  $\pi^* = 0.04$  exceeds the 2% that follows from fiscal policy, the steady-state nominal interest rate equals  $i^{ss} = \max(0.02 + 1.5(0.02 - 0.04), 0) = 0$  and inflation is still 2%.

policy. If this simple rule sets  $B'/B$  equal to 2 percent then the inflation rate equals 2 percent and setting  $B'/B$  equal to 4 percent implies an inflation rate of 4 percent. In contrast to the FTPL no game between the fiscal and the monetary authority needs to be specified. The result here is just a combination of a steady-state condition and that fiscal policy implements a constant debt growth-rate policy and monetary policy a constant nominal interest rate. These two policies are consistent as all equilibrium conditions are satisfied so that no game needs to be specified to resolve any inconsistencies. The only restriction is that the steady-state real interest rate has to satisfy the equilibrium condition (18), so that steady-state policies have to respect

$$\frac{1}{\beta} - \frac{\chi'(0)}{\beta u'(c^*)} < \frac{1 + i_{ss}}{1 + \pi_{ss}} < \frac{1}{\beta} - \frac{\chi'(\infty)}{\beta u'(c^*)}, \quad (28)$$

which I assume to be the case. Note that, under the standard assumption  $\chi'(\infty) = 0$ , the second inequality reduces to  $\frac{1+i_{ss}}{1+\pi_{ss}} < 1/\beta$  (Result 2).

In this sense fiscal policy can determine the long-run inflation rate (and the price level) through commitment to such a simple rule. Below, I will consider endogenous fiscal rules which respond to movements in prices and output and characterize which rules imply determinacy. Determinacy means that that causality runs from policy to prices and inflation and that reverse causality is ruled out. An example of indeterminacy and thus reverse causality would be a policy which keeps the nominal interest rate and real bonds constant. The steady-state equation  $1 + \pi_{ss} = \frac{B' - B}{B}$  would still hold but would be driven by inflation sunspots fully accommodated by fiscal policy through adjustments in nominal bonds.

The only purpose of adding the demand for money to the model is thus to determine the quantity that the central bank will need to supply in order to implement its nominal interest rate target. Here, the central bank exchanges money for consumption goods. An alternative, which does not alter my conclusions, is to exchange money for bonds in open-market operations. In a steady state, households hold  $B_t - M_t$  nominal bonds and money  $M_t$  which both grow at the same rate as the supply of nominal bonds  $B$  which again equals the inflation rate:

$$1 + \pi_{ss} = \frac{B' - B}{B} = \frac{(B - M)' - (B - M)}{B - M} = \frac{M' - M}{M}. \quad (29)$$

The steady-state price level  $P^*$  is determinate, jointly with money demand  $M$  as the solution to

the first-order conditions for bonds and money

$$u'(c^*) = \chi'\left(\frac{B-M}{P^*}\right) + \beta \frac{R}{1 + \pi_{ss}} u'(c^*), \quad (30)$$

$$\frac{\mu'(M/P^*)}{u'(c^*)} = \frac{R-1}{R}. \quad (31)$$

Again, the amount of money is endogenous and not a policy instrument, since the central bank controls the nominal interest rate. The demand for real money is non-zero since money and bonds are not perfect substitutes in the utility function. If they were perfect substitutes with period  $t$  utility

$$u(c_t) - v(h_t) + \chi\left(\frac{B_t + M_t}{P_t}\right), \quad (32)$$

then the demand for money would be zero for positive nominal interest rates,  $R = 1 + i > 1$ , since bonds return-dominate money. If  $R = 1$  then households are indifferent between holding bonds and money, leaving the ratio of money to bond holdings indeterminate. This indeterminacy is however irrelevant since the total supply of nominal assets still grows at rate  $\frac{B'-B}{B}$  equal to the inflation rate and the price level is determined using the first-order condition of bonds. I obtain

**Result 4.** *The steady-state price level is determinate when government bonds are net wealth.*

Consider a simple example in which  $u(c) = \log(c)$ ,  $v(h) = \rho^h \frac{h^{1+\theta}}{1+\theta}$ ,  $\mu(M/P) = \rho^\mu \log(M/P)$  and  $\chi(B/P) = \rho^\chi \log(B/P)$  and both the nominal interest rate  $R$  and the amount of government bonds  $B$  are constant. The steady-state price level  $P^*$  then solves

$$\frac{1}{c^*} = \frac{\rho^\chi}{B/P^*} + \frac{\beta R}{c^*}, \quad (33)$$

with an explicit solution

$$P^* = (1 - \beta R) \frac{B}{c^* \rho^\chi} \quad (34)$$

and

$$c^* = h^* = \left( \frac{(1 - \tau)(\epsilon - 1)}{\rho^h \epsilon} \right)^{\frac{1}{\theta-1}}. \quad (35)$$

The demand for real money is  $m^* = \frac{R \rho^\mu c^*}{R-1}$  so that the central bank has to provide cash  $M^* = m^* P^*$ .

An equivalent way to determine the price level, is to first solve for the steady-state demand

for real bonds from the FOC,

$$S(1+r) = \frac{\rho^x c^*}{1 - \beta R}, \quad (36)$$

and use the asset-market clearing condition

$$S(1+r) = \frac{\rho^x c^*}{1 - \beta R} = \frac{B}{P^*} \quad (37)$$

to solve for  $P^*$ , the same expression as in (34).

The steady-state price level  $P^*$  depends on policy as expected. Tighter monetary policy - a higher nominal interest rate - lowers the price level, whereas an expansionary fiscal policy - a higher debt level - increases it since, as explained above,  $\beta R = \beta R / (1 + \pi_{ss}) < 1$ . A larger demand for bonds - a higher  $\rho^x$  - increases the real value of bonds and thus leads to a lower price level.

If government debt is growing at rate  $\frac{B_t}{B_{t-1}} = 1 + \pi^{ss}$  not necessarily one, then

$$P^* = \left(1 - \frac{\beta R}{1 + \pi^{ss}}\right) \frac{B}{c \rho^x} \quad (38)$$

and the steady-state price level, taking into account the steady-state inflation rate  $1 + \pi_{ss}$ , is

$$P_t^* = P^* (1 + \pi_{ss})^t. \quad (39)$$

A positive inflation rate lowers, for a fixed nominal interest rate, the real return on bonds. Households are willing to accept this lower real return, only if the utility of holding bonds increases. Concavity implies that that the real value of bonds has to fall, that is, the price level must increase.

### Adding Capital

Adding investment  $I_t$  and capital  $K_t$  to the model does not change the conclusion of this Section. To see this, assume a production function  $Y_t = F(K_t, h_t)$  and that capital accumulates as

$$K_{t+1} = F(K_t, h_t) + (1 - \delta)K_t - c_t, \quad (40)$$

for a depreciation rate  $\delta$ . The household budget constraint changes to

$$M_t + B_t + P_t K_{t+1} = R_t B_{t-1} + P_t (1 + r_t^k) K_t + M_{t-1} + P_t (1 - \tau) w_t h_t - P_t (c_t + I_t) - T_t + P_t d_t, \quad (41)$$

for a real return on capital  $1 + r_t^k$  and yields the first-order conditions

$$u'(c_t) = \beta(1 + r_{t+1}^k)u'(c_{t+1}), \quad (42)$$

$$F_K(K_t, h_t) + (1 - \delta) = 1 + r_t^k, \quad (43)$$

for which the first takes into account the fact that capital does not provide any extra services as do bonds. In a steady state, these FOCs simplify to

$$(1 + r^k) = 1/\beta, \quad (44)$$

$$F_K(K^*, h^*) + (1 - \delta) = 1/\beta \quad (45)$$

and hours  $h^*$  now solve

$$\frac{v'(h^*)}{u'(h^*)} = (1 - \tau) \frac{\epsilon - 1}{\epsilon} F_h(K^*, h^*), \quad (46)$$

which allows solving for the steady-state level of capital  $K^*$ , investment  $I^*$  and hours  $h^*$  as a function of parameters such as  $\beta$  and  $\delta$ , but independently of the price level. The steady-state price level  $P^*$  still solves

$$\frac{1}{c^*} = \frac{\rho^x}{B/P^*} + \frac{\beta R}{c^*}, \quad (47)$$

but taking into account that consumption  $c^* = F(K^*, h^*) - (1 - \delta)K^*$ , which in this simple setup is again independent of the price level. This independence of the price level simplifies the exposition but is irrelevant for determinacy. What is relevant is that the FOC for bonds depends on the price level and can be used to solve for  $P^*$ .

This completes the first part of showing determinacy in a steady state. This is a fundamental, but not the only step in establishing global determinacy. A proof also requires showing determinacy when the economy is not in a steady state, either because policy is not in a steady state or because non-steady-state beliefs move the economy away from steady state. In the next step, I therefore turn to local determinacy.

## 2.3 Local Determinacy

The steady state is locally unique, if there is no other equilibrium in which all variables are within a neighborhood of their steady-state values. To check local determinacy of the steady state, it is sufficient to check it for the log-linearized economy (Woodford, 2003). Assessing local determinacy

in the linearized model is quite simple, as it amounts to checking the properties of the eigenvalues of the transition matrix.

I first assume that fiscal policy follows a stationary exogenous policy and sets a fixed amount of nominal government debt, and that monetary policy sets a constant nominal interest rate, and I allow for feedback policy rules below. For the ease of exposition, I first assume that the costs of setting a price  $p_{jt}$  is in terms of the deviation from the previous period's price level  $P_{t-1}$ ,  $\Phi(\frac{p_{jt}}{P_{t-1}})$ , but I obtain the same conclusion when I consider the standard New Keynesian Phillips curve below. I also assume that government spending  $g = G = 0$  and that utility derived from bonds and money holdings is separable, and I show in the appendix that a mild parameter restriction delivers the same result.

The (around the steady-state) linearized equations are

$$\kappa \hat{Y}_t = (\hat{p}_t - \hat{p}_{t-1}) \quad (48)$$

$$\hat{Y}_t = \tilde{\beta} E_t \hat{Y}_{t+1} - \psi \hat{p}_t - \sigma (\hat{i}_{t+1} - (E_t \hat{p}_{t+1} - \hat{p}_t)), \quad (49)$$

where  $\hat{Y}_t = \log(Y_t/Y^*)$  is the log deviation of output from steady state (also equal to the output gap, since the natural rate of output is constant in the absence of real disturbances),  $\hat{p}_t = \log(P_t/P_t^*)$  for the steady-state price  $P_t^* = P^*(1 + \pi_{ss})^t$  and  $\hat{i}_{t+1} = \log\left(\frac{R_{t+1}}{R_{ss}}\right)$ . The parameters  $\sigma \equiv \frac{\tilde{\beta} u'(c^*)}{-u''(c^*)c^*} > 0$ ,  $\psi = \frac{-\sigma b^* \chi''(b^*)}{u'(c^*)\tilde{\beta}} > 0$ ,  $\kappa = \frac{\epsilon\varphi}{\Phi''(0)} > 0$  is the slope of the Phillips curve,  $\hat{w}_t = \varphi \hat{Y}_t$ ,  $b^* = B/P^*$  is the steady-state real value of bonds and  $\tilde{\beta} = \beta(1 + r_{ss}) < 1$  which is less than one when government bonds are net wealth. This is one difference between a model with or without valued government bonds. The other more important difference is that households derive utility from holding bonds, which adds the term  $-\psi \hat{p}_t$  to the Euler equation. The interpretation is that a higher real value of bonds requires a higher real interest rate, so that  $\psi > 0$ . A higher price level  $\hat{p}_t > 0$  lowers the real value of bonds, leading to a negative coefficient  $-\psi < 0$ .

Substitution of (48) into (49) and using  $\hat{i}_t = 0$  yields a second-order difference equation in  $\hat{p}_t$ ,

$$E_t \hat{p}_{t+1} = \underbrace{\frac{1 + \tilde{\beta} + \kappa(\sigma + \psi)}{\tilde{\beta} + \kappa\sigma}}_{=:a_1} \hat{p}_t + \underbrace{\frac{-1}{\tilde{\beta} + \kappa\sigma}}_{=:a_0} \hat{p}_{t-1}, \quad (50)$$

and in matrix form

$$\begin{pmatrix} E_t p_{t+1} \\ p_t \end{pmatrix} = \begin{pmatrix} \frac{1 + \tilde{\beta} + \kappa(\sigma + \psi)}{\tilde{\beta} + \kappa\sigma} & \frac{-1}{\tilde{\beta} + \kappa\sigma} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_t \\ p_{t-1} \end{pmatrix}$$

with one eigenvalue larger and one smaller than one,<sup>11</sup>

$$\lambda_{1,2} = \frac{a_1 \pm \sqrt{a_1^2 + 4a_0}}{2}. \quad (51)$$

A two-dimensional model with one predetermined endogenous state variable,  $\hat{p}_{t-1}$ , and one non-predetermined one,  $\hat{p}_t$  is determinate if one eigenvalue is outside and the other inside the unit circle.<sup>12</sup>

If  $g, G > 0$  and the utility derived from money and bonds is not separable, so that  $\epsilon_m := \frac{\sigma \chi_{mb} m^*}{u'(c^*) \tilde{\beta}} \neq 0$ , I reach the same conclusion if  $1 + \epsilon_m \eta_y > 0$ , where log-linearized money-demand is

$$\hat{m}_t = \eta_y \hat{Y}_t - \eta_i \hat{i}_{t+1} \quad (52)$$

for an output elasticity  $\eta_y$  and an interest-rate elasticity  $\eta_i > 0$ . I therefore obtain

**Result 5.** *The steady state is locally determinate when government bonds are net wealth ( $\psi > 0$ ).*

I also obtain the same conclusion if I assume that the cost of setting a price  $p_{jt}$  is in terms of the deviation from the firm's own previous period price  $p_{jt-1}$ ,  $\Phi\left(\frac{p_{jt}}{p_{jt-1}}\right)$ , which leads to the standard linearized New Keynesian Phillips curve,

$$(\hat{p}_t - \hat{p}_{t-1}) = \kappa \hat{Y}_t + \beta E_t(\hat{p}_{t+1} - \hat{p}_t). \quad (53)$$

In matrix form, the economy is described through

$$\begin{pmatrix} E_t p_{t+1} \\ p_t \\ E_t Y_{t+1} \end{pmatrix} = \frac{1}{\beta} \begin{pmatrix} 1 + \beta & -1 & -\kappa \\ \beta & 0 & 0 \\ \frac{\beta\psi - \sigma}{\tilde{\beta}} & \frac{\sigma}{\tilde{\beta}} & \frac{\beta + \sigma\kappa}{\tilde{\beta}} \end{pmatrix} \begin{pmatrix} p_t \\ p_{t-1} \\ Y_t \end{pmatrix}.$$

This three-dimensional model with one predetermined endogenous state variable,  $\hat{p}_{t-1}$ , and two non-predetermined ones,  $\hat{Y}_t$  and  $\hat{p}_t$ , is determinate if two eigenvalues are outside and the other inside the unit circle. Applying the characterization in Woodford (2003, Proposition C.2 in Appendix C) shows that this is the case. The economy is locally determinate.

<sup>11</sup>If government bonds have no extra value,  $\psi = 0$ , then  $\lambda_1 = 1$  and the second eigenvalue  $\lambda_2 = \frac{1}{1 + \kappa\sigma} < 1$ , since  $\psi = 0$  implies  $\tilde{\beta} = 1$ , as in the model in which government bonds are not an argument of the utility function.

<sup>12</sup>In the boundary case  $\psi = 0$ , the eigenvalue is one and the linearized model cannot be used to assess local determinacy. The previous analysis implies however, that the steady state is locally indeterminate if  $\psi = 0$ .

## 2.4 Local Determinacy: Policy Rules

The previous section establishes price-level determinacy when monetary and fiscal policy are constant. I now extend the analysis and allow for policies responding to deviations of prices and output from their respective steady-state values and establish conditions for the policy rules which deliver local determinacy. Interestingly, in models in which government bonds have a real value, policy rules do not overcome indeterminacy, but instead may induce it.

I assume an interest rate rule

$$\hat{i}_{t+1} = \varphi^i \hat{p}_t \quad (54)$$

where  $\varphi^i$  is the response to price deviations. Since prices are the state-variables, it is convenient to specify the rule in terms of prices and not in terms of inflation. Additionally adding a response to output deviations is typically important for quantitative assessments, but less relevant for local determinacy. I report these results below, but start with the simpler rule of price deviations only, as this conveys the intuition more effectively. Similarly, I assume a rule for nominal debt

$$\hat{B}_t = \varphi^B \hat{p}_t \quad (55)$$

with a price response  $\varphi^B$ . Taxes are still set to balance the government budget constraint. The model is now described through

$$\begin{pmatrix} E_t p_{t+1} \\ p_t \end{pmatrix} = \begin{pmatrix} \frac{1+\tilde{\beta}+\kappa(\sigma+\tilde{\psi})}{\tilde{\beta}+\kappa\sigma} & \frac{-1}{\tilde{\beta}+\kappa\sigma} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_t \\ p_{t-1} \end{pmatrix},$$

which differs from the previous model as

$$\tilde{\psi} = \psi(1 - \varphi^B) + \sigma\varphi^i$$

replaces  $\psi$ . The local determinacy condition with policy response functions follows immediately from the previous analysis with constant policies:

**Result 6.** *The steady state is locally determinate when government bonds have value ( $\psi > 0$ ) and*

$$\tilde{\psi} = \psi(1 - \varphi^B) + \sigma\varphi^i > 0. \quad (56)$$

It is easy to see the conditions for local determinacy for two special cases. When monetary policy is constant ( $\varphi^i = 0$ ) and only fiscal policy responds, or when fiscal policy is constant



( $\varphi^B = 0$ ) and only monetary policy is responding

**Result 7 (Special Cases).** *If fiscal policy is constant ( $\varphi^B = 0$ ), the economy is locally determinate if government bonds are net wealth ( $\psi > 0$ ) and*

$$\varphi^i \geq 0. \quad (57)$$

*If monetary policy is constant ( $\varphi^i = 0$ ), the economy is locally determinate if government bonds are net wealth ( $\psi > 0$ ) and*

$$\varphi^B < 1. \quad (58)$$

The intuition as to why determinacy can be ensured only if parameter restrictions on fiscal policy are imposed, is straightforward. Consider debt policy first and suppose that  $P_t > P^*$ . If debt policy does not respond to prices,  $\varphi^B = 0$ ,  $P_t > P^*$  implies a fall in the real value of debt, so that households require a lower real interest rate or equivalently, a higher inflation rate ( $P_{t+1} > P_t$ ) to absorb less real debt. That is, prices move further away from the steady state with dynamics governed by the eigenvalue larger than one. In contrast, if debt policy is aggressive,  $\varphi^B > 1$ , this reasoning does not work, since in this case,  $P_t > P^*$  implies a policy-induced increase in the real value of debt. Households then require a higher real interest rate to be willing to absorb more real debt. If the nominal interest rate does not respond to prices, this requires a fall in prices,  $P_{t+1} < P_t$ , that is, the eigenvalue is smaller than one.

Note that there are no restrictions on monetary policy. Even a negative response to price hikes,  $\varphi^i \geq -1/\sigma$ , would still imply determinacy. This is not surprising, since a constant nominal interest rate already implies determinacy. A higher price  $P_t > P^*$  implies a fall in real debt and requires a decrease in the real interest rate, which is equivalent to  $P_{t+1} > P_t$ , that is, the eigenvalue is larger than one. If the nominal interest rate increases in response to higher prices, then an even larger increase in prices is necessary to lower the real interest rate, that is, the eigenvalue becomes even larger, again implying determinacy.

Note that the intuition when fiscal policy induces local determinacy independently of how monetary policy is conducted did not rely on many model details and can thus be expected to be more generally valid. As I show in Section 3, this is the case in richer incomplete markets models. Result 6 derives the criterion that combines conditions for monetary and fiscal policy. A key number in this endeavour is naturally the intertemporal elasticity of substitution, as this allows comparing the demand effects of monetary and fiscal policy. An expansionary fiscal policy ( $\varphi^B > 1$ ) now induces determinacy, but monetary policy has to be sufficiently contractionary, that

is,  $\varphi^i$  has to be sufficiently high,

$$\varphi^i > \frac{\psi(\varphi^B - 1)}{\sigma}. \quad (59)$$

The intuition behind this result builds on the above explanations for the determinacy of monetary and fiscal policy. Suppose again  $P_t > P^*$  and  $\varphi^B > 1$ . Again, real debt increases so that households require a higher real interest rate to be willing to absorb more real debt. If monetary policy were passive, this would require a fall in  $P_{t+1}$  (relative to  $P_t$ ). But if the nominal interest rate increases more than the required real interest rate, then  $P_t = P_{t+1}$  would imply that the real interest rate is too high. As a consequence, the next period's price  $P_{t+1}$  must again increase to bring the real interest rate down. That is prices move away from the steady-state as the relevant eigenvalue is larger than one.

For policy rules which respond not only to price but also to output deviations,

$$\hat{i}_{t+1} = \varphi_p^i \hat{p}_t + \varphi_y^i \hat{Y}_t, \quad (60)$$

$$\hat{B}_t = \varphi_p^B \hat{p}_t + \varphi_y^B \hat{Y}_t \quad (61)$$

and natural sign restrictions -  $\varphi_y^i > 0$  (higher interest rate in a boom) and  $\varphi_y^B < 0$  (expansionary fiscal policy in a recession) the same condition,

$$\psi(1 - \varphi_p^B) + \sigma\varphi_p^i > 0, \quad (62)$$

ensures local determinacy.<sup>13</sup>

For a cost function  $\Phi(p_{jt}/p_{jt-1})$  which induces a New Keynesian Phillips curve, and under the same sign restrictions the local determinacy criterion is again unchanged,

$$\psi(1 - \varphi_p^B) + \sigma\varphi_p^i > 0. \quad (63)$$

## 2.5 Hyperinflations and Hyperdeflations

As a last step in establishing global determinacy, it is necessary to show that prices are in steady state when monetary and fiscal policy are stationary, which is equivalent to excluding inflationary and deflationary price spirals. Note that this argument rules out hyperinflations only if nominal variables are growing at a constant rate, but not when policy decides on an explosive path of

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<sup>13</sup>The derivation in the appendix provides a sufficient condition without sign restrictions, which rules out all eigenvalues having a modulus larger than one (meaning that the system is not stable), so that the economy is locally determinate.

nominal variables. The theory thus allows for policy-induced hyperinflations, but not for belief-induced ones. It is important to point out that depending on the fiscal and monetary policy implemented, the economy may experience a hyperinflation, a deflation or price stability, but in each scenario, the price level is determinate.

The starting point is the non-linear pricing equation

$$\left(\frac{v'(h_t)}{u'(c_t)} - \frac{\epsilon - 1}{\epsilon}\right) = \frac{P_t}{\epsilon P_{t-1}} \Phi'\left(\frac{P_t}{P_{t-1}} - \pi_{ss}\right), \quad (64)$$

which defines output as a function  $Y_t(\frac{P_t}{P_{t-1}})$  so that consumption

$$c_t = C\left(\frac{P_t}{P_{t-1}}, P_t\right) = Y_t\left(\frac{P_t}{P_{t-1}}\right) - g - \frac{G_t}{P_t} \quad (65)$$

is a function of inflation  $\frac{P_t}{P_{t-1}}$  and the price  $P_t$ . Substituting this into the FOC for bonds

$$\frac{u'(c_t)}{P_t} = \frac{\chi'\left(\frac{B_t}{P_t}\right)}{P_t} + \beta R \frac{u'(c_{t+1})}{P_{t+1}} \quad (66)$$

allows describing the nonlinear equilibrium dynamics of prices through two functions  $H$  and  $\Gamma$ , which relate the price in period  $t + 1$  to the prices in periods  $t$  and  $t - 1$ ,

$$H(\tilde{P}_t, \tilde{P}_{t-1}) := \frac{u'(C(\frac{\tilde{P}_t}{\tilde{P}_{t-1}}, \tilde{P}_t))}{\tilde{P}_t} - \frac{\chi'\left(\frac{B}{\tilde{P}_t}\right)}{\tilde{P}_t} = \frac{\beta R}{1 + \pi_{ss}} \frac{u'(C(\frac{\tilde{P}_{t+1}}{\tilde{P}_t}, \tilde{P}_{t+1}))}{\tilde{P}_{t+1}} =: \Gamma(\tilde{P}_{t+1}, \tilde{P}_t), \quad (67)$$

where the detrended price is defined as

$$\tilde{P}_t = \frac{P_t}{(1 + \pi_{ss})^t}, \quad (68)$$

so that  $\frac{B_t}{P_t}$  can be written as  $\frac{B}{\tilde{P}_t}$  and  $\tilde{P} = P^*$  in steady state.

I now show that (67) is satisfied only if  $\tilde{P} = P^*$ , that is neither  $\tilde{P}_t > P^*$  nor  $\tilde{P}_t < P^*$  can constitute an equilibrium. My first step is to prove that for  $\tilde{P}_t > P^*$  to be an equilibrium price, prices have to be monotonically increasing and that for  $\tilde{P}_t < P^*$  to be an equilibrium price, prices have to be monotonically decreasing. These two results are not surprising, given that the local determinacy analysis implies that the model has sufficiently many eigenvalues with a modulus exceeding one. In a second step, I then show that both cases violate the equilibrium conditions.

### 2.5.1 Ruling out Speculative Hyperinflations

First assume that  $\tilde{P}_t > P^*$  is an equilibrium price. Suppose that  $\tilde{P}_{t-1} \leq \tilde{P}_t$  (I consider  $\tilde{P}_{t-1} > \tilde{P}_t$  below) then using that  $\frac{\chi'(\frac{B}{\tilde{P}_t})}{\tilde{P}_t} > \frac{\chi'(\frac{B}{P^*})}{\tilde{P}_t} = (1 - \frac{\beta R}{1 + \pi_{ss}}) \frac{u'(C(1, P^*))}{\tilde{P}_t}$ ,

$$H(\tilde{P}_t, \tilde{P}_{t-1}) - \Gamma(\tilde{P}_t, \tilde{P}_t) \leq H(\tilde{P}_t, \tilde{P}_t) - \Gamma(\tilde{P}_t, \tilde{P}_t) \quad (69)$$

$$= \frac{u'(C(\frac{\tilde{P}_t}{\tilde{P}_t}, \tilde{P}_t))}{\tilde{P}_t} (1 - \frac{\beta R}{1 + \pi_{ss}}) - \frac{\chi'(\frac{B}{\tilde{P}_t})}{\tilde{P}_t} \quad (70)$$

$$< \frac{u'(C(1, \tilde{P}_t))}{\tilde{P}_t} (1 - \frac{\beta R}{1 + \pi_{ss}}) - \frac{\chi'(\frac{B}{P^*})}{\tilde{P}_t} \quad (71)$$

$$= \frac{u'(C(1, \tilde{P}_t))}{\tilde{P}_t} (1 - \frac{\beta R}{1 + \pi_{ss}}) - \frac{u'(C(\pi_{ss}, P^*))}{\tilde{P}_t} (1 - \frac{\beta R}{1 + \pi_{ss}}) \quad (72)$$

$$= (\frac{u'(C(1, \tilde{P}_t))}{\tilde{P}_t} - \frac{u'(C(1, P^*))}{\tilde{P}_t}) (1 - \frac{\beta R}{1 + \pi_{ss}}) \quad (73)$$

$$\leq 0. \quad (74)$$

Since  $\Gamma(\tilde{P}_{t+1}, \tilde{P}_t)$  is falling in  $\tilde{P}_{t+1}$ , I obtain  $\tilde{P}_{t+1} > \tilde{P}_t$ . Iterating this argument delivers a monotonically increasing sequence of prices. The uniqueness of the steady state implies that this sequence is unbounded, since a monotone bounded function converges. The intuition is straightforward. A price level  $\tilde{P}_t$  higher than  $P^*$  implies a lower real value of government bonds and therefore a higher marginal utility  $\chi'$ . Households are then willing to accept a lower real interest rate, requiring an even higher price  $\tilde{P}_{t+1} > \tilde{P}_t$  since the nominal interest rate is constant here. It is easy to show that this argument generalizes to environments with non-separable utility functions or when the value of bonds is stochastic, as long as the trade-off between real interest rates and the amount of bonds is still operative: A higher real value of bonds requires a higher real interest rate.

The literature (Obstfeld and Rogoff, 1983, 2017) provides a very elegant solution aimed at ruling out hyper-inflations in monetary models which can be adopted here as well.<sup>14</sup> A slight adaptation is necessary, since the central bank controls the money supply in Obstfeld and Rogoff (1983, 2017), whereas it sets the nominal interest here, so that their arguments on money demand cannot be applied here. Suppose the government provides some fractional real backing of debt/transfers, that is the government trades bonds/transfers for real consumption at a (very high) price  $\bar{P}$ . Prices cannot then rise above  $\bar{P}$ , since households always trade with the government and not at a price higher than  $\bar{P}$ . But then, the increasing price sequence must unravel backward, simply because the price level cannot converge to infinity as an equilibrium would require.<sup>15</sup>

<sup>14</sup>Wallace (1981) made the same argument.

<sup>15</sup>Strictly speaking, the arguments rely on perfect foresight, but they are sufficient to rule out sunspots as well. Indeed the same arguments show that  $E_t \tilde{P}_{t+1}$  converges to infinity and that again this cannot be an equilibrium.

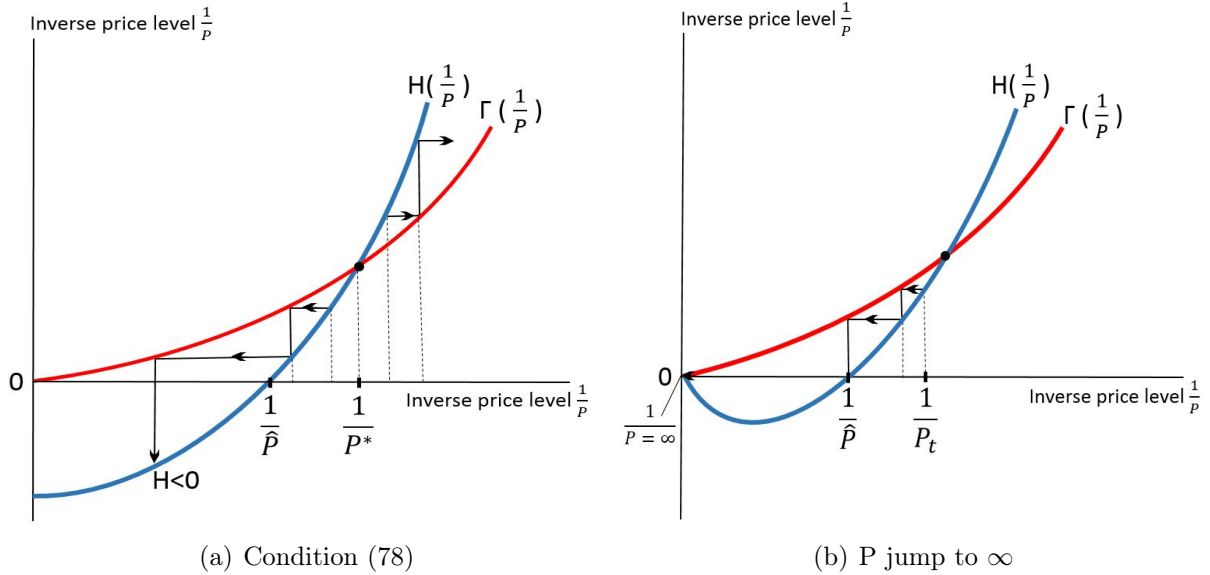


Figure 3: Dynamics of the Price Level

**Result 8.** (Obstfeld and Rogoff, 1983, 2017) *backing of nominal government obligations at a sufficiently high price  $\tilde{P}$  rules out hyperinflations.*

Hyperinflations can also be ruled out without such a backing. This is where sticky prices become relevant. To better understand the role of price rigidities, I first assume that prices are flexible, which allows for a diagrammatic analysis as in Obstfeld and Rogoff (1983). With flexible prices, the functions  $H$  and  $\Gamma$  simplify to

$$H(\tilde{P}_t) = \frac{u'(C(\tilde{P}_t))}{\tilde{P}_t} - \frac{\chi'(\frac{B}{\tilde{P}_t})}{\tilde{P}_t} \quad (75)$$

$$\Gamma(\tilde{P}_{t+1}) = \frac{\beta R}{1 + \pi_{ss}} \frac{u'(C(\tilde{P}_{t+1}))}{\tilde{P}_{t+1}} \quad (76)$$

Figure 3 illustrates the dynamics of the price level,  $\tilde{P}_t$ , which is implied by

$$H(\tilde{P}_t) = \Gamma(\tilde{P}_{t+1}), \quad (77)$$

which defines  $\tilde{P}_{t+1}$  as a function of  $\tilde{P}_t$ . To render the comparison with Obstfeld and Rogoff (1983) easier, the diagram is presented in terms of the inverse price level,  $1/P$  and I assume a constant level of debt so that  $\tilde{P}_t = P_t$  in the diagram. Since both functions  $H$  and  $\Gamma$  are downward sloping in  $P$ , they are upward sloping in  $1/P$  and from the left panel of the diagram, it is apparent that

there is a unique steady state at  $P^*$ , since  $H$  is steeper than  $\Gamma$ . There is a  $\hat{P}$  such that

$$H(\hat{P}) = 0 \text{ and } H(\tilde{P}_t) < 0 \text{ for all } \tilde{P}_t > \hat{P}. \quad (78)$$

A negative value  $H < 0$  does not enable iterating on (77), since  $\Gamma(\tilde{P}) > 0$  for all finite  $\tilde{P}$ . The only remaining possibility is that  $\tilde{P}$  jumps to infinity.

The proof for ruling out hyperinflations therefore boils down, in the absence of a government backing, to excluding an infinite price level,  $P = \infty$ . If

$$\lim_{\tilde{P} \rightarrow \infty} \frac{\chi'(\frac{B}{\tilde{P}_t})}{\tilde{P}_t} > 0 \quad (79)$$

then  $P = \infty$  cannot be an equilibrium, since

$$\lim_{\tilde{P} \rightarrow \infty} H(\tilde{P}) = - \lim_{\tilde{P} \rightarrow \infty} \frac{\chi'(\frac{B}{\tilde{P}_t})}{\tilde{P}_t} < 0 = \lim_{\tilde{P} \rightarrow \infty} \Gamma(\tilde{P}). \quad (80)$$

This is what the left panel of Figure 3 shows. There is no equilibrium at  $P = \infty$  ( $1/P = 0$ ), since the curve  $H$  is strictly below  $\Gamma$  when  $1/P \rightarrow 0$ ,  $H(P = \infty) < \Gamma(P = \infty) = 0$ .

**Result 9.** *Condition (79) rules out hyperinflations.*

In the context of their monetary models, Obstfeld and Rogoff (1983, 2017) consider condition (79) to be implausible, because it implies money to be an absolute necessity, although in fact money is thought of as merely reducing some trading frictions. Here however, this condition refers to the value of bonds which arises not because they reduce trading frictions, but, for example, because bonds enable consumption smoothing in response to uninsurable idiosyncratic risk. Heterogeneous agent incomplete markets economies such as Huggett (1993) constitute a model class in which bonds have value and which I consider in Section 3. In such models, condition (79) seems less implausible.

But fortunately, this question need not to be resolved here, since price rigidities offer an alternative which requires much weaker assumptions. Panel b) of Figure 3 shows the dynamics when condition (79) is not met and  $H(P = \infty) = \Gamma(P = \infty) = 0$ . The previous argument for ruling out  $P = \infty$  no longer applies and the diagram suggests that prices could now jump to infinity once they have reached the level  $\hat{P}$ .

Whereas all the previous arguments work if prices are flexible or sticky, it is straightforward to show that  $P = \infty$  is not possible in models with price rigidity. The argument is quite simple. In contrast to flexible price economies, prices are set by profit-maximizing firms in models with

sticky prices and are not just the magic outcome of a fictitious auctioneer. This simple insight is all that is needed. It is never optimal for a firm to set  $P = \infty$ . The pricing equation (64) is not satisfied, since  $\frac{P_t}{\epsilon P_{t-1}} \Phi'(\frac{P_t}{P_{t-1}} - \pi_{ss}) = \infty$  if  $\frac{P_t}{P_{t-1}} = \infty$ .

Setting prices to infinity would also incur large or even infinite price adjustment costs (Rotemberg, 1982), which is clearly not an equilibrium. A different way of modeling price rigidities, but drawing the same conclusion, is to assume Calvo price setting. The proportion of non-adjusting firms would charge a finite price and would absorb all demand, whereas the “ $P = \infty$  firms” would face no demand, implying that setting  $P = \infty$  is not optimal and that the output-weighted price level would be finite. Note that the only assumption needed is that there be a  $\hat{P}$  such that  $H(\hat{P}) = 0$  and  $H(\tilde{P}_t) < 0$  for all  $\tilde{P}_t > \hat{P}$ . This is an assumption which seems quite weak, certainly much more so than condition (79), and which is satisfied and not questioned in Obstfeld and Rogoff (1983, 2017). The only role of price rigidities is to rule out the price jumping to infinity. This is the case for the two leading models of price stickiness in the literature - price adjustment costs and Calvo price setting - and is a reasonable property of any mechanism that imposes price rigidities on firms. I obtain:

**Result 10.** *Price rigidities and condition (78) rule out hyperinflations.*

This completes the proof, since  $\tilde{P}_{t-1} > \tilde{P}_t > P^*$  can be ruled out as well, using the same arguments as above and showing that the price sequence increases when moving back in time, ( $\tilde{P}_{t-1} < \tilde{P}_{t-2} < \tilde{P}_{t-3} < \dots$ ). If we assume this is not so and that  $\tilde{P}_{t-2} < \tilde{P}_{t-1}$ , the above arguments imply that  $\tilde{P}_t > \tilde{P}_{t-1}$ , a contradiction. Iterating backwards yields a monotonically increasing function which again by means of the same arguments cannot be an equilibrium.<sup>16</sup>

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<sup>16</sup>The dynamics of the price level out of steady state can be characterized as monotone here, in contrast to the analysis of models with money in Woodford (1994). The reason why such a concise characterization is available is that I can use the result that at equilibrium, a higher level of real bonds requires a higher real interest. This is the only property used and as long as this is satisfied, the results in this Section hold more generally, for example for non-separable preferences.

## 2.5.2 Ruling Out Speculative Hyperdeflations

Now assume that  $\tilde{P}_t < P^*$  is an equilibrium price. Suppose that  $\tilde{P}_{t-1} \geq \tilde{P}_t$  then, using  $\frac{\chi'(\frac{B}{\tilde{P}_t})}{\tilde{P}_t} < \frac{\chi'(\frac{B}{P^*})}{P^*} = (1 - \frac{\beta R}{1 + \pi_{ss}}) \frac{u'(C(1, P^*))}{P^*}$ ,

$$H(\tilde{P}_t, \tilde{P}_{t-1}) - \Gamma(\tilde{P}_t, \tilde{P}_t) \geq H(\tilde{P}_t, \tilde{P}_t) - \Gamma(\tilde{P}_t, \tilde{P}_t) \quad (81)$$

$$= \frac{u'(C(\frac{\tilde{P}_t}{\tilde{P}_t}, \tilde{P}_t))}{\tilde{P}_t} (1 - \frac{\beta R}{1 + \pi_{ss}}) - \frac{\chi'(\frac{B}{\tilde{P}_t})}{\tilde{P}_t} \quad (82)$$

$$> \frac{u'(C(1, \tilde{P}_t))}{\tilde{P}_t} (1 - \frac{\beta R}{1 + \pi_{ss}}) - \frac{\chi'(\frac{B}{P^*})}{P^*} \quad (83)$$

$$= \frac{u'(C(1, \tilde{P}_t))}{\tilde{P}_t} (1 - \frac{\beta R}{1 + \pi_{ss}}) - \frac{u'(C(\pi_{ss}, P^*))}{P^*} (1 - \frac{\beta R}{1 + \pi_{ss}}) \quad (84)$$

$$= (\frac{u'(C(1, \tilde{P}_t))}{\tilde{P}_t} - \frac{u'(C(1, P^*))}{P^*}) (1 - \frac{\beta R}{1 + \pi_{ss}}) \quad (85)$$

$$\geq 0. \quad (86)$$

Since  $\Gamma(\tilde{P}_{t+1}, \tilde{P}_t)$  declines in  $\tilde{P}_{t+1}$ , I obtain  $\tilde{P}_{t+1} < \tilde{P}_t$ . Iterating this argument delivers a monotonically decreasing sequence of prices. Steady state uniqueness of the steady state implies that this sequence is not bounded by a number larger than zero, since a monotone bounded function converges. The intuition is again straightforward. A price level  $\tilde{P}_t$  lower than  $P^*$  implies a higher real value of government bonds and therefore, a lower marginal utility  $\chi'$ . Households then demand higher real interests, requiring an even lower price  $\tilde{P}_{t+1} < \tilde{P}_t$ , since the nominal interest rate is constant here.

I now show that such speculative deflationary spirals do not constitute an equilibrium. A simple proof is to recognize that ultimately the price level will be so low that government demand  $g + \frac{G}{P}$  exceeds output, which clearly cannot be an equilibrium. An equivalent way to think of this result is that the government sets a nominal anchor  $G$ , which serves as a lower bound for nominal aggregate demand and thus rules out expectations of prices converging to zero. But, as noted above, when discussing local determinacy above, indeterminacy may arise even if government bonds have value if fiscal policy is too accommodative, for example altering government spending more than one-for-one with prices. The same applies here. If  $G$  decreases more than prices,  $\frac{G}{P}$  is not explosive and this simple proof does not then rule out hyperdeflations.

**Result 11.** *Hyperdeflations are ruled out if nominal government spending  $G > 0$ .*

The proof in Aiyagari (1994a) that the steady-state real interest rate is strictly smaller than  $1/\beta$  offers another approach. Aiyagari (1994a) considers a heterogeneous agent incomplete markets



model, but his arguments can be used in the reduced form model, as well as subject to one small caveat which I discuss below. In the incomplete markets model in Section 3, the arguments apply one-for-one without such a caveat.

I now show that the arguments in Aiyagari (1994a) can be adopted here under the reasonable assumption that

$$\lim_{P \rightarrow 0} \chi' \left( \frac{B}{P} \right) = 0. \quad (87)$$

The FOC for bonds (66) is equivalent to

$$1 = \frac{\chi' \left( \frac{B_t}{P_t} \right)}{u'(c_t)} + \beta R \frac{P_t}{P_{t+1}} \frac{u'(c_{t+1})}{u'(c_t)}, \quad (88)$$

so that (87) implies for  $\lim_{t \rightarrow \infty} \tilde{P}_t = 0$  and thus  $\lim_{t \rightarrow \infty} B_t/P_t = \infty$ ,

$$\lim_{t \rightarrow \infty} \beta R \frac{P_t}{P_{t+1}} \frac{u'(c_{t+1})}{u'(c_t)} = 1. \quad (89)$$

Assumption (87) thus delivers the result, which also holds in all incomplete markets models, that asset demand is infinite if the real interest rate is equal to  $1/\beta$ . Thus, for a sufficiently large  $T$

$$\bar{\gamma}_t := \prod_{j=T}^{T+t-1} \frac{1}{1+r_{j+1}} \approx \beta^t \frac{u'(c_{T+t})}{u'(c_T)}, \quad (90)$$

so that the transversality condition implies, since the capital stock  $K_t$  is bounded,

$$\lim_{t \rightarrow \infty} \bar{\gamma}_t \left( \frac{B_{T+t}}{P_{T+t}} + K_t \right) = \lim_{t \rightarrow \infty} \bar{\gamma}_t \frac{B_{T+t}}{P_{T+t}} = \lim_{t \rightarrow \infty} \beta^t u'(c_{T+t}) \frac{B_{T+t-1}}{P_{T+t}} = 0. \quad (91)$$

The government budget constraint (for  $G = 0$ ) implies

$$\frac{B_t}{P_t} \leq \frac{\frac{B_{t+1}}{P_{t+1}}}{1+r_t} + \frac{Y_t - g + S^M}{1+r_t}. \quad (92)$$

which uses the fact that period  $t$  tax revenue is bounded from above by  $Y_t$  and that seignorage is bounded by some number  $S^M$  in a hyperdeflation. This is a result in Aiyagari (1994a), based on a standard Laffer curve logic and the fact that lump-sum taxes are zero or must be very small (less than the lowest labor income). This is where the caveat applies. In a representative agent economy, lump-sum taxes could be large enough to fully cover all interest rate payments. Thus, in this section, it is an assumption, but only a result in Section 3, that tax revenue is bounded.

Combining the government budget inequality (92) with (91) shows that  $K_t + \frac{B_t}{P_t}$  is bounded, contradicting that  $\frac{B_t}{P_t} \rightarrow \infty$  in a hyperdeflation.

**Result 12.** *Hyperdeflations are ruled out if government tax revenue is bounded.*

A standard proof in the literature shows that deflationary price paths violate the transversality condition. The same arguments apply here if  $\frac{R}{1+\pi_{ss}} \leq 1$ . It follows from (66) that  $\frac{u'(c_t)}{P_t}$  eventually grows at rate  $1/(\beta R)$ , so that  $u'(c_t)\frac{B_t}{P_t}$  grows at rate  $\frac{1+\pi_{ss}}{\beta R} \geq \frac{1}{\beta}$ , implying that the transversality condition

$$\lim_{t \rightarrow \infty} \beta^t u'(c_t) \frac{B_t}{P_t} > 0 \quad (93)$$

is violated. Mechanically, it is clear that the transversality condition is not violated if  $\frac{1+\pi_{ss}}{\beta R} < \frac{1}{\beta}$ . Economically, this is related to a point made by Obstfeld and Rogoff (1986) for instance. A hyperdeflation allows for a utility gain from selling a bond worth \$1 and rolling it over indefinitely.<sup>17</sup> As Obstfeld and Rogoff (1986) point out, selling a bond worth \$1 must be feasible for the model to work. Here this is the case only if rolling over this \$1 forever is feasible, requiring the asset position to remain nonnegative. Selling one \$1 lowers the asset position by  $R^t$  within  $t$  periods, which keeps assets positive, since  $B$  grows at rate  $1 + \pi_{ss} \geq R$ . If, on the other hand,  $1 + \pi_{ss} < R$  real debt would ultimately become negative and then converge to minus infinity at rate  $1/\beta$ , this would violate the transversality condition or equivalently the No-Ponzi condition. Therefore selling \$1 and rolling it over is not feasible if  $1 + \pi_{ss} < R$ .<sup>18</sup>

**Result 13.** *If  $\frac{R}{1+\pi_{ss}} \leq 1$ , hyperdeflations can be ruled out using a transversality condition argument.*

In combination, the results for steady-state determinacy, local determinacy and hyperinflations and hyperdeflations imply that the only price sequence which forms an equilibrium is one in which the price is constant and equal to the steady-state price level  $P^*$ , that is, the price at time  $t$  equals  $P^*(1 + \pi_{ss})^t$ :

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<sup>17</sup>Iterating on the first-order condition for bonds yields

$$\frac{u'(c_t)}{P_t} = \sum_{s=t}^{\infty} (\beta R)^{s-t} \frac{\chi'(\tilde{P}_s)}{P_s} + \lim_{T \rightarrow \infty} (\beta R)^T \frac{u'(c_{t+T})}{P_{t+T}},$$

for which the latter limit term is positive as shown above, so that the LHS is larger than the sum on the RHS.

<sup>18</sup>The condition  $1 + \pi_{ss} \geq 1$  in the monetary model of Obstfeld and Rogoff (1986) is a special case, since money has a nominal return  $R = 1$ .

**Proposition 1 (Global Determinacy).** *The sufficient conditions in the text imply a globally determinate and unique price level:*

$$\begin{aligned}\tilde{P}_t &= P^* \\ P_t &= P^*(1 + \pi_{ss})^t.\end{aligned}$$

### 3 A Heterogeneous Agent Incomplete Markets Model

In this Section, I develop a model in which markets are incomplete as in Aiyagari (1994b, 1995) and which, in reduced form, resembles the model of Section 2.2, so that the determinacy arguments carry over from the reduced form model to my incomplete markets model. A key difference is that now, the value of bonds and of money arise endogenously and are not assumed to be an argument of the utility function. I follow Lucas and Stokey (1987) to generate a value for money, which requires assumptions on the timing of events and on payment arrangements. The main challenge is to integrate them with the trading restrictions of incomplete markets models, such that the properties of both model classes are preserved. The main changes to the previous model then concern the household sector which I describe in detail first. The labor market and firms' price setting are unaffected, and I therefore do not repeat those model features here.

#### 3.1 The Model

To obtain a reduced form that resembles the model of Section 2.2, I follow Lucas and Stokey (1987) and assume that households derive utility from three different consumption goods, and leisure:

$$U = E_0 \sum_{t=0}^{\infty} \beta^t \left[ u^k(c_t^k) + u^b(c_t^b) + u^m(c_t^m) - v(h_t) \right] \quad (94)$$

where  $c_t^k$  are “capital goods”,  $c_t^b$  are “bond goods” and  $c_t^m$  are “money goods”. The only difference between these goods, besides different utility functions, is in terms of which specific period  $t$  assets can be used to purchase them. All assets including capital can be used to acquire capital goods, bonds and money can be used to buy bond goods and only money can be used to purchase money goods.

Agents' labor productivity  $\{e_t\}_{t=0}^{\infty}$  is stochastic and characterized by an  $N$ -state Markov chain that can take on values  $e_t \in \mathcal{E} = \{e_1, \dots, e_N\}$  with transition probability characterized by  $p(e'|e)$  and  $\int e = 1$ . Agents rent out their labor,  $eh$ , to firms for a real wage  $w_t$ . The resource constraint

is

$$c_t^k + c_t^b + c_t^m + g_t + \frac{G_t}{P_t} = F(K_t, h_t) + (1 - \delta)K_t - I_t, \quad (95)$$

where  $K_t$  and  $I_t$  are capital and investment. As in Lucas and Stokey (1987), sellers receive payment at the beginning of the following period, implying that all three goods sell at the same nominal price.

Household  $i$ 's beginning-of-period- $t$  capital is  $k_{it-1}$ , nominal bond holdings is  $P_{t-1}b_{it-1}$  and nominal money is  $P_{t-1}m_{it-1}$ , so that the real value of bonds and money in terms of period  $t - 1$  prices are  $b_{it-1}$  and  $m_{it-1}$ , respectively. Before period- $t$  productivity  $e_t$  is known, asset markets open and the household purchases capital goods  $c_t^k \geq 0$ , capital  $k_{it}$ , bonds  $b_{it}$  and money  $m_{it}$ , such that the credit constraint for a credit limit  $\bar{b}$

$$k_{it} + b_{it} \geq -\bar{b} \quad (96)$$

and the budget constraint

$$\begin{aligned} & P_t k_{it} + P_t b_{it} + P_t m_{it} R_{t+1} + P_t c_t^k \\ = & P_t (1 + r_t^k) k_{it-1} + R_t P_{t-1} b_{it-1} + P_{t-1} (m_{it-1} - c_{it-1}^m) - P_{t-1} c_{it-1}^b \\ + & P_{t-1} d_{it-1} + P_{t-1} e_{t-1} h_{t-1} (1 - \tau) w_{t-1} - T_t \end{aligned} \quad (97)$$

are satisfied. Instead of engaging in open market operations, the central bank here loans money  $m_{it}$  to households at the nominal interest rate  $R_{t+1}$ , so that households bear cost  $m_{it} R_{t+1}$  to have  $m_{it}$  units of real money available.<sup>19</sup> As I show below, this leads to the same money demand function as in the reduced form model. The first two terms on the left (first line) are the nominal value of capital and bonds, and the fourth term is the payment for capital goods. The first two terms on the right (second line) are the income of assets bought in the preceding period, the third term is unspent cash and the fourth term is payments for bond goods. The first term on the third line is dividends, the second term is after-tax receipts from labor income and the last term is lump-sum taxes.

After this trading, each household splits into a money good shopper and a Huggett consumer. The shopper must use money  $m_{it}$  to buy money goods that satisfy a cash-in-advance constraint

$$c_{it}^m \leq m_{it} \quad (98)$$

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<sup>19</sup>Equivalently, one can assume that the household acquires  $m_{it}$ , but has only  $m_{it}/R_{it+1}$  available for spending.

and the Huggett consumer learns the income shock  $e_{it}$ . While the labor income  $P_t e_{it} h_t (1 - \tau) w_t$  is paid at the beginning of the next period when sellers receive the payment for goods sales, consumption of  $c_t^b$  by the Huggett consumer results in invoices which are settled during next period. Following the ideas in Lucas and Stockey (1987), it is the possibility to use invoices and not cash to pay for bond goods that distinguishes those goods from money goods. But in Lucas and Stockey (1987), households are not restricted in their use of invoices to pay for non-cash goods. Such an assumption would circumvent the credit-constraints which are essential in incomplete markets models. To reconcile both models, I allow for payment by invoices, but impose an upper bound which ensures that households can definitely pay back, using labor income and bonds,

$$P_t c_t^b \leq P_t e_{it} h_t (1 - \tau) w_t + P_t (b_{it} - \bar{b}) R_{t+1}. \quad (99)$$

Note that this is not a “bond-in-advance” constraint, since there is no exchange of goods for bonds here, while the cash-in-advance constraint involves an exchange of goods for money. Instead, more bonds relax the invoice constraint, which is equivalent to a collateral constraint, and payments are settled at the beginning of the next period. There are various other ways to allow households a more generous credit line, which would certainly be quantitatively appealing but less tractable and would not alter my conclusions. The only relevant aspect here is that the invoice constraint implies that the bond goods consumption  $c^b$  depends on the amount of bonds. Also, none of the conclusions would change if I added capital  $k_{it}$  to the invoice constraint. Capital and bonds would then be perfect substitutes and, as I explain below, the determinacy results are unaffected.

To formulate the household choice problem, it is convenient to follow the timing assumptions just described and write the household’s problem recursively, both for the beginning of the period before the income shock  $e$  has been realized and only  $e_{-1}$  is known,

$$V(k, b, m; \Omega) = \max_{c^k \geq 0, k', b', m'} u^k(c^k) + u^m(c^m) - v(h) + \sum_{e \in \mathcal{E}} p(e|e_{-1}) V^1(k', b', m'; \Omega) \quad (100)$$

subject to

$$\begin{aligned} & Pk' + Pb' + PR'm' + Pc^k \\ &= P(1 + r^k)k + P_{-1}Rb + P_{-1}(m - c_{-1}^m) - P_{-1}c_{-1}^b + P_{-1}d_{-1} + P_{-1}e_{-1}h_{-1}(1 - \tau)w_{-1} - T \\ & c^m \leq m' \\ & k' + b' \geq -\bar{b} \end{aligned}$$

and for the decision after the income shock  $e$  has been learned and expectations are now about

the next periods' shock  $e'$

$$V^1(k', b', m'; \Omega) = \max_{c^b \geq 0} u^b(c^b) + \beta \sum_{e \in \mathcal{E}} p(e'|e) V(k', b', m'; \Omega') \quad (101)$$

$$\begin{aligned} \text{subj. to} \quad & Pc^b \leq Peh(1 - \tau)w + P(b' - \bar{b})R' \\ & \Omega' = \mathcal{T}(\Omega), \end{aligned}$$

where  $\Omega(k, b, m, e) \in \mathcal{M}$  is the distribution on the space  $X = \mathcal{K} \times \mathcal{B} \times \mathcal{M} \times \mathcal{E}$ , agents' capital holdings  $k \in \mathcal{K}$ , bond holdings  $b \in \mathcal{B}$ , money holdings  $m \in \mathcal{M}$  and labor productivity  $e \in \mathcal{E}$ , across the population, which, together with the policy variables, determine the equilibrium prices.  $\mathcal{T}$  is an equilibrium object that specifies the evolution of the distribution  $\Omega$ .

The household decision problem yields optimal consumption choices  $c^b(b', e)$ , due to the invoice constraint, and  $c^m(m') = m'$ , due to the binding cash-in-advance constraint, so that utility can be rewritten as

$$U = E_0 \sum_{t=0}^{\infty} \beta^t \left[ \underbrace{u^k(c_t^k) - v(h_t) + \underbrace{u^m(m_{it})}_{=: \mu(m_{it})} + \underbrace{E_{t-1} u^b(c^b(b_{it}, e))}_{=: \bar{u}^b(b_{it})}}_{=: \Upsilon_t(k, b, m, k', b', m')} \right], \quad (102)$$

resembling utility in the reduced form model. Denote by  $\Upsilon_t(k, b, m, k', b', m')$  the indirect period  $t$  utility of household  $i$  as a function of period  $t$  state variables  $(k, b, m)$  and the next period's state variables  $(k', b', m')$ , so that  $U = E_0 \sum_{t=0}^{\infty} \beta^t \Upsilon_t(k_{it-1}, b_{it-1}, m_{it-1}, k_{it}, b_{it}, m_{it})$  and the transversality condition for bonds can be written compactly as

$$\lim_{T \rightarrow \infty} E_0 \left[ -\beta^T \frac{\partial \Upsilon_T}{\partial b_{iT}} b_{iT} \right] = 0, \quad (103)$$

and those for capital and money are

$$\begin{aligned} \lim_{T \rightarrow \infty} E_0 \left[ -\beta^T \frac{\partial \Upsilon_T}{\partial k_{iT}} k_{iT} \right] &= 0, \\ \lim_{T \rightarrow \infty} E_0 \left[ -\beta^T \frac{\partial \Upsilon_T}{\partial m_{iT}} m_{iT} \right] &= 0. \end{aligned} \quad (104)$$

Market clearing conditions for capital and bonds are

$$K_t = \int k_{it} di \quad [\text{Capital}] \quad (105)$$

$$\frac{B_t}{P_t} = \int b_{it} di \quad [\text{Bonds}] \quad (106)$$

and the central bank provides money

$$M_t = \int P_t m_{it} di. \quad (107)$$

As in the reduced form model above, the central bank has to provide just enough money to achieve the interest rate target.<sup>20</sup> Depending on the price level  $P$  and real money demand, this may require expanding or shrinking the money stock. The government budget constraint is as before

$$T_t := R_t B_{t-1} + P_t g + G_t - B_t + M_{t-1} - M_t R_{t+1} - \tau P_{t-1} w_{t-1} h_{t-1}. \quad (108)$$

In an incomplete markets model, lump sum taxes are bounded from below by the lowest income  $e_1 w_t h_t$ , requiring  $\tau$  to be large enough so that  $T_t \leq 0$  is a transfer and not a tax.

A *competitive equilibrium* is then a sequence of prices  $P_t$ ,  $R_t$ ,  $r_t$  and  $w_t$ , taxes  $T_t$ , bonds  $B_t$ , money  $M_t$ , and value functions  $V$  and  $V^1$  with policy functions  $c^k$ ,  $c^b$ ,  $c^m$ ,  $k'$ ,  $b'$ ,  $m'$  such that:

1. Households maximize utility, taking prices and policies as given.
2. Firms maximize profits.
3. The government budget constraint is satisfied.
4. The resource constraint is satisfied.
5. The transversality conditions for bonds (103) and capital and money (104) hold.
6. All markets clear.

As in the reduced form model, the FOC for bonds is the key equation for price level determinacy. To understand the choice of bonds, it is useful to start with the choice of  $c^b(b', e)$  after the income shock  $e$  has realized, which maximizes

$$u^b(c^b(b', e)) + \beta E_t u^k(c_{it+1}^k), \quad (109)$$

taking into account the income and the budget constraints. Let  $\tilde{\chi}(b', e)$  be the indirect utility function of this maximization problem, such that bond goods satisfy

$$u_c^b(c_{it}^b(b_{it}, e_{it})) \leq \frac{\beta P_t}{P_{t+1}} E_t u_c^k(c_{it+1}^k) \quad (110)$$

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<sup>20</sup>If I considered a cashless economy (Woodford, 2003) instead, I could dispense altogether with cash goods, money entering the utility function and cash-in-advance constraints. I do not choose this route, since adding (a small amount of) money makes it easier to understand how the central bank can set the nominal interest rate.

with equality if the invoice constraint (99) is not binding and

$$\frac{\partial \tilde{\chi}(b_t, e)}{\partial b'} = \begin{cases} 0 & \text{if (99) is not binding} \\ R_{t+1}[u_c^b(c_{it}^b) - \beta \frac{P_t}{P_{t+1}} E_t u_c^k(c_{it+1}^k)] > 0 & \text{if (99) is binding} \end{cases}$$

The marginal value of relaxing the invoice constraint, which is the derivative of the indirect utility function, is zero if the constraint is not binding and positive if it is binding. In the latter case, more bonds allows the household to acquire more bond goods  $c^b$  to increase utility by  $R_{t+1}u_c^b$ , which outweighs the loss  $\beta R_{t+1} \frac{P_t}{P_{t+1}} E_t u_c^k$  incurred in the next period when the invoice is settled. Defining the expected value of relaxing the constraint as

$$\chi(b') = E_{t-1} \tilde{\chi}(b', e_t), \quad (111)$$

the FOC of bonds is (the appendix provides a detailed derivation)

$$u_c^k(c_{it}^k) \geq \frac{\partial \chi(b_{it})}{\partial b'} + E_{t-1} \frac{\beta R_{t+1}}{P_{t+1}/P_t} u_c^k(c_{it+1}^k), \quad (112)$$

with equality if  $k' + b' > -\bar{b}$  and where  $E_{t-1}$  indicates that households do not know the period  $t$  income shock when making their asset decisions. Bonds provide additional utility in states in which the invoice constraint is binding, which adds the term  $\frac{\partial \chi(b_{it})}{\partial b'}$  to the FOC. This again resembles the reduced form model, but now this is the FOC of a household which in addition faces idiosyncratic income risk.

The capital decision is described through the FOC

$$u_c^k(c_{it}^k) \geq E_{t-1} \beta (1 + r_{t+1}^k) u_c^k(c_{it+1}^k) \quad (113)$$

with equality if the credit constraint is not binding,  $k' + b' > -\bar{b}$ . Money demand is determined in a standard manner through

$$\frac{u^m(c_{it}^m)}{u^k(c_{it}^k)} = R_{t+1}. \quad (114)$$

The similarity between the FOC for bonds in the reduced form and the incomplete markets model suggests that the determinacy results carry over from the first to the latter model. Building on this similarity and the previous analysis, I now discuss determinacy in the same order as before, starting with the steady state.



## 3.2 Steady State

The arguments in the reduced form and in the incomplete markets model are basically identical.<sup>21</sup> The steady-state demand for bonds is now the aggregation of all individual asset demands  $b_i$

$$S(1+r) = \int b_i di, \quad (115)$$

the Fisher relation (23) between the steady-state nominal interest  $i_{ss}$ , the real interest rate  $r_{ss}$  and inflation  $\pi_{ss}$  holds:

$$1+r_{ss} = \frac{1+i_{ss}}{1+\pi_{ss}}, \quad (116)$$

and the steady-state inflation rate satisfies (24),

$$1+\pi_{ss} = \frac{T' - T}{T} = \frac{B' - B}{B} = \frac{G' - G}{G}. \quad (117)$$

The steady-state price level  $P^*$  is again determined through the asset-market-clearing equation (22),

$$S\left(\frac{1+i_{ss}}{1+\pi_{ss}}\right) = \frac{B}{P^*}. \quad (118)$$

The central bank has to provide nominal money

$$M_t = P^* \int m_i di \quad (119)$$

to implement the nominal interest rate  $R = 1 + i_{ss}$ , where  $\int m_i di$  is the steady-state aggregate real money demand. Steady-state capital  $K_{ss}$  is determined by combining market clearing and the FOC for firms, such that  $1 + r^k$  is the market clearing price and solves

$$F_K\left(\int k_i, h\right) + (1 - \delta) = 1 + r^k, \quad (120)$$

where individual demand  $k_i$  depends on  $1 + r^k$  and aggregates to  $K_{ss}$ ,

$$K_{ss} = \int k_i di, \quad (121)$$

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<sup>21</sup>For a textbook treatment of incomplete markets models and their steady states, see Ljungqvist and Sargent (2012).

so that both  $r^k$  and  $K_{ss}$  are determinate. If capital and bonds were perfect substitutes, for example if both enter the invoice constraint (99) symmetrically, then total household demand for  $k + b$  is

$$S(1 + r) = \int (b_i + k_i) di, \quad (122)$$

and the price level is determined through

$$S(1 + r) = K_{ss} + \frac{B}{P^*}, \quad (123)$$

where  $K_{ss}$  satisfies  $F_K(K_{ss}, h) + (1 - \delta) = 1 + r^k = 1 + r_{ss}$ .

The assumption here is that households exchange consumption goods for money. If one assumes instead that households obtain money through open market operations, then  $P^*$  and  $M$  solve

$$\frac{M}{P^*} = \int m_i \quad (124)$$

$$K_{ss} + \frac{B - M}{P^*} = \int (b_i + k_i) di, \quad (125)$$

again two equations with two unknowns. Clearly, equation (124) alone does not determine the price level, since the central bank sets  $i$  and not  $M$ , which adjusts endogenously to satisfy the quantity equation. It is the asset market-clearing condition that determines the price level, which depends on the fiscal variables  $G$ ,  $T$  and  $B$  and on  $i$ .

As in the reduced form model, the incomplete markets model delivers an aggregate demand for bonds which depends on the real interest rate. And as in the reduced form model, the associated asset-market-clearing condition delivers a non-redundant additional equation which then determines the price level. Furthermore both models allow for the same graphical representation of price level determination through panel a) of Figure 2.<sup>22</sup>

### 3.3 Local Determinacy

While the analysis of the steady state in the incomplete markets model is straightforward to analyze, dynamics outside of steady states are quite intractable, and I therefore consider an economy without capital and assume a natural borrowing limit and log utility or CRRA with risk aversion larger than one, so that the FOC for bonds holds with equality for all households (see Bhandari,

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<sup>22</sup>Incomplete markets models may have multiple *real* equilibria, which is orthogonal to the nominal multiplicity issues here. The results in this paper show that that each real equilibrium is associated with a unique price. That is adding a nominal element to the model does not add any multiplicities. Of course, adding nominal elements cannot remove any real multiplicities.

Evans, Golosov, and Sargent, 2015, 2016a,b, 2017, for the same simplification),

$$u_c^k(c_{it}^k) = \frac{\partial \chi(b', e)}{\partial b'} + E_{t-1} \frac{\beta R_{t+1}}{P_{t+1}/P_t} u_c^k(c_{it+1}^k), \quad (126)$$

which allows for a linear approximation around the median household's values as

$$\tilde{\sigma}^{-1} \hat{c}_{it}^k = \chi \hat{b}_{it} - \tilde{\beta} (\hat{i}_{t+1} - (E_{t-1} \hat{p}_{t+1} - \hat{p}_t)) + \tilde{\sigma}^{-1} \tilde{\beta} E_{t-1} \hat{c}_{it+1}^k, \quad (127)$$

where  $\chi = \frac{-\tilde{\sigma} \chi_{bb} \bar{b}}{u_c^k} > 0$ ,  $\tilde{\sigma} \equiv \frac{u_c^k(c^{k,*})}{-u_{cc}^k(c^{k,*}) c^{k,*}} > 0$ , and  $\tilde{\beta} = \beta(1 + r_{ss})$ . Aggregating across households and assuming that capital good consumption changes are proportional to output changes<sup>23</sup> yields for  $\sigma := \tilde{\beta} \tilde{\sigma}$

$$\hat{Y}_t = \tilde{\beta} E_t \hat{Y}_{t+1} + \underbrace{\tilde{\sigma} \chi (\hat{B}_t - \hat{p}_t)}_{=: \psi(\hat{B} - \hat{p}_t)} - \sigma (\hat{i}_{t+1} - (E_t \hat{p}_{t+1} - \hat{p}_t)), \quad (128)$$

which is identical to (49) in the reduced form model for  $\hat{B} = 0$ . Combining this again with the Phillips curve (48) and using  $\hat{i} = 0$  yields the same second-order difference equation for prices as in (50) in the reduced form model,

$$E_t \hat{p}_{t+1} = \frac{1 + \tilde{\beta} + \kappa(\sigma + \psi)}{\tilde{\beta} + \kappa\sigma} \hat{p}_t + \frac{-1}{\tilde{\beta} + \kappa\sigma} \hat{p}_{t-1}. \quad (129)$$

I therefore come here to the same conclusion that the steady-state is locally determinate, since government bonds are indeed net wealth ( $\psi > 0$ ). Allowing for policy reaction functions also yields the same conclusion as before, that the economy is locally determinate if

$$\tilde{\psi} = \psi(1 - \varphi^B) + \sigma \varphi^i > 0, \quad (130)$$

where  $\varphi^B$  is the bond and  $\varphi^i$  is the interest rate response to prices.

### 3.4 Ruling Out Hyperinflations and Hyperdeflations

As a last step in establishing global determinacy, I ruled out hyperinflations and hyperdeflations in the reduced-form model. I now show that the same arguments apply subject to a caveat. In the reduced form model, I showed first that for  $\tilde{P}_t > P^*$  to be an equilibrium price, prices have to increase monotonically and that for  $\tilde{P}_t < P^*$  to be an equilibrium price, prices have

<sup>23</sup>The appendix shows that the results become slightly stronger if one takes into account that a lower price and thus a higher real value of bonds leads to higher  $c^b$  consumption and thus lower  $c^k$  consumption

to decrease monotonically. Prices moving away from the steady-state is a consequence of local determinacy, since the eigenvalue larger than one eventually dominates the dynamics. As the incomplete markets model is also locally determinate, the same result can be expected to apply here. But the intractability of outside steady-state dynamics in incomplete markets models does not allow the provision of conditions on primitives that would rule out cycles, that is, prices moving away from the steady-state level, but neither converging to infinity nor to zero but instead forming a cycle.

However, inspecting the derivations in the reduced form model shows that the only necessary property is that a lower real value of bonds requires a lower real interest rate. A price  $P_t > P^*$  leads to a lower real value of bonds and thus requires a lower real interest rate, which in turn requires prices to increase even more,  $P_{t+1} > P_t$ . Similarly, a price  $P_t < P^*$  leads to a higher real value of bonds and thus requires a higher real interest rate, which in turn requires prices to decrease even more,  $P_{t+1} < P_t$ . In numerical applications, the aggregate savings curve is typically increasing in the real interest rate, as in the left panel of Figure 2, but this cannot be ensured theoretically. The remaining main arguments for ruling out hyperinflations and hyperdeflations still hold, as I argue below.

### 3.4.1 Hyperinflations

In the reduced form model, I provided three different proofs for ruling out hyperinflation - the Obstfeld and Rogoff (1983, 2017) type backing of nominal government, Condition (79) holds and sticky prices - which I show to carry over to the incomplete markets model as well.

The Obstfeld and Rogoff (1983, 2017) argument obviously holds, since trading bonds/transfers for real consumption at a (very high) price  $\bar{P}$  prevents prices from rising above  $\bar{P}$ , since households always trade with the government and not at a price higher than  $\bar{P}$ , thus eliminating any hyperinflation.

The second proof used condition (79),

$$\lim_{P \rightarrow \infty} \frac{\chi'(\frac{B}{P})}{P} > 0. \quad (131)$$

It is easy to provide a sufficient condition for this condition to hold here as well. Assume that a state  $e^0 = 0$  exists, in which households have no labor income, so that all consumption  $c^b$  needs to be financed through savings, and that the utility of zero consumption is minus infinity (satisfied by the standard assumption of log utility and CRRA with risk aversion large than one). In state

$$e^0 = 0,$$

$$\lim_{P \rightarrow \infty} \frac{u_c^b(0 + \frac{B}{P})}{P} > 0, \quad (132)$$

which holds, for example, if  $u^b(c) = \log(c)$ . Furthermore  $\frac{\partial c^b}{\partial b} = R$ , so that  $\lim_{P \rightarrow \infty} \frac{u_c^b(0 + \frac{B}{P})}{P} \frac{\partial c^b}{\partial b} > 0$ , implying Condition (131), since  $\chi'$  is an expectation of nonnegative terms with at least one strictly positive. In Obstfeld and Rogoff (1983, 2017), condition (79) implies that money is an absolute necessity for everyone. The corresponding assumption in the incomplete markets model means that bonds are an absolute necessity in some state of the world for someone. One may argue that households could hold capital instead, so as to avoid zero consumption in state  $e^0$ . Adding a fixed cost of capital adjustment (e.g. as in Kaplan, Moll, and Violante, 2016) would eliminate this possibility and make bonds without adjustment costs a necessity.

A third possibility with much weaker assumptions with respect to income, adjustment costs etc. arises if prices are sticky. Again, the arguments from the reduced form model hold here as well. If condition (78) holds, that is for large enough  $P$  for some household in at least one state of the world,

$$\frac{u_c^k(c^k)}{P} < \frac{\chi'(\frac{B}{P})}{P}, \quad (133)$$

then hyperinflations can be ruled out, for a simple reason. A hyperinflation requires the price to jump to infinity at some point, but no firm subject to pricing frictions would set a price  $P = \infty$ . Clearly, condition (131) is sufficient, as it implies that the RHS is bounded from below, whereas the LHS converges to zero. But it is way too strong, as (133) does not require a state  $e^0 = 0$ . Condition (133) just requires slower convergence of the RHS than the LHS, which is likely to be the case for some households, since the real value of bonds converges to zero, depriving households of any consumption smoothing abilities. If the real value of bonds is low enough, consumption smoothing is so poor that for some households, the marginal utility from bond goods exceeds the marginal utility of capital goods, which they would like to correct by accumulating more bonds, but they cannot do so. I therefore obtain

**Result 14.** *Hyperinflations can be ruled out if either*

1. *An Obstfeld & Rogoff backing of nominal government obligations is available, or*
2. *Condition (131) holds, or*
3. *Price are rigid and condition (133) holds.*

### 3.4.2 Hyperdeflations

I also provided three different proofs in the reduced form model, in order to rule out hyperinflation - nominal government spending  $G > 0$ , bounded tax revenue and the transversality condition - which I again argue carry over to the incomplete markets model as well.

If nominal government expenditure  $G > 0$ , a price converging to 0 implies that  $\frac{G}{P}$  ultimately exceeds output, which is neither an equilibrium in the reduced form model nor in my or any incomplete markets model.

As a second proof, I showed that hyperdeflations are ruled out if government tax revenue is bounded. The boundedness of tax revenue is, as I argued above, a result in Aiyagari (1994a) based on standard Laffer curve logic and that lump-sum taxes are zero or very small (less than the lowest labor income). In the reduced-form model, the proof also required the (weak) assumption (87),  $\lim_{P \rightarrow 0} \chi'(\frac{B}{P}) = 0$ , in order to show that the real interest rates converges to  $1/\beta$ . Such an additional assumption is not necessary now, since a result in Aiyagari (1994a) is that asset demand converges to infinity if the real interest rate approaches  $1/\beta$ , so that the value of bonds converges to zero. This shows that hyperdeflations can be ruled out here without further assumptions.

Finally, the result that the real interest rate converges to  $1/\beta$  means that I can also use the same transversality condition argument as before. The same arguments apply here if  $\frac{R}{1+\pi_{ss}} \leq 1$ .

Prices eventually grow at rate  $\beta R$ , so that  $\frac{B_i}{P_i}$  grows at rate  $\frac{1+\pi_{ss}}{\beta R} \geq \frac{1}{\beta}$  and  $-\frac{\partial \Upsilon_T}{\partial b_{iT}}$  is positive from the FOC (112) for bonds, implying that the transversality condition,

$$\lim_{T \rightarrow \infty} E_0[-\beta^T \frac{\partial \Upsilon}{\partial b_{iT}} b_{iT}] > 0 \quad (134)$$

is violated for a positive mass of households. I therefore obtain

**Result 15.** *Hyperdeflations can be ruled out, since government tax revenue is bounded or*

1. *If nominal government spending  $G > 0$  or*
2. *If  $\frac{R}{1+\pi_{ss}} \leq 1$  using a transversality condition argument.*

## 4 Conclusion

This paper shows that the price level is globally determinate in models in which government bonds are net wealth, including Bewley-Imrohoroglu-Huggett-Aiyagari heterogenous agents incomplete markets models. A key finding is that the price level is determined jointly by monetary and fiscal policy, with long-run inflation determined by the growth rate of nominal government debt, even

if monetary policy is operating an interest rate rule with a different inflation target. The nominal anchor - nominal fiscal variables - is controlled by fiscal policy, which therefore has the power to set the long-run inflation rate.

Building on these findings, Hagedorn (2016) and Hagedorn et al. (2018b,a, 2017) use incomplete markets models, in which government bonds are net wealth, so as to investigate several hotly debated topics in monetary economics. Not surprisingly, the novel way of thinking proposed in this paper yields different answers to those emerging from the conventional view.

In this research, we consider forward guidance, which refers to the idea that central bank commitment to keep the nominal interest rate low during a period after the liquidity trap, leads to large output gains even during the liquidity trap. We show that the large output gains from forward guidance do not occur in incomplete markets models. Independently of whether the nominal interest rate is raised earlier or later than in the benchmark, the output path is almost unchanged. The same conclusion is reached, if instead of raising the nominal interest rate according to some exogenous rule to its steady-state level, an interest rate rule is applied.

We also calculate fiscal multipliers, the output response to a government spending increase. In New Keynesian models, fiscal multipliers increase as price stickiness is reduced. By contrast, we show that output declines and so does the fiscal multiplier, if price stickiness is reduced in incomplete markets models.

Furthermore, New Keynesian models also imply that technological regress is expansionary in a liquidity trap. Again by contrast, we show that this is not the case in incomplete markets models. Technological progress is expansionary and technological regress contractionary.

Given the findings in Hagedorn (2016) and Hagedorn et al. (2018b,a, 2017), we therefore conclude that in models where government bonds are net wealth, such as incomplete markets models, the forward guidance puzzle disappears as commitment to future monetary policy has only negligible effects, technological regress decreases output and the size of the fiscal multiplier declines if prices are less sticky.

We also numerically compute impulse responses to monetary and fiscal policy shocks, as well as to technology and discount-factor shocks. We find that all impulse responses are in line with their empirical counterparts. While price rigidities are not needed for monetary policy to have an effect on prices and inflation,<sup>24</sup> for these effects to be quantitatively consistent with empirical findings, nominal rigidities are probably necessary (Hagedorn et al., 2017). Having established a framework with a determinate price level allows for a rigorous study of these and many more policy and empirical questions, many times with new and different answers.

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<sup>24</sup>Garriga et al. (2013), Sterk and Tenreyro (2015) and Buera and Nicolini (2016) among others, also find that monetary policy has real effects in incomplete markets models with flexible prices.

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# APPENDIX

## A.1 Monetary and Fiscal Policy and Ljungqvist and Sargent's (2012) Ten Monetary Doctrines

In this Section, I explain the differences between the FTPL and my theory in more detail. The starting point is chapter 26 on “Fiscal-Monetary Theories of Inflation” in Ljungqvist and Sargent (2012), which considers a policy designed to differentiate between the initial period  $t = 0$  (“short run”) and the remaining periods  $t \geq 1$  (“long run”). I adopt this setting in my reduced form model and assume that prices are flexible, that labor is supplied inelastically with labor income  $\omega$ , that nominal government expenditure  $G = 0$ , and that

$$\begin{aligned}g_t &= g \quad \forall t \geq 0 \\ \tau_t &= \tau \quad \forall t \geq 0 \\ B_t &= B \quad \forall t \geq 0 \\ T_t &= T \quad \forall t \geq 1, \\ R_t &= R \quad \forall t \geq 0,\end{aligned} \tag{A1}$$

where I permit initial bonds  $B_{-1} \neq B$  and initial lump-sum taxes  $T_0 \neq T$ . I allow  $G > 0$  and elastically supplied labor below without any consequences for my conclusions. Money is endogenous and the central bank supplies  $M_t$  to satisfy demand

$$\frac{\mu'(M_t/P_t)}{u'(\omega - g)} = \frac{R - 1}{R}. \tag{A2}$$

Since  $c = \omega - g$ , the first-order condition for bonds is

$$u'(\omega - g) = \chi'\left(\frac{B_t}{P_t}\right) + \beta R u'(\omega - g), \tag{A3}$$

so that the steady-state price level  $P^*$  is the solution to

$$u'(\omega - g) = \chi'\left(\frac{B}{P^*}\right) + \beta R u'(\omega - g). \tag{A4}$$

The steady-state nominal lump-sum tax  $T$  is set to satisfy the steady-state government budget constraint

$$T := B(R - 1) + P^*(g - \tau\omega) + M_{t-1} - M_t = B(R - 1) + P^*(g - \tau\omega), \tag{A5}$$

where the endowment  $\omega$  is taxed at rate  $1 - \tau$  and money  $M$  is constant.

Iterating the first-order condition (A3) yields

$$P_t = P^* \quad \forall t \geq 0 \quad (\text{A6})$$

and therefore  $M_t = M$  for all  $t \geq 0$  using (A2). If initial bonds  $B_{-1} \neq B$  or  $M_{-1} \neq M$ , lump sum taxes in the first period are adjusted to satisfy the initial government budget constraint

$$T_0 = (RB_{-1} + M_{-1}) - (B + M) + P^*(g - \tau\omega). \quad (\text{A7})$$

I therefore obtain an equilibrium with a determinate price level in a model where government bonds are net wealth.

The Fiscal Theory of the Price Level (FTPL) takes a different approach. To clarify the difference I assume now that bonds are not an argument of the utility function,  $\chi \equiv 0$ , and  $T = 0$ , so that  $T$  cannot adjust anymore to balance the government budget what is key to the FTPL.

The period  $t$  government budget constraint in real term reads

$$\frac{B_t}{P_t} = (1 + r_t) \frac{B_{t-1}}{P_{t-1}} + \frac{M_{t-1} - M_t}{P_t} + (g - \tau\omega), \quad (\text{A8})$$

where  $(1 + r_t) = R \frac{P_{t-1}}{P_t} = 1/\beta$  is the real interest rate and the inflation rate is  $1 + \pi_t = \frac{P_t}{P_{t-1}} = R\beta$ . Money demand (A2) allows us to write  $M_t/P_t = f(R)$  so that the per period seigniorage

$$\frac{M_t - M_{t-1}}{P_t} = f(R) \left(1 - \frac{1}{R\beta}\right) =: \tilde{f}(R). \quad (\text{A9})$$

The intertemporal government budget constraint says that

$$\frac{B_0}{P_0} = \sum_{t=1}^{\infty} \beta^t (\tau\omega + \tilde{f}(R) - g) \quad (\text{A10})$$

and thus we obtain for the initial debt level  $B_{-1}$

$$\frac{B_{-1}}{P_0} = \frac{1}{R} \left[ \frac{B_0}{P_0} + (\tau\omega + \tilde{f}(R) - g) \right] = \frac{1}{R} \sum_{t=0}^{\infty} \beta^t (\tau\omega + \tilde{f}(R) - g) = \frac{\tau\omega + \tilde{f}(R) - g}{R(1 - \beta)}. \quad (\text{A11})$$

The FTPL assumes that  $g$  and  $\tau$  are exogenous so that the initial price level  $P_0$  is determined as the ratio of outstanding nominal debt  $B_{-1}$  to the present value of the primary surplus and seigniorage.

Apparently, the determination of the price level when government bonds are net wealth or when the FTPL is operating are different. In the first case, the first-order condition for bonds is

used so that the price level depends on the valuation of bonds but not on  $B_{-1}$ . In the second case, the government budget constraint is used so that the price level depends on the initial outstanding government debt level  $B_{-1}$ , seigniorage and the fiscal variables  $g$  and  $\tau$ .

Using  $u(c) = \log(c)$  and  $\chi(B/P) = \rho^\chi \log(B/P)$  from the example of the main text, the steady-state price level  $P^*$  and thus the initial price level when government bonds are net wealth equals

$$P^* = P_0 = (1 - \beta R) \frac{B}{(\omega - g)\rho^\chi}, \quad (\text{A12})$$

whereas the FTPL implies a different initial price level

$$P_0^{FTPL} = \frac{B_{-1}R(1 - \beta)}{\tau\omega + \tilde{f}(R) - g}. \quad (\text{A13})$$

The reason for the difference is that  $P^*$  is determined as clearing the asset market, that is satisfying the FOC for bonds, and that the price  $P_0^{FTPL}$  is determined as satisfying the government budget constraint. The government budget constraint also has to be satisfied in the first case but it is not the price but the lump-sum tax  $T$  which ensures this, such that both the FOC and the budget constraint are satisfied when government bonds are net wealth. Both constraints are also satisfied when the FTPL is operating. The government budget constraint is satisfied by construction. The FOC for bonds is satisfied as well but in the context of the FTPL government bonds are not net wealth so that the FOC

$$u'(\omega - g) = (1 + r)\beta u'(\omega - g) \quad (\text{A14})$$

just pins down the real interest rate  $1 + r = 1/\beta$ . This is a consequence of Ricardian equivalence since the private sector is willing to absorb any equilibrium amount of real bonds. Using the FTPL in different environment in which Ricardian equivalence does not hold, for example if government bonds are net wealth, requires the private sector to be willing to absorb the real value of debt which satisfies the government budget constraint. The difference in prices  $P_0^{FTPL} \neq P^*$  suggests that this is a binding equilibrium restriction.

Finally, it is easy to see that allowing nominal government expenditure  $G > 0$  and elastically supplied labor in the model in which government bonds are net wealth does not affect my conclusions. The steady-state price level  $P^*$  and hours  $h^*$  are determined as the solution to the FOCs

for bonds and hours

$$u'(h^* - g - G/P^*) = \chi'(\frac{B}{P^*}) + \beta Ru'(h^* - g - G/P^*), \quad (\text{A15})$$

$$\frac{v'(h^*)}{u'(h^* - g - G/P^*)} = (1 - \tau) \quad (\text{A16})$$

and the steady-state nominal lump-sum tax  $T$  is

$$T := B(R - 1) + G + P^*(g - \tau h^*) + M_{t-1} - M_t = B(R - 1) + G + P^*(g - \tau h^*). \quad (\text{A17})$$

Iterating the first-order condition for bonds and hours backwards shows that

$$P_t = P^* \quad \forall t \geq 0, \quad (\text{A18})$$

$$h_t = h^* \quad \forall t \geq 0, \quad (\text{A19})$$

$$T_t = T \quad \forall t \geq 1, \quad (\text{A20})$$

$$T_0 = (RB_{-1} + M_{-1}) - (B + M) + G + P^*(g - \tau \omega), \quad (\text{A21})$$

is the unique equilibrium, satisfying all FOCs and government budget constraints, with a uniquely determined price level.

## A.II Proofs and Derivations

### A.II.1 Derivations of Section 2.3

#### Local Determinacy

As shown in the main text, the economy is described through

$$\begin{pmatrix} E_t p_{t+1} \\ p_t \end{pmatrix} = A \begin{pmatrix} p_t \\ p_{t-1} \end{pmatrix},$$

where

$$A \equiv \begin{pmatrix} \frac{1 + \tilde{\beta} + \kappa(\sigma + \psi)}{\tilde{\beta} + \kappa\sigma} & \frac{-1}{\tilde{\beta} + \kappa\sigma} \\ 1 & 0 \end{pmatrix}.$$

Since

$$\text{trace}(A) - \det(A) = \frac{\tilde{\beta} + \kappa(\sigma + \psi)}{\tilde{\beta} + \kappa\sigma} = 1 + \frac{\kappa\psi}{\tilde{\beta} + \kappa\sigma} > 1, \quad (\text{A22})$$

the characteristic polynomial  $q(x)$  is strictly negatively evaluated at 1,  $q(1) < 0$ , if  $\psi > 0$ . Since  $q$

eventually increases to infinity in  $x$  one eigenvalue,  $\lambda_1$  is real and greater than one. If  $\det(A) < 1$ , the other eigenvalue,  $\lambda_2$ , is smaller than one, since  $\det(A) = \lambda_1\lambda_2$ . If  $\det(A) > 1$ ,  $q(0) = \det(A) > 0$ , so that  $0 < \lambda_2 < 1$  by the intermediate value theorem.

Local Determinacy with non-separable utility and  $G > 0$ .

Non-separability of  $\chi(B/P, M/P)$  and nominal government expenditure  $G > 0$  require two changes. First, it is necessary to log-linearize money-demand as in (52),

$$\hat{m}_t = \eta_y \hat{Y}_t - \eta_i \hat{i}_{t+1} \quad (\text{A23})$$

for an output elasticity  $\eta_y$  and an interest-rate elasticity  $\eta_i > 0$ , and consumption as

$$\hat{C}_t = s_Y \hat{Y}_t - s_G (\hat{G}_t - \hat{p}_t), \quad (\text{A24})$$

where  $s_Y = \frac{Y^*}{c^*}$  and  $s_G = \frac{g^*}{c^*}$  and  $Y^*$  and  $g^*$  are steady-state real output and real government expenditure respectively. Plugging this into the linearized Euler Equation yields

$$s_Y \hat{Y}_t - s_G (\hat{G}_t - \hat{p}_t) = s_Y \tilde{\beta} E_t \hat{Y}_{t+1} - s_G \tilde{\beta} E_t (\hat{G}_{t+1} - \hat{p}_{t+1}) - \psi \hat{p}_t - \epsilon_m (\eta_y \hat{Y}_t - \eta_i \hat{i}_{t+1}) - \sigma (\hat{i}_{t+1} - (E_t \hat{p}_{t+1} - \hat{p}_t)),$$

where  $\epsilon_m = \frac{\sigma \chi_{mb} m^*}{u'(c^*) \tilde{\beta}}$ . On the basis that  $\hat{G}_t = \hat{i}_{t+1} = 0$  and collecting terms yields

$$(s_Y + \epsilon_m \eta_y) \hat{Y}_t = s_Y \tilde{\beta} E_t \hat{Y}_{t+1} - s_G \tilde{\beta} (\hat{p}_t - E_t \hat{p}_{t+1}) - \psi \hat{p}_t + \sigma (E_t \hat{p}_{t+1} - \hat{p}_t). \quad (\text{A25})$$

Finally, using (48),  $\kappa \hat{Y}_t = (\hat{p}_t - \hat{p}_{t-1})$ , and rearranging the terms,

$$(\tilde{\beta}(s_Y + \kappa s_G) + \kappa \sigma) E_t \hat{p}_{t+1} = (s_Y(1 + \tilde{\beta}) + \epsilon_m \eta_y + \kappa(\sigma + \psi + \tilde{\beta} s_G)) \hat{p}_t - (s_Y + \epsilon_m \eta_y) \hat{p}_{t-1}.$$

Equivalently the transition matrix is

$$A \equiv \begin{pmatrix} \frac{s_Y(1+\tilde{\beta})+\epsilon_m\eta_y+\kappa(\sigma+\psi+s_G)}{\tilde{\beta}(s_Y+\kappa s_G)+\kappa\sigma} & \frac{-(s_Y+\epsilon_m\eta_y)}{\tilde{\beta}(s_Y+\kappa s_G)+\kappa\sigma} \\ 1 & 0 \end{pmatrix}.$$

Since

$$\text{trace}(A) - \det(A) = \frac{s_Y \tilde{\beta} + \kappa(\sigma + \psi + s_G)}{\tilde{\beta}(s_Y + \kappa s_G) + \kappa \sigma} = 1 + \frac{\kappa \psi}{\tilde{\beta}(s_Y + \kappa s_G) + \kappa \sigma} > 1, \quad (\text{A26})$$

the characteristic polynomial  $q(x)$  is strictly negatively evaluated at 1,  $q(1) < 0$ , if  $\psi > 0$ . Since  $q$  eventually increases to infinity in  $x$  one eigenvalue,  $\lambda_1$ , is real and greater than one. If  $|\det(A)| < 1$ , the other eigenvalue,  $\lambda_2$ , is smaller than one, since  $\det(M) = \lambda_1\lambda_2$ . If  $\det(A) > 1$ ,  $q(0) = \det(A) >$



0, so that  $0 < \lambda_2 < 1$  by the intermediate value theorem.

The parameter restriction  $\frac{s_Y + \epsilon_m \eta_y}{\bar{\beta}(s_Y + \kappa s_G) + \kappa \sigma} > -1$  for which  $s_Y + \epsilon_m \eta_y \geq 1 + \epsilon_m \eta_y > 0$  is sufficient, means that  $\det(A) > -1$ . A violation of the parameter restriction would deliver two eigenvalues with a modulus larger than one. The economy is locally determinate.

### Derivation of Phillips Curve

The firm's pricing problem is

$$V_t(p_{jt-1}) \equiv \max_{p_{jt}} \frac{p_{jt}}{P_t} y(p_{jt}; P_t, Y_t) - w_t y(p_{jt}; P_t, Y_t) - \Phi \left( \frac{p_{jt}}{p_{jt-1}} - \pi_{ss} \right) Y_t + \beta V_{t+1}(p_{jt}),$$

subject to the constraints  $n_{jt} = y(p_{jt}; P_t, Y_t) = \left( \frac{p_{jt}}{P_t} \right)^{-\epsilon} Y_t$ .

Equivalently

$$V_t(p_{jt-1}) \equiv \max_{p_{jt}} \frac{p_{jt}}{P_t} \left( \frac{p_{jt}}{P_t} \right)^{-\epsilon} Y_t - w_t \left( \frac{p_{jt}}{P_t} \right)^{-\epsilon} Y_t - \Phi \left( \frac{p_{jt}}{p_{jt-1}} - \pi_{ss} \right) Y_t + \beta V_{t+1}(p_{jt})$$

The FOC w.r.t  $p_{jt}$

$$(1 - \epsilon) \left( \frac{p_{jt}}{P_t} \right)^{-\epsilon} \frac{Y_t}{P_t} + \epsilon w_t \left( \frac{p_{jt}}{P_t} \right)^{-\epsilon-1} \frac{Y_t}{P_t} - \Phi' \left( \frac{p_{jt}}{p_{jt-1}} - \pi_{ss} \right) \frac{Y_t}{p_{jt-1}} + \beta V'_{t+1}(p_{jt}) = 0 \quad (\text{A27})$$

and the envelope condition

$$V'_{t+1}(p_{jt}) = \Phi' \left( \frac{p_{jt+1}}{p_{jt}} - \pi_{ss} \right) \frac{p_{jt+1} Y_{t+1}}{p_{jt} p_{jt}}. \quad (\text{A28})$$

Combining the FOC and and the envelope condition

$$(1 - \epsilon) \left( \frac{p_{jt}}{P_t} \right)^{-\epsilon} \frac{Y_t}{P_t} + \epsilon w_t \left( \frac{p_{jt}}{P_t} \right)^{-\epsilon-1} \frac{Y_t}{P_t} - \Phi' \left( \frac{p_{jt}}{p_{jt-1}} - \pi_{ss} \right) \frac{Y_t}{p_{jt-1}} + \beta \Phi' \left( \frac{p_{jt+1}}{p_{jt}} - \pi_{ss} \right) \frac{p_{jt+1} Y_{t+1}}{p_{jt} p_{jt}} = 0$$

Finally, on the basis that all firms choose the same price in equilibrium, that  $\frac{p_{jt+1}}{p_{jt}} = \pi_{t+1}$ , and dividing by  $Y_t/P_t = Y_t/p_{jt}$  yields the non-linear Phillips curve:

$$(1 - \epsilon) + \epsilon w_t - \Phi'(\pi_t - \pi_{ss}) \pi_t + \beta \Phi'(\pi_{t+1} - \pi_{ss}) \pi_{t+1} \frac{Y_{t+1}}{Y_t} = 0. \quad (\text{A29})$$

### Linearization of the Phillips Curve

A linearization of (A29) around the steady state yields

$$\epsilon\varphi\hat{Y}_t - \theta\hat{\pi}_t + \beta\theta\hat{\pi}_{t+1} = 0 \quad (\text{A30})$$

where  $\theta = \Phi''(0)$ ,  $\Phi'(0) = 0$  and  $\hat{w}_t = \varphi\hat{Y}_t$  and  $\hat{\pi}$  is the deviation of inflation from its steady-state value  $\pi_{ss}$ . Equivalently, for  $\kappa = \frac{\epsilon\varphi}{\theta}$

$$\hat{\pi}_t = \kappa\hat{Y}_t + \beta\hat{\pi}_{t+1} \quad (\text{A31})$$

or in terms of prices

$$\hat{p}_t - \hat{p}_{t-1} = \kappa\hat{Y}_t + \beta(\hat{p}_{t+1} - \hat{p}_t), \quad (\text{A32})$$

where  $\hat{p}_t$  is the deviation from the steady-state price  $P_t^* = P^*(1 + \pi_{ss})^t$ .

#### Local Determinacy with New Keynesian Phillips Curve

The economy is described through two dynamic equations,

$$(\hat{p}_t - \hat{p}_{t-1}) = \kappa\hat{Y}_t + \beta E_t(\hat{p}_{t+1} - \hat{p}_t) \quad (\text{A33})$$

$$\hat{Y}_t = \tilde{\beta} E_t \hat{Y}_{t+1} - \psi\hat{p}_t - \sigma(\hat{i}_{t+1} - (E_t\hat{p}_{t+1} - \hat{p}_t)). \quad (\text{A34})$$

The first allows solving for expected inflation

$$E_t(\hat{p}_{t+1} - \hat{p}_t) = (\hat{p}_t - \hat{p}_{t-1})/\beta - \frac{\kappa}{\beta}\hat{Y}_t, \quad (\text{A35})$$

which can be plugged into the second equation, which can then be solved for the only remaining period  $t + 1$  variable  $E_t\hat{Y}_{t+1}$ , using  $\hat{i}_{t+1} = 0$ ,

$$E_t\hat{Y}_{t+1} = \frac{1}{\tilde{\beta}}\left\{\left(1 + \frac{\kappa\sigma}{\beta}\right)\hat{Y}_t + \left(\psi - \frac{\sigma}{\beta}\right)\hat{p}_t + \frac{\sigma}{\beta}\hat{p}_{t-1}\right\}. \quad (\text{A36})$$

The period  $t + 1$  price is

$$E_t\hat{p}_{t+1} = \left(1 + \frac{1}{\beta}\right)\hat{p}_t - \frac{1}{\beta}\hat{p}_{t-1} - \frac{\kappa}{\beta}\hat{Y}_t, \quad (\text{A37})$$

which together yield the matrix in the main text.

The characteristic equation of the matrix

$$\begin{pmatrix} E_t p_{t+1} \\ p_t \\ E_t Y_{t+1} \end{pmatrix} = \frac{1}{\beta} \begin{pmatrix} 1 + \beta & -1 & -\kappa \\ \beta & 0 & 0 \\ \beta\psi - \sigma & \sigma & \beta/\tilde{\beta} + \sigma\kappa \end{pmatrix} \begin{pmatrix} p_t \\ p_{t-1} \\ Y_t \end{pmatrix}.$$

is

$$x^3 + \underbrace{\left(-\frac{1 + \kappa\sigma}{\beta} - 1 - \frac{1}{\tilde{\beta}}\right)}_{=:a_2} x^2 + \underbrace{\left(\frac{(\psi + \sigma)\kappa}{\beta} + \frac{1 + \beta}{\beta\tilde{\beta}} + \frac{1}{\beta}\right)}_{=:a_1} x + \underbrace{\frac{-1}{\beta\tilde{\beta}}}_{=:a_0}, \quad (\text{A38})$$

where for simplicity,  $\frac{\sigma}{\tilde{\beta}}$  is relabelled as  $\sigma$  and  $\frac{\psi}{\tilde{\beta}}$  as  $\psi$ .

Checking the conditions in Proposition C.2 in Appendix C of Woodford (2003) requires computing

$$1 + a_2 + a_1 + a_0 = \frac{\kappa\psi}{\beta} > 0, \quad (\text{A39})$$

$$-1 + a_2 - a_1 + a_0 = -\frac{(\psi + 2\sigma)\kappa + 2(1 + \beta)}{\beta} - 2\frac{1 + \beta}{\beta\tilde{\beta}} < 0, \quad (\text{A40})$$

where  $\psi > 0$  implies that the first is positive and the second negative. The conditions of (Case III) are satisfied, since  $\tilde{\beta} < 1$  and thus  $|a_2| > \frac{1}{\tilde{\beta}} + \frac{1}{\beta} + 1 > 3$  and the equilibrium is locally determinate. Note that the condition  $a_0^2 - a_0 a_2 + a_1 - 1 < 0$  is not needed for (Case III).

### Local Determinacy with Policy Rules Responding to Prices and Output

With policy rules, the economy is described through

$$\kappa \hat{Y}_t = (\hat{p}_t - \hat{p}_{t-1}) \quad (\text{A41})$$

$$\hat{Y}_t = \tilde{\beta} E_t \hat{Y}_{t+1} + \psi(\hat{B}_t - \hat{p}_t) - \sigma(\hat{i}_{t+1} - (E_t \hat{p}_{t+1} - \hat{p}_t)), \quad (\text{A42})$$

where policy rules for  $i$  and  $B$  are

$$\hat{i}_{t+1} = \varphi_p^i \hat{p}_t + \varphi_y^i \hat{Y}_t, \quad (\text{A43})$$

$$\hat{B}_t = \varphi_p^B \hat{p}_t + \varphi_y^B \hat{Y}_t. \quad (\text{A44})$$

Plugging the policy rules into the Euler equation yields

$$\hat{Y}_t = \tilde{\beta} E_t \hat{Y}_{t+1} + \psi(\varphi_p^B \hat{p}_t + \varphi_y^B \hat{Y}_t - \hat{p}_t) - \sigma(\varphi_p^i \hat{p}_t + \varphi_y^i \hat{Y}_t - (E_t \hat{p}_{t+1} - \hat{p}_t)), \quad (\text{A45})$$

which, after using  $\kappa\hat{Y}_t = \hat{p}_t - \hat{p}_{t-1}$ , is equal to

$$\hat{p}_t - \hat{p}_{t-1} = \tilde{\beta}E_t(\hat{p}_{t+1} - \hat{p}_t) + \psi\kappa(\varphi_p^B\hat{p}_t + \frac{\varphi_y^B}{\kappa}(\hat{p}_t - \hat{p}_{t-1}) - \hat{p}_t) - \sigma\kappa(\varphi_p^i\hat{p}_t + \frac{\varphi_y^i}{\kappa}(\hat{p}_t - \hat{p}_{t-1}) - (E_t\hat{p}_{t+1} - \hat{p}_t)).$$

Rearranging the terms yields

$$\begin{aligned} & \left(\tilde{\beta} + \kappa\sigma\right)E_t\hat{p}_{t+1} \\ &= \left(1 + \tilde{\beta} + \kappa\underbrace{\left[\psi\left(1 - \varphi_p^B - \frac{\varphi_y^B}{\kappa}\right)\right]}_{=:\hat{\psi}} + \sigma\underbrace{\left(1 + \varphi_p^i + \frac{\varphi_y^i}{\kappa}\right)}_{=:\hat{\sigma}}\right)\hat{p}_t \\ &- \left(1 + \sigma\varphi_y^i - \psi\varphi_y^B\right)\hat{p}_{t-1} \end{aligned}$$

so that the economy is now described through the matrix

$$A \equiv \begin{pmatrix} E_t p_{t+1} \\ p_t \end{pmatrix} = \begin{pmatrix} \frac{1+\tilde{\beta}+\kappa(\hat{\psi}+\hat{\sigma})}{\tilde{\beta}+\kappa\sigma} & \frac{-(1+\sigma\varphi_y^i-\psi\varphi_y^B)}{\tilde{\beta}+\kappa\sigma} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_t \\ p_{t-1} \end{pmatrix}$$

where  $\hat{\psi} = \psi(1 - \varphi_p^B - \frac{\varphi_y^B}{\kappa})$  and  $\hat{\sigma} = \sigma(1 + \varphi_p^i + \frac{\varphi_y^i}{\kappa})$ . It follows that

$$\det A = \frac{1 + \sigma\varphi_y^i - \psi\varphi_y^B}{\tilde{\beta} + \kappa\sigma} \quad (\text{A46})$$

$$\text{trace} A - \det A = \frac{\tilde{\beta} + \kappa\psi(1 - \varphi_p^B) + \kappa\sigma(1 + \varphi_p^i)}{\tilde{\beta} + \kappa\sigma} = 1 + \frac{\kappa\tilde{\psi}}{\tilde{\beta} + \kappa\sigma}. \quad (\text{A47})$$

The latter is smaller than  $-1$  if  $\tilde{\psi} = \psi(1 - \varphi_p^B) + \sigma\varphi_p^i > 0$  (the condition in the text). The two eigenvalues  $\lambda_{1,2}$  therefore satisfy

$$(\lambda_1 - 1)(\lambda_2 - 1) < 0, \quad (\text{A48})$$

implying that both eigenvalues are real (otherwise  $\lambda_1 - 1$  and  $\lambda_2 - 1$  would be a complex pair with a positive norm) and that one eigenvalue is greater than one. The second eigenvalue can then be either smaller or larger than one in modulus.

Proposition C.1 in Appendix C of Woodford (2003) shows that if the condition

$$\det A + \text{trace} A = \frac{2 + \tilde{\beta} + \psi(\kappa - \kappa\varphi_p^B - 2\varphi_y^B) + \sigma(\kappa + \kappa\varphi_p^i + 2\varphi_y^i)}{\tilde{\beta} + \kappa\sigma} < -1 \quad (\text{A49})$$

is satisfied, then the modulus of both eigenvalues is greater than one and if the condition is not

satisfied, the second eigenvalue is smaller than one in modulus. Thus, the latter case is satisfied ( $|\lambda_2| < 1$ ), using  $\tilde{\psi} = \psi(1 - \varphi_p^B) + \sigma\varphi_p^i > 0$ , if

$$\frac{2 + \tilde{\beta} - 2\psi\varphi_y^B + 2\sigma\varphi_y^i}{\tilde{\beta} + \kappa\sigma} > -2 \quad (\text{A50})$$

and is thus sufficient for local determinacy. Accordingly under natural sign restrictions, the latter condition is satisfied, since

$$\frac{2 + \tilde{\beta} - 2\psi\varphi_y^B + 2\sigma\varphi_y^i}{\tilde{\beta} + \kappa\sigma} > \frac{2 + \tilde{\beta}}{\tilde{\beta} + \kappa\sigma} > 0 \quad (\text{A51})$$

and the economy is locally determinate.

### Local Determinacy with New Keynesian Phillips Curve and Policy Responses

The economy is described by

$$(\hat{p}_t - \hat{p}_{t-1}) = \kappa\hat{Y}_t + \beta E_t(\hat{p}_{t+1} - \hat{p}_t) \quad (\text{A52})$$

$$\hat{Y}_t = \tilde{\beta} E_t \hat{Y}_{t+1} + \psi(\hat{B}_t - \hat{p}_t) - \sigma(\hat{i}_{t+1} - (E_t \hat{p}_{t+1} - \hat{p}_t)) \quad (\text{A53})$$

together with policy rules for  $i$  and  $B$

$$\hat{i}_{t+1} = \varphi_p^i \hat{p}_t + \varphi_y^i \hat{Y}_t, \quad (\text{A54})$$

$$\hat{B}_t = \varphi_p^B \hat{p}_t + \varphi_y^B \hat{Y}_t. \quad (\text{A55})$$

Plugging the policy rules into the Euler equation yields

$$\hat{Y}_t = \tilde{\beta} E_t \hat{Y}_{t+1} + \psi(\varphi_p^B \hat{p}_t + \varphi_y^B \hat{Y}_t - \hat{p}_t) - \sigma(\varphi_p^i \hat{p}_t + \varphi_y^i \hat{Y}_t - (E_t \hat{p}_{t+1} - \hat{p}_t)). \quad (\text{A56})$$

Using expected inflation

$$E_t(\hat{p}_{t+1} - \hat{p}_t) = (\hat{p}_t - \hat{p}_{t-1})/\beta - \frac{\kappa}{\beta} \hat{Y}_t, \quad (\text{A57})$$

in the Euler equation, which can then be solved for the only remaining period  $t+1$  variable  $E_t \hat{Y}_{t+1}$ ,

$$\tilde{\beta} E_t \hat{Y}_{t+1} \tag{A58}$$

$$= \left(1 + \frac{\sigma \kappa}{\beta} + \sigma \varphi_y^i - \psi \varphi_y^B\right) \hat{Y}_t \tag{A59}$$

$$+ \left(\psi(1 - \varphi_p^B) + \sigma(\varphi_p^i - \frac{1}{\beta})\right) \hat{p}_t \tag{A60}$$

$$+ \left(\frac{\sigma}{\beta}\right) \hat{p}_{t-1}. \tag{A61}$$

The period  $t+1$  price is

$$E_t \hat{p}_{t+1} = \left(1 + \frac{1}{\beta}\right) \hat{p}_t - \frac{1}{\beta} \hat{p}_{t-1} - \frac{\kappa}{\beta} \hat{Y}_t, \tag{A62}$$

so that in matrix form:

$$\begin{pmatrix} E_t p_{t+1} \\ p_t \\ E_t Y_{t+1} \end{pmatrix} = \frac{1}{\beta} \begin{pmatrix} 1 + \beta & -1 & -\kappa \\ \beta & 0 & 0 \\ \beta \hat{\psi} - \sigma & \sigma & \beta/\tilde{\beta} + \sigma \kappa + \beta \hat{\epsilon} \end{pmatrix} \begin{pmatrix} p_t \\ p_{t-1} \\ Y_t \end{pmatrix}.$$

where  $\hat{\psi} = \psi(1 - \varphi_p^B) + \sigma \varphi_p^i$  and  $\hat{\epsilon} = \sigma \varphi_y^i - \psi \varphi_y^B \geq 0$  and for simplicity,  $\frac{\sigma}{\tilde{\beta}}$  is relabelled as  $\sigma$  and  $\frac{\psi}{\tilde{\beta}}$  as  $\psi$ .

The characteristic equation is

$$x^3 + \underbrace{\frac{(-\hat{\epsilon} - 1 - 1/\tilde{\beta})\beta - \kappa\sigma - 1}{\beta}}_{=:a_2} x^2 + \underbrace{\frac{1 + \hat{\epsilon}(1 + \beta) + (\psi + \sigma)\kappa + (1 + \beta)/\tilde{\beta}}{\beta}}_{=:a_1} x - \underbrace{\frac{1 + \hat{\epsilon}\tilde{\beta}}{\beta\tilde{\beta}}}_{=:a_0}. \tag{A63}$$

Checking the conditions in Proposition C.2 in Appendix C of Woodford (2003) requires computing

$$1 + a_2 + a_1 + a_0 = \frac{\kappa \hat{\psi}}{\beta} > 0 \tag{A64}$$

$$-1 + a_2 - a_1 + a_0 = \frac{(-2\hat{\epsilon} - 2)(1 + \beta) + (-\hat{\psi} - 2\sigma)\kappa - (2\beta + 2)/\tilde{\beta}}{\beta} \tag{A65}$$

The first term (A64) is positive, since  $\hat{\psi} > 0$ . The second term (A65) is negative, since  $\hat{\epsilon} \geq 0$ . Furthermore  $|a_2| > 3$ , since  $\hat{\epsilon} \geq 0$  implies that  $a_2 < \frac{(-1-1/\tilde{\beta})\beta-1}{\beta} < -3$ , which completes the proof.

The proof also shows that the natural sign restrictions can be replaced with the assumption that  $\hat{\epsilon} \geq 0$ .

## A.II.2 Derivations of Section 3

### Incomplete Market Household Problem: Derivation of FOCs

In the main text I substituted  $c^b$  and  $c^m$  as the optimal choices first and then calculated FOCs. Here instead, I choose a Langrange approach, omitting indexing by household  $i$ . The household maximizes

$$U = E_0 \sum_{t=0}^{\infty} \beta^t \left[ u^k(c_t^k) + u^b(c_t^b) + u^m(c_t^m) - v(h_t) \right] \quad (\text{A66})$$

subject to credit constraints

$$k_t + b_t \geq -\bar{b}, \quad (\text{multiplier } \beta^t \mu_t^c) \quad (\text{A67})$$

budget constraints

$$\begin{aligned} & P_t k_t + P_t b_t + P_t m_t R_{t+1} + P_t c_t^k \\ = & P_t (1 + r_t^k) k_{t-1} + R_t P_{t-1} b_{t-1} + P_{t-1} (m_{t-1} - c_{t-1}^m) - P_{t-1} c_{t-1}^b \\ + & P_{t-1} d_{t-1} + P_{t-1} e_{t-1} h_{t-1} (1 - \tau) w_{t-1} - T_t, \quad (\text{multiplier } \beta^t \lambda_t) \end{aligned} \quad (\text{A68})$$

invoice constraints

$$P_t c_t^b \leq P_t e_t h_t (1 - \tau) w_t + P_t (b_t - \bar{b}) R_{t+1}, \quad (\text{multiplier } \beta^t \mu_t^I) \quad (\text{A69})$$

and cash-in-advance constraints

$$c_t^m \leq m_t. \quad (\text{multiplier } \beta^t \mu_t^m) \quad (\text{A70})$$

The first-order conditions are

$$c_t^k : u_c^k(c_t^k) = -P_t \lambda_t \quad (\text{A71})$$

$$c_t^b(e_t) : u_c^b(c_t^b(e_t)) = -P_t E_t \lambda_{t+1} - P_t \mu_t^I \quad (\text{A72})$$

$$c_t^m : u_c^m(c_t^m) = -\beta P_t E_{t-1} \lambda_{t+1} - \mu_t^m \quad (\text{A73})$$

$$b_t : -P_t \lambda_t = -\beta R_{t+1} P_t E_{t-1} \lambda_{t+1} + \mu_t^c - R_{t+1} P_t E_{t-1} \mu_t^I \quad (\text{A74})$$

$$k_t : -P_t \lambda_t = -\beta (1 + r_{t+1}^k) P_{t+1} E_{t-1} \lambda_{t+1} + \mu_t^c \quad (\text{A75})$$

$$m_t : R_{t+1} P_t \lambda_t = \beta P_t E_{t-1} \lambda_{t+1} + \mu_t^m \quad (\text{A76})$$

Combining the FOC for  $c_t^k$  and  $k_t$  yields the FOC for capital,

$$u_c^k(c_t^k) = \beta (1 + r_{t+1}^k) E_{t-1} u_c^k(c_{t+1}^k) + \mu_t^c, \quad (\text{A77})$$

or equivalently

$$u_c^k(c_t^k) \geq E_{t-1}\beta(1+r_{t+1}^k)u_c^k(c_{t+1}^k) \quad (\text{A78})$$

with equality, if the credit constraint is not binding,  $\mu_t^c = 0$ .

Combining the FOC for  $c_t^m$  and  $m_t$  and using the FOC for  $c_t^k$  yields

$$u_c^m(c_t^m) = -R_{t+1}P_t\lambda_t = R_{t+1}u_c^k(c_t^k) \quad (\text{A79})$$

or equivalently

$$\frac{u_c^m(c_t^m)}{u_c^k(c_t^k)} = R_{t+1} \quad (\text{A80})$$

Combining the FOCs for  $c_t^b$  and  $c_{t+1}^k$  yields

$$u_c^b(c_t^b(b_t, e_t)) = E_t \frac{\beta P_t}{P_{t+1}} u_c^k(c_{t+1}^k) - P_t \mu_t^I \quad (\text{A81})$$

or equivalently

$$u_c^b(c_t^b(b_t, e_t)) \geq \frac{\beta P_t}{P_{t+1}} E_t u_c^k(c_{t+1}^k) \quad (\text{A82})$$

with equality, if the invoice constraint is not binding,  $\mu_t^I = 0$ . Note that since the invoice constraint is a “ $\geq$ ” constraint, the multiplier  $\mu_t^I \leq 0$ , whereas the multiplier for the credit constraint is a “ $\leq$ ” constraint and the multiplier  $\mu_t^c \geq 0$ .

Finally, the FOC for bonds also using the FOC for  $c^k$  and  $c^b$  to obtain  $\mu_t^I$ ,

$$\begin{aligned} u_c^k(c_t^k) &= E_{t-1} \frac{\beta R_{t+1}}{P_{t+1}/P_t} u_c^k(c_{t+1}^k) + \mu_t^c - R_{t+1}P_t E_{t-1} \mu_t^I \\ &= E_{t-1} \frac{\beta R_{t+1}}{P_{t+1}/P_t} u_c^k(c_{t+1}^k) + \mu_t^c + R_{t+1} E_{t-1} \{u_c^b(c_t^b(b_t, e_t)) - \frac{\beta P_t}{P_{t+1}} u_c^k(c_{t+1}^k)\} \\ &= E_{t-1} \frac{\beta R_{t+1}}{P_{t+1}/P_t} u_c^k(c_{t+1}^k) + E_{t-1} \frac{\partial \tilde{\chi}(b_t, e)}{\partial b'} + \mu_t^c \\ &= E_{t-1} \frac{\beta R_{t+1}}{P_{t+1}/P_t} u_c^k(c_{t+1}^k) + \frac{\partial \chi(b_t)}{\partial b'} + \mu_t^c \end{aligned} \quad (\text{A83})$$

or equivalently

$$u_c^k(c_t^k) \geq \frac{\partial \chi(b_t)}{\partial b'} + E_{t-1} \frac{\beta R_{t+1}}{P_{t+1}/P_t} u_c^k(c_{t+1}^k), \quad (\text{A84})$$



with equality, if the credit constraint is not binding,  $\mu_t^c = 0$ .

Incomplete Market Household Problem: Linearized Euler Equation

In the main text, I assumed that  $\int \hat{c}_{it}^k di$  is approximated by  $\hat{Y}_t$ . Starting instead from the aggregate resource constraint, omitting government expenditure,

$$\int (c_{it}^k + c_{it}^b + c_{it}^m) di = Y_t \quad (\text{A85})$$

leads to approximating  $\int c_{it}^k di$  as

$$\hat{Y}_t + \omega \hat{p}_t, \quad (\text{A86})$$

since  $\int c_{it}^b di$  is increasing in the real value of bonds and thus decreasing in  $\hat{p}_t$ . Similarly, an interest rate rise in response to a price increase lowers  $\int c_{it}^m di$ . Thus, an increase in  $\hat{p}_t$  leads to lower consumption of both bonds and money goods and thus to higher capital good consumption, implying that  $\omega > 0$ , resulting in similar derivations as in the reduced form model when nominal government expenditure  $G > 0$ . Using this approximation yields

$$\hat{Y}_t + \omega \hat{p}_t = \tilde{\beta} E_t(\hat{Y}_{t+1} + \omega \hat{p}_{t+1}) + \psi(\hat{B}_t - \hat{p}_t) - \sigma(\hat{i}_t - (E_t \hat{p}_{t+1} - \hat{p}_t)). \quad (\text{A87})$$

Substitution of the optimal pricing rule (48) and using  $\hat{i}_t = 0$  and  $\hat{B}_t = 0$  yields a second-order difference equation

$$E_t \hat{p}_{t+1} = \frac{1 + \tilde{\beta} + \kappa(\sigma + \psi + \omega)}{\tilde{\beta}(1 + \kappa\omega) + \kappa\sigma} \hat{p}_t + \frac{-1}{\tilde{\beta}(1 + \kappa\omega) + \kappa\sigma} \hat{p}_{t-1}, \quad (\text{A88})$$

and equivalently, the transition matrix is

$$A \equiv \begin{pmatrix} \frac{1 + \tilde{\beta} + \kappa(\sigma + \psi + \omega)}{\tilde{\beta}(1 + \kappa\omega) + \kappa\sigma} & \frac{-1}{\tilde{\beta}(1 + \kappa\omega) + \kappa\sigma} \\ 1 & 0 \end{pmatrix}.$$

Since

$$\text{trace}(A) - \det(A) = \frac{\tilde{\beta} + \kappa(\sigma + \psi + \omega)}{\tilde{\beta}(1 + \kappa\omega) + \kappa\sigma} = 1 + \frac{\kappa\psi + (1 - \tilde{\beta})\kappa\omega}{\tilde{\beta}(1 + \kappa\omega) + \kappa\sigma} > 1, \quad (\text{A89})$$

the characteristic polynomial  $q(x)$  is strictly negatively evaluated at 1,  $q(1) < 0$ , since  $\psi > 0$ . Since  $q$  eventually rises to infinity in  $x$ , one eigenvalue,  $\lambda_1$ , is real and larger than one. If  $\det(A) < 1$  the other eigenvalue,  $\lambda_2$ , is smaller than one since  $\det(A) = \lambda_1 \lambda_2$ . If  $\det(A) > 1$ ,  $q(0) = \det(A) > 0$  so that  $0 < \lambda_2 < 1$  by the intermediate value theorem.

### A.II.3 Models in which the Price Level is Indeterminate

#### Price Level Indeterminacy: Hand-to-Mouth Consumers

The same basic arguments for the economy in which government bonds are not net wealth apply to models where a fraction of households always lives hand-to-mouth and the remaining ones behave according to the permanent income hypothesis (PIH). Since hand-to-mouth consumers do not participate in the asset market, the real interest rate is determined by the discount factor of PIH households only,  $(1 + r_{ss})\beta = 1$ , and equilibrium in the asset market is again characterized by

$$\frac{1 + i_{ss}}{1 + \pi_{ss}} = 1 + r_{ss} = 1/\beta, \quad (\text{A90})$$

which does not depend on the price level, implying that the price level is indeterminate. This model shows that it is not any form of heterogeneity by itself that delivers the result. Rather, it is the combination of heterogeneity and market incompleteness that makes government bonds net wealth, which leads to a well-defined aggregate savings function and implies price-level determinacy. By the same argument, permanent heterogeneity in productivity, but otherwise complete markets, will not lead to price level determinacy either, since again  $(1 + r_{ss})\beta = 1$  in a steady state.

#### Price Level Indeterminacy: Perpetual youth model

Similar arguments hold in “perpetual youth” models (Yaari (1965), Blanchard (1985)), since the steady-state interest rate is again equal to the discount rate, but now adjusted for the probability of death or retirement, so that  $(1 + r_{ss})\tilde{\beta} = 1$  in a steady state for the adjusted discount rate  $\tilde{\beta}$ . Again, the steady-state real interest rate is independent of the price level, which is not determinate.

In this class of models, this is however, not the only equilibrium if the Samuelson dynamic inefficiency condition is satisfied. In this case, both a bubbleless as well as a continuum of bubbly equilibria exist, a scenario explored in recent work by Galí (2017). Whereas most papers assume that the bubble is a real asset affecting the stock market or housing, a monetary bubble may coexist, so that money has value as in Samuelson’s work. As a result, there is a continuum of equilibria, each associated with a different value of money (= different size of the monetary bubble) and each is associated with a different price level. As an example, suppose that there is a bubble which has a real value of one. In one equilibrium, nominal money has a value of one, the price level is one and thus there are no real bubbles. In another equilibrium, the price level is two and a real bubble with a value of one half exists. Or the price level is three and the real bubble has a value of two thirds. Or the real bubble has a value of one and money has no value.

Bénassy (2005, 2008) make a specific choice as to the size of the monetary bubble through ruling out real bubbles (the first case in the previous example) and conditional on this choice, find a unique bubbly price level. This approach however, does not overcome the indeterminacy problem in the “bubbleless” equilibrium and it rules out by assumption other bubbly equilibria

with different price levels.<sup>25</sup> Bénassy (2005, 2008) need to make this equilibrium selection to obtain a well-defined demand for money (or more generally for nominal government liabilities), since the Samuelson logic only ensures the existence of a monetary equilibrium, but not uniqueness. This shows again, as in the Hand-to-Mouth economy, that the failure of Ricardian equivalence is a necessary but not sufficient condition for price-level determinacy.

### Price Level Indeterminacy: Representative Agent and Aggregate Risk

Price-level indeterminacy also arises in representative agent economies with aggregate risk. Suppose there are  $n$  aggregate states  $s_1, \dots, s_n$  with associated consumption levels of the representative household  $c_1, \dots, c_n$  and marginal utilities of consumption  $u_1, \dots, u_n$ . The FOC for nominal bonds are therefore

$$v_i := \frac{u_i}{\tilde{P}_i} = \beta \frac{1 + i_{ss}}{1 + \gamma} \sum_{j=1}^n q_{ij} \frac{u_j}{\tilde{P}_j},$$

where  $q_{ij} = \text{Prob}(s_j | s_i)$ ,  $\tilde{P}_i$  is the price level in state  $s_i$  as a deviation from the trend inflation rate  $\gamma = \pi_{ss}$ , which is equal to the constant growth rate of nominal debt. In matrix form the FOCs read

$$\underbrace{\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}}_v = \beta \frac{1 + i_{ss}}{1 + \gamma} \underbrace{\begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ \vdots & \ddots & & \vdots \\ q_{n1} & \cdots & & q_{nn} \end{pmatrix}}_Q \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

The vector  $v$  is an eigenvector with eigenvalue one of the matrix  $\beta \frac{1+i_{ss}}{1+\gamma} Q$  and the largest eigenvalue of  $Q$  is one with eigenvector  $(1, 1, \dots, 1)^{tr}$ . Since  $\beta \frac{1+i_{ss}}{1+\gamma} \leq 1$ , it follows that  $\beta \frac{1+i_{ss}}{1+\gamma} = 1$ . Therefore, for each  $\kappa > 0$ , the vector

$$\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} \kappa \\ \vdots \\ \kappa \end{pmatrix}$$

is a solution to the FOCs. This continuum of solutions corresponds to different price levels  $\tilde{P}_i > 0$ , establishing the indeterminacy, not surprisingly, since government bonds do not become net wealth just by adding aggregate uncertainty. Adding aggregate risk to an economy with PIH-households and hand-to-mouth households also fails to overcome the indeterminacy problem. The same arguments for the representative agent now apply to the PIH households.

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<sup>25</sup>Bénassy (2005, 2008) implicitly assume a particularly strong dynamic inefficiency condition - the population growth rate exceeds the real interest rate (which exceeds  $1/\beta$ ) - since consumption by the initial generation would eventually exceed GDP otherwise.