

Biasing Unbiased Dynamic Contests*

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Abstract

We consider a best-of-three Tullock contest between two ex-ante identical players. An effort-maximizing designer commits to a vector of player-specific biases (advantages or disadvantages). In our benchmark model the designer chooses victory-dependent biases (i.e., the biases depend on the record of matches won by players); the effort-maximizing biases eliminate the discouragement effect, leaving players equally likely to win each match and the overall contest. This is in contrast with the common “favor-the-leader” result in the literature. We compare our benchmark model with one where the designer chooses victory-independent biases; the effort-maximizing biases leave players unequally likely to win each match and the overall contest. This result holds in Tullock contests and all-pay auctions, as well as under maximization of total effort and winner’s effort. The appeal of this result comes from the players being ex-ante identical; thus, it challenges the conventional wisdom of the optimality of unbiased contests. This result also has an applied interest, as it shows that alternating biases, as when teams alternate home and away games, may increase total effort as opposed to an unbiased contest.

1 Introduction

A contest is a situation where players exert costly efforts to win a prize. We call a contest between players X and Y “unbiased” if swapping the efforts of X and Y implies swapping of the probabilities of winning.¹ Unbiased contests are prevalent in the literature, in part because of the “conventional wisdom” that unbiased contests are effort-maximizing when players X and Y are symmetric.² In particular, an unbiased contest between symmetric players leaves players equally likely to win in equilibrium, hence maximizing competition and thus efforts. Our contribution is to analyse whether and how the conventional wisdom of unbiased contest optimality carries over to a *dynamic* contest

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¹This property is alternatively called “anonymity” in the seminal axiomatization of Skaperdas (1996).

²This conventional wisdom has been extended to asymmetric players; if X is stronger than Y , effort maximization is achieved by giving a disadvantage to X so as to restore a level playing-field. It is in fact very intuitive that a level playing-field maximizes competition, and therefore efforts, in a one-shot symmetric environment. This conventional wisdom has a few exceptions that we discuss in the literature review.

between symmetric players. The workhorse of our analysis is a best-of-three unbiased contest between two ex-ante symmetric players; we ask if an effort-maximizing designer wants to introduce biases in an ex-ante unbiased contest, and how.

One of the most important aspects of dynamic contests is the *discouragement effect*; for instance, a player’s loss in the first match of the game gives her a one-match disadvantage in the second match, and when a contestant is sufficiently disadvantaged, competition — and thus effort — suffers.³ It is the asymmetry of the second-match continuation payoffs due to the first-match loss that generates the discouragement effect, which is thus highly intertwined with the conventional wisdom of unbiased contest optimality. Thus, as we investigate the conventional wisdom in a dynamic setting, we ask whether it is possible to mitigate the discouragement effect by means of biased matches, and if so, how.

Our theoretical analysis has relevant applied implications, as there are many real-life contests with the two features we analyse, namely; (i) dynamics, so that the victory of the contest occurs only if a player wins a sufficient number of matches, and (ii) varying biases, so that in each match a player may have or may be given a competitive advantage, or a pre-existing competitive advantage may be endogenously mitigated. Applications range from sports to business, from politics to procurements to R&D races.

Applications. In most of these applications the location of each match provides an advantage, and it is to some extent endogenous. *Sports*: home-field advantage is the benefit that the home team has over the visiting team because of psychological effects (supporting fans, referees, ...), physiological effects (the advantage home teams have playing near home in familiar situations, or the disadvantages away teams suffer from travelling), or strategic effects (such as the home team batting second in baseball).⁴ *Business*: Brown and Baer (2011) find that business professionals enjoy a similar home-field advantage in negotiations that take place in the comfort of their own offices. Local knowledge of market, social, economic and legal conditions is among the reasons that local e-retailers such as Lazada in Indonesia are able to hold off competition from the international giants Amazon and Alibaba, as reported by The Economist (“Home-field advantage,” 2015). Dahl and Sorenson (2012) find that entrepreneurs that locate in regions in which they have deep roots (“home” regions) perform better. *Politics*: an accepted fact of US vice presidential elections is that candidates win extra votes in their home states (e.g., Heersink and Peterson, 2016). One initial evidence of home-state advantage is provided by Lewis-Beck and Rice (1983). Heersink and Peterson (2016) find that the vice-presidential home-state advantage “could have swung four presidential elections since 1960, if presidential candidates had chosen running mates from strategically optimal states.” Moreover, a presidential candidate born and raised in one state is likely to have a home advantage in that state, as shown by the fact that four candidates (James K. Polk, Woodrow Wilson, Richard Nixon, and Donald Trump) have won the presidency despite losing their state of residence.⁵ *Procurement*: a source of home-field advantage is the governments’ preference for domestic over foreign suppliers, ceteris paribus; e.g., Brullhart and Trionfetti (2004) and references therein. This is considered to be a serious and pervasive issue; as reported by Laffont and Tirole

³For empirical evidence of discouragement effect see, e.g., Malueg and Yates (2010).

⁴Empirical evidence abounds for instance in soccer (Nevill and Holder, 1999) and for the NFL (Vergin and Sosik, 1999). Jamieson (2010) provides a general meta-study of home-field advantages.

⁵Besides presidential elections in the United States, evidence of home-state advantage has also been provided in “gubernatorial, senatorial, and statewide judicial elections in the United States, parliamentary elections in Ireland, Brazil, and the United Kingdom and local elections in New Zealand and the United States” (see p. 1, Meredith, 2013, original references omitted).

(1991): “[...]the European Economic Commission, alarmed by the abnormally large percentage (above 95% in most countries) of government contracts awarded to domestic firms is trying to design rules that would foster fairer competition between domestic and foreign suppliers.” *R&D races*: a source of advantage arises from players’ special expertise. The Joint Strike Fighter contest is a famous instance; Boeing and Lockheed Martin, the two main contenders, competed on the development of the components needed to submit a winning prototype of a Joint Strike Fighter. Lockheed had a substantial advantage in warplane expertise, while Boeing had an advantage in developing wings (see, NOVA, 2003). Lockheed eventually won, and its warplane expertise played a key role; “Lockheed may have clinched the deal because of its experience with the development and production of the F-22.” (Frost & Sullivan Market Insight, 2001).

In the above applications, competition often unfolds over time, through a series of interim matches where players’ advantages vary. At each match, the *extent* to which the contest designer has control over the biases varies. For example, in sports, a home-field advantage can be mitigated by introducing instant replays to weaken the referees’ discretion or by reserving a certain proportion of tickets for the visiting team. In procurement, the advantage to home suppliers might be quite explicitly mandated and carefully targeted, as for “[...]the United States’s ‘Buy American Act’, which in different cases requires U.S. suppliers to be chosen despite cost disadvantages of up to 6 (normal), 12 (small businesses and firms in regions of high unemployment) or 50 percent (military equipment). Explicit national preferences are also applied in Canada, Australia and New Zealand [...]” (Vagstad, 1995).

In order to account for such varying control over biases across applications, we consider two alternative settings. First, in our benchmark model the designer has full control over the biases (Section 4), in the sense that biases are contingent on the outcome of previous matches; for instance, if X wins (or loses) today, she will be given a disadvantage (or advantage) tomorrow. We find that the effort-maximizing designer leaves the first and (if necessary) third matches unbiased, and biases the second match in favor of the loser of the first match, so as to compensate for her disadvantage of lagging one match behind. Such a structure of biases leaves players equally likely to win each match and the entire contest. In this sense, we conclude that the conventional wisdom of equalizing players’ equilibrium winning probabilities *carries over* to a dynamic setting when the designer can tailor the biases to the outcome of previous matches. This result contributes to our understanding of whether, in subsequent matches, one should favor the winner or the loser of early matches. In particular, this result contrasts with the common “favor-the-leader” result in the literature that we discuss below.

Second, in the more realistic case of limited control (Section 5), the designer cannot tailor the biases to the outcome of previous matches; for instance, whoever plays at home today plays away tomorrow, regardless of who wins today. We find that the effort-maximizing designer alternates the player receiving the advantage, rather than leaving the contest unbiased. To understand the intuition behind our result, recall that in dynamic contests early victories distort future matches so that the laggard gives up and the front-runner eases up.⁶ This well-known result is mitigated in our context by alternating biases, which gives one player the advantage in the first match and the other player the advantage in the second. Such an alternation of biases, despite creating a bias in the first match — thus reducing first-match efforts — balances the second match since the most likely second-match laggard is given an advantage — thus increasing second-match expected efforts.

⁶See the “momentum effect” (Klumpp and Polborn, 2006), the “discouragement effect” (Konrad, 2009), or “avalanche effect” (Beviá and Corchón, 2013).

Most importantly, it increases the probability of the game reaching the tie-breaker — thus increasing third-match efforts.⁷ We show that the second and third positive effects overcome the first negative effect. The optimality of introducing (alternating) biases into a symmetric contest between ex-ante identical players is a key finding of the present paper, hence we challenge its robustness and show that it holds when matches are modeled both as Tullock contests and all-pay auctions, and both under maximization of total effort and winner’s effort.⁸ We conclude that the conventional wisdom of unbiased contest optimality *does not carry over* to a dynamic setting when the designer cannot tailor biases to the outcome of previous matches.

Literature. There are two strands of the literature that are usually — but not exclusively — kept apart; dynamic contests and biased contests. On *dynamic* contests, one of the groundbreaking theoretical contribution is Klumpp and Polborn (2006); they model the US primaries as a best-of- n contest between two candidates, where the battlefield in each state takes the form of a Tullock contest. As it is typical in tournaments, they find that the outcome of the very first match creates an asymmetry between ex-ante symmetric players that is endogenously carried over to later periods. This momentum boosts efforts in the first matches and make the latest matches less relevant. This finding resemble the so-called “New Hampshire effect”; candidates who win early primaries are more likely to win later primaries, too.⁹ Similar findings are those of Konrad and Kovenock (2009), who model stage-battlefields as all-pay auctions, and of Ferrall and Smith (1999) who adopt rank-order tournaments of Lazear and Rosen (1981). Malueg and Yates (2006) generalize Klumpp and Polborn’s (2006) results to a general symmetric contest success function and derive results for a three-battle contest assuming the existence of a pure strategy equilibrium.¹⁰ All the above models, as the vast majority of the literature, restrict attention to either an ex-ante level playing-field or exogenous advantages. Our main point of departure from this strand of the literature is the introduction of endogenous biases.

The literature on *biases* in static contest is extensive. The conventional wisdom typically drawn from this strand of the literature is the optimality of an unbiased contest if the two players are symmetric.¹¹ A prominent recent exception to this wisdom of static contests arises in Drugov and Ryvkin (2017);¹² they characterize properties of the contest success function and of the cost of effort that determine whether a biased or unbiased contest is optimal. In the standard Tullock contest with linear costs of effort, multiplicative bias and static competition, Drugov and Ryvkin (2017)

⁷Note that the third match is virtually certain to occur if an arbitrarily high advantage is given in the first period to a player and in the second period to her rival.

⁸For the all-pay auction we need to add the technical assumption of private information because otherwise rents would be fully dissipated, thus making our exercise uninteresting.

⁹A victory in the New Hampshire primary increases a candidate’s expected share of total primary votes by 26.6% (Mayer, 2004).

¹⁰Empirical tests of theoretical predictions with sports data is provided for best-of-three contests by Malueg and Yates (2010) and for best-of- n by Ferrall and Smith (1999). In the experimental literature, a test of best-of-three Tullock contests is provided by Mago et al. (2013).

¹¹A non-exhaustive list of papers is: Dukerich et al., 1990; Schotter and Weigelt, 1992; Nti, 2004; Fu, 2006; Fain, 2009; Epstein et al., 2011; Franke, 2012; Franke et al., 2013; Lee, 2013, and in general references in Drugov and Ryvkin (2017). Serena (2017) shows that the conventional wisdom holds not only for maximization of total effort, but also winner’s effort.

¹²Other exceptions to the conventional wisdom can be derived from an extension of the model to an ex-ante heterogenous n -player setting (e.g., Franke et al., 2013), to a private information setting (e.g., Pérez-Castrillo and Wettstein, 2016), and to maximization of the probability of a high-ability winner (e.g., Kawamura and Moreno de Barreda, 2014). In the present paper, we keep the standard two-player complete information setting under effort-maximization.

show that the conventional wisdom of unbiased contest optimality is not robust to dropping the assumption of multiplicative bias. In contrast, we maintain the multiplicative biases, but challenge the conventional wisdom in a dynamic, rather than static, setting. We find that when the designer can (cannot) tailor the biases to the outcome of previous matches, players are optimally left equally (unequally) likely to win each match and the overall contest.

We are not the first to consider biases in dynamic contests. A first strand of the literature focuses on dynamic effort elicitation and on a dynamic inference problem due to the extra information that may or may not be disclosed as the dynamic contest unravels (see Meyer, 1991, 1992; Lizzeri et al., 1999, 2002; Höfler and Sliwka, 2003; Aoyagi, 2010; Ederer, 2010). A common finding is the “favor-the-leader” result; for example, Meyer (1991, 1992) shows that biasing the second match in favor of whoever performed better in the first match tends to be beneficial for the principal in terms of better information and of larger efforts, as one obtains only a second-order negative effect of bias in the second period, but a first-order positive effect of bias in the first period. The different result we obtain is driven by our best-of-three setting, rather than the typical two sequential contests in this strand of the literature. Another issue that this set of papers analyses is whether disclosure of interim performance is beneficial or detrimental to the expected aggregate effort; the trade-off here is that if the first-period outcome leaves the players sufficiently symmetric (asymmetric), then disclosure levels (unlevels) the playing field. More recently, Beviá and Corchón (2013) consider a two-period Tullock contest where the first-period effort yields a player’s second-period advantage; in particular, they propose an exogenous mapping between first-period efforts and second-period advantage so as to capture situations such as wars in which the strength of a country depends on the fraction of the territory owned by this country. Similarly to Beviá and Corchón (2013), Clark et al. (2012), Möller (2012), and Esteve-González (2016) analyse a dynamic setting where the outcome of the first period generates an asymmetry between players in a second Tullock contest. In contrast to this strand of the literature, in our model the victory of any match only gives the winner a one match advantage, and does not give her more advantageous future battlefields (modeled by the above literature as a greater probability of victory or as a lower marginal cost for given efforts). In our model, more advantageous future battlefields are determined ex-ante by the contest designer.

2 Model

Two risk-neutral and ex-ante identical players, X and Y , play in a best-of-three contest. That is, they play at most three matches, and the first player who wins two matches is the contest winner and obtains a prize equal to $V > 0$. The game begins at node $(0, 0)$, where no player has won a match; here efforts are denoted as $x^{(0,0)}$ and $y^{(0,0)}$. The game then moves to node $(1, 0)$ or $(0, 1)$ according to whether X or Y , respectively, wins the first match in $(0, 0)$; here efforts are denoted as $x^{(1,0)}$ and $y^{(1,0)}$, or $x^{(0,1)}$ and $y^{(0,1)}$. If the first two matches are won by the same player the game ends, otherwise the game reaches node $(1, 1)$ and the third match is played; here efforts are denoted as $x^{(1,1)}$ and $y^{(1,1)}$. In each node (i, j) , if reached, players simultaneously choose efforts $(x^{(i,j)}, y^{(i,j)})$, and the probability of victory of player X in that match depends on the contest technology as follows:

$$p_X^{(i,j)}(x^{(i,j)}, y^{(i,j)}) = \frac{\alpha_{(i,j)} x^{(i,j)}}{\alpha_{(i,j)} x^{(i,j)} + y^{(i,j)}}, \quad (1)$$

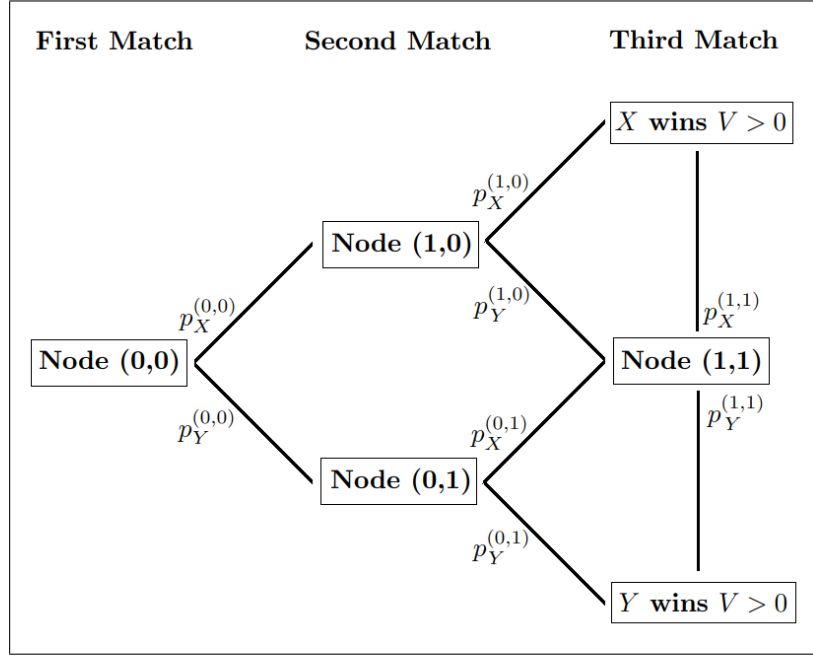


Figure 1: Nodes and matches in a best-of-three contest between players X and Y .

and $p_Y^{(i,j)}(x^{(i,j)}, y^{(i,j)}) = 1 - p_X^{(i,j)}(x^{(i,j)}, y^{(i,j)})$.¹³ In Figure 1 we draw the structure of the best-of-three contest we analyse. We refer to $\alpha_{(i,j)} > 1$ ($\alpha_{(i,j)} < 1$) as an *advantage* (*disadvantage*) given to X in node (i, j) . It is without loss of generality to multiply only x , and not y , with a bias. A vector of biases $\{\alpha_{(0,0)}, \alpha_{(1,0)}, \alpha_{(0,1)}, \alpha_{(1,1)}\} \in [0, \infty)^4$ becomes commonly known at the beginning of the game. There is complete information and the marginal cost of effort equals 1 for both players. We analyse how the expected total effort (henceforth, TE) varies with the α 's. We define TE as follows

$$\begin{aligned}
 TE \equiv & \left[x^{(0,0)} + y^{(0,0)} \right] \\
 & + p_X^{(0,0)} \left[x^{(1,0)} + y^{(1,0)} \right] + p_Y^{(0,0)} \left[x^{(0,1)} + y^{(0,1)} \right] \\
 & + \left(p_X^{(0,0)} p_Y^{(1,0)} + p_Y^{(0,0)} p_X^{(0,1)} \right) \left[x^{(1,1)} + y^{(1,1)} \right].
 \end{aligned} \tag{2}$$

For simplicity, in (2) we omitted the arguments of probabilities and efforts, as we will do throughout the paper whenever this does not yield confusion.

Structure of the paper. In Section 3 we provide some general preliminary results that apply to each node and that we specialize node-by-node in Appendix A. In our benchmark model of

¹³ Any $p_X^{(i,j)}(0,0) \in (0,1)$ specified using one of the “usual” tie-breaking rules for no efforts will leave our results unchanged.

Section 4 α 's are *victory-dependent*; that is, a possibly different α is chosen for each node, so that four biases $\{\alpha_{(0,0)}, \alpha_{(1,0)}, \alpha_{(0,1)}, \alpha_{(1,1)}\}$ are chosen in order to maximize TE . We find that the optimal α 's leave the two players equally likely to win in every node and in the entire best-of-three contest.¹⁴ In Section 5, instead, α 's are *victory-independent*; that is, α 's cannot be conditioned on the outcome of the previous matches, so that three biases $\{\alpha_1, \alpha_2, \alpha_3\}$ are chosen in order to maximize TE ; in the first match (i.e., node $(0, 0)$) player X is given bias α_1 , in the second match (i.e., node $(1, 0)$ or $(0, 1)$) she is given bias α_2 , and in the third match (i.e., node $(1, 1)$), if played, she is given bias α_3 . The extra constraint of victory-independent α 's is inspired by the applications, as discussed in the Introduction. We find that the optimal α 's leave players unequally likely to win in every node and in the entire best-of-three contest; in particular, it is optimal to give a large advantage to a player, say X , in the first match, and balance it out with a medium advantage to player Y in the second match and a small advantage to player Y in the third match, if necessary.¹⁵ Since we deem this result the most interesting of the paper, we show that it is robust to:

1. Replacing the objective function (2) with the expected winner's effort (Appendix D);
2. Replacing the Tullock model (1) with an all-pay auction model with multiplicative bias (Appendix E); and
3. Combining the above two replacements (Appendix F).

The interest in total effort or the expected winner's effort crucially depends on the specific application one has in mind. In sport contests the audience might find a lackluster of performance of the teams disappointing, thus total effort maximization is a suitable objective. In contrast, in a research contest only the winner's project is typically implemented and hence only the winner's effort is beneficial for the contest designer (see also Serena, 2017).

3 Preliminaries

Denoting with u_X^W and u_Y^W the expected (continuation) payoff of player X and Y in case of winning, and denoting with u_X^L and u_Y^L the expected (continuation) payoff of player X and Y in case of losing, the individual payoff u_X of player X in a general node with bias α reads

$$u_X = \frac{\alpha x}{\alpha x + y} u_X^W + \left(1 - \frac{\alpha x}{\alpha x + \alpha y}\right) u_X^L - x = \frac{\alpha x}{\alpha x + y} (u_X^W - u_X^L) + u_X^L - x.$$

Defining the "effective prize spread" as

$$\Delta u_X \equiv u_X^W - u_X^L,$$

we obtain

$$u_X = \frac{\alpha x}{\alpha x + y} \Delta u_X + u_X^L - x.$$

Similarly, for player Y , defining the effective prize spread as $\Delta u_Y \equiv u_Y^W - u_Y^L$, we obtain

$$u_Y = \frac{y}{\alpha x + y} \Delta u_Y + u_Y^L - y.$$

¹⁴The complete analytical derivation is provided in Appendix B.

¹⁵The complete analytical derivation is provided in Appendix C.

The equilibrium is uniquely identified by the FOCs, which give the typical property

$$y = \frac{\Delta u_Y}{\Delta u_X} x;$$

the equilibrium efforts

$$x = \frac{\alpha \Delta u_X^2 \Delta u_Y}{(\alpha \Delta u_X + \Delta u_Y)^2}, \quad (3)$$

$$y = \frac{\alpha \Delta u_X \Delta u_Y^2}{(\alpha \Delta u_X + \Delta u_Y)^2}; \quad (4)$$

the equilibrium probabilities of victory

$$p_X = \frac{\alpha \Delta u_X}{\alpha \Delta u_X + \Delta u_Y}, \quad (5)$$

$$p_Y = \frac{\Delta u_Y}{\alpha \Delta u_X + \Delta u_Y}; \quad (6)$$

and the equilibrium payoffs

$$u_X = \Delta u_X p_X^2 + u_X^L, \quad (7)$$

$$u_Y = \Delta u_Y p_Y^2 + u_Y^L. \quad (8)$$

Summing (3) and (4), and using (5) and (6), we obtain the following property, which we use repeatedly in our proofs,

$$x + y = (\Delta u_X + \Delta u_Y) \cdot p_X p_Y. \quad (9)$$

In Appendix A, we specialize the above analysis for each node; (0, 0), (1, 0), (0, 1), and (1, 1), substituting the appropriate continuation value in each node.

4 Benchmark: Victory-dependent biases

The problem of maximizing TE when a possibly different α is chosen at each node has four choice variables, $\{\alpha_{(0,0)}, \alpha_{(1,0)}, \alpha_{(0,1)}, \alpha_{(1,1)}\}$. We obtain the following result.

Proposition 1 *Consider a best-of-three Tullock contest between two ex-ante identical players. With victory-dependent biases, the point $\{\alpha_{(0,0)}, \alpha_{(1,0)}, \alpha_{(0,1)}, \alpha_{(1,1)}\} = \{1, 1/3, 3, 1\}$ is the unique global maximum for TE in $\mathbb{R}_{>0}^4$.*

Proof. See Appendix B. ■

The “asymmetric” nodes (1, 0) and (0, 1), where one player is leading by one match, are affected by the discouragement effect typically present in dynamic contests without endogenous biases (e.g., Konrad, 2009). With endogenous biases, Proposition 1 shows that it is optimal in these asymmetric nodes to give the laggard an advantage; namely, if X loses the first match, the game reaches node (0, 1) and X is given an advantage of $\alpha_{(0,1)} = 3$, while if X wins the first match, the game reaches

node $(1, 0)$ and X is given a disadvantage of $\alpha_{(1,0)} = 1/3$. These α 's eliminate the competitive unbalancedness due to the discouragement effect and leave players equally likely to win the second-match, as Corollary 2 shows. The “symmetric” nodes $(0, 0)$ and $(1, 1)$, where players have identical continuation values and have won an identical number of matches, are optimally left unbiased by setting $\alpha_{(0,0)} = \alpha_{(1,1)} = 1$. Therefore, the unique global maximum of Proposition 1 leaves players equally likely to win each match and the entire contest;

Corollary 2 *The optimal vector of biases of Proposition 1 yields $p_X^{(i,j)} = 1/2$ with $(i, j) \in \{0, 1\}^2$.*

Proof. See Appendix B. ■

The result in Proposition 1 and Corollary 2 may at first appear intuitive. However, further reflection reveals it to be surprising. Intuitively, if today’s victory grants an advantage tomorrow, a player fights fiercely today so as to require less effort to win tomorrow. This simple intuition has been extensively analysed by the literature. The general finding, stemming from Meyer (1992), is that the second match should be biased in favor of the winner of the first match in order to increase total efforts because, “starting with no bias, the introduction of a small amount in favor of the first-period winner generates a first-order increase in first-period incentives, but only a second-order reduction in second-period incentives.”¹⁶ This strand of the literature would suggest leaving some small advantage in the second match to the winner of the first match by setting $\alpha_{(0,1)} = 3 - \varepsilon$ and symmetrically $\alpha_{(1,0)} = 1/(3 - \varepsilon)$, which would result in $p_X^{(0,1)} < \frac{1}{2}$ and $p_X^{(1,0)} > \frac{1}{2}$. Indeed, one can verify that in our setup this would result in an increase in first-period efforts and a negligible decrease in second-period efforts, in line with the literature. However, in our *best-of-three* setup, a new force arises: one needs to account for the probability of reaching node $(1, 1)$, which decreases with ε . Proposition 1 shows that this effect overwhelms the well-known increase in first-period efforts that setting $\varepsilon > 0$ generates.

Thus, the conventional wisdom of static contests, namely to leave players equally likely to win in equilibrium, carries over to each node and to the entire contest in our dynamic model. However, this is clearly possible since the designer can tailor the biases to the outcome of previous matches, so as to keep competition fierce at all nodes and to eliminate the discouragement effect. In the remainder of the paper we characterize and discuss the optimal vector of biases when the designer cannot tailor the biases to the outcome of previous matches.

5 Victory-independent biases

Under victory-independent α 's the model has the constraint $\alpha_{(i,j)} = \alpha_{i+j+1}$ with $i, j \in \{0, 1\}$; that is, α_{i+j+1} is given to player X in the $(i + j)^{th}$ -match *regardless* of the outcome of previous matches. Thus, the vector of biases used to maximize TE boils down to $\{\alpha_1, \alpha_2, \alpha_3\}$; one bias per match. Under such a constraint, it is clearly impossible to induce an equal equilibrium probability of winning across players at each node of the contest. An easy way to see this is that since α_2 cannot depend on the outcome of the first match, one can set α_2 so as to achieve at most one between $p_X^{(1,0)} = 1/2$ and $p_X^{(0,1)} = 1/2$. In this sense, the discouragement effect cannot be eliminated as in Section 4, but can be at most mitigated.

¹⁶The idea that biasing tomorrow’s playing field in favor of today’s winner enhances efforts has been explored by many scholars after Meyer, including some recent contributions; e.g., Clark et al. (2012), Möller (2012), Beviá and Corchón (2013), Ridlon and Shin (2013), Esteve-González (2016) and Klein and Schmutzler (2017).

An equal equilibrium probability of winning across players at each node implies an equal equilibrium probability of winning the entire contest. Since the former can no longer be achieved, it is natural to ask whether the latter is optimal; that is, whether effort-maximization is achieved at $\alpha_1 = \alpha_2 = \alpha_3 = 1$. For this purpose, we define,

Definition 3 A contest is **fully unbiased** if $\{\alpha_1, \alpha_2, \alpha_3\} = \{1, 1, 1\}$.

The main result of this section, formally proven in Proposition 5, is that a fully unbiased contest is not optimal. In order to pinpoint the economic forces behind the result, we focus on a particular contest structure that improves upon a fully unbiased contest, namely,

Definition 4 A contest is **alternating** if $\{\alpha_1, \alpha_2, \alpha_3\} = \{\alpha, 1/\alpha, 1\}$ with $\alpha \neq 1$.

This structure resembles a typical practice observed, for instance, in sports; namely, the alternation of home matches between X and Y , followed by an unbiased tie-breaker (i.e., the final match).

Intuition. Before the formal statement of the result, we provide the intuition behind the suboptimality of a fully unbiased contest by showing that TE increases as we move from a fully unbiased contest to an alternating contest (i.e., setting $\alpha > 1$ wlog). When we do so, three effects arise,

1. The *first effect* concerns the first match: node $(0,0)$. In a fully unbiased contest the first-match continuation value is identical across players. Thus, since players are ex-ante identical, an alternating contest creates an asymmetry in the continuation value that unbalances the first match, hence reducing efforts. Thus, the first effect on efforts is *negative*.
2. The *second effect* concerns the second match: node $(1,0)$ or $(0,1)$. In an alternating contest player X is given an advantage in the first match, and thus she is the most likely winner of the first match, but she is also disadvantaged in the second match. Hence, the effect of her second-match disadvantage is more likely to attenuate than exacerbate her lead; in other words, the second-match bias is more likely to help the second-match laggard rather than the second-match leader. Thus, the second effect on efforts is *positive*.
3. The *third effect* concerns the third match: node $(1,1)$. This node is not necessarily reached, and its probability of being reached is what drives its effect on TE ; a greater bias given first to X and then to Y increases the probability of reaching $(1,1)$. This is easy to see for extreme biases ($\alpha \rightarrow \infty$), where node $(1,1)$ is reached with certainty. Total effort increases with the existence of $(1,1)$. Thus, the third effect on efforts is *positive*.

All in all, the second and third effects are positive and the first is negative. In our intuition we neglect the second effect since the third effect by itself suffices to overcome the first. In words, we show that, when moving from a fully unbiased contest to an alternating contest, the beneficial effect on efforts of increasing the probability of existence of node $(1,1)$ overcomes the decrease in efforts at node $(0,0)$.

Unfortunately, a local, first-order intuition around $\alpha = 1$ in an alternating contest is not helpful to draw conclusions. The reason is that both the first and the third effects have zero derivative

at $\alpha = 1$.¹⁷ In fact, the probability of existence of node (1,1) and the total efforts in (0,0) have zero derivative at $\alpha = 1$. While the latter fact is well known, the former can be intuitively seen, disregarding effort changes, in the probability of existence of node (1,1),

$$\forall (i, j) \in \{0, 1\}^2, \Pr \left\{ \exists(1, 1) \mid x^{(i, j)} = y^{(i, j)} \right\} = \underbrace{\frac{\alpha}{\alpha + 1}}_{p_X^{(0,0)}} \underbrace{\frac{\alpha}{\alpha + 1}}_{p_Y^{(1,0)}} + \underbrace{\frac{1}{\alpha + 1}}_{p_Y^{(0,0)}} \underbrace{\frac{1}{\alpha + 1}}_{p_X^{(0,1)}} = \frac{\alpha^2 + 1}{(\alpha + 1)^2},$$

which has zero first derivative at $\alpha = 1$.

Since neither a first-order derivation nor intuition suffices to show that the third effect dominates the first, in the analytical proofs we derive and compare second-order effects (Appendix C-D-E-F-G), and in the intuition we compare a fully unbiased contest to an alternating contest that is significantly away from a fully unbiased contest, but for which calculations remain simple; that is, an alternating contest with $\alpha = 2$. We visualize the difference between $\alpha = 1$ and $\alpha = 2$ in Figure 2 and Figure 3. The formulae of Section 3 help the reader follow the mathematics behind the reasoning. Consider, for example, node (1,0) in both figures. The values of $\Delta u_X^{(1,0)}$ and $\Delta u_Y^{(1,0)}$ are simply the differences in utilities at nodes (2,0) and (1,1), while the values of $p_X^{(1,0)}$, $u_X^{(1,0)}$ and $u_Y^{(1,0)}$ are computed using (5), (7) and (8). The values at all the other nodes are similarly computed.¹⁸ At the bottom of each figure, the total efforts at node (0,0) are calculated using (9) and are labeled as “first effect.” The label “third effect” describes the total efforts at node (1,1) weighted by the probability of existence of node (1,1). Comparing figures 2 and 3, it is now easy to verify that, when moving from a fully unbiased to an alternating contest with $\alpha = 2$, the beneficial effect on efforts of increasing the probability of existence of node (1,1) is $\frac{1}{7} - \frac{1}{8} \simeq 0.018$, and the decrease in efforts at node (0,0) is $\frac{21}{64} - \frac{26}{81} \simeq 0.007$. Thus, the former is greater than the latter.

Optimal Alternating Contest. While an alternating contest with $\alpha = 2$ sufficed to show that a fully unbiased contest can be improved upon by an alternating contest, deriving the optimal alternating contest is of independent interest. The TE -maximizing alternating contest has $\alpha \simeq 4.21$. To stress once again that this result relies on the probability of existence of (1,1), consider the effort difference between the optimal alternating contest and the fully unbiased contest, *conditional* on reaching each node. This effort difference is $-0.063V$ in node (0,0), $+0.055V$ in node (1,0), $-0.119V$ in node (0,1), and clearly 0 in node (1,1). Hence, if we consider efforts *regardless* of probabilities, the effect of biasing a fully unbiased contest would always be negative since node (0,0) happens with probability 1. This fact highlights once again the key role played by the probabilities of reaching node (1,1) in driving our result. In particular, to get an intuitive feeling of the magnitude of probability changes, note that moving from a fully unbiased to the optimal alternating contest, the probability of existence of node (1,1) increases from 0.25 to 0.41.

The qualitative conclusions drawn from the numerical examples of Figure 2 and Figure 3 are proven generally true in Proposition 5, which shows that $\{\alpha_1, \alpha_2, \alpha_3\} = \{1, 1, 1\}$ is a saddle rather than a maximum for TE . In the proof we directly make use of the alternating contests to show

¹⁷Also, the second effect has zero derivative at $\alpha = 1$, but neglecting the second effect suffices for our purposes, as we have mentioned.

¹⁸In Figure 3, we approximated the values in first and second matches, without losing the qualitative features of any comparisons.

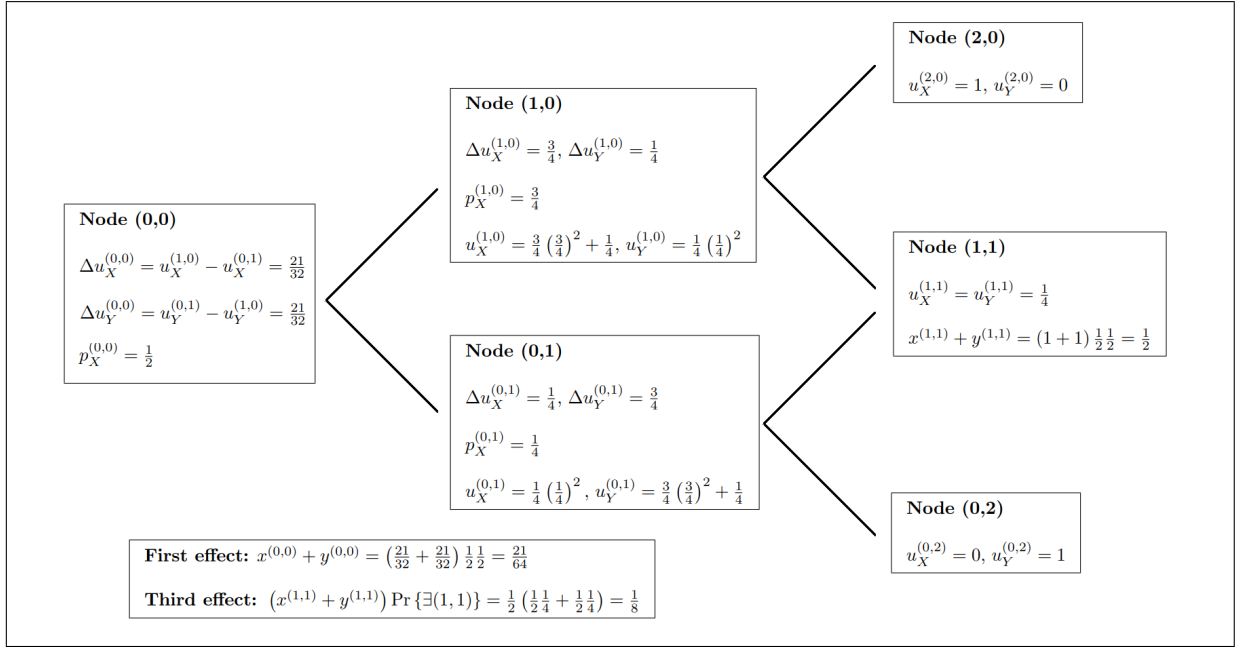


Figure 2: Fully unbiased contest under $V = 1$.

that a fully unbiased contest can be improved upon.¹⁹

Proposition 5 Consider a best-of-three Tullock contest between two ex-ante identical players. With victory-independent biases, the point $\{\alpha_1, \alpha_2, \alpha_3\} = \{1, 1, 1\}$ is a saddle for TE in $\mathbb{R}_{>0}^3$.

In Appendix C we provide an extensive analytical derivation of the result. Here, we provide a short proof that calculates an analytical expression for TE that is best checked by a computer software, such as Mathematica.²⁰

Proof. We consider moving away from $\{\alpha_1, \alpha_2, \alpha_3\} = \{1, 1, 1\}$ in two directions. The first direction is $\{\alpha_1, \alpha_2, \alpha_3\} = \{1 + b, 1, 1\}$; since the continuation values of the first match are identical between players, the players' incentives in the first match are the same as in a one-shot contest, where it is well known that $b = 0$ gives a local maximum for total effort (e.g., Franke, 2012). The second direction is $\{\alpha_1, \alpha_2, \alpha_3\} = \{1 + b, 1 - b, 1\}$;²¹ we show below that along this direction $b = 0$ is a local minimum. This establishes that in the $\{\alpha_1, \alpha_2, \alpha_3\}$ -space, $\{1, 1, 1\}$ is a saddle.

¹⁹Consider how TE changes with α in an alternating contest. The first derivative of each of the three effects is 0, thus the second derivative is needed to shed light on the ranking of the three effects, as we do in the proof of Proposition 5. We can quantify the three effects by plugging $\alpha = 1$ in the second derivative with respect to α of the first effect, obtaining $-0.071V$, of the second effect, obtaining $0.017V$, and of the third effect, obtaining $0.087V$. Therefore, we obtain that; i) the overall second derivative is positive, proving our claim, and ii) the third effect suffices to overcome the first effect, which is the only negative one.

²⁰The code is available from the authors upon request. *Note for the referees: the code is attached to our submission.*

²¹This is an algebraically convenient linearization of the alternating contest structure.

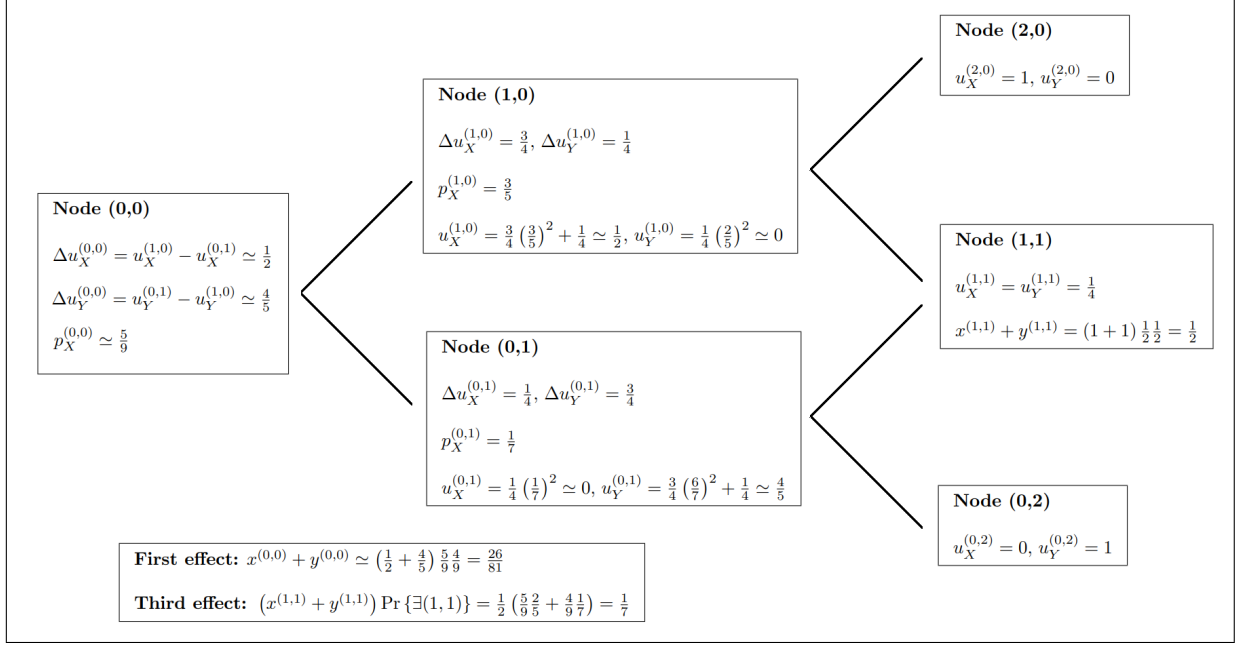


Figure 3: Alternating contest under $V = 1$ and $\alpha = 2$.

Plugging $\{\alpha_1, \alpha_2, \alpha_3\} = \{1 + b, 1 - b, 1\}$ into the formulae in Section 3, the expected total effort is:

$$TE(b) = \frac{4592 - 22960b + 45873b^2 + o(b^2)}{256(28 - 140b + 279b^2 + o(b^2))}.$$

Now define the numerator $n(b) \equiv 4592 - 22960b + 45873b^2 + o(b^2)$ and the denominator $d(b) \equiv 256(28 - 140b + 279b^2 + o(b^2))$, then

$$TE'(b) = \frac{n'(b)d(b) - n(b)d'(b)}{[d(b)]^2}.$$

Thus,

$$\begin{aligned} TE'(0) &= \frac{n'(0)d(0) - n(0)d'(0)}{[d(0)]^2} \\ &= \frac{22960 * 256 * 28 - 4592 * 256 * 140}{[d(0)]^2} \\ &= 0, \end{aligned}$$

so $b = 0$ is a critical point in this direction. We then consider the second derivative and obtain

$$\begin{aligned}
TE''(0) &= \frac{[n''(0)d(0) - n(0)d''(0)] [d(0)]^2 - 2d(0) [n'(0)d(0) - n(0)d'(0)]}{[d(0)]^4} \\
&= \frac{[n''(0)d(0) - n(0)d''(0)]}{[d(0)]^2} \\
&= 2 \frac{45873 * 256 * 28 - 4592 * 256 * 279}{[d(0)]^2} \\
&= 2 \frac{256 * 3276}{[d(0)]^2} > 0.
\end{aligned}$$

Thus, $b = 0$ is a local minimum. ■

Robustness of Proposition 5. The optimality of biasing a contest between symmetric players (Proposition 5) is the result of the present paper that mostly diverges from the existing literature. This is a call for robustness tests. First, while we followed the vast majority of the literature on contests and maximize TE , in Appendix D we show that Proposition 5 carries over to the maximization of the expected winner's effort. Second, while the Tullock contest is broadly used, another well-studied model of contest is that of all-pay auctions; in Appendix E we show the robustness of Proposition 5 to an all-pay-auction model. Third, in Appendix F we merge the two previous robustness tests and show that Proposition 5 carries over to the maximization of the expected winner's effort in an all-pay-auction model.

Globally Optimal Contest. Define the *globally optimal* contest as the best-of-three contest with TE -maximizing biases $\{\alpha_1, \alpha_2, \alpha_3\}$. In the remainder of this section, we provide numerical features of the globally optimal contest, which is of independent interest.

Numerical simulations show that the globally optimal contest has

$$\{\alpha_1, \alpha_2, \alpha_3\} \simeq \{5.22, 0.33, 0.75\}.$$

Thus, it is optimal to give a large (approx. 5) advantage to player X in the first match, and balance it out with a medium (approx. 3) advantage to player Y in the second match and a small (approx. 4/3) advantage to player Y in the third match, if necessary. This structure resembles an alternating contest. Moreover, the optimal alternating contests already attains 81% of the improvement achieved by the globally optimal contest over the fully unbiased contest.²² Reasonably, in the optimal alternating contest $\alpha \simeq 4.21$, which is in-between the first-match advantage (approx. 5) and the second-match advantage (approx. 3) of the globally optimal contest.

Next, we analyze the globally optimal contest through the light of the equilibrium winning probabilities. In particular, evaluating p_X at each node, we obtain

$$p_X^{(0,0)} \simeq 0.696, \quad p_X^{(0,1)} \simeq 0.082, \quad p_X^{(1,0)} \simeq 0.450, \quad p_X^{(1,1)} \simeq 0.429.$$

In $(0, 0)$, on the one hand, the optimal bias yields a substantial departure from $p_X = 0.5$. On the other hand, it is less than what would happen without dynamics; in fact, if we apply the globally optimal α_1 to a one-shot contest, we obtain $p_X \simeq 5.22/6.22 \simeq 0.839$. Instead, with dynamics, one

²²In Appendix E we show that the analogous figure for an all-pay auction is 94%.

needs to account for the future advantages of Y . In $(0, 1)$, it is not at all surprising to obtain a small p_X . In fact, two effects point in the same direction: Y is both one-match ahead and advantaged by the bias. Instead, in $(1, 0)$, while Y is still advantaged, she is lagging one match behind. The bias favoring Y more than compensates for her disadvantage of lagging one match behind — i.e., $p_X^{(1,0)} < 1/2$. In the last node $(1, 1)$ the stakes are the same for X and Y . Thus, what accounts for $p_X^{(1,1)}$ being different than $1/2$ is solely the mechanical effect of $\alpha_3 \neq 1$. Combining the above findings, we calculate that the ex-ante probability of victory in the best-of-three contest for player X is 0.488.

To summarize, an interesting contrast arises comparing sections 4 and 5. In Section 4 we found that when the designer can tailor the biases to the outcome of previous matches, an optimal best-of-three contest between two ex-ante identical players equalizes probabilities of winning across players, both at each match and in the overall contest. In Section 5 we found that when the designer can not tailor the biases to the outcome of previous matches, an optimal best-of-three contest between ex-ante identical players leaves players unequally likely to win, both at each match and in the overall contest. In this sense, we conclude that the static-contest conventional wisdom of equalizing equilibrium winning probabilities across players does (not) carry over to a setting where the designer can (cannot) tailor the biases to the outcome of previous matches.

6 Conclusions

We analyze the effort-maximizing biases in a best-of-three contest. The first contribution is to show that the conventional wisdom of optimality of unbiased contest carries over to a setting where an effort-maximizing designer *can* tailor the biases to the outcome of previous matches; that is, by giving a player a different advantage or disadvantage tomorrow whether she won or lost today. Specifically, we characterize the optimal vector of biases, and show that it eliminates the well-known “discouragement effect”, and leaves the two ex-ante identical players equally likely to win at each match, regardless of who is leading in terms of past matches won, and therefore in the entire contest; the conventional wisdom of the optimality of unbiased contest still holds.

The second contribution of the paper is to show that such optimality of unbiased contest does not carry over when the effort-maximizing designer *cannot* tailor the biases to the outcome of the previous matches. We characterize the optimal vector of biases and show that it resembles an observed pattern in real life; namely, the alternation of advantages between players, followed by an unbiased tie-breaker. At the optimum, the two ex-ante identical players are not equally likely to win in equilibrium, neither at a node nor for the entire contest; a biased contest stimulates more efforts than an unbiased contest. We show that this result holds when matches are modeled both as Tullock contests and all-pay auctions, and both under maximization of total effort and winner’s effort; the conventional wisdom of optimality of unbiased contest fails.

This study is, to the best of our knowledge, the first cut into challenging the conventional wisdom of the optimality of unbiased contest in a best-of-three contest. Being the first cut, the present paper leaves sizeable room for future extensions. First, we analyzed the optimal vectors of victory-dependent and victory-independent biases, assuming that the designer has full control over the size of such biases; an interesting extension is that of exogenous-value biases to be allocated by the designer either to player X or to Y . Second, while our analysis focused on best-of-three

contests, we conjecture that the suboptimality of an unbiased contest between ex-ante identical players carries over to best-of- n contests, since the same logic we explored appears to hold when comparing the vector of victory-independent biases $\{\alpha, 1/\alpha, 1, \dots, 1\}$ and $\{1, \dots, 1\}$ because of two effects; the symmetry of the continuation values from the third match onwards, and the higher probability of reaching node where players have the same number of victories. Nevertheless, the structure of the *globally* optimal biases in a best-of- n contest is not a priori clear, and despite the present analysis tempting us to make conjectures, we flag it as an open research question.

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APPENDIX

A Preliminary node-by-node results

We specialize the general analysis of Section 3, i.e. equations (3)–(8), to each node.

Node (1,1). Since (1,1) is the last match, $\Delta u_X^{(1,1)} = \Delta u_Y^{(1,1)} = V$ and $u_X^L = u_Y^L = 0$. Thus,

$$\begin{aligned} x^{(1,1)} &= y^{(1,1)} = \frac{\alpha_3 V}{(\alpha_3 + 1)^2}, \\ p_X^{(1,1)} &= \frac{\alpha_3}{\alpha_3 + 1}, p_Y^{(1,1)} = \frac{1}{\alpha_3 + 1}, \\ u_X^{(1,1)} &= \frac{\alpha_3^2 V}{(\alpha_3 + 1)^2}, u_Y^{(1,1)} = \frac{V}{(\alpha_3 + 1)^2}. \end{aligned} \tag{10}$$

Node (1,0). Recall that at (1,0), if player X wins the game ends, otherwise the game moves to node (1,1). Thus,

$$\Delta u_X^{(1,0)} = V - u_X^{(1,1)} = \frac{(2\alpha_3 + 1)V}{(\alpha_3 + 1)^2}, \Delta u_Y^{(1,0)} = u_Y^{(1,1)} = \frac{V}{(\alpha_3 + 1)^2}, u_X^L = u_X^{(1,1)} = \frac{\alpha_3^2 V}{(\alpha_3 + 1)^2}, u_Y^L = 0. \tag{11}$$

Therefore, plugging the above into (3)–(8), we obtain

$$\begin{aligned} x^{(1,0)} &= \frac{\alpha_2 (2\alpha_3 + 1)^2}{(\alpha_3 + 1)^2 (\alpha_2 + 2\alpha_2\alpha_3 + 1)^2} V, y^{(1,0)} = \frac{\alpha_2 (2\alpha_3 + 1)}{(\alpha_3 + 1)^2 (\alpha_2 + 2\alpha_2\alpha_3 + 1)^2} V, \\ p_X^{(1,0)} &= \frac{2\alpha_2\alpha_3 + \alpha_2}{\alpha_2 + 2\alpha_2\alpha_3 + 1}, p_Y^{(1,0)} = \frac{1}{\alpha_2 + 2\alpha_2\alpha_3 + 1}, \\ u_X^{(1,0)} &= \frac{\alpha_2^2 (2\alpha_3 + 1)^3 + \alpha_2^2 (\alpha_2 + 2\alpha_2\alpha_3 + 1)^2}{(\alpha_3 + 1)^2 (\alpha_2 + 2\alpha_2\alpha_3 + 1)^2} V, u_Y^{(1,0)} = \frac{1}{(\alpha_3 + 1)^2 (\alpha_2 + 2\alpha_2\alpha_3 + 1)^2} V. \end{aligned}$$

Node (0,1). Symmetrically to the above analysis for node (1,0),

$$\Delta u_X^{(0,1)} = u_X^{(1,1)} = \frac{\alpha_3^2 V}{(\alpha_3 + 1)^2}, \Delta u_Y^{(0,1)} = V - u_Y^{(1,1)} = \frac{\alpha_3^2 + 2\alpha_3}{(\alpha_3 + 1)^2} V, u_X^L = 0, u_Y^L = u_Y^{(1,1)} = \frac{V}{(\alpha_3 + 1)^2}. \tag{12}$$

Thus,

$$\begin{aligned} x^{(0,1)} &= \frac{(\alpha_3 + 2) \alpha_2 \alpha_3^3}{(\alpha_3 + 1)^2 (\alpha_3 + \alpha_2\alpha_3 + 2)^2} V, y^{(0,1)} = \frac{(\alpha_3 + 2)^2 \alpha_2 \alpha_3^2}{(\alpha_3 + 1)^2 (\alpha_3 + \alpha_2\alpha_3 + 2)^2} V, \\ p_X^{(0,1)} &= \frac{\alpha_2 \alpha_3}{\alpha_3 + \alpha_2\alpha_3 + 2}, p_Y^{(0,1)} = \frac{\alpha_3 + 2}{\alpha_3 + \alpha_2\alpha_3 + 2}, \\ u_X^{(0,1)} &= \frac{\alpha_2^2 \alpha_3^4}{(\alpha_3 + 1)^2 (\alpha_3 + \alpha_2\alpha_3 + 2)^2} V, u_Y^{(0,1)} = \frac{\alpha_3 (\alpha_3 + 2)^3 + (\alpha_3 + \alpha_2\alpha_3 + 2)^2}{(\alpha_3 + 1)^2 (\alpha_3 + \alpha_2\alpha_3 + 2)^2} V. \end{aligned}$$

Node (0,0). All equilibrium efforts, probabilities and payoffs can be obtained as above, plugging into (3), (4), (5), (6), (7) and (8), the following

$$\Delta u_X^{(0,0)} = u_X^{(1,0)} - u_X^{(0,1)}, \quad \Delta u_Y^{(0,0)} = u_Y^{(0,1)} - u_Y^{(1,0)}, \quad u_X^L = u_X^{(0,1)}, \quad u_Y^L = u_Y^{(1,0)}.$$

For the sake of space, we do not report here the explicit formulae.

B Benchmark; victory-dependent biases

Proof of Proposition 1. We iteratively use the formulae (7) and (8) into (9) to characterize the total effort node-by-node, to then plug them into TE . We express TE , rather than as a function of α 's, as a function of $p_X^{(0,0)}, p_X^{(1,0)}, p_X^{(0,1)}$ and $p_X^{(1,1)}$. Denoting $A = p_X^{(0,0)}$, $B = p_X^{(1,0)}$, $C = p_X^{(0,1)}$ and $D = p_X^{(1,1)}$, we obtain

$$\begin{aligned} TE(A, B, C, D) = & 2A(1-B)(1-D)(B+D) + 2(1-A)CD(2-C-D) + \\ & + 2(1-A)A[B(1-D)^2 + (1-C)(1+C(-1+D))D + B^2(1-D)D]. \end{aligned}$$

Consider an optimal solution $\{A^*, B^*, C^*, D^*\}$ to the problem

$$\max_{A, B, C, D} TE(A, B, C, D).$$

Note first that TE is a polynomial, so on $\{A, B, C, D\} \in [0, 1]^4$, TE admits a global maximum by Weierstrass' theorem. Fix now A and D at arbitrary levels and consider now TE only as a function of B and C , a function we denote with $TE^{AD}(B, C)$. This function is strictly concave for $\{A, D\} \in (0, 1)^2$. To see this, note that

$$\begin{aligned} \frac{\partial^2 TE^{AD}(B, C)}{\partial B \partial C} &= 0 \\ \frac{\partial^2 TE^{AD}(B, C)}{\partial B^2} &= 4A(1-D)[D(1-A) - 1] < 0, \\ \frac{\partial^2 TE^{AD}(B, C)}{\partial C^2} &= 4D(1-A)[A(1-D) - 1] < 0. \end{aligned}$$

Therefore, the Hessian matrix of $TE^{AD}(B, C)$ is a diagonal, negative definite matrix for $\{A, D\} \in (0, 1)^2$. In this case, we can solve for B and C with the FOCs, which give

$$B = \beta(A, D) \equiv \frac{1(2-A)(1-D)}{2(1-D)(1-A)} \in (0, 1) \quad \text{and} \quad C = \gamma(A, D) \equiv \frac{1(1-A)(2-D)}{2(1-A)(1-D)} \in (0, 1). \quad (13)$$

Thus, (13) has to hold in the maximum $\{A^*, B^*, C^*, D^*\}$ if $\{A^*, D^*\} \in (0, 1)^2$, i.e., we must have $B^* = \beta(A^*, D^*)$ and $C^* = \gamma(A^*, D^*)$. Note that we can use one of the formulations in (13) as soon as $1 - A(1 - D) \neq 0$ or $1 - D(1 - A) \neq 0$, and this is useful to rule out the possibility of a corner global maximum for $TE(A, B, C, D)$. Consider for instance $A = 1$ and $D < 1$. Then TE does not depend on C as node (0,1) is never reached, and thus it only depends on B and D . We can repeat the reasoning leading to (13) to show that the optimal B in this case must solve

$B^* = \beta(1, D) = \frac{1-D}{2}$ by concavity and (13). Thus, evaluating TE at $A = 1$ and $B = \frac{1-D}{2}$, we obtain $TE = \frac{(1-D)(1+D)^2}{2}$, which is maximized at $D = 1/3$. These values yield $TE(1, 1/3, c, 1/3) = \frac{16}{27} < \frac{11}{16} = TE(1/2, 1/2, 1/2, 1/2)$, thus this corner cannot be optimal. Similar reasoning rules out the other corners for which we can use one of the two formulae in (13), which also give $TE = \frac{16}{27}$. To finish the proof that the maximum must be interior, notice that $TE(0, B, C, 0) = TE(1, B, C, 1) = 0$, so that $\{A, D\} = \{0, 0\}$ and $\{A, D\} = \{1, 1\}$ cannot be optimal. The above establishes that in any global maximum for TE , we must have $\{A^*, B^*, C^*, D^*\} \in (0, 1)^4$, $B^* = \beta(A^*, D^*)$, and $C^* = \gamma(A^*, D^*)$.

To show that the only possibility is $\{A^*, B^*, C^*, D^*\} = \{1/2, 1/2, 1/2, 1/2\}$, we proceed in several steps. We begin by establishing that in our search for a global maximum we must satisfy the constraint $A + D = 1$. To see this, we show that, for any $\{A, D\} \in (0, 1)^2$, we have

$$TE(A, \beta(A, D), \gamma(A, D), D) \leq TE\left(\frac{1+A-D}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1-A+D}{2}\right),$$

with equality only if $A + D = 1$. Indeed, after a few algebraic steps,²³ we have

$$\begin{aligned} & TE(A, \beta(A, D), \gamma(A, D), D) - TE\left(\frac{1+A-D}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1-A+D}{2}\right) \\ &= \frac{(1-A-D)^2}{16(1-A(1-D))(1-D(1-A))} r(A, D), \end{aligned} \quad (14)$$

where

$$\begin{aligned} r(A, D) \equiv & -A^4(1-D)D + A^3(-1+D+7D^2-6D^3) + A^2(-9+38D-49D^2+7D^3+D^4) \\ & + A(1-D)(21-38D+D^3) - (1-D)^2(11+D). \end{aligned}$$

We now show that $r(A, D) < 0$ by showing that

$$r(A, D) \leq r\left(\frac{1+A-D}{2}, \frac{1-A+D}{2}\right) < 0.$$

Proceeding as above, we obtain

$$r(A, D) - r\left(\frac{1+A-D}{2}, \frac{1-A+D}{2}\right) = -\frac{1}{8}(1-A-D)^2 q(A, D), \quad (15)$$

where

$$\begin{aligned} q(A, D) \equiv & A^4 + A^3(2-8D) + 2A^2(-10-5D+11D^2) - 2A(17-58D+5D^2+4D^3) \\ & + 67 - 34D - 20D^2 + 2D^3 + D^4. \end{aligned}$$

Note that $s(A, D) \equiv q(A, D) - [A^4 + A^3(2-8D)]$ is a strictly concave function of A alone, as $-10-5D+11D^2 < 0$, so $s(A, D)$ admits a minimum at either $A = 0$ or $A = 1$. We have

$$s(0, D) = D^4 + 2D^3 - 20D^2 - 34D + 67,$$

²³For the referee: we provide a check for (14), (15) and (16) with Mathematica.

with a minimum at $D = 1$ and $s(0, 1) = 16$. Similarly,

$$s(1, D) = 13 + 72D - 8D^2 - 6D^3 + D^4,$$

with a minimum at $D = 0$ and $s(1, 0) = 72$. Therefore,

$$q(A, D) = A^4 + A^3(2 - 8D) + s(A, D) \geq 0 - 6 + 16 > 0,$$

Thus, $q(A, D) > 0$ and (15) yield

$$r(A, D) < r\left(\frac{1+A-D}{2}, \frac{1-A+D}{2}\right).$$

Since

$$r\left(\frac{1+A-D}{2}, \frac{1-A+D}{2}\right) = \frac{1}{8} \left(1 - (A-D)^2\right)^2 \left(-21 + (A-D)^2\right) < 0, \quad (16)$$

we have verified that $r(A, D) < 0$, and hence, using (14), that

$$TE(A, \beta(A, D), \gamma(A, D), D) \leq TE\left(\frac{1+A-D}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1-A+D}{2}\right),$$

with equality only if $A + D = 1$.

We now proceed to look for the global maximum of $TE(A, \beta(A, D), \gamma(A, D), D)$ by maximizing $TE\left(\frac{1+A-D}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1-A+D}{2}\right)$ for A and D . We obtain

$$TE\left(\frac{1+A-D}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1-A+D}{2}\right) = \frac{11 - (A-D)^2(2 + (A-D)^2)}{16} \leq \frac{11}{16} = TE\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right).$$

with equality only if $A = D$.

Concatenating our previous inequalities, we have that for any $\{A, B, C, D\} \in (0, 1)^4$,

$$\begin{aligned} TE(A, B, C, D) &\leq TE(A, \beta(A, D), \gamma(A, D), D) \\ &\leq TE\left(\frac{1+A-D}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1-A+D}{2}\right) \\ &\leq \frac{11}{16} \\ &= TE\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), \end{aligned}$$

with equality only if $A + D = 1$ and $A = D$, or equivalently $A = D = 1/2$. Since $\beta(1/2, 1/2) = \gamma(1/2, 1/2) = 1/2$, there is a unique global maximum at $\{A^*, B^*, C^*, D^*\} = \{1/2, 1/2, 1/2, 1/2\}$. ■

Proof of Corollary 2. Clearly, $D = 1/2$ implies $\alpha_{(1,1)} = 1$. Then, at node $(1,0)$, $\Delta u_X^{(1,0)} = \frac{3}{4}V$ and $\Delta u_Y^{(1,0)} = \frac{1}{4}V$. From (5) and $C = 1/2$ we obtain,

$$\frac{1}{2} = p_X^{(1,0)} = \frac{\alpha_{(1,0)} \frac{3}{4}}{\alpha_{(1,0)} \frac{3}{4} + \frac{1}{4}} \Rightarrow \alpha_{(1,0)} = \frac{1}{3}.$$

Similarly, we obtain $\alpha_{(0,1)} = 3$. Thus, at node $(0,0)$, $\Delta u_X^{(0,0)} = \Delta u_Y^{(0,0)}$, so $\alpha_{(0,0)} = 1$ follows. ■

C Victory-independent biases

Proof of Proposition 5. Since we know that along $\{\alpha_1, \alpha_2, \alpha_3\} = \{1 + b, 1, 1\}$ $b = 0$ gives a local maximum for TE , we focus here only on showing that along $\{\alpha_1, \alpha_2, \alpha_3\} = \{1 + b, 1 - b, 1\}$ $b = 0$ gives a local minimum for TE , so that $\{\alpha_1, \alpha_2, \alpha_3\} = \{1, 1, 1\}$ is a saddle.

We divide the proof into the three effects spelled out in the main text, the first (second, third) effect referring to the TE in the first (second, third) match. We analyse from the third to the first effect.

Third effect. We quantify the effect on TE of node (1,1) only, thus weighted by the probability of getting to node (1,1). Using (10) with $\alpha_3 = 1$, we obtain that such effect on TE equals

$$\left(p_X^{(0,0)} p_Y^{(1,0)} + p_Y^{(0,0)} p_X^{(0,1)} \right) \left(x^{(1,1)} + y^{(1,1)} \right) = \left[p_X^{(0,0)} \left(1 - p_X^{(1,0)} - p_X^{(0,1)} \right) + p_X^{(0,1)} \right] \frac{V}{2}. \quad (17)$$

Setting $\alpha_1 = 1 + b$ and $\alpha_2 = 1 - b$, we compute

$$\frac{dp_X^{(0,1)}}{db} = \frac{\partial p_X^{(0,1)}}{\partial \Delta u_X^{(0,1)}} \frac{\partial \Delta u_X^{(0,1)}}{\partial b} + \frac{\partial p_X^{(0,1)}}{\partial \Delta u_Y^{(0,1)}} \frac{\partial \Delta u_Y^{(0,1)}}{\partial b} + \frac{\partial p_X^{(0,1)}}{\partial \alpha_2} \frac{\partial \alpha_2}{\partial b}.$$

Notice that $\frac{\partial \Delta u_X^{(0,1)}}{\partial b} = \frac{\partial \Delta u_Y^{(0,1)}}{\partial b} = 0$ since they only depend on α_3 which we keep fixed at 1. Thus, the above displayed equation reads

$$\begin{aligned} \frac{dp_X^{(0,1)}}{db} &= - \frac{\partial p_X^{(0,1)}}{\partial \alpha_2} \\ &= - \frac{\Delta u_X^{(0,1)} \Delta u_Y^{(0,1)}}{\left(\alpha_2 \Delta u_X^{(0,1)} + \Delta u_Y^{(0,1)} \right)^2} \\ &= - \frac{p_X^{(0,1)} p_Y^{(0,1)}}{\alpha_2}, \end{aligned}$$

where in the last step we used (5) and (6). Similarly,

$$\frac{dp_X^{(1,0)}}{db} = - \frac{p_X^{(1,0)} p_Y^{(1,0)}}{\alpha_2}.$$

Finally,

$$\begin{aligned} \frac{dp_X^{(0,0)}}{db} &= \frac{\partial p_X^{(0,0)}}{\partial \Delta u_X^{(0,0)}} \frac{\partial \Delta u_X^{(0,0)}}{\partial b} + \frac{\partial p_X^{(0,0)}}{\partial \Delta u_Y^{(0,0)}} \frac{\partial \Delta u_Y^{(0,0)}}{\partial b} + \frac{\partial p_X^{(0,0)}}{\partial \alpha_1} \frac{\partial \alpha_1}{\partial b} \\ &= \frac{\alpha_1 \Delta u_Y^{(0,0)}}{\left(\alpha_1 \Delta u_X^{(0,0)} + \Delta u_Y^{(0,0)} \right)^2} \frac{\partial \Delta u_X^{(0,0)}}{\partial b} - \frac{\alpha_1 \Delta u_X^{(0,0)}}{\left(\alpha_1 \Delta u_X^{(0,0)} + \Delta u_Y^{(0,0)} \right)^2} \frac{\partial \Delta u_Y^{(0,0)}}{\partial b} + \frac{\Delta u_X^{(0,0)} \Delta u_Y^{(0,0)}}{\left(\alpha_1 \Delta u_X^{(0,0)} + \Delta u_Y^{(0,0)} \right)^2} \\ &= p_X^{(0,0)} p_Y^{(0,0)} \left[\frac{\partial \ln \Delta u_X^{(0,0)}}{\partial b} - \frac{\partial \ln \Delta u_Y^{(0,0)}}{\partial b} + \frac{\partial \ln \alpha_1}{\partial b} \right] \\ &= p_X^{(0,0)} p_Y^{(0,0)} \left(\frac{\partial}{\partial b} \ln \frac{\alpha_1 \Delta u_X^{(0,0)}}{\Delta u_Y^{(0,0)}} \right). \end{aligned} \quad (18)$$

Thus, the derivative of (17) equals

$$\begin{aligned}
\frac{d \left[p_X^{(0,0)} \left(1 - p_X^{(1,0)} - p_X^{(0,1)} \right) + p_X^{(0,1)} \right] V}{db} \frac{1}{2} &= \left[\frac{dp_X^{(0,0)}}{db} \left(1 - p_X^{(1,0)} - p_X^{(0,1)} \right) + p_X^{(0,0)} \left(-\frac{dp_X^{(1,0)}}{db} - \frac{dp_X^{(0,1)}}{db} \right) + \frac{dp_X^{(0,1)}}{db} \right] \frac{V}{2} \\
&= \left[p_X^{(0,0)} p_Y^{(0,0)} \left(\frac{\partial}{\partial b} \ln \frac{\alpha_1 \Delta u_X^{(0,0)}}{\Delta u_Y^{(0,0)}} \right) \left(1 - p_X^{(1,0)} - p_X^{(0,1)} \right) \right. \\
&\quad \left. + p_X^{(0,0)} \left(\frac{p_X^{(1,0)} p_Y^{(1,0)}}{\alpha_2} + \frac{p_X^{(0,1)} p_Y^{(0,1)}}{\alpha_2} \right) - \frac{p_X^{(0,1)} p_Y^{(0,1)}}{\alpha_2} \right] \frac{V}{2}.
\end{aligned}$$

We evaluate the above at $b = 0$ using formulae of Appendix A and obtain²⁴

$$\left[\frac{1}{4} \frac{3}{7} \left(1 - \frac{3}{4} - \frac{1}{4} \right) + \frac{1}{2} \left(\frac{3}{16} + \frac{3}{16} \right) - \frac{3}{16} \right] \frac{V}{2}.$$

Since the above equals 0, the first derivative of the third effect evaluates to 0. Thus, we have to consider the second derivative of (17). Using $p_Y^{(0,0)} = 1 - p_X^{(0,0)}$,

$$\frac{d^2 \left[p_X^{(0,0)} \left(1 - p_X^{(1,0)} - p_X^{(0,1)} \right) + p_X^{(0,1)} \right] V}{db^2} \frac{1}{2}$$

equals

$$\begin{aligned}
&\left(\left(1 - 2 \left(p_X^{(0,0)} \right) \right) \frac{dp_X^{(0,0)}}{db} \right) \left(\frac{\partial}{\partial b} \ln \frac{\alpha_1 \Delta u_X^{(0,0)}}{\Delta u_Y^{(0,0)}} \right) \left(1 - p_X^{(1,0)} - p_X^{(0,1)} \right) \frac{V}{2} \\
&+ \left(p_X^{(0,0)} - \left(p_X^{(0,0)} \right)^2 \right) \left(\frac{\partial^2}{\partial b^2} \ln \frac{\alpha_1 \Delta u_X^{(0,0)}}{\Delta u_Y^{(0,0)}} \right) \left(1 - p_X^{(1,0)} - p_X^{(0,1)} \right) \frac{V}{2} \\
&+ \left(p_X^{(0,0)} - \left(p_X^{(0,0)} \right)^2 \right) \left(\frac{\partial}{\partial b} \ln \frac{\alpha_1 \Delta u_X^{(0,0)}}{\Delta u_Y^{(0,0)}} \right) \left(-\frac{dp_X^{(1,0)}}{db} - \frac{dp_X^{(0,1)}}{db} \right) \frac{V}{2} \\
&+ \frac{dp_X^{(0,0)}}{db} \left(\frac{p_X^{(1,0)} p_Y^{(1,0)}}{\alpha_2} + \frac{p_X^{(0,1)} p_Y^{(0,1)}}{\alpha_2} \right) \frac{V}{2} \\
&+ p_X^{(0,0)} \left(\frac{\left(p_X^{(1,0)} - \left(p_X^{(1,0)} \right)^2 \right)}{\alpha_2^2} + \frac{\left(1 - 2p_X^{(1,0)} \right) dp_X^{(1,0)}}{\alpha_2 db} + \frac{\left(p_X^{(0,1)} - \left(p_X^{(0,1)} \right)^2 \right)}{\alpha_2^2} + \frac{\left(1 - 2p_X^{(0,1)} \right) dp_X^{(0,1)}}{\alpha_2 db} \right) \frac{V}{2} \\
&- \frac{\left(p_X^{(0,1)} - \left(p_X^{(0,1)} \right)^2 \right)}{\alpha_2^2} \frac{V}{2} - \frac{\left(1 - 2p_X^{(0,1)} \right) dp_X^{(0,1)}}{\alpha_2 db} \frac{V}{2}.
\end{aligned}$$

²⁴For example, in order to obtain $p_X^{(0,0)}$, one can plug the expression for $\Delta u_X^{(0,0)}$ and $\Delta u_Y^{(0,0)}$ provided in Appendix A (**Note (0,0)**) into (5). Indeed, with $b = 0$, $\Delta u_X^{(0,0)} = \Delta u_Y^{(0,0)}$ and thus $p_X^{(0,0)} = \frac{1}{2}$.

Evaluating the above at $b = 0$ we obtain²⁵

$$\left[\frac{1}{4} \frac{3}{7} \frac{6}{16} + \frac{1}{4} \frac{3}{7} \left(\frac{3}{16} + \frac{3}{16} \right) + \frac{1}{2} \left(\frac{3}{16} + \frac{1}{2} \frac{3}{16} + \frac{3}{16} - \frac{1}{2} \frac{3}{16} \right) - \frac{3}{16} + \frac{1}{2} \frac{3}{16} \right] \frac{V}{2} = \frac{39}{224} \frac{V}{2} \quad (19)$$

Therefore we obtain that the overall third effect is positive and quantified by (19).

Second effect. We quantify the effect on TE of nodes (1,0) and (0,1), thus weighted by the probability of getting to those two nodes.

$$p_X^{(0,0)} \left(x^{(1,0)} + y^{(1,0)} \right) + p_Y^{(0,0)} \left(x^{(0,1)} + y^{(0,1)} \right) \quad (20)$$

At node (1,0), by $\alpha_3 = 1$ and (11), we have $\Delta u_X^{(1,0)} = \frac{3}{4}V$, $\Delta u_Y^{(1,0)} = \frac{1}{4}V$, and $p_X^{(1,0)} = \frac{3\alpha_2}{3\alpha_2+1}$.

At node (0,1), by $\alpha_3 = 1$ and (12), we have $\Delta u_X^{(0,1)} = \frac{1}{4}V$, $\Delta u_Y^{(0,1)} = \frac{3}{4}V$, and $p_X^{(0,1)} = \frac{\alpha_2}{\alpha_2+3}$.

We differentiate (20) and obtain

$$\frac{dp_X^{(0,0)}}{db} \left(x^{(1,0)} + y^{(1,0)} \right) + p_X^{(0,0)} \frac{d \left(x^{(1,0)} + y^{(1,0)} \right)}{db} + \frac{dp_Y^{(0,0)}}{db} \left(x^{(0,1)} + y^{(0,1)} \right) + p_Y^{(0,0)} \frac{d \left(x^{(0,1)} + y^{(0,1)} \right)}{db} \quad (21)$$

Since $\frac{dp_Y^{(0,0)}}{db} = -\frac{dp_X^{(0,0)}}{db}$ and since from (3) and (4) we obtain that, when $b \rightarrow 0$, $(x^{(1,0)} + y^{(1,0)}) = (x^{(0,1)} + y^{(0,1)})$, then, at $b = 0$, the above first derivative evaluates to

$$\frac{1}{2} \cdot \left(\frac{d \left(x^{(1,0)} + y^{(1,0)} \right)}{db} + \frac{d \left(x^{(0,1)} + y^{(0,1)} \right)}{db} \right). \quad (22)$$

Now rewrite (9) as $x^{(1,0)} + y^{(1,0)} = \left(\Delta u_X^{(1,0)} + \Delta u_Y^{(1,0)} \right) \cdot \left(p_X^{(1,0)} - \left(p_X^{(1,0)} \right)^2 \right)$, and use this to obtain

$$\frac{d \left(x^{(1,0)} + y^{(1,0)} \right)}{db} = \left(\Delta u_X^{(1,0)} + \Delta u_Y^{(1,0)} \right) \cdot \frac{dp_X^{(1,0)}}{d\alpha_2} \frac{d\alpha_2}{db} \left(1 - 2p_X^{(1,0)} \right). \quad (23)$$

Since $\frac{d\alpha_2(b)}{db} = -1$, evaluating the above at $b = 0$ we have

$$-V \cdot \frac{3}{16} \left(1 - 2 * \frac{3}{4} \right) = \frac{3}{32}V.$$

Similarly,

$$\frac{d \left(x^{(0,1)} + y^{(0,1)} \right)}{db} = -V \cdot \frac{3}{16} \left(1 - 2 * \frac{1}{4} \right) = -\frac{3}{32}V.$$

Therefore, as for the third effect, the first derivative evaluates to 0 and we thus move to the analysis of the second derivative. We differentiate (21) and obtain

$$\begin{aligned} & \frac{d^2 p_X^{(0,0)}}{db^2} \left(x^{(1,0)} + y^{(1,0)} \right) + 2 \frac{dp_X^{(0,0)}}{db} \frac{d \left(x^{(1,0)} + y^{(1,0)} \right)}{db} + p_X^{(0,0)} \frac{d^2 \left(x^{(1,0)} + y^{(1,0)} \right)}{db^2} \\ & + \frac{d^2 p_Y^{(0,0)}}{db^2} \left(x^{(0,1)} + y^{(0,1)} \right) + 2 \frac{dp_Y^{(0,0)}}{db} \frac{d \left(x^{(0,1)} + y^{(0,1)} \right)}{db} + p_Y^{(0,0)} \frac{d^2 \left(x^{(0,1)} + y^{(0,1)} \right)}{db^2}. \end{aligned}$$

²⁵The procedure is similar to the one outlined in footnote 24.

When we evaluate this expression at $b = 0$, the first and fourth summands cancel out, as efforts are $x^{(1,0)} + y^{(1,0)} = x^{(0,1)} + y^{(0,1)}$ and $\frac{d^2 p_X^{(0,0)}}{db^2} = -\frac{d^2 p_Y^{(0,0)}}{db^2}$. Thus, we are left with

$$\begin{aligned} & 2 \frac{dp_X^{(0,0)}}{db} \left(\frac{3}{32} V \right) + \frac{1}{2} \frac{d^2 (x^{(1,0)} + y^{(1,0)})}{db^2} + 2 \frac{dp_Y^{(0,0)}}{db} \left(-\frac{3}{32} V \right) + \frac{1}{2} \frac{d^2 (x^{(0,1)} + y^{(0,1)})}{db^2} \\ &= 4 \frac{dp_X^{(0,0)}}{db} \left(\frac{3}{32} V \right) + \frac{1}{2} \left(\frac{d^2 (x^{(1,0)} + y^{(1,0)})}{db^2} + \frac{d^2 (x^{(0,1)} + y^{(0,1)})}{db^2} \right). \end{aligned} \quad (24)$$

Using (18) evaluated at $b = 0$ we obtain

$$\frac{dp_X^{(0,0)}}{db} = \frac{1}{4} \frac{3}{7}.$$

Now in (24) we focus on the term $\frac{d^2 (x^{(1,0)} + y^{(1,0)})}{db^2}$. Differentiating (23), we obtain

$$\begin{aligned} & \frac{d^2 (x^{(1,0)} + y^{(1,0)})}{db^2} = \left(\Delta u_X^{(1,0)} + \Delta u_Y^{(1,0)} \right) \cdot \frac{d^2 p_X^{(1,0)}}{d\alpha_2^2} \frac{d\alpha_2}{db} \left(2p_X^{(1,0)} - 1 \right) + 2 \left(\Delta u_X^{(1,0)} + \Delta u_Y^{(1,0)} \right) \cdot \left(\frac{dp_X^{(1,0)}}{d\alpha_2} \right)^2 \frac{d\alpha_2}{db} \\ &= \left(\Delta u_X^{(1,0)} + \Delta u_Y^{(1,0)} \right) \cdot \left(\frac{d^2 p_X^{(1,0)}}{d\alpha_2^2} \left(1 - 2p_X^{(1,0)} \right) - 2 \left(\frac{dp_X^{(1,0)}}{d\alpha_2} \right)^2 \right), \end{aligned}$$

which at $b = 0$ evaluates to

$$V \cdot \left(-\frac{9}{32} \left(1 - 2 * \frac{3}{4} \right) - 2 \left(\frac{3}{16} \right)^2 \right) = \frac{9}{128} V.$$

Similarly,

$$\frac{d^2 (x^{(0,1)} + y^{(0,1)})}{db^2} = \left(\Delta u_X^{(0,1)} + \Delta u_Y^{(0,1)} \right) \cdot \left(\frac{d^2 p_X^{(0,1)}}{d\alpha_2^2} \left(1 - 2p_X^{(0,1)} \right) - 2 \left(\frac{dp_X^{(0,1)}}{d\alpha_2} \right)^2 \right),$$

which at $b = 0$ evaluates to

$$V \cdot \left(-\frac{6}{64} \left(1 - 2 * \frac{1}{4} \right) - 2 \left(\frac{3}{16} \right)^2 \right) = -\frac{15}{128} V.$$

So the whole second derivative (24) evaluates to

$$4 * \frac{1}{4} * \frac{3}{7} * \left(\frac{3}{32} V \right) + \frac{1}{2} \left(\frac{9}{128} - \frac{15}{128} \right) V = \frac{15}{896} V. \quad (25)$$

Therefore we obtain that the overall second effect is positive and quantified by (25).

First effect. We quantify the effect on TE of node $(0,0)$. We rewrite the sum of efforts using (9) as

$$x^{(0,0)} + y^{(0,0)} = \left(\Delta u_X^{(0,0)} + \Delta u_Y^{(0,0)} \right) \cdot \left(p_X^{(0,0)} - \left(p_X^{(0,0)} \right)^2 \right). \quad (26)$$

So,

$$\frac{d(x^{(0,0)} + y^{(0,0)})}{db} = \left(\frac{d\Delta u_X^{(0,0)}}{db} + \frac{d\Delta u_Y^{(0,0)}}{db} \right) \cdot \left(p_X^{(0,0)} - \left(p_X^{(0,0)} \right)^2 \right) + \left(\Delta u_X^{(0,0)} + \Delta u_Y^{(0,0)} \right) \cdot \frac{dp_X^{(0,0)}}{db} \left(1 - 2p_X^{(0,0)} \right).$$

Note that $\left(2p_X^{(0,0)} - 1 \right)$ evaluates to 0 at $b = 0$. Consider now $\frac{d\Delta u_X^{(0,0)}}{db}$ and $\frac{d\Delta u_Y^{(0,0)}}{db}$. Defining $h(x) = \frac{27x^2}{(3x+1)^2} + 1 - \frac{x^2}{(x+3)^2}$, and using the formulae of Appendix A, we can write²⁶ $\Delta u_X^{(0,0)} = h(\alpha_2) \frac{V}{4}$ and $\Delta u_Y^{(0,0)} = h(1/\alpha_2) \frac{V}{4}$ so that

$$\frac{d\Delta u_X^{(0,0)}}{d\alpha_2} \frac{d\alpha_2}{db} + \frac{d\Delta u_Y^{(0,0)}}{d\alpha_2} \frac{d\alpha_2}{db} = - \left(h'(\alpha_2) - h'(1/\alpha_2) \frac{1}{\alpha_2^2} \right) \frac{V}{4},$$

which evaluates to 0. Therefore,

$$\left. \frac{d(x^{(0,0)} + y^{(0,0)})}{db} \right|_{b=0} = 0.$$

Thus, we move to evaluate the second derivative of (26);

$$\begin{aligned} \frac{d^2(x^{(0,0)} + y^{(0,0)})}{db^2} &= \left(\frac{d^2\Delta u_X^{(0,0)}}{d(\alpha_2(b))^2} + \frac{d^2\Delta u_Y^{(0,0)}}{d(\alpha_2(b))^2} \right) \cdot \left(p_X^{(0,0)} - \left(p_X^{(0,0)} \right)^2 \right) \\ &+ 2 \left(\frac{d\Delta u_X^{(0,0)}}{d\alpha_2(b)} + \frac{d\Delta u_Y^{(0,0)}}{d\alpha_2(b)} \right) \cdot \frac{dp_X^{(0,0)}}{db} \left(1 - 2p_X^{(0,0)} \right) \\ &- \left(\Delta u_X^{(0,0)} + \Delta u_Y^{(0,0)} \right) \cdot \left(-\frac{d^2p_X^{(0,0)}}{db^2} \left(1 - 2p_X^{(0,0)} \right) + 2 \left(\frac{dp_X^{(0,0)}}{db} \right)^2 \right). \end{aligned}$$

Using the definition of $h(\cdot)$ above, we write

$$\frac{d^2\Delta u_X^{(0,0)}}{db^2} + \frac{d^2\Delta u_Y^{(0,0)}}{db^2} = \left(h''(\alpha_2) + h''(1/\alpha_2) \frac{1}{\alpha_2^4} + 2h'(1/\alpha_2) \frac{1}{\alpha_2^3} \right) \frac{V}{4},$$

which evaluated at $\alpha_2 = 1$, gives $2h''(1) + 2h'(1) = 2 * \left(-\frac{69}{64} \right) V + 2 * \frac{3}{4} V = -\frac{21}{32} V$.

Using (18), $\left. \frac{dp_X^{(1,1)}}{db} \right|_{\alpha_2=1} = \frac{1}{4} \frac{3}{7}$, then,

$$\left. \frac{d^2(x^{(0,0)} + y^{(0,0)})}{db^2} \right|_{b=0} = -\frac{21}{32} \cdot \frac{V}{4} \cdot \frac{1}{4} - 0 - \left(\frac{54}{16} + 2 - \frac{1}{8} \right) \frac{V}{4} * \left(0 + 2 \left(\frac{1}{4} * \frac{3}{7} \right)^2 \right) = -\frac{255}{3584} V. \quad (27)$$

²⁶ For the referee: we provide a check for this expression with Mathematica.

Therefore we obtain that the overall first effect is negative and quantified by (27).

Overall second derivative. We can finally put together the three effects, and conclude that along the direction $\{\alpha_1, \alpha_2, \alpha_3\} = \{1+b, 1-b, 1\}$, $b = 0$ is a local minimum since the first derivative of the total effort with respect to b is 0, and the second derivative equals $\frac{39}{224}V + \frac{15}{896}V - \frac{255}{3584}V = \frac{117}{3584}V > 0$. This establishes that in the $\{\alpha_1, \alpha_2, \alpha_3\}$ -space, $\{1, 1, 1\}$ is a saddle. ■

D Robustness to maximization of winner's effort

The expected winner's effort (WE) is defined as follows

$$WE \equiv \tag{28}$$

$$p_X^{(0,0)} \left\{ p_X^{(1,0)} \cdot [x^{(0,0)} + x^{(1,0)}] + p_Y^{(1,0)} p_X^{(1,1)} \cdot [x^{(0,0)} + x^{(1,0)} + x^{(1,1)}] \right\} + p_Y^{(0,0)} p_X^{(0,1)} p_X^{(1,1)} \cdot [x^{(0,0)} + x^{(0,1)} + x^{(1,1)}]$$

$$+ p_Y^{(0,0)} \left\{ p_Y^{(0,1)} \cdot [y^{(0,0)} + y^{(0,1)}] + p_X^{(0,1)} p_Y^{(1,1)} \cdot [y^{(0,0)} + y^{(0,1)} + y^{(1,1)}] \right\} + p_X^{(0,0)} p_Y^{(1,0)} p_Y^{(1,1)} \cdot [y^{(0,0)} + y^{(1,0)} + y^{(1,1)}].$$

The top line of WE is player X 's overall effort considering all instances when she wins, and the bottom line does the same for player Y .

Proposition 6 *Consider a best-of-three Tullock contest between two ex-ante identical players. With victory-dependent biases, the point $\{\alpha_1, \alpha_2, \alpha_3\} = \{1, 1, 1\}$ is a saddle for WE in $\mathbb{R}_{>0}^3$.*

Proof. We follow the structure of Proof of Proposition 5 provided in the main text.²⁷

$$WE(b) = p_X^{(0,0)} p_X^{(1,0)} [x^{(0,0)} + x^{(1,0)}] + p_Y^{(0,0)} p_Y^{(0,1)} [y^{(0,0)} + y^{(1,0)}]$$

$$+ p_X^{(0,0)} p_Y^{(1,0)} \left\{ p_X^{(1,1)} [x^{(0,0)} + x^{(1,0)} + x^{(1,1)}] + p_Y^{(1,1)} [y^{(0,0)} + y^{(1,0)} + y^{(1,1)}] \right\}$$

$$+ p_Y^{(0,0)} p_X^{(1,0)} \left\{ p_X^{(1,1)} [x^{(0,0)} + x^{(1,0)} + x^{(1,1)}] + p_Y^{(1,1)} [y^{(0,0)} + y^{(1,0)} + y^{(1,1)}] \right\}$$

$$= \frac{2548 - 19110b + 61989b^2 + o(b^2)}{256(28 - 210b + 681b^2) + o(b^2)}.$$

As we did for $TE(b)$, we separate numerator and denominator and one could verify that

$$\begin{aligned} n'(0)d(0) - n(0)d'(0) &= 0 \\ n''(0)d(0) - n(0)d''(0) &= 256 * (61989 * 28 - 2548 * 681) \\ &= 256 * 504 > 0 \end{aligned}$$

Again, $b = 0$ is a local minimum under the constraint $\{\alpha_1, \alpha_2, \alpha_3\} = \{1+b, 1-b, 1\}$. This establishes the result. ■

²⁷The code is available from the authors upon request. *Note for the referees: the code is attached to our submission.*

E Robustness to the all-pay auction

In this last Section of the Appendix we provide robustness tests of the Tullock contest analysed in the main text to the all-pay auction. That is, we replace (1) with the following;

$$p_X^{(i,j)}(x, y) = \begin{cases} 1 & \text{if } \alpha_{(i,j)}x > y \\ 1/2 & \text{if } \alpha_{(i,j)}x = y \\ 0 & \text{if } \alpha_{(i,j)}x < y \end{cases}$$

In an all-pay auction (APA), we assume private information with marginal costs distributed on $[1, +\infty)$ with density $f(c) = 1/c^2$ and cumulative $F(c) = 1 - 1/c$.²⁸ In every match, a new pair of realizations of marginal costs are independently drawn. Our choice of adding private information to the APA is driven by the fact that with full information rents would be fully dissipated, thus making our exercise non-interesting.

In this Section we keep the same notation for $u_X^W, u_X^L, u_Y^W, u_Y^L, \Delta u_X, \Delta u_Y$ as for the Tullock contest. In an APA, a strategy is a function $x(c)$ for player X and $y(c)$ for player Y . We look for a strictly increasing equilibrium so ties do not happen with positive probability in equilibrium. Following Amann and Leininger (1996) we define a function $k(c)$ that matches cost-type c of player X with cost-type $k(c)$ of player Y that bids the same effective amount taking into account the bias α . Thus,

$$k(c) = y^{-1}(\alpha x(c)).$$

Expected individual payoff of cost-type c of player X who behaves as c'

$$\begin{aligned} u_X(c, c') &= (1 - F(k(c'))u_X^W + F(k(c'))u_X^L - cx(c') \\ &= (1 - F(k(c'))\Delta u_X + u_X^L - cx(c'). \end{aligned}$$

Its first derivative reads

$$\frac{\partial u_X(c, c')}{\partial c'} = -f(k(c'))k'(c')\Delta u_X - cx'(c').$$

Since the FOC must hold at $c' = c$, we obtain the differential equation

$$cx'(c) = -f(k(c))k'(c)\Delta u_X. \tag{29}$$

Similarly,

$$cy'(c) = -\frac{f(k^{-1}(c))}{k'(k^{-1}(c))}\Delta u_Y,$$

which evaluated at $k(c)$ gives

$$k(c)y'(k(c)) = -\frac{f(c)}{k'(c)}\Delta u_Y.$$

Since $k'(c) = \frac{\alpha x'(c)}{y'(k(c))}$, the above displayed equation reads

$$k(c)\alpha x'(c) = -f(c)\Delta u_Y. \tag{30}$$

²⁸The choice of distribution of marginal costs is equivalent to a uniform distribution over valuations.

Putting together (29) and (30) we obtain

$$\begin{aligned} -\frac{cf(c) [u_Y^W - u_Y^L]}{k(c)\alpha} &= -f(k(c))k'(c)\Delta u_X \\ cf(c) &= f(k(c))k'(c)k(c)\alpha \frac{\Delta u_X}{\Delta u_Y}. \end{aligned}$$

For our chosen PDF, we obtain

$$\frac{1}{c} = \frac{k'(c)}{k(c)} \alpha \frac{\Delta u_X}{\Delta u_Y}.$$

Integrating with the boundary condition $k(1) = 1$ we have

$$k(c) = c^{\frac{\Delta u_Y}{\alpha \Delta u_X}}. \quad (31)$$

Therefore the ex-ante winning probability of X is

$$\begin{aligned} p_X &= \int_1^{+\infty} \frac{1}{c^2} (1 - F(k(c))) dc \\ &= \int_1^{+\infty} \frac{1}{c^2} \left(\frac{1}{k(c)} \right) dc \\ &= \int_1^{+\infty} c^{-\frac{\Delta u_Y}{\alpha \Delta u_X} - 2} dc \\ &= \frac{\alpha \Delta u_X}{\alpha \Delta u_X + \Delta u_Y}. \end{aligned} \quad (32)$$

Naturally, $p_Y = 1 - p_X$.

Substituting (31) into (29), we obtain

$$x'(c) = -\frac{\Delta u_Y}{\alpha} c^{-\frac{\Delta u_Y}{\alpha \Delta u_X} - 2},$$

and integrating, we obtain

$$x(c) = \frac{\Delta u_X \Delta u_Y}{\alpha \Delta u_X + \Delta u_Y} c^{-\frac{\Delta u_Y + \alpha \Delta u_X}{\alpha \Delta u_X}}.$$

Similar steps yield the following

$$y(c) = \frac{\alpha \Delta u_X \Delta u_Y}{\alpha \Delta u_X + \Delta u_Y} c^{-\frac{\Delta u_Y + \alpha \Delta u_X}{\Delta u_Y}}.$$

Thus, the expected efforts read

$$\begin{aligned}
x &= \int_1^{+\infty} \frac{\Delta u_X \Delta u_Y}{\alpha \Delta u_X + \Delta u_Y} c^{-\frac{\Delta u_Y + \alpha \Delta u_X}{\alpha \Delta u_X} - 2} dc \\
&= - \frac{\alpha \Delta u_X^2 \Delta u_Y}{(\alpha \Delta u_X + \Delta u_Y)(\Delta u_Y + 2\alpha \Delta u_X)} c^{-\frac{\Delta u_Y + 2\alpha \Delta u_X}{\alpha \Delta u_X}} \Big|_1^{+\infty} \\
&= \frac{\alpha \Delta u_X^2 \Delta u_Y}{(\alpha \Delta u_X + \Delta u_Y)(\Delta u_Y + 2\alpha \Delta u_X)}. \tag{33}
\end{aligned}$$

Similarly,

$$y = \frac{\alpha \Delta u_X \Delta u_Y^2}{(\alpha \Delta u_X + \Delta u_Y)(2\Delta u_Y + \alpha \Delta u_X)}. \tag{34}$$

Finally, payoffs are

$$\begin{aligned}
u_X &= \frac{\alpha \Delta u_X^2}{\alpha \Delta u_X + \Delta u_Y} + u_X^L - \int_1^{+\infty} c \frac{\Delta u_X \Delta u_Y}{\alpha \Delta u_X + \Delta u_Y} c^{-\frac{\Delta u_Y + \alpha \Delta u_X}{\alpha \Delta u_X} - 2} dc \\
&= \frac{\alpha \Delta u_X^2}{\alpha \Delta u_X + \Delta u_Y} + u_X^L - \frac{\alpha \Delta u_X^2 \Delta u_Y}{(\alpha \Delta u_X + \Delta u_Y)^2} \\
&= \frac{\alpha^2 \Delta u_X^3}{(\alpha \Delta u_X + \Delta u_Y)^2} + u_X^L. \tag{35}
\end{aligned}$$

And similarly,

$$u_Y = \frac{\Delta u_Y^3}{(\alpha \Delta u_X + \Delta u_Y)^2} + u_Y^L.$$

The APA equivalent of the Tullock property (9) is

$$x + y = \Delta u_Y \cdot p_X \left(\frac{p_X}{\alpha(1 + p_X)} + \frac{p_Y}{1 + p_Y} \right). \tag{36}$$

We now specialize the above general analysis to each node, as we did in Appendix A for the Tullock contest. While at each node the equilibrium expected payoffs and probabilities of victory have the same functional forms as in the Tullock contest (e.g., compare (32) and (35) with (5) and (7)), the equilibrium expected efforts are not (compare (33) and (3)). Therefore, in what follows, we spell out only efforts.

Node (1,1). Since (1,1) is the last match, $\Delta u_X = \Delta u_Y = V$ and $u_X^L = u_Y^L = 0$. Thus,

$$\begin{aligned}
x^{(1,1)} &= \frac{\alpha_3 V}{(\alpha_3 + 1)(2\alpha_3 + 1)} \\
y^{(1,1)} &= \frac{\alpha_3 V}{(\alpha_3 + 1)(\alpha_3 + 2)}
\end{aligned}$$

Node (1,0). Plugging Δu_X and Δu_Y into (33) and (34), we obtain

$$\begin{aligned} x^{(1,0)} &= \frac{\alpha_2 (2\alpha_3 + 1)^2}{(\alpha_3 + 1)^2 (\alpha_2 + 2\alpha_2\alpha_3 + 1) (2\alpha_2 + 4\alpha_2\alpha_3 + 1)} V \\ y^{(1,0)} &= \frac{\alpha_2 (2\alpha_3 + 1)}{(\alpha_3 + 1)^2 (\alpha_2 + 2\alpha_2\alpha_3 + 2) (\alpha_2 + 2\alpha_2\alpha_3 + 1)} V \end{aligned}$$

Node (0,1). Symmetrically to the above analysis for node (1,0),

$$\Delta u_X = u_X^{(1,1)} = \frac{\alpha_3^2 V}{(\alpha_3 + 1)^2}, \quad \Delta u_Y = V - u_Y^{(1,1)} = \frac{\alpha_3^2 + 2\alpha_3}{(\alpha_3 + 1)^2} V, \quad u_X^L = 0, \quad u_Y^L = u_Y^{(1,1)} = \frac{V}{(\alpha_3 + 1)^2}.$$

Thus,

$$\begin{aligned} x^{(0,1)} &= \frac{\alpha_2 \alpha_3^3 (\alpha_3 + 2)}{(\alpha_3 + 1)^2 (\alpha_3 + \alpha_2\alpha_3 + 2) (\alpha_3 + 2\alpha_2\alpha_3 + 2)} \\ y^{(0,1)} &= \frac{\alpha_2 \alpha_3^2 (\alpha_3 + 2)^2}{(\alpha_3 + 1)^2 (\alpha_3 + \alpha_2\alpha_3 + 2) (2\alpha_3 + \alpha_2\alpha_3 + 4)} \end{aligned}$$

Node (0,0). Equilibrium efforts can be obtained as above.

From the above, we retrieve the main result for APA, which mirrors Proposition 5 for Tullock contests; namely, a fully unbiased contest is not optimal. In fact, the following proposition shows that $\{\alpha_1, \alpha_2, \alpha_3\} = \{1, 1, 1\}$ is a saddle rather than a maximum for TE . In Appendix F we show that the result carries over to the maximization of expected winner's effort WE .

Proposition 7 *Consider a best-of-three APA between two ex-ante identical players. With victory-independent biases, the point $\{\alpha_1, \alpha_2, \alpha_3\} = \{1, 1, 1\}$ is a saddle for TE in $\mathbb{R}_{>0}^3$.*

In Appendix G we provide an extensive analytical derivation of the result. Here, we provide a short proof that calculates an analytical expression for TE which is best checked by a computer software, such as Mathematica.²⁹

Proof. We consider moving away from $\{\alpha_1, \alpha_2, \alpha_3\} = \{1, 1, 1\}$ in two directions. The first direction is $\{\alpha_1, \alpha_2, \alpha_3\} = \{1 + b, 1, 1\}$; since the continuation values of the first match are identical between players, the players' incentives in the first match are the same as in a one-shot contest, where it is well known that $b = 0$ gives a local maximum for total effort (Clark and Riis, 2000). The second direction draws an intuition from a typical practice in sports; namely, the alternation of home matches between X and Y , followed by an unbiased tie-breaker (i.e., the final match). In fact, after setting $\{\alpha_1, \alpha_2, \alpha_3\} = \{1 + b, 1 - b, 1\}$, we show below that along this direction $b = 0$ is a local minimum. This establishes that in the $\{\alpha_1, \alpha_2, \alpha_3\}$ -space, $\{1, 1, 1\}$ is a saddle.

Plugging $\{\alpha_1, \alpha_2, \alpha_3\} = \{1 + b, 1 - b, 1\}$ into the formulae in the beginning of Appendix E, the expected total effort is:

$$TE(b) = \frac{5(39508 - 335818b + 1265867b^2 + o(b^2))}{24(19600 - 166600b + 627029b^2 + o(b^2))}$$

²⁹The code is available from the authors upon request. *Note for the referees: the code is attached to our submission.*

Now define the numerator $n(b) \equiv 5(39508 - 335818b + 1265867b^2 + o(b^2))$ and the denominator $d(b) \equiv 24(19600 - 166600b + 627029b^2 + o(b^2))$, then

$$TE'(b) = \frac{n'(b)d(b) - n(b)d'(b)}{[d(b)]^2}$$

Thus,

$$\begin{aligned} TE'(0) &= \frac{n'(0)d(0) - n(0)d'(0)}{[d(0)]^2} \\ &= \frac{5}{24} \frac{-335818 * 19600 + 166600 * 39508}{[d(0)]^2} \\ &= 0 \end{aligned}$$

so $b = 0$ is a critical point in this direction. We then consider the second derivative and obtain

$$\begin{aligned} TE''(0) &= \frac{[n''(0)d(0) - n(0)d''(0)] [d(0)]^2 - 2d(0) [n'(0)d(0) - n(0)d'(0)]}{[d(0)]^4} \\ &= \frac{[n''(0)d(0) - n(0)d''(0)]}{[d(0)]^2} \\ &= \frac{5}{12} \frac{1265867 * 19600 - 39508 * 627029}{[d(0)]^2} \\ &= 5 \frac{3^3 * 7 * 16901}{[d(0)]^2} > 0 \end{aligned}$$

Thus, $b = 0$ is a local minimum. ■

Optimal Contest. Numerical simulations show that the globally optimal contest requires $\{\alpha_1, \alpha_2, \alpha_3\} \simeq \{9.21, 0.21, 0.76\}$. Thus, it is optimal to give a large (approx. 9) advantage to player X in the first match, and balance it out with a medium (approx. 5) advantage to player Y in the second match and a small (approx. 4/3) advantage to player Y in the third match, if necessary. This structure suggests an alternating contest, as it was the case for the globally optimal Tullock contest. Moreover, the optimal alternating contest already attains 94% of the improvement achieved by the unconstrained maximum over the fully unbiased contest.³⁰

The intuition why the alternating contest improves upon the fully unbiased contest and the qualitative features of the equilibrium winning probabilities are identical to the ones already given for Tullock contests.³¹

³⁰The globally optimal contest improves TE vs. the fully unbiased contest by 8.80%. The optimal alternating contest (achieved at $\alpha \simeq 7.13$) improves TE vs. the fully unbiased contest by 8.29%.

³¹All qualitative comparisons we discussed for Tullock contests go through despite a few numerical differences. First, the effort difference between the optimal alternating contest vs. the fully unbiased contest, conditional on reaching each node, is $-0.059V$ in node (0,0), $+0.033V$ in node (1,0), $-0.074V$ in node (0,1), and clearly 0 in node (1,1). Second, the second derivative of TE with respect to α evaluated at $\alpha = 1$ for an alternating contest is $-0.034V$ for the first effect, $0.020V$ for the second effect, and $0.058V$ for the third. More precisely, these number are $\{-\frac{65}{1792}V, \frac{54363}{2744000}V, \frac{13}{224}V\}$. Third, the probability of existence of node (1,1) increases from 0.25 to 0.52. Fourth, the ex-ante probability of victory of the best-of-three contest by player X is 0.487, and the probabilities of victory node-by-node are, $p_X^{(0,0)} = 0.767$, $p_X^{(0,1)} = 0.054$, $p_X^{(1,0)} = 0.343$, and $p_X^{(1,1)} = 0.433$.

F Robustness to maximization of winner's effort in all-pay auction

The definition of WE is (28). We obtain,

Proposition 8 *Consider a best-of-three APA between two ex-ante identical players. With victory-independent biases, the point $\{\alpha_1, \alpha_2, \alpha_3\} = \{1, 1, 1\}$ is a saddle for WE in $\mathbb{R}_{>0}^3$.*

Proof. We follow the structure of Proof of Proposition 7 provided in Appendix E.³²

$$WE(b) = \frac{7(243040 - 2430400b + 10943997b^2 + o(b^2))}{384(19600 - 196000b + 881529b^2 + o(b^2))}.$$

As we did for $TE(b)$, we separate numerator and denominator and one could verify that

$$\begin{aligned} n'(0)d(0) - n(0)d'(0) &= 0 \\ n''(0)d(0) - n(0)d''(0) &= 35280 * 7243 > 0. \end{aligned}$$

Again, $b = 0$ is a local minimum under the constraint $\{\alpha_1, \alpha_2, \alpha_3\} = \{1 + b, 1 - b, 1\}$. This establishes the result. ■

G Extensive analytical proof of Proposition 7

Proof of Proposition 7. Since we know that along $\{\alpha_1, \alpha_2, \alpha_3\} = \{1 + b, 1, 1\}$ $b = 0$ gives a local maximum for TE , we focus here only on showing that along $\{\alpha_1, \alpha_2, \alpha_3\} = \{1 + b, 1 - b, 1\}$ $b = 0$ gives a local minimum for TE , so that $\{\alpha_1, \alpha_2, \alpha_3\} = \{1, 1, 1\}$ is a saddle. We divide the proof into the three effects spelled out in the main text. Notice that we only point out here the difference with respect to the Tullock model analyzed in the proof of Proposition 5.

Third effect. The only difference with respect to Tullock for the third effect is that $x^{(1,1)} + y^{(1,1)} = V/3$ rather than $V/2$. Thus, the equivalent of (17) is

$$\left(p_X^{(0,0)} p_Y^{(1,0)} + p_Y^{(0,0)} p_X^{(0,1)} \right) \left(x^{(1,1)} + y^{(1,1)} \right) = \left[p_X^{(0,0)} \left(1 - p_X^{(1,0)} - p_X^{(0,1)} \right) + p_X^{(0,1)} \right] \frac{V}{3}. \quad (37)$$

This difference carries over throughout the entire proof. Thus, while the first derivative of (37) evaluates to 0 at $b = 0$, its second derivative is $2/3$ of (19), which is the corresponding Tullock expression. Thus, the second derivative of (37) evaluates to $\frac{39}{224} \frac{V}{3} = 0.058V > 0$.

Second effect. Nothing changes with respect to the proof of Proposition 5 until expression (22), including for efforts the fact that $b \rightarrow 0$ $(x^{(1,0)} + y^{(1,0)}) = (x^{(0,1)} + y^{(0,1)})$. From there, we obtain the same equivalence as in Tullock, namely,

$$\left. \frac{d \left[p_X^{(0,0)} (x^{(1,0)} + y^{(1,0)}) + p_Y^{(0,0)} (x^{(0,1)} + y^{(0,1)}) \right]}{db} \right|_{b=0} = \frac{1}{2} \cdot \left(\frac{d(x^{(1,0)} + y^{(1,0)})}{db} + \frac{d(x^{(0,1)} + y^{(0,1)})}{db} \right).$$

³²The code is available from the authors upon request. *Note for the referees: the code is attached to our submission.*

But now instead of using (9) we use its APA homologous (36), and obtain

$$\begin{aligned} \frac{d(x^{(1,0)} + y^{(1,0)})}{db} &= \Delta u_Y^{(1,0)} \cdot \left(\frac{dp_X^{(1,0)}}{d\alpha_2} \frac{d\alpha_2}{db} \right) \left(\frac{p_X^{(1,0)}}{\alpha_2 (1 + p_X^{(1,0)})} + \frac{p_Y^{(1,0)}}{1 + p_Y^{(1,0)}} \right) \\ &\quad + \Delta u_Y^{(1,0)} \cdot p_X^{(1,0)} \left(\frac{1}{\alpha_2} \frac{1}{(1 + p_X^{(1,0)})^2} \left(\frac{dp_X^{(1,0)}}{d\alpha_2} \frac{d\alpha_2}{db} \right) \right. \\ &\quad \left. - \frac{1}{\alpha_2^2} \frac{p_X^{(1,0)}}{1 + p_X^{(1,0)}} \frac{d\alpha_2(b)}{db} - \frac{1}{(1 + p_Y^{(1,0)})^2} \left(\frac{dp_X^{(1,0)}}{d\alpha_2} \frac{d\alpha_2}{db} \right) \right). \end{aligned}$$

Since $\frac{d\alpha_2(b)}{db} = -1$, recall from (36) that

$$x^{(1,0)} + y^{(1,0)} = \Delta u_Y^{(1,0)} \cdot p_X^{(1,0)} \left(\frac{p_X^{(1,0)}}{\alpha_2 (1 + p_X^{(1,0)})} + \frac{p_Y^{(1,0)}}{1 + p_Y^{(1,0)}} \right).$$

Thus, we obtain

$$\begin{aligned} \frac{d(x^{(1,0)} + y^{(1,0)})}{db} &= \left(-\frac{dp_X^{(1,0)}}{d\alpha_2(b)} \right) \frac{x^{(1,0)} + y^{(1,0)}}{p_X^{(1,0)}} + \\ &\quad \Delta u_Y^{(1,0)} \cdot p_X^{(1,0)} \left(\frac{1}{\alpha_2} \frac{1}{(1 + p_X^{(1,0)})^2} \left(-\frac{dp_X^{(1,0)}}{d\alpha_2(b)} \right) + \frac{1}{\alpha_2^2} \frac{p_X^{(1,0)}}{1 + p_X^{(1,0)}} + \frac{1}{(1 + p_Y^{(1,0)})^2} \left(\frac{dp_X^{(1,0)}}{d\alpha_2(b)} \right) \right) \\ &= (x^{(1,0)} + y^{(1,0)}) \cdot \left(-\frac{d \ln p_X^{(1,0)}}{d\alpha_2(b)} \right) \\ &\quad + \Delta u_Y^{(1,0)} \cdot (p_X^{(1,0)})^2 \left(\left(\frac{1}{(1 + p_Y^{(1,0)})^2} - \frac{1}{\alpha_2} \frac{1}{(1 + p_X^{(1,0)})^2} \right) \frac{d \ln p_X^{(1,0)}}{d\alpha_2(b)} + \frac{1}{\alpha_2^2} \frac{1}{1 + p_X^{(1,0)}} \right). \end{aligned} \tag{38}$$

Evaluating the above at $b = 0$ it becomes

$$\left(\frac{33}{280} \right) \cdot \left(-\frac{1}{4} \right) + V \left(\frac{1}{4} \right) \cdot \left(\frac{3}{4} \right)^2 \left(\left(\frac{1}{(1 + \frac{1}{4})^2} - \frac{1}{(1 + \frac{3}{4})^2} \right) \frac{1}{4} + \frac{1}{1 + \frac{3}{4}} \right) V = \frac{2427}{39200} V.$$

Similarly,

$$\begin{aligned} \frac{d(x^{(0,1)} + y^{(0,1)})}{db} &= (x^{(0,1)} + y^{(0,1)}) \cdot \left(-\frac{d \ln p_X^{(0,1)}}{d\alpha_2(b)} \right) + \\ &\quad \Delta u_Y \cdot (p_X^{(0,1)})^2 \left(\left(\frac{1}{(1 + p_Y^{(0,1)})^2} - \frac{1}{\alpha_2} \frac{1}{(1 + p_X^{(0,1)})^2} \right) \frac{d \ln p_X^{(0,1)}}{d\alpha_2(b)} + \frac{1}{\alpha_2^2} \frac{1}{1 + p_X^{(0,1)}} \right), \end{aligned} \tag{39}$$

which evaluates at $b = 0$ to

$$\left(\frac{33}{280}\right) \cdot \left(-\frac{3}{4}\right) + V \cdot \frac{3}{4} \cdot \left(\frac{1}{4}\right)^2 \left(\left(\frac{1}{\left(1 + \frac{3}{4}\right)^2} - \frac{1}{\left(1 + \frac{1}{4}\right)^2} \right) \frac{3}{4} + \frac{1}{1 + \frac{1}{4}} \right) V = -\frac{2427}{39200}V.$$

thus, the first derivative of the second effect equals zero. Therefore, we move to the analysis of the second derivative.

Similarly to how we reached (24), the second derivative of the second effect equals

$$4 \frac{dp_X^{(0,0)}}{db} \left(\frac{2427}{39200}V \right) + \frac{1}{2} \left(\frac{d^2(x^{(1,0)} + y^{(1,0)})}{db^2} + \frac{d^2(x^{(0,1)} + y^{(0,1)})}{db^2} \right). \quad (40)$$

Notice that (18) applies equally well to Tullock and APA because probabilities and Δ 's are identical. Therefore,

$$\left. \frac{dp_X^{(0,0)}}{db} \right|_{b=0} = \frac{1}{4} \frac{3}{7}. \quad (41)$$

Now we evaluate in what follows the two remaining terms of (40) separately namely $\frac{d^2(x^{(1,0)} + y^{(1,0)})}{db^2}$ and $\frac{d^2(x^{(0,1)} + y^{(0,1)})}{db^2}$, which we respectively compute by differentiating again (38) and (39). Thus, we obtain

$$\begin{aligned} \frac{d^2(x^{(1,0)} + y^{(1,0)})}{db^2} &= \frac{d(x^{(1,0)} + y^{(1,0)})}{db} \cdot \left(-\frac{d \ln p_X^{(1,0)}}{d\alpha_2} \right) + (x^{(1,0)} + y^{(1,0)}) \cdot \left(-\frac{d^2 \ln p_X^{(1,0)}}{d\alpha_2^2} \frac{d\alpha_2}{db} \right) \\ &+ \Delta u_Y^{(1,0)} \cdot 2 \left(p_X^{(1,0)} \right) \frac{dp_X^{(1,0)}}{d\alpha_2} \frac{d\alpha_2}{db} \left(\left(\frac{1}{\left(1 + p_Y^{(1,0)}\right)^2} - \frac{1}{\alpha_2 \left(1 + p_X^{(1,0)}\right)^2} \right) \frac{d \ln p_X^{(1,0)}}{d\alpha_2(b)} \right. \\ &\left. + \frac{1}{\alpha_2^2} \frac{1}{1 + p_X^{(1,0)}} \right) \\ &+ \Delta u_Y^{(1,0)} \cdot \left(p_X^{(1,0)} \right)^2 \frac{d}{db} \left(\left(\frac{1}{\left(1 + p_Y^{(1,0)}\right)^2} - \frac{1}{\alpha_2 \left(1 + p_X^{(1,0)}\right)^2} \right) \frac{d \ln p_X^{(1,0)}}{d\alpha_2(b)} + \frac{1}{\alpha_2^2} \frac{1}{1 + p_X^{(1,0)}} \right). \end{aligned} \quad (42)$$

Isolating the last terms, we obtain

$$\begin{aligned}
& \frac{d}{db} \left(\left(\frac{1}{(1+p_Y^{(1,0)})^2} - \frac{1}{\alpha_2} \frac{1}{(1+p_X^{(1,0)})^2} \right) \frac{d \ln p_X^{(1,0)}}{d\alpha_2(b)} + \frac{1}{\alpha_2^2} \frac{1}{1+p_X^{(1,0)}} \right) \quad (43) \\
&= -2 \frac{1}{(1+p_Y^{(1,0)})^3} \frac{dp_Y^{(1,0)}}{db} \frac{d \ln p_X^{(1,0)}}{d\alpha_2(b)} \\
&+ \frac{1}{(1+p_X^{(1,0)})^2} \left(\frac{1}{\alpha_2^2} \frac{d\alpha_2}{db} \right) \frac{d \ln p_X^{(1,0)}}{d\alpha_2(b)} \\
&+ 2 \frac{1}{\alpha_2} \frac{1}{(1+p_X^{(1,0)})^3} \frac{dp_X^{(1,0)}}{db} \frac{d \ln p_X^{(1,0)}}{d\alpha_2(b)} \\
&+ \left(\frac{1}{(1+p_Y^{(1,0)})^2} - \frac{1}{\alpha_2} \frac{1}{(1+p_X^{(1,0)})^2} \right) \frac{d^2 \ln p_X^{(1,0)}}{d\alpha_2^2} \frac{d\alpha_2}{db} \\
&- 2 \frac{1}{\alpha_2^3} \frac{d\alpha_2}{db} \frac{1}{1+p_X^{(1,0)}} \\
&- \frac{1}{\alpha_2^2} \frac{1}{(1+p_X^{(1,0)})^2} \frac{dp_X^{(1,0)}}{db}.
\end{aligned}$$

and if we evaluate at $b = 0$ (43), we obtain,

$$-2 \frac{1}{(1+\frac{1}{4})^3} \frac{3}{16} \frac{1}{4} - \frac{1}{(1+\frac{3}{4})^2} \frac{1}{4} - 2 \frac{1}{(1+\frac{3}{4})^3} \frac{3}{16} \frac{1}{4} + \left(\frac{1}{(1+\frac{1}{4})^2} - \frac{1}{(1+\frac{3}{4})^2} \right) \frac{7}{16} + 2 \frac{1}{1+\frac{3}{4}} + \frac{1}{(1+\frac{3}{4})^2} \frac{3}{16} = \frac{51197}{42875}.$$

We can finally go back to expression (42) evaluated at $b = 0$ to obtain

$$\begin{aligned}
\frac{d^2 (x^{(1,0)} + y^{(1,0)})}{db^2} &= \frac{2427}{39200} V \cdot \left(-\frac{1}{4} \right) + V \left(\frac{33}{280} \right) \cdot \left(-\frac{7}{16} \right) \\
&- \frac{1}{4} V \cdot 2 \left(\frac{3}{4} \right) \frac{3}{16} \left(\left(\frac{1}{(1+\frac{1}{4})^2} - \frac{1}{(1+\frac{3}{4})^2} \right) \frac{1}{4} + \frac{1}{1+\frac{3}{4}} \right) + \frac{1}{4} V \cdot \left(\frac{3}{4} \right)^2 \frac{51197}{42875} \\
&= \frac{151443}{2744000}. \quad (44)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{d^2 (x^{(0,1)} + y^{(0,1)})}{db^2} &= \frac{d (x^{(0,1)} + y^{(0,1)})}{db} \cdot \left(-\frac{d \ln p_X^{(0,1)}}{d\alpha_2} \right) + (x^{(0,1)} + y^{(0,1)}) \cdot \left(-\frac{d^2 \ln p_X^{(0,1)}}{d\alpha_2^2} \frac{d\alpha_2}{db} \right) \\
&+ \Delta u_Y^{(0,1)} \cdot 2 \left(p_X^{(0,1)} \right) \frac{dp_X^{(0,1)}}{d\alpha_2} \frac{d\alpha_2}{db} \left(\left(\frac{1}{(1+p_Y^{(0,1)})^2} - \frac{1}{\alpha_2} \frac{1}{(1+p_X^{(0,1)})^2} \right) \frac{d \ln p_X^{(0,1)}}{d\alpha_2(b)} + \frac{1}{\alpha_2^2} \frac{1}{1+p_X^{(0,1)}} \right) \\
&+ \Delta u_Y^{(0,1)} \cdot \left(p_X^{(0,1)} \right)^2 \frac{d}{db} \left(\left(\frac{1}{(1+p_Y^{(0,1)})^2} - \frac{1}{\alpha_2} \frac{1}{(1+p_X^{(0,1)})^2} \right) \frac{d \ln p_X^{(0,1)}}{d\alpha_2(b)} + \frac{1}{\alpha_2^2} \frac{1}{1+p_X^{(0,1)}} \right). \quad (45)
\end{aligned}$$

Isolating the last terms, we obtain

$$\begin{aligned}
&\frac{d}{db} \left(\left(\frac{1}{(1+p_Y^{(0,1)})^2} - \frac{1}{\alpha_2} \frac{1}{(1+p_X^{(0,1)})^2} \right) \frac{d \ln p_X^{(0,1)}}{d\alpha_2(b)} + \frac{1}{\alpha_2^2} \frac{1}{1+p_X^{(0,1)}} \right) \quad (46) \\
&= -2 \frac{1}{(1+p_Y^{(0,1)})^3} \frac{dp_Y^{(0,1)}}{db} \frac{d \ln p_X^{(0,1)}}{d\alpha_2(b)} \\
&\quad + \frac{1}{(1+p_X^{(0,1)})^2} \left(\frac{1}{\alpha_2^2} \frac{d\alpha_2}{db} \right) \frac{d \ln p_X^{(0,1)}}{d\alpha_2(b)} \\
&\quad + 2 \frac{1}{\alpha_2} \frac{1}{(1+p_X^{(0,1)})^3} \frac{dp_X^{(0,1)}}{db} \frac{d \ln p_X^{(0,1)}}{d\alpha_2(b)} \\
&\quad + \left(\frac{1}{(1+p_Y^{(0,1)})^2} - \frac{1}{\alpha_2} \frac{1}{(1+p_X^{(0,1)})^2} \right) \frac{d^2 \ln p_X^{(0,1)}}{d\alpha_2^2} \frac{d\alpha_2}{db} \\
&\quad - 2 \frac{1}{\alpha_2^3} \frac{d\alpha_2}{db} \frac{1}{1+p_X^{(0,1)}} \\
&\quad - \frac{1}{\alpha_2^2} \frac{1}{(1+p_X^{(0,1)})^2} \frac{dp_X^{(0,1)}}{db}.
\end{aligned}$$

and if we evaluate at $b = 0$ (46), we obtain,

$$-2 \frac{1}{(1+\frac{3}{4})^3} \frac{3}{16} \frac{3}{4} - \frac{1}{(1+\frac{1}{4})^2} \frac{3}{4} - 2 \frac{1}{(1+\frac{1}{4})^3} \frac{3}{16} \frac{3}{4} + \left(\frac{1}{(1+\frac{3}{4})^2} - \frac{1}{(1+\frac{1}{4})^2} \right) \frac{15}{16} + 2 \frac{1}{1+\frac{1}{4}} + \frac{1}{(1+\frac{1}{4})^2} \frac{3}{16} = \frac{32141}{42875}.$$

We can finally go back to expression (45) evaluated at $b = 0$ to obtain

$$\begin{aligned}
\frac{d^2 (x^{(0,1)} + y^{(0,1)})}{db^2} &= -\frac{2427}{39200}V \cdot \left(-\frac{3}{4}\right) + V \left(\frac{33}{280}\right) \cdot \left(-\frac{15}{16}\right) \\
&- \frac{3}{4}V \cdot 2 \left(\frac{1}{4}\right) \frac{3}{16} \left(\left(\frac{1}{\left(1 + \frac{3}{4}\right)^2} - \frac{1}{\left(1 + \frac{1}{4}\right)^2} \right) \frac{3}{4} + \frac{1}{1 + \frac{1}{4}} \right) + \frac{3}{4}V \cdot \left(\frac{1}{4}\right)^2 \frac{32141}{42875} \\
&= -\frac{188337}{2744000}V.
\end{aligned} \tag{47}$$

We finally plug (41), (44) and (47) into (40), and obtain that the second derivative of the second effect equals

$$4 \frac{1}{4} \frac{3}{7} \left(\frac{2427}{39200}V \right) + \frac{1}{2} \left(\frac{151443}{2744000} - \frac{188337}{2744000} \right) V = \frac{54363}{2744000}V > 0.$$

First effect. We quantify the effect on TE of node $(0,0)$. We rewrite (36) as

$$x^{(0,0)} + y^{(0,0)} = \Delta u_Y^{(0,0)} \cdot p_X^{(0,0)} \left(\frac{p_X^{(0,0)}}{\alpha_1 (1 + p_X^{(0,0)})} + \frac{p_Y^{(0,0)}}{1 + p_Y^{(0,0)}} \right). \tag{48}$$

Thus,

$$\begin{aligned}
\frac{d(x^{(0,0)} + y^{(0,0)})}{db} &= \frac{d\Delta u_Y^{(0,0)}}{d\alpha_2} \frac{d\alpha_2}{db} \cdot p_X^{(0,0)} \left(\frac{1}{\alpha_1} \frac{p_X^{(0,0)}}{1 + p_X^{(0,0)}} + \frac{p_Y^{(0,0)}}{1 + p_Y^{(0,0)}} \right) \\
&+ \Delta u_Y^{(0,0)} \cdot \left(\frac{dp_X^{(0,0)}}{d\alpha_1} \frac{d\alpha_1}{db} + \frac{dp_X^{(0,0)}}{d\alpha_2} \frac{d\alpha_2}{db} \right) \left(\frac{1}{\alpha_1} \frac{p_X^{(0,0)}}{1 + p_X^{(0,0)}} + \frac{p_Y^{(0,0)}}{1 + p_Y^{(0,0)}} \right) \\
&+ \Delta u_Y^{(0,0)} \cdot p_X^{(0,0)} \left(\frac{1}{\alpha_1} \frac{1}{\left(1 + p_X^{(0,0)}\right)^2} \left(\frac{dp_X^{(0,0)}}{d\alpha_1} \frac{d\alpha_1}{db} + \frac{dp_X^{(0,0)}}{d\alpha_2} \frac{d\alpha_2}{db} \right) \right. \\
&\left. - \frac{1}{\alpha_1^2} \frac{p_X^{(0,0)}}{1 + p_X^{(0,0)}} \frac{d\alpha_1}{db} - \frac{1}{\left(1 + p_Y^{(0,0)}\right)^2} \left(\frac{dp_X^{(0,0)}}{d\alpha_1} \frac{d\alpha_1}{db} + \frac{dp_X^{(0,0)}}{d\alpha_2} \frac{d\alpha_2}{db} \right) \right).
\end{aligned}$$

Since, $\frac{d\alpha_1(b)}{db} = 1$, $\frac{d\alpha_2(b)}{db} = -1$, and using (48), we obtain

$$\begin{aligned}
& \frac{d(x^{(0,0)} + y^{(0,0)})}{db} = -\frac{d\Delta u_Y^{(0,0)}}{d\alpha_2(b)} \cdot \frac{x^{(0,0)} + y^{(0,0)}}{\Delta u_Y^{(0,0)}} + \frac{x^{(0,0)} + y^{(0,0)}}{p_X^{(0,0)}} \cdot \left(\frac{dp_X^{(0,0)}}{d\alpha_1(b)} - \frac{dp_X^{(0,0)}}{d\alpha_2(b)} \right) \\
& + \Delta u_Y^{(0,0)} \cdot p_X^{(0,0)} \left(\left(\frac{1}{\alpha_1} \frac{1}{(1+p_X^{(0,0)})^2} - \frac{1}{(1+p_Y^{(0,0)})^2} \right) \left(\frac{dp_X^{(0,0)}}{d\alpha_1(b)} - \frac{dp_X^{(0,0)}}{d\alpha_2(b)} \right) - \frac{1}{\alpha_1^2} \frac{p_X^{(0,0)}}{1+p_X^{(0,0)}} \right) \\
= & (x^{(0,0)} + y^{(0,0)}) \cdot \left(\frac{d \ln p_X^{(0,0)}}{d\alpha_1(b)} - \frac{d \ln p_X^{(0,0)}}{d\alpha_2(b)} - \frac{d \ln \Delta u_Y^{(0,0)}}{d\alpha_2(b)} \right) \\
& + \Delta u_Y^{(0,0)} \cdot (p_X^{(0,0)})^2 \left(\left(\frac{1}{\alpha_1} \frac{1}{(1+p_X^{(0,0)})^2} - \frac{1}{(1+p_Y^{(0,0)})^2} \right) \left(\frac{d \ln p_X^{(0,0)}}{d\alpha_1(b)} - \frac{d \ln p_X^{(0,0)}}{d\alpha_2(b)} \right) - \frac{1}{\alpha_1^2} \frac{1}{1+p_X^{(0,0)}} \right).
\end{aligned}$$

Evaluating the above at $b = 0$ we obtain

$$\left. \frac{d(x^{(0,0)} + y^{(0,0)})}{db} \right|_{b=0} = \frac{7}{32} \cdot \left(\frac{1}{2} - \frac{2}{7} + \frac{2}{7} \right) + \frac{21}{32} \cdot \frac{1}{4} \left(\left(\frac{4}{9} - \frac{4}{9} \right) \left(\frac{1}{2} - \frac{2}{7} \right) - \frac{2}{3} \right) = 0.$$

Thus, we move to analyze the second derivative,

$$\begin{aligned}
\frac{d^2 (x^{(0,0)} + y^{(0,0)})}{db^2} &= \frac{d (x^{(0,0)} + y^{(0,0)})}{db} \cdot \left(\frac{d \ln p_X^{(0,0)}}{d\alpha_1(b)} - \frac{d \ln p_X^{(0,0)}}{d\alpha_2(b)} - \frac{d \ln \Delta u_Y^{(0,0)}}{d\alpha_2(b)} \right) \\
&+ (x^{(0,0)} + y^{(0,0)}) \cdot \left(\frac{\partial^2 \ln p_X^{(0,0)}}{\partial \alpha_1^2} \frac{d\alpha_1}{db} + 2 \frac{\partial^2 \ln p_X^{(0,0)}}{\partial \alpha_1 \partial \alpha_2} \frac{d\alpha_2}{db} \right. \\
&\quad \left. - \frac{\partial^2 \ln p_X^{(0,0)}}{\partial \alpha_2^2} \frac{d\alpha_2}{db} - \frac{d^2 \ln \Delta u_Y^{(0,0)}}{d\alpha_2^2} \frac{d\alpha_2}{db} \right) \\
&+ \left(\frac{d \Delta u_Y^{(0,0)}}{d\alpha_2} \frac{d\alpha_2}{db} \cdot (p_X^{(0,0)})^2 + 2 p_X^{(0,0)} \Delta u_Y^{(0,0)} \left(\frac{dp_X^{(0,0)}}{d\alpha_1(b)} - \frac{dp_X^{(0,0)}}{d\alpha_2(b)} \right) \right) \\
&* \left(\left(\frac{1}{\alpha_1} \frac{1}{(1+p_X^{(0,0)})^2} - \frac{1}{(1+p_Y^{(0,0)})^2} \right) \left(\frac{d \ln p_X^{(0,0)}}{d\alpha_1(b)} - \frac{d \ln p_X^{(0,0)}}{d\alpha_2(b)} \right) - \frac{1}{\alpha_1^2} \frac{1}{1+p_X^{(0,0)}} \right) \\
&+ \Delta u_Y^{(0,0)} \cdot (p_X^{(0,0)})^2 \left[\left(-\frac{1}{\alpha_1^2} \frac{1}{(1+p_X^{(0,0)})^2} - 2 \frac{1}{\alpha_1} \frac{1}{(1+p_X^{(0,0)})^3} \left(\frac{dp_X^{(0,0)}}{d\alpha_1(b)} - \frac{dp_X^{(0,0)}}{d\alpha_2(b)} \right) \right. \right. \\
&\quad \left. \left. - 2 \frac{1}{(1+p_Y^{(0,0)})^3} \left(\frac{dp_X^{(0,0)}}{d\alpha_1(b)} - \frac{dp_X^{(0,0)}}{d\alpha_2(b)} \right) \right) \left(\frac{d \ln p_X^{(0,0)}}{d\alpha_1(b)} - \frac{d \ln p_X^{(0,0)}}{d\alpha_2(b)} \right) \right. \\
&\quad \left. + \left(\frac{1}{\alpha_1} \frac{1}{(1+p_X^{(0,0)})^2} - \frac{1}{(1+p_Y^{(0,0)})^2} \right) \left(\frac{\partial^2 \ln p_X^{(0,0)}}{\partial \alpha_1^2} \frac{d\alpha_1}{db} + 2 \frac{\partial^2 \ln p_X^{(0,0)}}{\partial \alpha_1 \partial \alpha_2} \frac{d\alpha_2}{db} \right. \right. \\
&\quad \left. \left. - \frac{\partial^2 \ln p_X^{(0,0)}}{\partial \alpha_2^2} \frac{d\alpha_2}{db} \right) + 2 \frac{1}{\alpha_1^3} \frac{1}{1+p_X^{(0,0)}} + \frac{1}{\alpha_1^2} \frac{1}{(1+p_X^{(0,0)})^2} \left(\frac{dp_X^{(0,0)}}{d\alpha_1(b)} - \frac{dp_X^{(0,0)}}{d\alpha_2(b)} \right) \right].
\end{aligned}$$

Thus,

$$\begin{aligned}
\left. \frac{d^2 (x^{(0,0)} + y^{(0,0)})}{db^2} \right|_{b=0} &= 0 + \frac{7}{32} \cdot \left(-\frac{3}{4} + \frac{2}{7} - \frac{18}{49} + \frac{31}{49 \cdot 8} \right) V + \left(\frac{3}{16} \cdot \frac{1}{4} + \frac{21}{32} \left(\frac{1}{4} - \frac{1}{7} \right) \right) \left(-\frac{2}{3} \right) V + \\
&\quad + \frac{21}{32} \cdot \frac{1}{4} \left[\left(-\frac{4}{9} - 2 \frac{8}{27} \left(\frac{1}{4} - \frac{1}{7} \right) - 2 \frac{8}{27} \left(\frac{1}{4} - \frac{1}{7} \right) \right) \left(\frac{1}{2} - \frac{2}{7} \right) + 2 \frac{2}{3} + \frac{4}{9} \left(\frac{1}{4} - \frac{1}{7} \right) \right] V \\
&= -\frac{65}{1792} V.
\end{aligned}$$

Overall second derivative. We can finally put together the three effects, and conclude that along the direction $\{\alpha_1, \alpha_2, \alpha_3\} = \{1+b, 1-b, 1\}$, $b=0$ is a local minimum since the first derivative of the total effort with respect to b is 0, and the second derivative equals $\frac{39}{224} \frac{V}{3} + \frac{54363}{2744000} V - \frac{65}{1792} V = \frac{456327}{10976000} V > 0$. This establishes that in the $\{\alpha_1, \alpha_2, \alpha_3\}$ -space, $\{1, 1, 1\}$ is a saddle. ■