# Debt Sustainability in a Low Interest Rate World*

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**Abstract**

Historic levels of public debt and conditions of $r < g$ for advanced economies have prompted a reassessment of debt sustainability. Using a continuous-time model in which the debt-to-GDP ratio is stochastic and $r < g$ on average, we find that theoretical conditions for sustainability are not closely tied to common metrics of sustainability: the level of debt or whether $r < g$. However, when the primary surplus is bounded, a state-dependent threshold level of public debt determines sustainability. Secular stagnation factors like slow population growth, low productivity growth, or higher output risk carry differing implications for debt sustainability.

Keywords: secular stagnation, public debt, debt sustainability, low interest rates

JEL Classification: E43, E62, H68

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# 1 Introduction

A decade after the end of the Great Recession, output growth in advanced economies remains historically slow with economies saddled with increased levels of public debt. The 2020 coronavirus pandemic will leave public debt levels far higher.

The combination of slow growth and historically high levels of debt would appear to be problematic for debt sustainability. However, advanced economies have benefited from historically low real interest rates. Even prior to the pandemic, short-term rates were close to the zero lower bound and long-term rates fell over the decade, keeping the difference between interest rates and output growth, which we call the cost of debt servicing, low for these economies. This combination of low growth and low real interest rates—sometimes labeled the secular stagnation hypothesis—carry contradictory implications for debt dynamics.

Among fiscal authorities and in much of the applied literature (Reinhart and Rogoff, 2010), discussions of debt sustainability center on two approaches. The first approach, which we label *the stock approach*, focuses on the level of the debt-to-GDP ratio or its distribution. Under the stock approach, debt-to-GDP ratios must remain below a threshold level or converge to a stable distribution.

The second approach, which we label *the flow approach*, focuses on the trajectory of the debt-to-GDP ratio (Ball, Elmendorf, and Mankiw, 1998; Barrett, 2018; Blanchard, 2019). The flow approach recognizes that debt dynamics depend on the difference between the interest on government debt $r$ and the growth rate of the economy $g$. As Figure 1 shows, many advanced economies today enjoy conditions of $r < g$ and, consequently, a negative cost of servicing the debt. These favorable dynamics have prompted a reassessment of the risks of high public debt.

In this paper, we analyze debt sustainability from a more theoretically grounded perspective. We define debt sustainability as conditions under which forward-looking investors willingly hold public debt. We label this *the economic approach* to debt sustainability. The central message of this paper is that these three approaches to debt sustainability need not coincide, and the drivers affecting $r$ and $g$ may have diverging effects on debt sustainability defined in these three ways.

We start by summarizing some basic facts about the historical relationship between interest rates on government debt and economic growth. Using data on 17 advanced economies, we find that cases in which the real interest rate is less than GDP growth are fairly common. Taking 5-year averages, $r < g$ for nearly half the country-period observations. If $r - g$ was permanently negative, a government in primary balance would see its debt-to-GDP ratio shrink to zero. However, the cost of servicing the debt exhibits substantial variability. The interquartile range for $r - g$ is approximately five percentage points, with substantial and rising persistence in the postwar period.

Consistent with these empirical facts, we analyze a continuous-time general equilibrium endowment economy that produces closed-form solutions even with aggregate risk. In the baseline model, output follows an exogenous geometric Brownian motion. The government is assumed to follow a fiscal rule; we separately consider rules that ensure the safety of public debt in all future states and rules that result in default in some states. Households are willing to hold public debt if fiscal policy satisfies the household's transversality condition and the present value of the government's primary surpluses equals the outstanding value of the public debt. These conditions define debt sustainability under the economic approach.

The transversality condition and the government budget constraint are satisfied so long as the primary surplus responds at least linearly to the debt-to-GDP ratio. This requirement on fiscal policy turns out to be quite mild—the size of the (positive) primary surplus can be arbitrarily small at any particular debt-to-GDP ratio but must nevertheless scale linearly, implying unboundedly large primary surpluses as the debt-to-GDP ratio becomes large. Crucially, neither the level of the debt-to-GDP ratio nor whether $r < g$ matters for debt sustainability; put starkly, the stock and flow approaches of debt sustainability are unrelated to the economic approach.

The debt-to-GDP ratio has a stationary distribution that can be characterized in closed form if the fiscal surplus reacts in a stronger than linear fashion to the debt-to-GDP ratio. In this case, fiscal policy satisfies both the stock and economic approach. Conversely, we can construct cases where the flow approach and stock approach are satisfied, but the economic approach is not. In the former case, the debt-to-GDP ratio evolves as an Ornstein-Uhlenbeck process and converges to a log-normal distribution. In the latter case, the debt-to-GDP ratio converges to a Pareto distribution.

When the stationary distribution exists, the mean debt-to-GDP ratio is *decreasing* in population growth and *increasing* in productivity growth when the elasticity of intertemporal substitution (EIS) is less than one.[1] Lower population growth leaves the borrowing rate unchanged while directly lowering output growth, shifting the average debt-to-GDP ratio higher. By contrast, when the EIS is less than one, a decline in productivity growth has a more than a one-for-one effect on the real interest rate, lowering $r - g$ and thereby reducing the average debt-to-GDP ratio. A rise in output uncertainty raises the variance of the debt-to-GDP ratio but may well reduce the mean by lowering the interest rate $r$.

The assumption that the primary surplus (as a share of GDP) can be unboundedly large is a strong one and separates completely theoretical conditions for sustainability from the flow and stock notions of sustainability. The stock view—specifically the idea of a debt threshold—can be resurrected if the primary surplus is bounded above and default becomes possible, a property Ghosh, Kim, Mendoza, Ostry, and Qureshi (2013) labeled as fiscal fatigue. To preserve tractability and highlight key results, we assume that the only force that generates $r < g$ is aggregate risk due to rare disasters. The possibility of a rare disasters ensures $r < g$, but once aggregate uncertainty is realized, $r > g$. This time-varying probability of disasters generates interest rate variation that can force a government into default.

In the presence of a maximum primary surplus over GDP, an endogenous maximum debt-

[1] In the Pareto case, parameters must be such that higher moments of the distribution are finite.

to-GDP ratio emerges, labeled the *fiscal limit.* The fiscal limit is closer to the stock approach of debt sustainability, but is state-dependent as it depends on the time-varying probability of disasters. For levels of debt below the limit, the interest rate on government debt rises with the debt-to-GDP ratio reflecting default risk. In contrast to the no default case, stagnation forces like slower productivity and population growth affect debt sustainability. Perhaps counterintuitively, lower productivity growth relaxes the fiscal limit (when EIS $< 1$), while lower population growth tightens the fiscal limit.

Our model also gives rise to a *flipping point* level of debt—a debt-to-GDP ratio above which $r$ becomes greater than $g$. An increase in disaster risk has *opposing* effects on the flipping point and the fiscal limit, raising the former and lowering the latter, showing again that flow and stock notions of debt sustainability need not be closely linked. When disaster risk rises, the fiscal limit falls compressing fiscal space. However, because the flipping point has *increased,* policymakers may enjoy a greater range of the debt-to-GDP over which $r < g$.

Calibrating our model to pre-pandemic (2019) US data and making somewhat pessimistic assumptions: i) relatively high disaster risk (probability of 6.5 percent annually), ii) zero recovery value in default, and iii) a maximum fiscal surplus of 5 percent of GDP, we nevertheless find substantial fiscal space. The fiscal limit is 222 percent of GDP in low risk state and 144 percent of GDP in high risk state, and the flipping point is 106 percent of GDP.[2] Furthermore, our calibration highlights the potential benefits of higher population growth. Reverting back to the postwar average growth rate would push these various thresholds higher by more than 50 percentage points.

The paper is laid out as follows. Section 2 presents basic statistics on the average cost of servicing the public debt and its variability. Section 3 presents and solves the model without default, while Section 4 analyzes the model with fiscal fatigue and sovereign default. Section 5 concludes.

[2]The recent papers that estimate fiscal limits in advanced countries are Ghosh, Kim, Mendoza, Ostry, and Qureshi (2013), Collard, Habib, and Rochet (2015), and Pallara and Renne (2019).

**Related Literature.** This paper builds on the literature analyzing fiscal policy when interest rates are low. Darby (1984) shows that the condition of $r < g$ turns the famous unpleasant monetarist arithmetic in Sargent and Wallace (1981) into a pleasant one. Abel, Mankiw, Summers, and Zeckhauser (1989) argue that low or negative rates do not necessarily imply under-consumption and excess saving. Blanchard and Weil (2001) show that even in cases when $r < g$ due to the effects of uncertainty, it may not be feasible for the government to operate a debt Ponzi game. Bohn (1995) establishes that empirical tests of debt sustainability that rely on $r < g$ are not appropriate in stochastic economies—a finding that we build on in our analytical framework.

More recently, Reis (2021) shows how monetary and fiscal policies affect public debt liquidity and, ultimately, debt sustainability when $r < g$. Relatedly, Jiang, Lustig, Van Nieuwerburgh, and Xiaolan (2019) find that the market value of US public debt exceeds the expected present value of future government surpluses.

The paper also builds on a recent literature that seeks to explain the decline in safe interest rates. Eggertsson and Mehrotra (2014) and Eggertsson, Mehrotra, and Robbins (2018) emphasize factors like low population and productivity growth. Caballero and Farhi (2017) stress a shortage of safe assets and an elevated risk premium in accounting for low interest rates. The model here is closer to the literature that cites the risk premium as the chief factor behind low interest rates.[3]

# 2 Stylized Facts on $r - g$

This section briefly summarizes some basic facts on the relationship between the return on government debt and the growth rate of the economy. We show that, for advanced countries, $r$ is frequently less than $g$ and $r - g$ exhibits substantial variability over time.[4]

[3] See, for example, Del Negro, Giannone, Giannoni, and Tambalotti (2017) and Farhi and Gourio (2018) for a quantitative analysis of how risk and liquidity premia account for low interest rates on government debt.

[4] Mehrotra (2018) provides further empirical analysis on the risk of reverting to periods of $r > g$ and the historical relationship of $r - g$ with both productivity and population growth.

The importance of $r$ relative to $g$ for debt dynamics can easily be seen by inspecting the government's flow budget constraint, expressed here in continuous time with no shocks:

$$dB_t = [r_t B_t + (G_t - T_t)]dt, \tag{1}$$

where $T_t$ is real tax revenue (net of any transfers), $G_t$ is real government expenditures, $B_t$ is real government debt, and $r_t$ is the effective real interest rate paid on government debt. $Y_t$ is real gross domestic product (GDP), hence the debt-to-GDP ratio is $B_t/Y_t$. Along the balanced growth path with constant interest rate and output growth, equation (1) gives the primary surplus that keeps the debt-to-GDP ratio constant:

$$\frac{T_t - G_t}{Y_t} = (r - g)\,\frac{B_t}{Y_t}.$$

The difference between the return on public debt $r$ and output growth rate $g$ represents the unit fiscal cost of servicing public debt. When $r > g$, the fiscal authority must raise real resources to keep the debt-to-GDP ratio constant. Equivalently, when $r > g$, the debt-to-GDP ratio will be on an ever increasing trajectory even in primary balance, justifying the flow approach focus on whether $r > g$ or $r < g$.

To analyze the behavior of $r - g$, we draw on the historical macroeconomic dataset of Jordà, Schularick, and Taylor (2016). This dataset provides macroeconomic and financial variables for seventeen advanced economies including the US from 1870 to 2016.

To compute measures of the cost of servicing the debt $r - g$, we need a measure of the ex-ante real interest rate. A three-year moving average of inflation is used as a proxy for expected inflation in line with the approach in Hamilton, Harris, Hatzius, and West (2016). The real interest rate is then the nominal measure less the three-year moving average of inflation. When using annual data, we drop extreme observations of $r - g$ above ten percent and below negative ten percent. The resulting dataset is an unbalanced panel of 2145 observations.

Table 1 provides basic summary statistics for the real interest rate, population growth rate and real GDP per capita growth rate in the dataset. Values are shown for all countries and for just the US. For all countries, the median nominal long-term rate is 4.6 percent with a median inflation rate of 2.1 percent. For the US, both interest rates and inflation rates are slightly lower than the global median. Population growth is somewhat higher in the US, as is per capita real GDP growth. Public debt-to-GDP ratios are, on average, slightly lower in the US.[5]

Real interest rates (negative in the US) and population growth were in the bottom quartile of the distribution of historical observations prior to the start of the COVID-19 pandemic. By contrast, the value of the US debt-to-GDP ratio before the pandemic (80% of GDP) was in the top quartile.

Figure 2 plots the debt servicing cost for the US—long-term real interest rates less GDP growth. The blue and more variable line shows this measure for the US and the smoother red line is a five-year moving average of the blue line to smooth out business cycle fluctuations. As the figure shows, the cost of servicing the debt has frequently been negative historically and for a large part of the postwar period. Indeed, the period between 1980 and 2000 is the exception, namely one of the few periods where real interest rates on government debt consistently exceeded real GDP growth. In the postwar period, US $r - g$ displays less volatility and greater persistence than the prewar or interwar periods.

Table 2 presents statistics on the fiscal cost of servicing the debt: $r - g$. We take averages over five year periods (non-overlapping) of $r - g$ for the US and 16 other advanced economies, presenting median values and ranges.[6] As the table shows, over the full sample of advanced economies, the median value of $r - g$ is nearly zero (eight basis points). In the US, that

[5] The macrohistory data set contains general government gross debt when data are available and central government gross debt further back in time when data are scarce. These data do not net government debt-like assets holdings, which is a more appropriate measure of public debt for sustainability purposes. Using the data from the IMF World Economic Outlook, which contains gross and net debt after 1980 for these countries, we document that the difference between gross and net debt-to-GDP ratios is on average 15 percent. However, gross debt variance over time—the statistic that we use for calibration of the model—is almost identical to net debt variance.

Finally, the data do not include unfunded future government liabilities, such as Social Security.

[6] Five-year periods with fiscal cost above ten percent or below minus ten percent are winsorized at these levels.

median value ($-16$ basis points) has been negative over the past 140 years. The finding that $r-g$ is close to zero is not a function of historically extreme periods. Excluding the world wars and the interwar years (including the Great Depression) leaves the median value slightly higher for all countries and unaffected for the US. Limiting the sample to only the postwar period, $r-g$ becomes *more* negative for both the full sample and the US.

Though the median value is negative, there remains substantial variability in the cost of servicing the debt. Table 2 provides the interquartile range of $r-g$ for both the full sample and the US. The 75th percentile is roughly one percent in the US, while the 25th percentile is substantially negative. These percentiles for the USdisplay the same level shift as the median; $r-g$ is lower at each quartile than the corresponding figure for the all country sample. An interquartile range of four to five percentage points demonstrates the substantial variability in $r-g$.

Table 2 also shows the fraction of observations with a negative debt servicing cost or a substantially negative cost (less than negative two percent). In the all-country sample, half of the observations are negative and between 20 to 35 percent of five-year periods feature a substantially negative value for the fiscal cost depending on the time period. In the case of the US, these percentages are somewhat higher than those for the global sample. Again, the percentage of years with a negative cost for the public debt are not driven by the interwar years and the Great Depression, or the world wars. If anything, the postwar period features a greater percentage of years with $r-g$ negative. Quite remarkably, 70 percent of five-year periods in the US and 55 percent of five-year periods across all advanced countries show negative net fiscal cost in the postwar period.

Nevertheless, values for $r-g$ at the 75th percentile illustrate the perils of carrying a high public debt level. Sustained periods of even relatively moderate levels of debt servicing costs would require a substantial primary surplus to stabilize the debt-to-GDP ratio, particularly among countries with debt-to-GDP ratios above 100 percent. This section treated $r-g$ as a statistical object; in the next section, we turn to the analysis of debt dynamics when $r-g$

is determined endogenously.

# 3 Debt Sustainability without Default

This section introduces a continuous-time general equilibrium model in which aggregate output follows an exogenous geometric Brownian motion process, and fiscal policy rules out default. The model allows closed-form solution for the real interest rate on government debt, the return on risky assets, and the distribution of the debt-to-GDP ratio. Here, we define and contrast the three approaches to public debt sustainability. We show that theeconomic approach to debt sustainability is not related to the stationarity of the debt-to-GDP distribution (the stock approach) and the sign of $r - g$ (the flow approach) and conclude with a numerical exercise that assesses the importance of secular stagnation forces that may account for low $r$ and low $g$.

## 3.1 Households

Time is continuous, infinite, and indexed by $t$. The economy is populated by a representative household with a continuum of identical infinitely-lived members whose measure $N_t$ grows deterministically at a constant rate $n$ with the initial value at time zero of one. The household members derive utility from consuming and from holding safe and liquid bonds that only the government can supply. The household is endowed with a traded Lucas tree that yields per capita dividends that follow geometric Brownian motion:

$$\frac{dy_t}{y_t} = g_y dt + \sigma_y dZ_t^y, \tag{2}$$

where $g_y$ is the growth rate of per capita endowment, $\sigma_y^2$ is the variance of shocks to the growth rate, and $dZ_t^y$ is standard Brownian motion. The initial value is $y_0$. Total output $Y_t$ equals to $N_t y_t$. We refer to $y_t$ as productivity. We draw a distinction between the growth rate of output $g$ that we have referred to earlier and the growth rate of output per capita (or productivity), $g_y$. In the model, average output growth $g$ is the sum of population growth $n$

and productivity growth $g_y$.

Financial markets are dynamically complete, i.e., agents have access to government-issued liquid bonds, safe bonds that are non-liquid, and risky securities.[7] The safe bonds are assumed to be in zero net supply. Equity—a claim on a Lucas tree that pays consumption goods at flow rate $y_t$—is in positive net supply that increases at the rate of population growth. There is no international trade in either goods or assets.[8]

A typical household chooses paths for consumption $\{c_t\}$, wealth $\{w_t\}$, investments in liquid, safe, and risky assets $\{b_t, s_t, x_t w_t\}$, where $x_t$ denotes a fraction of wealth invested in risky assets, to maximize

$$\mathbb{E}_0 \int_0^\infty e^{-(\rho-n)t} \left[ \frac{c_t^{1-\gamma}-1}{1-\gamma} + \bar{c}_t^{-\gamma} y_t u\left(\frac{b_t}{y_t}\right) \right] dt, \tag{3}$$

subject to the flow budget constraint

$$dw_t = (r_t^s s_t + r_t b_t - c_t - \tau_t - w_t n)dt + w_t x_t dr_t^x, \tag{4}$$

and subject to the wealth breakdown into safe, liquid, and risky assets

$$w_t = s_t + b_t + x_t w_t, \tag{5}$$

together with a no-Ponzi game condition expressed below. All the variables in the above optimization problem represent per member of household quantities: $c_t$ is consumption of a member of the household, $w_t$ is the financial wealth, $b_t$ is the purchases of safe and liquid government bonds, $s_t$ is purchases of safe assets that do not provide liquidity services, $\tau_t$ are taxes. Note that the part of the drift in financial wealth of a member of the household $-nw_t$ captures the fact that new members dilute the household level of wealth. The returns

[7] When the households are free to re-optimize their portfolios at each instant, their access to just three securities mentioned above and optimal portfolio choice is equivalent to the presence of complete markets, i.e., the access to state-contingent Arrow-Debreu securities.

[8] It is potentially important to allow international trade in assets given a high level of integration in financial markets. We leave this fruitful avenue of research for future work and focus our analysis on a closed economy.

on liquid, safe, and the risky assets over a short period of time $dt$ are $r_t dt, r_t^s dt$, and $dr_t^x$ respectively.[9] Finally, $\bar{c}_t$ is an economy-wide average consumption of members of households. It will be equal to $c_t$ in equilibrium, but an individual household does not internalize this fact.

The members of the household derive utility from consumption $\{c_t\}$ and from holding government bonds $\{b_t\}$. Formally, the utility function (3) consists of two terms that capture the utility from consumption and utility from holding government bonds. The assumption that households have non-pecuniary preferences over government debt is a non-structural way to represent special features of government debt such as safety and liquidity.[10] The main reason for having non-pecuniary preferences in our model is to introduce a deviation from Ricardian equivalence in a representative household setting. In this case, changes in the supply of government bonds affect the interest rate on these bonds contributing to changes in $r - g$ over time that facilitate bringing the model to the data.

The preferences over holding government bonds (second term in equation (3)) are additive over time. The instantaneous utility over liquid bonds depends on average consumption $\bar{c}_t$, the exogenous endowment, and liquid debt holdings relative to this endowment. As a result, demand for liquid debt over aggregate output is a function of the liquidity yield only. We choose the following structural form for tractability:

$$u\left(\frac{b_t}{y_t}\right) = \frac{b_t}{y_t}\left[\alpha_u + \beta_u - \beta_u \log\left(\frac{b_t}{y_t}\right)\right], \tag{6}$$

where $\alpha_u$ are $\beta_u$ are non-negative real numbers. Equation (6) implies that the marginal preferences from holding government debt, i.e., $u'(b_t/y_t) = \alpha_u - \beta_u \log(b_t/y_t)$, is decreasing in debt holdings. The fact that the marginal utility turns negative after a certain level of debt-to-GDP can be thought of as a reduced form way of capturing the idea that people may

[9] All equalities featuring random variables hold "a.s.", and all stochastic differential equations are assumed to have solutions.

[10] This assumption has been used to improve asset-pricing properties of the standard finance models (Krishnamurthy and Vissing-Jorgensen, 2012), explain business cycles (Fisher, 2015), solve for optimal government debt maturity (Greenwood, Hanson, and Stein, 2015), resolve New Keynesian puzzles (Michaillat and Saez, 2018), and fit the consumption response to income shocks (Auclert, Rognlie, and Straub, 2018).

worry about holding government debt when it is too large. It is worth emphasizing that our *qualitative* results do not depend on government debt providing liquidity services.

We guess and later verify that the equity price $q_t$ follows the process $dq_t/q_t = (\mu_t - y_t/q_t)dt + \sigma_t dZ_t^y$, where $\mu_t$ and $\sigma_t$ are determined in equilibrium. The return on risky equity is

$$dr_t^x = \frac{dq_t + y_t dt}{q_t} = \mu_t dt + \sigma_t dZ_t^y, \tag{7}$$

so that $\mu_t$ and $\sigma_t$ are interpreted as the drift and diffusion of the risky equity return. We next turn the flow budget constraint in equation (4) into an intertemporal budget constraint. To do this we assume (and verify in Appendix C) that there exists a unique continuous-time stochastic discount factor $\xi_t$ that follows a diffusion process of the form

$$\frac{d\xi_t}{\xi_t} = -r_t^s dt - \frac{\mu_t - r_t^s}{\sigma_t} dZ_t^y. \tag{8}$$

with some initial value $\xi_0$. The no-Ponzi game condition can be expressed using this stochastic discount factor as

$$\lim_{T\to\infty} \mathbb{E}_0[e^{nT}\xi_T w_T] \geq 0. \tag{9}$$

The following lemma presents the intertemporal budget constraint of the household.

**Lemma 1.** *The flow budget constraint* (4) *together with the no-Ponzi game condition imply the intertemporal budget constraint*

$$w_0 \geq \mathbb{E}_0 \int_0^\infty [c_t + \tau_t + (r_t^s - r_t)b_t] \frac{e^{nt}\xi_t}{\xi_0} dt. \tag{10}$$

The proof is in Appendix A.1. The inequality (10) states that the value of initial wealth, comprised of the value of Lucas trees and the stock of government debt, must be enough to cover expected integral of future discounted spending on consumption, taxes, and government debt holding costs. Note that the last term in the brackets on the right-hand side of the inequality (10) is like the costs of renting a durable good, e.g., housing. However, in our model, durable goods are government bonds that provide liquidity services. The inequality reflects the imposed no-Ponzi game condition. The number of members in the household, i.e., $e^{nt}$, on the right-hand side (10) reflects the fact that all expenditure terms are expressed per household member.

### 3.2 Government

The government issues instantaneous riskless real debt $B_t$ that satisfies the flow budget constraint (1). Our assumption that government debt is real can be taken at face value. Alternatively, one can assume that the debt is nominal and nominal prices are sticky in the short run, making debt effectively real. Yet, a third possibility is to imagine that monetary policy keeps inflation stable. This possibility will only add a constant term to our analysis that would not change the analysis.[11]

The primary deficit drift $D_t$, the difference between total spending and revenues, follows a fiscal rule of the form

$$\frac{D_t}{Y_t} = \alpha_D \frac{B_t}{Y_t} - \beta_D \frac{B_t}{Y_t} \log\left(\frac{B_t}{Y_t}\right), \tag{11}$$

where $\alpha_D$ is a real number and $\beta_D$ is a non-negative real number.[12] This fiscal rule has two important properties. First, the deficit reacts proportionally to the level of debt as captured by the first term in equation (11). Second, when $\beta_D$ is strictly positive, the government reacts to increases in the debt-to-GDP ratio by strongly reducing (i.e., more than one-for-one) the primary deficit-to-GDP ratio. The particular functional form of the second term implies that the law of motion for the log of the debt-to-GDP ratio is linear in the log of debt to GDP as in equation (13) below. This allows us to solve the model in closed form.

The fiscal rule (11) only specifies the behavior of the primary fiscal deficit. To complete the description of fiscal policy, we assume that the government engages in wasteful spending,

[11] Inflation risk would alter our analysis. However, Hilscher et al. (2014) show that the government cannot inflate a significant part of its debt away for plausible maturities of government debt.

[12] We assumed that the rule does not explicitly respond to changes in the interest rate on public debt. Nevertheless, this rule implicitly depends on the interest rate because the interest rate is related to the debt-to-GDP ratio in equilibrium.

which is a constant fraction of total output:

$$g_t = \gamma_G y_t. \tag{12}$$

where $g_t$ is government spending per capita. Public debt dynamics in equation (1) can be re-expressed in terms of the log debt-to-GDP ratio denoted as $\widehat{B}_t$. We summarize this useful intermediate result in the next lemma.

**Lemma 2.** *Given the productivity process (2) and the fiscal policy (1), (11), (12), the log debt-to-GDP ratio follows*

$$d\widehat{B}_t = (r_t - g_y - n + \alpha_D - \beta_D \widehat{B}_t + \sigma_y^2/2)dt - \sigma_y dZ_t^y. \tag{13}$$

See Appendix A.2 for details. Equation (13) states that the drift of the log of debt-to-GDP ratio depends positively on the interest rate, the deficit-to-debt ratio $\alpha_D - \beta_D \widehat{B}_t$, and the volatility of the productivity process. The drift depends negatively on the growth rate of total output $g_y + n$. A convenient property of expression (13) is that the diffusion is constant, allowing a closed-form solution.

## 3.3 Equilibrium Characterization

An equilibrium is a collection of interest rates $\{r_t dt, r_t^s dt, dr_t^x\}$ and allocations $\{c_t, w_t, b_t, s_t, x_t, g_t, \tau_t, \overline{c}_t\}$ such that households solve (3)-(5), (9), the government acts according to (1), (11), (12), average consumption equals per member consumption, i.e., $\overline{c}_t = c_t$, and the markets clear: (i) $N_t b_t = B_t$ (government bonds), $N_t s_t = 0$ (safe bonds), $c_t + g_t = y_t$ (goods).

Asset returns are

$$r^s = \rho + \gamma g_y - \frac{\gamma(\gamma+1)}{2}\sigma_y^2, \tag{14}$$

$$r_t = r^s - \alpha_u + \beta_u \widehat{B}_t, \tag{15}$$

and the risky asset return is characterized by the drift $\mu_t = r^s + \gamma\sigma_y^2$ and the diffusion $\sigma_t = \sigma_y$.

The standard derivation of these formulas are in Appendix C.

The safe real interest rate is constant while the return on government bonds varies with the debt-to-GDP ratio so long as $\beta_u$ is not equal to zero. This lemma shows how candidate explanations for secular stagnation affect the safe and liquid rates. A lower rate of trend output growth (equivalently, productivity growth) or an increase in the variance of the output process lower the safe rate of return. The elasticity of the real interest rate with respect to trend growth is governed solely by the intertemporal elasticity of substitution $1/\gamma$. For a coefficient smaller than unity, the real interest rate responds more than one-for-one with a change in trend growth. A rise in endowment volatility lowers the safe interest rate and raises the risk premium, with the strength of the effect rising with the coefficient of risk aversion $\gamma$. Importantly, neither the risk premium nor the safe interest rate depends on population growth. Lower population growth—modeled as a slower rate of birth of new members of the representative household—leaves the real interest rate unaffected. Equation (15) shows that as the debt-to-GDP ratio rises, the marginal benefit of liquid wealth decreases, raising the required rate of return on government debt.

Using the law of motion for the debt-to-GDP ratio in Lemma 2 together with the expression for the return on liquid government bonds in equation (15), we have:

**Proposition 1.** *In equilibrium, the log debt-to-GDP ratio is an Ornstein-Uhlenbeck process:*

$$d\widehat{B}_t = \left(\alpha - \beta \widehat{B}_t\right) dt - \sigma_y dZ_t^y,$$

*where* $\alpha \equiv \alpha_D - \alpha_u + \rho + (\gamma - 1)g_y - n - [\gamma(\gamma + 1) - 1]\, \sigma_y^2/2$ *and* $\beta \equiv \beta_D - \beta_u$.

The parameter $\alpha$ is a sufficient statistic for all of the forces that increase the drift of the debt-to-GDP ratio and that are independent of the level of debt-to-GDP, while $\beta$ is a net effect of the forces that reduce the drift and that are proportional to the level of the debt-to-GDP ratio. The law of motion is an Ornstein-Uhlenbeck (OU) process, which is the continuous-time analog of an AR(1) process.

Finally, we define the government transversality condition and the intertemporal budget constraint. A combination of the no-Ponzi game condition (9) and the household transversality condition (a necessary condition of optimal behavior) imply

$$\lim_{T\to\infty} \mathbb{E}_0[e^{nT}\xi_T w_T] = 0. \tag{16}$$

In equilibrium, total financial wealth of the household is a sum of the value of Lucas trees and the government debt, i.e., $w_t = q_t + b_t$. Because $q_t$ and $b_t$ are always non-negative, the condition (16) necessarily implies that in equilibrium:

$$\lim_{T\to\infty} \mathbb{E}_0[e^{nT}\xi_T b_T] = 0. \tag{17}$$

We will call the limit in equation (17) *the government transversality condition* (TVC).

The household intertemporal budget constraint (10), equation (16), and the market clearing conditions yield

$$b_0 = \mathbb{E}_0 \int_0^\infty [\tau_t - g_t + (r_t^s - r_t)b_t]e^{nt}\frac{\xi_t}{\xi_0}dt, \tag{18}$$

*the intertemporal budget constraint of the government.* Equations (17) and (18) place equilibrium restrictions on the set of feasible fiscal policies. Intuitively, fiscal policy must satisfy these equations if the households purchase government debt in equilibrium. If this is not the case, then equilibrium does not exist.

## 3.4 Three Approaches to Debt Sustainability

We are now ready to define and compare the three approaches to debt sustainability.

**Definition 1.** Fiscal policy is sustainable according to the *stock approach* if the debt-to-GDP ratio has a stationary distribution.

Because the debt-to-GDP ratio process in Proposition 1 is a continuous-time analog of an AR(1) process, it is straightforward to derive the evolution of the distribution of the log of debt-to-GDP ratio and the conditions for the existence of its stationary distribution by solving the Kolmogorov forward equation (see, e.g., Section 3.8 of Stokey, 2009). The following proposition does precisely this.

**Proposition 2.** *If $\beta > 0$, then in equilibrium, the stationary distribution of the log debt-to-GDP ratio exists and it is a normal distribution with mean $\alpha/\beta$ and variance $\sigma_y^2/2\beta$. In levels, the stationary distribution of the debt-to-GDP ratio is log-normal with mean $\exp[(\alpha+\sigma_y^2)/\beta]$ and variance $[\exp(\sigma_y^2/2\beta)-1]\exp(2\alpha/\beta+\sigma_y^2/2\beta)$.*

Proposition 1 states that the debt-to-GDP process has a stationary distribution when it is mean-reverting. When, instead, $\beta \leq 0$, the mean of the log debt-to-GDP ratio drifts to minus or plus infinity, and its variance becomes unbounded. Interestingly, the sufficient condition for the existence of the stationary distribution is also necessary if we exclude a trivial degenerate distribution with zero debt.

**Definition 2.** Fiscal policy is sustainable according to the *flow approach* if $r_t - g_y - n < 0$.

The interest rate net of aggregate output growth rate can be written as $r_t - g_y - n = \alpha - \alpha_D - \sigma_y^2/2 + \beta_u \widehat{B}_t$. This expression implies that there is a cutoff level of the debt-to-GDP ratio, which we call *the flipping point*, where the interest rate differential changes its sign. Formally, $\widehat{B}_t^{FP} = -(\alpha - \alpha_D - \sigma_y^2/2)/\beta_u$. Comparing the definitions of the stock and flow approaches makes clear that they are distinct. The stationarity of the debt-to-GDP distribution depends only on $\beta$, while the sign of the debt servicing cost depends on other parameters of the model and also on the endogenous level of the debt-to-GDP ratio.

**Definition 3.** Fiscal policy is sustainable according to the *economic approach* if it satisfies equations (17) and (18).

To be clear, these conditions are necessary for the existence of equilibrium in our model. The next proposition summarizes necessary and sufficient conditions.

**Proposition 3.** *Fiscal policy is sustainable according to the economic approach if and only if $\beta > 0$, or $\beta = 0$ and $\alpha_u - \alpha_D > 0$.*

The proof is in Appendix A.4. Proposition 3 collects several results in a compact form. First, when $\beta < 0$, debt to GDP explodes fast enough that the conditions of sustainability under the economic approach are violated. Second, when $\beta > 0$, public debt is sustainable. Combining it with the fact that the same condition is necessary for the stationarity of the debt-to-GDP ratio (excluding the degenerate distribution with a zero public debt), we conclude that the stock approach implies the economic approach conditional on the assumed fiscal rule. However, the opposite is not true. To explain why this is so, we proceed to the case when $\beta$ equals zero.

When $\beta$ is zero, the law of motion of the *level* of debt-to-GDP in Proposition 1 implies that it either shrinks to zero when $\alpha$ is below zero or grows to infinity when $\alpha$ is greater than zero. In the latter case, the debt-to-GDP is exponential in time. Importantly, Proposition 3 states that the transversality condition (17) can fail in the former case but still holds in the latter case. To understand this, consider two situations. First, even when the debt-to-GDP ratio grows without limit over time, i.e., $\alpha$ is positive, debt is sustainable when $\alpha_u - \alpha_D$ is above zero. This scenario is sustainable because the growth rate of debt is lower than the growth rate of the discount factor so that the transversality condition of the government is satisfied. In this case, the intertemporal budget constraint of the government is satisfied automatically. A critical parameter that ensures sustainability is the sum of the sensitivity of the primary surplus to the stock of government debt $-\alpha_D$ and the constant part of the liquidity yield $\alpha_u$. Interestingly, note that it is sufficient for this object to be positive; it does not have to be large. Furthermore, if the government surplus does not respond to the level of debt at all, i.e., $\alpha_D$ is zero, then it is still possible for the debt to be sustainable because of the remaining term $\alpha_u$, which reflects profits that the government collects due of its unique ability to issue liquid debt that households value for their non-pecuniary returns. This profit is analogous to the seigniorage that the central bank receives on printing money that provides liquidity services.

Second, even if the debt-to-GDP ratio is shrinking with time on average ($\alpha$ is negative), it is not necessary that the TVCs are satisfied. For example, consider the case when $\alpha_D$ and $\alpha_u$ equal zero, which implies that the primary surplus is zero and there are no seigniorage revenue. At the same time, assume that $\alpha$ is negative because, for example, the interest rate is low due to high volatility of shocks to output growth $\sigma_y^2$. If the government starts with a positive level of debt, the intertemporal budget constraint does not hold because the present value of future surpluses is zero. As a result, the economic approach implies that debt is not sustainable in this case.

### 3.5 Lower Bound on Public Debt

When $\alpha$ is negative, the debt-to-GDP ratio tends to zero. However, governments rarely repay their debt in full. Here, we impose a lower bound on the log of debt-to-GDP ratio, which we denote $\widehat{B}_{min}$, establishing that stationarity of debt-to-GDP distribution is not sufficient to satisfy the economic approach.[13]

Formally, we assume that the log of debt-to-GDP ratio $\widehat{B}_t$ follows a reflected Brownian motion with a lower reflecting barrier at $\widehat{B}_{min}$ (Harrison, 1985). Intuitively, $\widehat{B}_t$ is pushed in the positive direction every time it falls to its lower reflecting barrier. When we impose this assumption, the debt-to-GDP ratio admits a closed-form solution for the stationary distribution.

**Proposition 4** (Lower reflecting barrier). *If $\beta = 0$, $\alpha < 0$, and $\widehat{B}_t \geq \widehat{B}_{min} > -\infty$, then there is a stationary distribution of log debt-to-GDP ratio which is a negative exponential with rate parameter $\xi \equiv -2\alpha/\sigma_y^2$. In levels, the stationary distribution of the debt-to-GDP ratio is a Pareto distribution with shape parameter $\xi$. Moreover, the transversality condition* (17) *holds if and only if $\alpha_u - \alpha_D > 0$.*

The proof is in Appendix A.5. Given that the debt-to-GDP ratio has a Pareto distribution, its mean and variance depend crucially on the shape parameter. The mean and variance

[13] In 2001, Federal Reserve Chair Alan Greenspan famously worried that the possibility that the US may payoff the public debt would complicate the implementation of monetary policy.

of a Pareto distribution only exist when the shape parameter is larger than two. Notably, a higher negative drift $\alpha$ or lower volatility shocks to output raise the shape parameter, lowering both the mean and variance.

To understand why the stock approach does not imply economic approach in this case note first that a combination of a negative drift $\alpha$ and a lower reflecting barrier *mechanically* imply existence of the stationary distribution. Next, consider an example where $\alpha_u - \alpha_D = 0$, implying a zero present value of government revenues. In this case, the investors are not willing to hold government debt because no real dividend is ever paid. In this case, the stationary distribution exists in mechanical but not in equilibrium sense.

These results starkly highlight the importance of the primary surplus and how both the stock and flow approach to sustainability can differ from the economic approach to sustainability.

## 3.6 Output Disasters

Anticipating our calibration exercises in Section 3.7 where we match asset pricing moments with realistic values of risk aversion, we discuss the consequences of adding rare disaster shocks to the above setup with a lower reflecting barrier. Now we replace output process (2) with

$$\frac{dy_t}{y_t} = g_y dt + \sigma_y dZ_t^y + \left(e^{-Z_t} - 1\right) dJ_t, \tag{19}$$

where $J_t$ is a Poisson process with constant intensity $\lambda > 0$. $Z_t$ is a positive, independent, and identically distributed random variable that describes an instantaneous change in log output when a disaster occurs.

Following the steps in Wachter (2013), we compute the interest rate when the endowment jumps. This allows us to derive the law of motion and the corresponding stationary distribution of the debt-to-GDP ratio. We present all details in Appendix A.6 and summarize the results in the following proposition:

**Proposition 5** (Disasters)**.** *If $\beta = 0$ and there is a lower reflecting barrier $\widehat{B}_{min}$, then the* *law of motion of the log of the debt-to-GDP ratio for $\widehat{B}_t > \widehat{B}_{min}$ is*

$$d\widehat{B}_t = \widetilde{\alpha}dt - \sigma_y dZ_t^y + Z_t dJ_t,$$

*where $\widetilde{\alpha} \equiv \alpha - \lambda(\mathbb{E}_Z[e^{\gamma Z}] - 1)$. If $\widetilde{\alpha} < -\lambda\mathbb{E}_Z[Z]$, then there is a stationary distribution of the log debt-to-GDP ratio which is exponential with the rate parameter $\xi$ that solves*

$$\widetilde{\alpha}\xi + \frac{\sigma_y^2}{2}\xi^2 = \lambda(1 - \mathbb{E}_Z[e^{\xi Z}]). \tag{20}$$

*In levels, the stationary distribution of the debt-to-GDP ratio is a Pareto distribution with shape parameter $\xi$. Moreover, when $\widetilde{\alpha} \geq -\lambda\mathbb{E}_Z[Z]$, the transversality condition (17) holds if and only if $\alpha_u - \alpha_D > 0$; when $\widetilde{\alpha} < -\lambda\mathbb{E}_Z[Z]$, it holds if $\xi > 1$.*

The shape parameter $\xi$ solves equation (20), which depends on the disaster intensity $\lambda$, the distribution of disasters, the drift $\alpha$, and the diffusion $\sigma_y$ of the debt-to-GDP ratio. Thus our model can accommodate an output process where rare disasters lead to discontinuous jumps in the debt-to-GDP ratio—of particular relevance for the discrete jumps in debt-to-GDP experience in recent decades.

## 3.7 Calibration

This section quantifies effects of various secular stagnation forces on the stationary distribution of the debt-to-GDP ratio. There are three distinct calibrations corresponding to the model in Sections 3.1-3.4, its extension with a lower bound on debt in Section 3.5 and disasters in Section 3.6. Considering three calibrations allows us to analyze the effects of different fiscal policies and different kinds of shocks.

In these calibrations, we use a richer model than the one presented so far. First, instead of separable utility function (3), we use Epstein-Zin-Weil (EZW) preferences to separate the IES coefficient and the coefficient of relative risk aversion. Equations (B.1) and (B.2) in Appendix B formally present these preferences. Second, we add shocks to the flow budget constraint of the government to dampen the covariance between the interest rates and productivity growth. Formally, the modified budget constraint is

$$dB_t = r_t B_t dt + D_t dt + B_t \sigma_B dZ_t^B,$$

where $D_t$ is given by equation (11). The last two terms can be thought as the primary and random terms of the primary fiscal deficit. A more detailed description and the solution of this richer model is in Appendix B.

The parameters in Table 3 are fixed across the three calibrations. We set the intertemporal elasticity of substitution to $1/\theta$ to 0.75.[14] The average growth rate of annual GDP per capita since 1950 of 0.02 is used to set $g_y$, while annual postwar population growth of 0.0115 determines $n$. The diffusion term on output growth $\sigma_y$ is 0.025 to match the standard deviation of annual output growth in the US since 1950. The difference between AAA corporate debt and the US 10-year Treasury yield, and its elasticity to the debt-to-GDP ratio pin down both $\alpha_u$ and $\beta_u$ (Krishnamurthy and Vissing-Jorgensen, 2012).[15] As mentioned above, we introduce a fiscal policy shock $\sigma_B$ to match the correlation of the real interest rate and output growth for the 17 countries in our dataset since 1950. This correlation is $-0.055$, meaning that our model requires the relative variance of the fiscal policy shocks to be higher than that for productivity growth to dampen the correlation between $r_t$ and $dy_t/y_t$.[16]

The remaining parameters, presented in Table 4, differ across the three calibrations. Calibration 1 assumes that all shocks are Brownian and that fiscal policy strongly reacts to the level of debt-to-GDP ratio so that the stationary distribution is log-normal as described in Proposition 2. Calibration 2 changes a fiscal response relative to calibration 1 by introducing

[14] An extensive literature has attempted to measure the elasticity of substitution by examining how household's consumption growth responds to changes in the real interest rate faced by these households. The IES is commonly assumed to be less than one in macroeconomics literature, with some estimates suggesting that it is in fact substantially less than one (e.g., Hall, 1988; Campbell, 1999).

[15] A regression of the AAA-10 year Treasury spread on the log debt-to-GDP ratio determines $\beta_u$ and $\alpha_u$, which is the constant from this regression plus $\beta_u$.

[16] An alternative way to dampen correlation between the interest rate $r_t$ and the growth rate of productivity without setting a high variance of instantaneous shocks to the fiscal rule $\sigma_B$ is to assume that the fiscal rule parameter $\alpha_D$ is not constant but rather a mean-reverting Ornstein-Uhlenbeck process with a sufficiently high persistence.

a lower reflecting barrier and by setting $\beta_D - \beta_u$ to zero for levels of debt-to-GDP above the reflecting barrier as in the first extension in Section 3.5. Finally, calibration 3 adds disaster shocks to calibration 2, which allows us to lower the coefficient of relative risk aversion required to match the equity premium.

We now describe calibration-specific parameters in detail. The average postwar real return on long-term government bonds of 0.025 in the US is used to pin down the rate of time preference $\rho$. The equity premium of 0.052 in the US is used to determine the coefficient of relative risk aversion given the size of the shocks.[17] In calibration 1, the fiscal response parameters $\alpha_D$ and $\beta_D$ are set to match the mean log debt-to-GDP ratio in the US postwar period of -0.92, where we used gross federal debt held by the public from 1950 to 2016 as the measure for public debt, and match the variance of the log debt-to-GDP ratio across the full set of 17 countries in the Jordà, Schularick, and Taylor (2016) data set.[18] We use the full set of 17 countries given the high degree persistence in the debt-to-GDP ratio and absent strong evidence on how systematically fiscal policy responds to the debt-to-GDP ratio. In calibrations 2 and 3, the same empirical moments for the debt-to-GDP ratio determine $\alpha_D$ and the position of a lower reflecting barrier $\min(B_t/Y_t)$. $\beta_D$ is just set to offset the effect of liquidity parameter $\beta_u$ so that $\beta$ is 0 in calibrations 2 and 3. Finally, in calibration 3, we assume that productivity follows the stochastic process with disasters in equation (19). Following Barro and Jin (2011) and Rebelo, Wang, and Yang (2018), we assume that sizes of disasters (a log change in productivity) are distributed exponentially with the lowest value of zero and the mean of $\overline{z} > 0$. We set the values of $\lambda$ and $\overline{z}$ to their empirical counterparts in Barro (2006). Importantly, due to the presence of disasters, calibration 3 requires a coefficient of relative risk aversion of around 4 to match the equity premium relative to its value of 84 in calibrations 1 and 2.

Table 5 presents several moments from our three calibrations. "Baseline" shows the

[17] The bonds return and equity premium are from Jorda, Knoll, Kuvshinov, Schularick, and Taylor (2018).

[18] The data for gross federal debt held by public is from the Economic Report of the President by the Council of Economic Advisers.

moment for baseline parameters where the three calibrations are split into the three panels. Columns "Ex. 1," "Ex. 2," and "Ex. 3" contain the moments of three comparative statics. We see an important difference across the three calibrations in the baseline column. The stationary distribution of debt-to-GDP ratio is log-normal in calibration 1 and Pareto with the shape parameter between one and two in calibrations 2 and 3. As a result, the second moment of the stationary distribution does not exist in the latter two calibrations.

In column Ex. 1, we show the effect of a decline in population growth rate to 0.7 percent—a projected US population growth over the next decade. Slower population growth directly lowers GDP growth while leaving the real interest rate unchanged thereby worsening debt dynamics.[19] All rates of return are left unaffected by changes in population growth, so the effects on the debt-to-GDP ratio come solely through the effects on GDP. Calibration 1 shows a modest increase in the mean of the debt-to-GDP ratio relative to calibrations 2 and 3. This is because a change in population growth affects a tail of the Pareto distribution with a stronger effect on the means in calibrations 2 and 3, while this effect is limited under calibration 1 when the stationary distribution is log-normal. Quantitatively, the decline in population growth raises the mean debt-to-GDP ratio by 2, 16, and 16 percentage points in calibrations 1, 2, and 3, respectively.

Column Ex. 2 in the table presents the effects of a decline in productivity growth to 0.7 percent—in line with the post-2008 productivity growth. Slower productivity growth is beneficial for debt levels, shifting the debt-to-GDP ratio downward as the fall in $r_t$ outpaces the decline in GDP growth when the IES is below one. Under calibration 1, the mean debt to GDP ratio falls by 1 percentage points. A decline in productivity reduces the government interest rate to 0.7 percent. As before, the quantitative effects on the debt-to-GDP ratio are modest under calibration 1. Calibrations 2 and 3 feature 12 percentage points decline in the

[19] The finding that population growth does not affect the interest rates is specific to this model. As Eggertsson, Mehrotra, and Robbins (2018) show, in a quantitative life-cycle model, slower population growth generally lowers the real interest rate. However, this effect in unlikely to be strong enough in standard quantitative life-cycle models (Carvalho, Ferrero, and Nechio, 2016) to overturn a rightward shift in debt-to-GDP distribution.

mean debt-to-GDP ratio. This again results from the fact that the mean is quite sensitive to the changes in the tail of the Pareto distribution.

Finally, column "Ex. 3" presents the model moments after an increase in volatility of GDP such that the equity premium rises by 2 percentage points. In calibrations 1 and 2, we achieve this by increasing the value of $\sigma_y$ from 0.025 to 0.029, while in calibration 3, we change the arrival rate of disasters $\lambda$ from 0.02 to 0.028. A rise in volatility shifts the debt-to-GDP distribution leftward in all of the cases. The real rate of return on government debt falls to 0.1 percent under calibrations 1, 2, and 3. The mean debt to GDP ratio falls by 7, 36, and 33 percentage points in the three calibrations, respectively. Overall, our findings show that the indirect effects of output volatility on the interest rate and consequently the drift of the debt-to-GDP ratio dominate the direct effects of spreading out the distribution of output.

# 4 Debt Sustainability with Default

In this section, we resurrect the stock approach but show that the flow approach remains distinct. To do this, we assume that the primary surplus (as a share of GDP) is bounded above. In this case, there exist threshold levels of debt-to-GDP ratio, which we label as fiscal limits with and without risk, after which the government defaults. These thresholds are state-dependent, implying that there is no single critical level of the debt-to-GDP ratio.

## 4.1 Model

As the model is nearly identical to the previous section, we only outline the differences. Household preferences are the same as in equation (3) except that we dispense with non-pecuniary benefits of holding government bonds to focus solely on the riskiness of public debt as the level of debt varies. Second, productivity grows stochastically over time according to equation (19) that allows for jumps. We assume that the Brownian motion term is absent ($\sigma_y = 0$). Moreover, after a single realization of a disaster, there are no further disasters or any other sources of uncertainty. Third, the government's flow budget constraint follows equation (1) as before, but we assume that the primary deficit $D_t$ takes the form:

$$D_t = -Y_t s\left(\frac{B_t}{Y_t}\right). \tag{21}$$

Importantly, unlike in Section 3, we assume that the surplus function $s(\cdot)$ is bounded above by a positive maximum primary fiscal surplus $\overline{s}$. This limit would emerge in a production economy with distortionary labor taxation; the government would not be able to raise additional revenue beyond the peak of the Laffer curve. Ghosh, Kim, Mendoza, Ostry, and Qureshi (2013) label this property of the surplus function as *fiscal fatigue*. Moreover, we use a step function shown with green solid horizontal lines in Figure 3.[20] This function is positive and equals $\overline{s}$ for positive values of debt-to-GDP, is zero when debt is zero, and it equals a negative value of $\underline{s}$ when debt is negative. Together, these assumptions allow us to solve the model in closed form.

Combining the flow budget constraint of the government and the endowment process, we get the law of motion for the debt-to-GDP ratio:

$$d\left(\frac{B_t}{Y_t}\right) = \left[(r_t - g_y - n)\frac{B_t}{Y_t} - s\left(\frac{B_t}{Y_t}\right)\right]dt + \frac{B_t}{Y_t}(e^{Z_t} - 1)dJ_t. \tag{22}$$

The first term on the right-hand side is the standard law of motion of debt-to-GDP ratio in a deterministic setting. The second term incorporates the effect of uncertainty; when a rare disaster decreases output by $Z_t$ in log terms, the debt-to-GDP ratio jumps by $Z_t$ in log terms.

## 4.2 Equilibrium and the Three Approaches

The model is solved using backward induction as there is no uncertainty after the first disaster shock. We first solve the model after uncertainty has realized, and then describe the

[20] The qualitative results that we highlight in this section also hold for monotonically increasing sigmoid functions with the fiscal fatigue property. This can be verified by redrawing the phase diagram in Figure 3 below.

evolution of the economy before disaster shock hits the economy.

After the resolution of uncertainty, the interest rate on instantaneous risk-free debt is standard: $r = \rho + \gamma g_y$. To understand the dynamics of the debt-to-GDP ratio after the resolution of uncertainty, we plot the two forces affecting debt in equation (22) in Figure 3. The upward-sloping straight black line shows the difference between the interest rate and GDP growth rate times the debt-to-GDP ratio, while the green step-function is the primary surplus over GDP. The black line must be upward sloping. If this were not the case, the present value of output would be unbounded.

Without uncertainty, equation (22) admits two steady states: a stable one at zero debt-to-GDP and an unstable one at $\mathcal{B}^{nr}_{FL}$ where the black and green lines intersect at a positive debt-to-GDP ratio. We label the second steady state *the fiscal limit without risk*.[21] The black arrows on the horizontal axis show the dynamics of the debt-to-GDP ratio on either side of this fiscal limit. Debt-to-GDP grows without bound to the right of the unstable steady state. At a sufficiently high level of the debt-to-GDP ratio where the presence of constant surplus does not materially matter anymore, debt-to-GDP grows exponentially. Importantly, this path does not satisfy the transversality condition of the government in equation (17) because debt grows at the rate of interest that is also used for discounting in the case of no uncertainty.

Before the shock occurs, the interest rate on a riskless asset must reflect the fact that the discount factor jumps by $\gamma Z$ percent whenever the disaster occurs. Moreover, if the shock is large enough and the debt-to-GDP level jumps over the fiscal limit $\mathcal{B}^{nr}_{FL}$, the government defaults. For simplicity, we assume a complete default, but it is straightforward to relax this assumption as in Yue (2010) and Lorenzoni and Werning (2019). The riskless interest rate

[21] There is a third steady state with a negative value of the debt-to-GDP ratio. However, the presence of this steady state is inconsequential for our analysis because the debt to GDP can never become negative with only negative disasters and a positive initial level of debt.

and the interest rate on public debt are

$$r_t^s = \rho + \gamma g_y - \lambda(\mathbb{E}_Z[e^{\gamma Z}] - 1), \tag{23}$$

$$r_t = \rho + \gamma g_y - \lambda \left\{ \mathbb{E}_Z \left[ e^{\gamma z} \mathbb{I} \left( \frac{B_t}{Y_t} e^Z \leq \mathcal{B}_{FL}^{nr} \right) \right] - 1 \right\}. \tag{24}$$

The standard derivation is in Appendix A.7. Equation (23) states that, in the presence of disaster shocks, households willingness to save is higher, which reduces the safe interest rate. The interest rate falls when the intensity of disasters $\lambda$ is higher, risk aversion $\gamma$ is greater, or the distribution of disasters has a fatter right tail.

Equation (24) shows that government debt features an endogenous credit spread. Formally, the variable inside the expectations operator is positive only when a disaster does not trigger a default. The interest rate on government debt depends on the arrival rate of disasters $\lambda$, the probability of crossing the fiscal limit $\mathcal{B}_{FL}^{nr}$, and the output decline conditional on a disaster. The default premium rises as the debt-to-GDP ratio approaches the fiscal limit. Notice that a rise in the arrival rate of disasters $\lambda$ has an ambiguous effect on the interest rate on government debt. On the one hand, higher disaster risk increases the likelihood of default. On the other hand, elevated disaster risk increases precautionary saving demand for public debt.

Figure 3 plots the dynamics of the debt-to-GDP ratio before the arrival of a disaster shock. The two red dashed lines show the debt-servicing cost, the interest rate net of GDP growth rate multiplied by the debt-to-GDP ratio, in two cases. First, a downward sloping line corresponds to riskless debt that pays interest rate $r_t^s$, where we assumed that the risk of disasters is sufficiently high to make $r - g$ negative. Second, the upward sloping line shows the case of debt that always defaults when a disaster arrives.[22] The actual debt-servicing cost curve, a solid red line, lies between these two extremes.

The unstable steady state of this system—a point in which the solid red debt-servicing

[22]The interest rate on the debt that always defaults when a disaster occurs is $r_t^d = \rho + \gamma g_y + \lambda$. It follows from equation (24) by setting the fiscal limit to zero.

cost curve intersects the surplus line—is $\mathcal{B}^{r}_{FL}$, which we label *the fiscal limit with risk*. This fiscal limit is distinct from the fiscal limit without risk $\mathcal{B}^{nr}_{FL}$. Debt-to-GDP levels to the right of the fiscal limit with risk cannot be an equilibrium when risk is still present in the economy because points to the right of $\mathcal{B}^{r}_{FL}$ do not satisfy standard backward induction.[23] We summarize this and the earlier observations in the following lemma. Appendix A.8 provides details of the backward induction argument.

**Proposition 6.** *The government defaults if debt to GDP exceeds $\mathcal{B}^{r}_{FL}$ before the resolution of uncertainty and if it exceeds $\mathcal{B}^{nr}_{FL} > \mathcal{B}^{r}_{FL}$ after the resolution of uncertainty.*

This proposition shows that the government can default even at levels of debt-to-GDP below the fiscal limit without risk. This observation underscores that there is no *single* fiscal limit in our environment.[24]

The debt-servicing cost changes its sign at the flipping point $\mathcal{B}_{FP}$ shown in Figure 3. This point does not coincide with either of the two fiscal limits indicating highlighting the flow approach is once again distinct from the economic or stock approaches to debt sustainability.

## 4.3 Secular Stagnation Forces

In this section, we present comparative statics of the debt limits with respect to the probability of disasters and changes in productivity and population growth.[25] These results highlight that different approaches to debt sustainability can offer diverging assessments of how forces lowering $r$ and $g$ affect debt sustainability.

We start with an increase in disaster intensity $\lambda$. Figure 4 illustrates the dynamics of the debt-to-GDP ratio with a higher probability of disasters $\lambda$. The figure copies the curves

[23] We thank Fernando Broner, who pointed out that this logic was absent from the previous version of our paper.

[24] We note that the model in this section also has rollover crises (e.g., Cole and Kehoe, 2000) with a possibility of default at any level of debt. However, we focus on more gradual debt dynamics generated by a possibility of future default, which is also the approach taken in Ghosh, Kim, Mendoza, Ostry, and Qureshi (2013) and Lorenzoni and Werning (2019). Formally, we assume that the government does not default if it can roll over its debt at a finite interest rate. At the same time, our setup does not have multiplicity in the spirit of Calvo (1988) because we assume that the government first chooses the amount of bonds it sells to investors. Then the investors determine the interest rate on these bonds (Eaton and Gersovitz, 1981).

[25] Appendix A.9 also presents comparative statics with respect to the sizes of disasters.

presented in Figure 3 in pale red and adds the comparative statics of higher $\lambda$ in blue. The next proposition summarizes what is evident from the Figure.

**Proposition 7.** *A higher disaster intensity $\lambda$ does not change the fiscal limit without risk $\mathcal{B}_{FL}^{nr}$ but unambiguously moves the fiscal limit with risk $\mathcal{B}_{FL}^{r}$ to the left and the flipping point $\mathcal{B}_{FP}$ to the right.*

The proof is straightforward. A higher $\lambda$ has two opposing effects on the government bond interest rate because the interest rate is a weighted average of the riskless rate and the interest rate on the debt that always defaults conditional on a disaster. With higher $\lambda$, the riskless interest rate declines uniformly for debt levels because of a stronger precautionary saving motive. The corresponding downward-sloping debt-servicing cost rotates clockwise with $\lambda$ as shown with dashed blue line in Figure 4. At the same time, debt that always defaults conditional on disaster is riskier with higher $\lambda$, uniformly raising interest rates. The debt-servicing cost curve for this type of debt rotates counterclockwise with $\lambda$. The first effect dominates at low levels of debt, and the second effect dominates at higher levels because the probability of default increases with debt. This is shown with the solid blue line in the figure. Point $O$ is where the effect of a decline in the safe interest rate exactly balances the higher probability of default due to a higher arrival rate of disasters.[26] As a result, the blue solid debt-servicing cost is always higher to the right of point $O$ relative to its analog under a lower $\lambda$. The last observation implies that the new fiscal limit with risk $\mathcal{B}_{FL}^{r,'}$ is lower than the original one $\mathcal{B}_{FL}^{r}$.

At the same time, the flipping point $\mathcal{B}_{FP}$, where the debt-servicing cost changes its sign, shifts to the right because the debt servicing cost curve falls for levels of debt-to-GDP between

[26] The fact that the point $O$ lies on the black line can be deduced by observing that the only situation in which the interest rate on government debt does no depend on the intensity $\lambda$ is when the term in brackets of equation (24) is zero, implying that the interest rate is $\rho + \gamma g_y$. The equation that defines the debt-to-GDP level $\mathcal{B}^*$ that corresponds to this fixed point is

$$\int_0^{\log \frac{\mathcal{B}_{FL}^{nr}}{\mathcal{B}^*}} e^{\gamma z} dF(z) = 1.$$

Clearly $\mathcal{B}^* < \mathcal{B}_{FL}^{nr}$, otherwise the left-hand side of this equation is negative.

zero and $\mathcal{B}^*$. This result shows how periods of elevated disaster risk can simultaneously cause $r < g$ while *tightening* the fiscal limit.

A decline in productivity growth $g_y$ has unambiguous effects on the debt cutoffs. To understand the intuition, note first that the lower growth rate lowers the interest rate on bonds after uncertainty has resolved. As a result, the debt-servicing cost, represented by the black line in Figure 3, turns clockwise (under the assumption of a low IES, i.e., $\gamma^{-1} < 1$). Hence the fiscal limit without risk increases. Before disaster arrival, the safe interest rate and the interest rate on debt that always defaults also declines. Moreover, a higher fiscal limit without risk reduces default probability. The last two observations imply that the debt servicing cost, the solid red curve in Figure 3, shifts down for all positive debt levels. Hence the fiscal limit with risk and flipping points move to the right.

In our environment, a decline in the population growth rate is a mirror image of a decline in the growth rate of productivity just discussed. Lower population growth directly reduces GDP growth but leaves the riskless interest rates unchanged. Consequently, the fiscal limits with and without risk and the flipping points move to the left. Thus, higher output risk and slower population growth reduce fiscal space, but lower productivity growth may raise fiscal space. As our model reveals, the impact of secular stagnation on debt sustainability depends on assessing the drivers of low $r$ and $g$.

## 4.4 Calibration

We now assess the quantitative magnitudes of the effects discussed in the previous section. We keep the IES parameter $1/\theta$ and the average size of disasters $\overline{z}$ at the values used in calibration 3 presented in Section 3.7. We choose the remaining parameters to reflect the recent state of the US economy in 2019 before the COVID-19 pandemic. The growth rate of real GDP per capita $g_y$ is set to 1.7 percent, and population growth $n$ is 0.5 percent. We use a higher value for the arrival rate of disasters $\lambda$ of 0.065 based on the structural estimate in Farhi and Gourio (2018). The subjective discount factor $\rho$ is set to match the average yield on 10-year inflation-protected Treasury bonds of 0.004. When computing $\rho$, we take into account that government bonds are defaultable in this version of the model. Specifically, we match the interest rate at the net public debt-to-GDP ratio of 80 percent. As a result, the parameter $\rho$ varies with the maximum fiscal surplus in the calibration. We set the coefficient of relative risk aversion $\gamma$ to 2.74 to match the equity risk premium of 5.5 percent. Faced with some uncertainty about the maximum fiscal surplus (see Eichengreen and Panizza, 2016), we experiment with two values: 5 percent and 10 percent.

Table 6 reports values of the three debt-to-GDP thresholds discussed in this section under the baseline calibration and the three comparative statics experiments. These experiments are the opposite of those in Section 3.7. Specifically, we investigate how the fiscal thresholds change if one parameter reverts from its end of 2019 (baseline) value to its post-WWII average value.

Consider first Panel A in Table 6, where the maximum fiscal surplus to GDP ratio is 5 percent. In the baseline calibration, the fiscal limit without risk is 222 percent of GDP, a number more than twice larger the current value of US public debt. The fiscal limit with risk is at 144 percent, well above the 90 percent level considered unsustainable in Reinhart and Rogoff (2009). The flipping point at which $r > g$ is at 106 percent of GDP. A reversion to post-war rates of population growth (column Ex. 1) leads to a substantial increase in all three cutoffs, while an increase in the growth rate of productivity, column Ex. 2, somewhat reduces these thresholds. A decline in the arrival rate of disasters, column Ex. 3, does not change the fiscal limit without risk. However, it slightly lowers the flipping point and somewhat increases the fiscal limit with risk. The small changes in Experiment 3 reflect that the equity premium is only modestly impacted, falling from 5.5 percent to 5.2 percent.

Panel B demonstrates the considerable sensitivity of the thresholds to the maximum fiscal surplus. If the government can extract a surplus-to-GDP ratio of 10 percent, it can sustain the level of debt of nearly 300 percent when risk is absent and about 200 percent in the presence of risk. It is worth emphasizing that the numerical values that we presented here are likely to represent lower bounds. First, we have not included a liquidity yield or seigniorage revenue. Second, we assumed a zero recovery value of the public debt after default, ignoring substantial physical assets held by the US government. Third, we have modeled US debt as exclusively short-term debt.

# 5 Conclusion

For advanced economies, the previous decade has seen rising levels of public debt and low rates of growth. Public debt levels are set to rise sharply in the wake of the coronavirus pandemic. Despite this, interest rates on government debt remain remarkably low for most advanced economies. Conditions of $r < g$ have prompted a reassessment of debt sustainability for these economies.

In this paper, we contrast informal ideas of debt sustainability with a more theoretically grounded approach. Policy debates on sustainability typically center on the level or stability of the debt-to-GDP ratio (stock approach) or whether $r < g$ (flow approach). By contrast, we consider an economic approach to sustainability— will forward-looking investors willingly hold the public debt? Our key finding is that the economic approach to sustainability is not closely tied to either the stock or flow approach. Debt is sustainable so long as the primary surplus rises linearly with the level of debt.

The stock approach can be partially resurrected when the primary surplus is bounded. In this case, a state-dependent fiscal limit emerges. Secular stagnation forces—explanation for low $r$ and low $g$—have counterintuitive effects on sustainability. Slower population growth tightens the fiscal limit, but lower productivity growth could loosen limits; higher output risk can lower rates but tightens fiscal limits.

This paper does not consider the optimal level of debt.[27] We leave to future work the

[27] See Barro (1979) and Lucas and Stokey (1983) for optimal debt policy with distortionary taxes. See Bhandari, Evans, Golosov, and Sargent (2017) for optimal debt policy with heterogeneous agents and incomplete markets. Woodford (1990) showed how high levels of debt may be welfare improving and may crowd-in capital in the presence of financial frictions, while stressing the empirical fact of low $r$ relative to $g$ for the US. Angeletos, Collard, and Dellas (2016) consider optimal policy when interest rates on public debt

question of optimal debt policy when $r < g$ on average and taxation may be either economically or politically costly.

are low due to financial frictions.

# Online Appendix

# A Proofs

## A.1 Proof of Lemma 1

First, take the household flow budget constraint

$$dw_t = (r_t^s s_t + r_t b_t - c_t - \tau_t - n w_t)dt + w_t x_t dr_t^x,$$

together with $s_t + b_t + x_t w_t = w_t$.

Second, consider the process $\xi_t$ such that

$$\frac{d\xi_t}{\xi_t} = -r_t^s dt - \frac{\mu_t - r_t^s}{\sigma_t} dZ_t^y.$$

Third, compute $d(\xi_t e^{nt} w_t)$ to get

$$d(\xi_t e^{nt} w_t) = e^{nt}\xi_t \left[(r_t - r_t^s) b_t - c_t - \tau_t\right] dt + \left(x_t - \frac{\mu_t - r_t^s}{\sigma_t^2}\right) e^{nt}\sigma_t \xi_t w_t dZ_t^y.$$

Fourth, integrate the above process forward and take expectations

$$\xi_T e^{nT} w_T - \xi_0 w_0 = \int_0^T e^{nt}\xi_t \left[(r_t - r_t^s) b_t - c_t - \tau_t\right] dt + \int_0^T \left(x_t - \frac{\mu_t - r_t^s}{\sigma_t^2}\right) e^{nt}\sigma_t \xi_t w_t dZ_t^y,$$

take expectations

$$\mathbb{E}_0 \xi_T e^{nT} w_T - \xi_0 w_0 = \mathbb{E}_0 \int_0^T e^{nt}\xi_t \left[(r_t - r_t^s) b_t - c_t - \tau_t\right] dt,$$

and, finally, take the limit

$$\xi_0 w_0 = \lim_{T\to\infty} \mathbb{E}_0 \xi_T e^{nT} w_T + \mathbb{E}_0 \int_0^\infty e^{nt}\xi_t \left[(r_t^s - r_t) b_t + c_t + \tau_t\right] dt,$$

where in the last equation we assumed that limit and expectations are interchangeable.

Assuming that $\xi_T$ is the stochastic discount factor, the no-Ponzi game condition is $\lim_{T\to\infty} \mathbb{E}_0 \xi_T e^{nT} w_T \geq 0$, which results in

$$\xi_0 w_0 \geq \mathbb{E}_0 \int_0^\infty e^{nt}\xi_t \left[(r_t^s - r_t) b_t + c_t + \tau_t\right] dt.$$

## A.2 Proof of Lemma 2

Log GDP is

$$
\begin{aligned}
d\log y_t &= \frac{dy_t}{y_t} - \frac{1}{2}\left(\frac{dy_t}{y_t}\right)^2 \\
&= g_y dt + \sigma_y dZ_t^y - \frac{1}{2}\left(g_y dt + \sigma_y dZ_t^y\right)^2 \\
&= \left(g_y - \frac{\sigma_y^2}{2}\right) dt + \sigma_y dZ_t^y.
\end{aligned}
$$

Log debt is

$$
\begin{aligned}
d\log B_t &= \frac{\left(r_t B_t + G_t - T_t\right) dt}{B_t} - \frac{1}{2}\left[\frac{\left(r_t B_t + G_t - T_t\right) dt}{B_t}\right]^2 \\
&= \left(r_t + \frac{G_t - T_t}{B_t}\right) dt.
\end{aligned}
$$

The low of motion of $\widehat{B}_t \equiv \log\left[B_t/(N_t y_t)\right]$ is

$$
d\widehat{B}_t = \left(r_t - g_y - n + \alpha_D + \frac{\sigma_y^2 - \sigma_B^2}{2} - \beta_D \widehat{B}_t\right) dt - \sigma_y dZ_t^y.
$$

## A.3 Derivation of Asset Prices in Section 3.3

**Step 0: preliminaries.** We evaluate a part of the value function that depends on the stream of consumption

$$
V_t = \mathbb{E}_t \int_t^\infty e^{-(\rho-n)(u-t)} \frac{c_u^{1-\gamma}}{1-\gamma} du.
$$

Note that

$$
\frac{dc_t}{c_t} = g_y dt + \sigma_y dZ_t^y
$$

implies

$$
d\log c_t^{1-\gamma} = (1-\gamma)\left(g_y - \frac{\sigma_y^2}{2}\right) dt + (1-\gamma)\sigma_y dZ_t^y.
$$

Hence

$$
c_u^{1-\gamma} = c_t^{1-\gamma} \exp\left\{(1-\gamma)\left(g_y - \frac{\sigma_y^2}{2}\right)(u-t) + (1-\gamma)\sigma_y Z_{u-t}^y\right\}
$$

and

$$
\mathbb{E}_t \int_t^T e^{-(\rho-n)(u-t)} c_u^{1-\gamma} du = \frac{1 - e^{\left[-\rho+n+(1-\gamma)\left(g_y - \frac{\gamma\sigma_y^2}{2}\right)\right](T-t)}}{\rho - n - (1-\gamma)\left(g_y - \frac{\gamma\sigma_y^2}{2}\right)} c_t^{1-\gamma}.
$$

As long as $\rho - n - (1-\gamma)(g_y - \gamma\sigma_y^2/2) > 0$, we have

$$V_t = \frac{c_t^{1-\gamma}}{\rho - n - (1-\gamma)\left(g_y - \frac{\gamma\sigma_y^2}{2}\right)}.$$

Note that the condition $\rho - n - (1-\gamma)(g_y - \gamma\sigma_y^2/2) > 0$ is necessary for the household problem to have a solution.

**Step 1: household problem.** The household problem when the budget constraint takes the intertemporal form is

$$\begin{aligned}
&\max_{c_t,b_t} \mathbb{E}_0 \int_0^\infty e^{-(\rho-n)t}\left[\frac{c_t^{1-\gamma}-1}{1-\gamma} + \bar{c}_t^{-\gamma} y_t u\left(\frac{b_t}{y_t}\right)\right] dt \\
&s.t. : w_0 \geq \mathbb{E}_0 \int_0^\infty \left[c_t + \tau_t + (r_t^s - r_t)b_t\right] \frac{e^{nt}\xi_t}{\xi_0} dt,
\end{aligned}$$

The Lagrangian of this problem is

$$\mathcal{L}_0 = \mathbb{E}_0 \int_0^\infty e^{-(\rho-n)t}\left[\frac{c_t^{1-\gamma}-1}{1-\gamma} + \bar{c}_t^{-\gamma} y_t u\left(\frac{b_t}{y_t}\right)\right] - \kappa\left[\mathbb{E}_0 \int_0^\infty \left[c_t + \tau_t + (r_t^s - r_t)b_t\right] \frac{e^{nt}\xi_t}{\xi_0} dt - w_0\right].$$

Note that $\mathcal{L}_0$ is a functional such that $\mathcal{L}_0 : \mathbb{L} \times \mathbb{L} \to \mathbb{R}$, where $\mathbb{L}$ is a space of square integrable progressively measurable processes with values in $\mathbb{R}$.

**Step 2: optimal choices and liquidity yield.** The first order conditions for this optimization take the following form

$$\begin{aligned}
\partial c_t :& e^{-\rho t} c_t^{-\gamma} = \kappa \xi_t, \\
\partial b_t :& e^{-\rho t} \bar{c}_t^{-\gamma} u'\left(\frac{b_t}{y_t}\right) = \kappa \xi_t \left(r_t^s - r_t\right),
\end{aligned}$$

together with the transversality condition

$$\lim_{T\to\infty} \mathbb{E}_0 e^{nT} \xi_T w_T \leq 0.$$

Together the transversality condition and no-Ponzi game condition imply that

$$\lim_{T\to\infty} \mathbb{E}_0 e^{nT} \xi_T w_T = 0.$$

Individual optimality condition with respect to consumption can be solved to get consumption

$$c_t = \left(e^{\rho t} \xi_t \kappa\right)^{-1/\gamma},$$

and with respect to liquid bonds for bonds

$$\frac{b_t}{y_t} = (u')^{-1}\left[\bar{c}_t^{\gamma} c_t^{-\gamma}(r_t^s - r_t)\right] = (u')^{-1}\left[\bar{c}_t^{\gamma} e^{\rho t} \kappa \xi_t (r_t^s - r_t)\right].$$

Substituting out $\kappa\xi_t$ from the first order conditions, we obtain

$$\bar{c}_t^{-\gamma} u'\left(\frac{b_t}{y_t}\right) = c_t^{-\gamma}\left(r_t^s - r_t\right).$$

It can be rewritten as

$$r_t^s - r_t = \bar{c}_t^{-\gamma} c_t^{\gamma} u'\left(\frac{b_t}{y_t}\right).$$

In equilibrium, where $c_t = \bar{c}_t$, we will have

$$r_t^s - r_t = u'\left(\frac{b_t}{y_t}\right).$$

Note that the Lagrange multiplier $\kappa$ solves the intertemporal budget constraint after substituting out for optimal consumption and bond holdings:

$$w_0 = \mathbb{E}_0 \int_0^\infty \left\{ (e^{\rho t}\kappa\xi_t)^{-\frac{1}{\gamma}} + \tau_t + (r_t^s - r_t) y_t \left(u'\right)^{-1}\left[\bar{c}_t^{\gamma} e^{\rho t}\kappa\xi_t (r_t^s - r_t)\right] \right\} e^{nt} \frac{\xi_t}{\xi_0} dt.$$

**Step 3: SDF.** First, we apply Ito's lemma to the first order condition with respect to $c_t$ to get the law of motion of $\xi_t$:

$$\begin{aligned}
\kappa d\xi_t &= d\left(e^{-\rho t}\right) c_t^{-\gamma} + e^{-\rho t} d\left(c_t^{-\gamma}\right) + d\left(e^{-\rho t}\right) d\left(c_t^{-\gamma}\right) \\
&= -\rho e^{-\rho t} c_t^{-\gamma} dt - \gamma e^{-\rho t} c_t^{-\gamma-1} dc_t + \frac{\gamma(\gamma+1)}{2} e^{-\rho t} c_t^{-\gamma-2} dc_t^2.
\end{aligned}$$

Divide both sides by $\kappa\xi_t$

$$\frac{d\xi_t}{\xi_t} = -\rho dt - \gamma\frac{dc_t}{c_t} + \frac{\gamma(\gamma+1)}{2}\left(\frac{dc_t}{c_t}\right)^2.$$

Next, we use the goods market clearing condition $y_t = c_t + G_t$ to note that $(1-\gamma_G)y_t = c_t$ and $dy_t/y_t = dc_t/c_t$. As a result,

$$\begin{aligned}
\frac{d\xi_t}{\xi_t} &= -\rho dt - \gamma\frac{dy_t}{y_t} + \frac{\gamma(\gamma+1)}{2}\left(\frac{dy_t}{y_t}\right)^2 \\
&= -\rho dt - \gamma(g_y dt + \sigma_y dZ_t^y) + \frac{\gamma(\gamma+1)}{2}\sigma_y^2 dt \\
&= -\left[\rho + \gamma g_y - \frac{\gamma(\gamma+1)}{2}\sigma_y^2\right] dt - \gamma\sigma_y dZ_t^y. \qquad \text{(A.1)}
\end{aligned}$$

**Step 4: safe rate.** No arbitrage implies that the price $p_t$ of any security that pays dividends $d_s$ to its holder equals

$$p_t = \frac{1}{\xi_t}\mathbb{E}_t \int_t^\infty \xi_s d_s ds. \qquad \text{(A.2)}$$

The differential version of this equation is

$$0 = \xi_t d_t dt + \mathbb{E}_t\left[d(\xi_t p_t)\right]. \qquad \text{(A.3)}$$

The safe bond is a security with the price of 1 and the dividend $r_t^s$. As a result,

$$
\begin{aligned}
0 &= \xi_t r_t^s dt + \mathbb{E}_t d\xi_t, \\
r_t^s &= -\frac{1}{dt}\mathbb{E}_t\left(\frac{d\xi_t}{\xi_t}\right) = \rho + \gamma g_y - \frac{\gamma(\gamma+1)}{2}\sigma_y^2.
\end{aligned}
$$

This confirms the guess of the drift in equation (8).

**Step 5: equity price.** The value of Lucas trees $q_t$ to the household that growth at rate $n$ and collects $y_t e^{nt}$ in dividends is

$$
q_t = \frac{1}{\xi_t}\mathbb{E}_t \int_t^\infty \xi_s e^{nt} y_s ds.
$$

We can compute the last integral explicitly. First note that $f_t \equiv \xi_t e^{nt} y_t = e^{-(\rho-n)t} y_t^{1-\gamma}$ follows the geometric Brownian motion

$$
\frac{df_t}{f_t} = -\left[\rho - n + (\gamma-1)g_y - \frac{(\gamma-1)\gamma}{2}\sigma_y^2\right]dt - (\gamma-1)\sigma_y dZ_t^y,
$$

which has the following solution

$$
f_s = f_t \exp\left\{-\left[\rho - n + (\gamma-1)g_y - \frac{(\gamma-1)\sigma_y^2}{2}\right](s-t) - (\gamma-1)\sigma_y(Z_s^y - Z_t^y)\right\}.
$$

Thus, we obtain

$$
\begin{aligned}
q_t &= \frac{1}{\xi_t}\mathbb{E}_t \int_t^\infty f_s ds \\
&= \frac{y_t}{\rho - n + (\gamma-1)g_y - \frac{\gamma(\gamma-1)\sigma_y^2}{2}} \\
&= \frac{y_t}{r^s - g_y - n + \gamma\sigma_y^2}.
\end{aligned}
$$

The last formula is a version of Gordon's growth formula with risk. It implies that $dq_t/q_t = dy_t/y_t$, which, in turn, yields

$$
\begin{aligned}
\sigma_t &= \sigma_y, \\
\mu_t &= g_y + \frac{y_t}{q_t} = r^s + \gamma\sigma_y^2.
\end{aligned}
$$

The last finding allows us to verify the diffusion part of the guess in equation 8. Specifically,

$$
\frac{\mu_t - r_t^s}{\sigma_t} = \gamma\sigma_y,
$$

which is the same turn as we obtained in equation (A.1).

## A.4 Proof of Proposition 3

In the first part of the proof, we derive some preliminary results. Then, we show that fiscal policy is unsustainable according to the economic approach when $\beta < 0$. Afte that we show that when $\beta = 0$, fiscal policy is sustainable if and only if $\alpha_u - \alpha_D > 0$. Finally, we present the case with $\beta > 0$.

**Preliminaries: TVC.** First, consider the transversality condition

$$\begin{aligned}\mathbb{E}_t\left[e^{n(s-t)}\frac{\xi_s}{\xi_t}b_s\right] =& \mathbb{E}_t\left[e^{n(s-t)}e^{-\rho(s-t)}\left(\frac{y_s}{y_t}\right)^{-\gamma}b_s\right] \\ =& e^{-(1-\gamma)\log y_t+(\rho-n)t}\mathbb{E}_t\left[e^{-(\rho-n)s+(1-\gamma)\log y_s+\widehat{B}_s}\right].\end{aligned}$$

To compute the expectations, we note that the term in the exponent in every point of time $s$ is a normal random variable because it is a sum of the Brownian motion with drift—term $\log y_s$—and the Ornstein–Uhlenbeck process, which is also a normally distributed in a given point in time. As a result

$$\begin{aligned}\mathbb{E}_t\left[e^{n(s-t)}\frac{\xi_s}{\xi_t}b_s\right] &= e^{-(1-\gamma)\log y_t+(\rho-n)t}\cdot e^{\mathbb{E}_t\left[-(\rho-n)s+(1-\gamma)\log y_s+\widehat{B}_s\right]+\frac{1}{2}\mathbb{V}_t\left[-(\rho-n)s+(1-\gamma)\log y_s+\widehat{B}_s\right]} \\ &= e^{-(1-\gamma)\log y_t-(\rho-n)(s-t)+\mathbb{E}_t\left[(1-\gamma)\log y_s+\widehat{B}_s\right]+\frac{1}{2}\mathbb{V}_t\left[(1-\gamma)\log y_s+\widehat{B}_s\right]}. \end{aligned}\tag{A.4}$$

To compute the moments of $(1-\gamma)\log y_s + \widehat{B}_s$, we note that

$$\log y_s = \log y_t + \left(g_y - \frac{\sigma_y^2}{2}\right)(s-t) + \sigma_y\int_t^s dZ_u, \tag{A.5}$$

$$\widehat{B}_s = \begin{cases}\widehat{B}_t e^{-\beta(s-t)} + \frac{\alpha}{\beta}\left(1-e^{-\beta(s-t)}\right) - \sigma_y\int_t^s e^{-\beta(s-u)}dZ_u, & \text{if } \beta \neq 0,\\ \widehat{B}_t + \alpha(s-t) - \sigma_y\int_t^s dZ_u, & \text{if } \beta = 0,\end{cases} \tag{A.6}$$

where the second equation can be obtained by first expressing the law of motion for $e^{\beta s}\widehat{B}_s$ from the law of motion for $\widehat{B}_s$ and then by integrating it. Combining (A.5) and (A.5), we get

$$(1-\gamma)\log y_s + \widehat{B}_s = \begin{cases}(1-\gamma)\log y_t + \widehat{B}_t e^{-\beta(s-t)} + (1-\gamma)\left(g_y - \frac{\sigma_y^2}{2}\right)(s-t) \\ \qquad + \frac{\alpha}{\beta}\left(1-e^{-\beta(s-t)}\right) + \sigma_y\int_t^s\left(1-\gamma-e^{-\beta(s-u)}\right)dZ_u, & \text{if } \beta\neq 0,\\ (1-\gamma)\log y_t + \widehat{B}_t + \left[(1-\gamma)\left(g_y-\frac{\sigma_y^2}{2}\right)+\alpha\right](s-t) - \sigma_y\gamma\int_t^s dZ_u, & \text{if } \beta = 0.\end{cases}$$

Conditional expectations and variance of the last expressions are

$$\mathbb{E}_t\left[(1-\gamma)\log y_s + \widehat{B}_s\right] = \begin{cases}(1-\gamma)\log y_t + \widehat{B}_t e^{-\beta(s-t)} + (1-\gamma)\left(g_y - \frac{\sigma_y^2}{2}\right)(s-t) + \frac{\alpha}{\beta}\left(1-e^{-\beta(s-t)}\right), & \text{if } \beta\neq 0,\\ (1-\gamma)\log y_t + \widehat{B}_t + \left[(1-\gamma)\left(g_y - \frac{\sigma_y^2}{2}\right)+\alpha\right](s-t), & \text{if } \beta = 0.\end{cases}$$

and

$$\begin{aligned}\mathbb{V}_t\left[(1-\gamma)\log y_s + \widehat{B}_s\right] &= \sigma_y^2\int_t^s\left(1-\gamma-e^{-\beta(s-u)}\right)^2 du \\ &= \sigma_y^2\frac{2\beta(\gamma-1)^2(s-t) - 4(1-\gamma)\left(1-e^{-\beta(s-t)}\right) + \left(1-e^{-2\beta(s-t)}\right)}{2\beta},\end{aligned}$$

when $\beta \neq 0$ and

$$\mathbb{V}_t\left[(1-\gamma)\log y_s + \widehat{B}_s\right] = \sigma_y^2\gamma^2(s-t),$$

when $\beta = 0$.

We are ready to compute the exponent in the TVC condition (A.4). When $\beta \neq 0$

$$
\begin{aligned}
&-(1-\gamma)\log y_t-(\rho-n)(s-t)+\mathbb{E}_t\left[(1-\gamma)\log y_s+\widehat{B}_s\right]+\frac{1}{2}\mathbb{V}_t\left[(1-\gamma)\log y_s+\widehat{B}_s\right]\\
=&-(1-\gamma)\log y_t-(\rho-n)(s-t)+(1-\gamma)\log y_t+\widehat{B}_t e^{-\beta(t-s)}+(1-\gamma)\left(g_y-\frac{\sigma_y^2}{2}\right)(s-t)+\frac{\alpha}{\beta}\left(1-e^{-\beta(s-t)}\right)\\
&+\sigma_y^2\frac{2\beta(\gamma-1)^2(s-t)-4(1-\gamma)\left(1-e^{-\beta(s-t)}\right)+\left(1-e^{-2\beta(s-t)}\right)}{4\beta}\\
=&\widehat{B}_t e^{-\beta(t-s)}+\frac{\alpha}{\beta}\left(1-e^{-\beta(s-t)}\right)+\sigma_y^2\frac{-4(1-\gamma)\left(1-e^{-\beta(s-t)}\right)+\left(1-e^{-2\beta(s-t)}\right)}{4\beta}\\
&-\left[\rho-n-(1-\gamma)\left(g_y-\frac{\gamma\sigma_y^2}{2}\right)\right](s-t).
\end{aligned}
\tag{A.7}
$$

When $\beta = 0$, the last equation becomes

$$
\begin{aligned}
&-(1-\gamma)\log y_t-(\rho-n)(s-t)+\mathbb{E}_t\left[(1-\gamma)\log y_s+\widehat{B}_s\right]+\frac{1}{2}\mathbb{V}_t\left[(1-\gamma)\log y_s+\widehat{B}_s\right]\\
=&\widehat{B}_t+\alpha(s-t)+\sigma_y^2\frac{2\gamma-1}{2}(s-t)-\left[\rho-n-(1-\gamma)\left(g_y-\frac{\gamma\sigma_y^2}{2}\right)\right](s-t)\\
=&\widehat{B}_t+\left\{\alpha_D-\alpha_u+\rho+(\gamma-1)g_y-n-\left[\gamma(\gamma+1)-1\right]\sigma_y^2/2+\sigma_y^2\frac{2\gamma-1}{2}-\left[\rho-n-(1-\gamma)\left(g_y-\frac{\gamma\sigma_y^2}{2}\right)\right]\right\}(s-t)\\
=&\widehat{B}_t+(\alpha_D-\alpha_u)(s-t).
\end{aligned}
\tag{A.8}
$$

**The case of $\beta < 0$.** In this case, the Ornstein-Uhlenbeck process in Proposition 1 is exploding instead of being mean-reverting. As a result, the transversality condition of the government, equation (17), is not satisfied. The easiest way to see why the transversality condition fails is to note that conditional expectation of the *log* of debt-to-GDP ratio increases or decreases exponentially depending on initial conditions and its conditional variance *grows* exponentially, as can be seen by solving the differential equation in Proposition 1. At the same time, the *log* of the discount factor decreases only *linearly* with time according to equation (8). As a result, the expectations term in the transversality condition increases without bounds. To see this formally, take equation (A.7) and observe that as $s$ goes to infinity, this equation becomes approximately equal to

$$\sigma_y^2\frac{e^{-2\beta(s-t)}}{-4\beta},$$

which increases to infinity with $s$.

**The case of $\beta = 0$.** Equation (A.8) implies that the TVC holds if and only if $\alpha_D - \alpha_u < 0$.

The iBC of the government is

$$
\begin{aligned}
b_0 &= \mathbb{E}_0\int_0^\infty [\tau_t - g_t + (r_t^s - r_t)b_t]e^{nt}\frac{\xi_t}{\xi_0}dt\\
&= \mathbb{E}_0\int_0^\infty [\alpha_u - \alpha_D + \beta\log(b_t/y_t)]b_t e^{nt}\frac{\xi_t}{\xi_0}dt.
\end{aligned}
$$

After imposing $\beta = 0$, the budget constraint becomes

$$
\begin{aligned}
b_0 &= (\alpha_u - \alpha_D)\mathbb{E}_0 \int_0^\infty b_t e^{nt} \frac{\xi_t}{\xi_0} dt \\
&= b_0(\alpha_u - \alpha_D)\mathbb{E}_0 \int_0^\infty e^{\widehat{B}_t - \widehat{B}_0 + nt - \rho t + (1-\gamma)(\log y_t - \log y_0)} dt.
\end{aligned}
$$

Note that

$$
\begin{aligned}
\widehat{B}_t - \widehat{B}_0 &= \alpha t - \sigma_y \left(Z_t^y - Z_0^y\right), \\
\log y_t - \log y_0 &= \left(g_y - \frac{\sigma_y^2}{2}\right) t + \sigma_y \left(Z_t^y - Z_0^y\right).
\end{aligned}
$$

or

$$
\begin{aligned}
&\widehat{B}_t - \widehat{B}_0 - (\rho - n)\, t + (1-\gamma)\left(\log y_t - \log y_0\right) \\
&= \left[\alpha - \rho + n + (1-\gamma)\left(g_y - \frac{\sigma_y^2}{2}\right)\right] t - \gamma\sigma_y \left(Z_t^y - Z_0^y\right).
\end{aligned}
$$

As a result,

$$
\begin{aligned}
\mathbb{E}_0 \int_0^\infty e^{\widehat{B}_t + nt - \rho t - \gamma(\log c_t - \log c_0)} dt &= e^{\widehat{B}_0} \mathbb{E}_0 \int_0^\infty e^{\left[\alpha - \rho + n + (1-\gamma)\left(g_y - \frac{\sigma_y^2}{2}\right)\right] t - \gamma\sigma_y\left(Z_t^y - Z_0^y\right)} dt \\
&= e^{\widehat{B}_0} \int_0^\infty e^{\left[\alpha - \rho + n + (1-\gamma)\left(g_y - \frac{\sigma_y^2}{2}\right) + \frac{\gamma^2\sigma_y^2}{2}\right] t} dt \\
&= e^{\widehat{B}_0} \frac{\lim_{t\to\infty} e^{\left[\alpha - \rho + n + (1-\gamma)\left(g_y - \frac{\sigma_y^2}{2}\right) + \frac{\gamma^2\sigma_y^2}{2}\right] t} - 1}{\alpha - \rho + n + (1-\gamma)\left(g_y - \frac{\sigma_y^2}{2}\right) + \frac{\gamma^2\sigma_y^2}{2}}
\end{aligned}
$$

Replace $\alpha$ in the following expression

$$
\alpha - \rho + n + (1-\gamma)\left(g_y - \frac{\sigma_y^2}{2}\right) + \frac{\gamma^2\sigma_y^2}{2} = \alpha_D - \alpha_u.
$$

As a result, we get

$$
\mathbb{E}_0 \int_0^\infty e^{\widehat{B}_t + nt - \rho t - \gamma(\log c_t - \log c_0)} dt = e^{\widehat{B}_0} \frac{1}{\alpha_u - \alpha_D}.
$$

Plug this in the original equation

$$
b_0 = (\alpha_u - \alpha_D) b_0 \frac{1}{\alpha_u - \alpha_D}.
$$

The last equation holds for any initial value $b_0$ and for any $\alpha_u - \alpha_D > 0$.

**The case of $\beta > 0$.** When $\beta > 0$, equation A.7 is dominated by the last term which tends to minus infinity as $s$ goes to infinity. This is because for equilibrium to exist the parameters must satisfy $\rho - n - (1-\gamma)(g_y - \gamma\sigma_y^2/2) > 0$. It is easy to check that the government budget constraint also holds following steps similar to the case with $\beta = 0$.

## A.5 Proof of Proposition 4

It is a standard result in the analysis of continuous-time stochastic processes (e.g., Dixit, 1994) that the stationary distribution of a reflected Brownian motion is exponential. As a result, we only derive the conditions for which the TVC is satisfied. Specifically, we will show that a necessary and sufficient condition for $\lim_{T\to\infty} \mathbb{E}_t[e^{nT}\xi_T b_T] = 0$ is $\alpha_u - \alpha_D > 0$. The challenge in proving this result stems from the fact that the log of debt-to-GDP ratio is not just Brownian motion with drift, but a reflected Brownian motion with drift.

To slightly simplify the notation (but without any loss of generality), we assume that $\widehat{B}_0 = 0$, $\widehat{B}_{min} = 0$ and $\gamma = 1$. In the end of the proof, we comment on the consequences of dropping these assumptions.

First, note that

$$\mathbb{E}_t e^{nT}\xi_T b_T = \mathbb{E}_t e^{nT} e^{-\rho T}\frac{b_T}{(1-\gamma_G)y_T} = \frac{1}{1-\gamma_G}\mathbb{E}_t e^{-(\rho-n)T}e^{\widehat{B}_T}.$$

Second, we use the observation, which is straightforward to prove (see, for example, Harrison, 1985) using the so-called reflection principle from probability theory, that the cdf of the reflected Brownian motion with negative drift and a single (lower) reflecting barrier is

$$\mathbb{P}\left(\widehat{B}_t \le x | \widehat{B}_0 = 0\right) = \Phi\left(\frac{x-\alpha t}{\sigma_y\sqrt{t}}\right) - e^{\frac{2\alpha x}{\sigma_y^2}}\Phi\left(\frac{-x-\alpha t}{\sqrt{t}}\right) \equiv F(x), \tag{A.9}$$

where $\Phi(\cdot)$ is the cdf of the standard normal distribution. Equation (A.9) implies that the PDF of the $\widehat{B}_t$ is

$$f(x) = \frac{1}{\sqrt{t}}\phi\left(\frac{x-\alpha t}{\sigma_y\sqrt{t}}\right) + \frac{1}{\sigma\sqrt{t}}e^{\frac{2\alpha x}{\sigma_y^2}}\phi\left(\frac{-x-\alpha t}{\sigma_y\sqrt{t}}\right) - \frac{2\alpha}{\sigma_y^2}e^{\frac{2\alpha x}{\sigma_y^2}}\Phi\left(\frac{-x-\alpha t}{\sigma_y\sqrt{t}}\right). \tag{A.10}$$

When $2\alpha/\sigma_y^2 + 1 \neq 0$ (we will consider the special case when $2\alpha/\sigma_y^2 + 1 = 0$ separately), we have

$$\begin{aligned}
\mathbb{E}_0 e^{\widehat{B}_t} &= \int_0^\infty e^x\left[\frac{1}{\sigma_y\sqrt{t}}\phi\left(\frac{x-\alpha t}{\sigma_y\sqrt{t}}\right) + \frac{1}{\sigma_y\sqrt{t}}e^{\frac{2\alpha x}{\sigma_y^2}}\phi\left(\frac{-x-\alpha t}{\sigma_y\sqrt{t}}\right) - \frac{2\alpha}{\sigma_y^2}e^{\frac{2\alpha x}{\sigma_y^2}}\Phi\left(\frac{-x-\alpha t}{\sigma_y\sqrt{t}}\right)\right]dx \\
&= \underbrace{\int_0^\infty e^x\frac{1}{\sigma_y\sqrt{t}}\phi\left(\frac{x-\alpha t}{\sigma_y\sqrt{t}}\right)dx}_{A_1} + \underbrace{\int_0^\infty e^{\left(\frac{2\alpha}{\sigma_y^2}+1\right)x}\frac{1}{\sigma_y\sqrt{t}}\phi\left(\frac{-x-\alpha t}{\sigma_y\sqrt{t}}\right)dx}_{A_2} \\
&\quad + \underbrace{\frac{\frac{-2\alpha}{\sigma_y^2}}{\frac{2\alpha}{\sigma_y^2}+1}\left[-\Phi\left(\frac{-\alpha t}{\sigma_y\sqrt{t}}\right) + \frac{1}{\sigma_y\sqrt{t}}\int_0^\infty e^{\left(\frac{2\alpha}{\sigma_y^2}+1\right)x}\phi\left(\frac{-x-\alpha t}{\sigma_y\sqrt{t}}\right)dx\right]}_{A_3}.
\end{aligned} \tag{A.11}$$

To compute $A_1, A_2$ and $A_3$, note that

$$e^{ax}\phi(bx+c) = e^{-\frac{c^2-\left(c-\frac{a}{b}\right)^2}{2}}\phi\left(bx+c-\frac{a}{b}\right).$$

As a result,

$$A_1 = e^{\frac{\sigma_y^2\left(\frac{2\alpha}{\sigma_y^2}+1\right)}{2}t}\Phi\left(\frac{\sigma_y^2+\alpha}{\sigma_y}\sqrt{t}\right), \tag{A.12}$$

$$A_2 = e^{\frac{\sigma_y^2\left(\frac{2\alpha}{\sigma_y^2}+1\right)}{2}t}\Phi\left(\frac{\sigma_y^2+\alpha}{\sigma_y}\sqrt{t}\right) = A_1, \tag{A.13}$$

$$A_3 = \frac{\frac{-2\alpha}{\sigma_y^2}}{\frac{2\alpha}{\sigma_y^2}+1}\left\{-\Phi\left(\frac{-\alpha}{\sigma_y}\sqrt{t}\right)+e^{\frac{\sigma_y^2\left(\frac{2\alpha}{\sigma_y^2}+1\right)}{2}t}\Phi\left(\frac{\sigma_y^2+\alpha}{\sigma_y}\sqrt{t}\right)\right\}. \tag{A.14}$$

Plugging equations (A.12)-(A.14) into (A.11) and rearranging, we get

$$\begin{aligned}
\mathbb{E}_0 e^{\widehat{B}_t} =& e^{\frac{\sigma_y^2\left(\frac{2\alpha}{\sigma_y^2}+1\right)}{2}t}\Phi\left(\frac{\sigma_y^2+\alpha}{\sigma_y}\sqrt{t}\right)+e^{\frac{\sigma_y^2\left(\frac{2\alpha}{\sigma_y^2}+1\right)}{2}t}\Phi\left(\frac{\sigma_y^2+\alpha}{\sigma_y}\sqrt{t}\right)\\
&+\frac{\frac{-2\alpha}{\sigma_y^2}}{\frac{2\alpha}{\sigma_y^2}+1}\left\{-\Phi\left(\frac{-\alpha}{\sigma_y}\sqrt{t}\right)+e^{\frac{\sigma_y^2\left(\frac{2\alpha}{\sigma_y^2}+1\right)}{2}t}\Phi\left(\frac{\sigma_y^2+\alpha}{\sigma_y}\sqrt{t}\right)\right\}\\
=&2\frac{\frac{\alpha}{\sigma_y^2}+1}{\frac{2\alpha}{\sigma_y^2}+1}e^{\frac{\sigma_y^2\left(\frac{2\alpha}{\sigma_y^2}+1\right)}{2}t}\Phi\left[\sigma_y\left(1+\frac{\alpha}{\sigma_y^2}\right)\sqrt{t}\right]-\frac{\frac{-2\alpha}{\sigma_y^2}}{\frac{2\alpha}{\sigma_y^2}+1}\Phi\left(\frac{-\alpha}{\sigma_y}\sqrt{t}\right).
\end{aligned} \tag{A.15}$$

When $2\alpha/\sigma_y^2+1>0$, then $\alpha/\sigma_y^2+1>-\alpha/\sigma_y^2>0$, so that the first term in equation (A.15) is positive. Moreover, when again $2\alpha/\sigma_y^2+1>0$, the first term on the last line always goes to infinity when $t$ tends to infinity dwarfing the second term in equation (A.15). At the same time, when $2\alpha/\sigma_y^2+1<0$, then as $t\to\infty$ the first term in equation (A.15) disappears leaving only the second term to be non-negligible. We now use these properties to compute the transversality condition.

Now, we can compute the limit

$$\begin{aligned}
\lim_{t\to\infty} e^{-(\rho-n)t}\mathbb{E}_0 e^{\widehat{B}_t} &= \lim_{t\to\infty}\left\{2\frac{\frac{\alpha}{\sigma_y^2}+1}{\frac{2\alpha}{\sigma_y^2}+1}e^{\left[\frac{\sigma_y^2\left(\frac{2\alpha}{\sigma_y^2}+1\right)}{2}-\rho+n\right]t}\Phi\left[\sigma_y\left(1+\frac{\alpha}{\sigma_y^2}\right)\sqrt{t}\right]-e^{-(\rho-n)t}\frac{\frac{-2\alpha}{\sigma_y^2}}{\frac{2\alpha}{\sigma_y^2}+1}\Phi\left(\frac{-\alpha}{\sigma_y}\sqrt{t}\right)\right\}\\
&= \lim_{t\to\infty}\left\{2\frac{\frac{\alpha}{\sigma_y^2}+1}{\frac{2\alpha}{\sigma_y^2}+1}e^{\left[\frac{\sigma_y^2\left(\frac{2\alpha}{\sigma_y^2}+1\right)}{2}-\rho+n\right]t}\Phi\left[\sigma_y\left(1+\frac{\alpha}{\sigma_y^2}\right)\sqrt{t}\right]\right\}\\
&= 2\frac{\frac{\alpha}{\sigma_y^2}+1}{\frac{2\alpha}{\sigma_y^2}+1}\lim_{t\to\infty} e^{\left[\frac{\sigma_y^2\left(\frac{2\alpha}{\sigma_y^2}+1\right)}{2}-\rho+n\right]t}
\end{aligned}$$

where the second equality took into account the fact that $\lim_{t\to\infty} e^{-(\rho-n)t}\Phi\left(\frac{-\alpha}{\sigma_y}\sqrt{t}\right)=0$. Finally, we obtain that

$$\lim_{t\to\infty} e^{-(\rho-n)t}\mathbb{E}_0 e^{\widehat{B}_t} = \begin{cases} 0, & \frac{\sigma_y^2}{2} < -\alpha+\rho-n, \\ 2\frac{\frac{\alpha}{\sigma_y^2}+1}{\frac{2\alpha}{\sigma_y^2}+1}, & \frac{\sigma_y^2}{2} = -\alpha+\rho-n, \\ +\infty, & \frac{\sigma_y^2}{2} > -\alpha+\rho-n. \end{cases}$$

After noting that

$$\alpha-\rho+n+\frac{\sigma_y^2}{2} = \alpha_D-\alpha_u,$$

we can write

$$\lim_{t\to\infty} e^{-(\rho-n)t}\mathbb{E}_0 e^{\widehat{B}_t} = \begin{cases} 0, & \alpha_u-\alpha_D > 0, \\ 2\frac{\frac{\alpha}{\sigma_y^2}+1}{\frac{2\alpha}{\sigma_y^2}+1}, & \alpha_u-\alpha_D = 0, \\ +\infty, & \alpha_u-\alpha_D < 0. \end{cases}$$

**Remark 1.** So far, we have considered the case when $2\alpha/\sigma_y^2+1 \neq 0$. When, instead, $2\alpha/\sigma_y^2+1 = 0$, the above calculations simplify considerably. The key difference starts from equation (A.15), which will not feature $2\alpha/\sigma_y^2+1$ in the denominator anymore.

**Remark 2.** While the assumptions that $\widehat{B}_0 = \widehat{B}_{min} = 0$ are clearly non-consequential. The assumption of $\gamma = 1$ can look suspicious. In fact, all of the calculation go through. The only non-standard feature is that we need to deal with a joint distribution of correlated Brownian motion (i.e., $\log y_T$) and reflected Brownian motion (i.e., $\widehat{B}_T$) because we have to compute the following object

$$\mathbb{E}_t e^{nT}\xi_T b_T = \mathbb{E}_t e^{-(\rho-n)T}\left[(1-\gamma_G)y_T\right]^{-\gamma} b_T = (1-\gamma_G)^{-\gamma} e^{-(\rho-n)T}\mathbb{E}_t e^{(1-\gamma)\log y_T+\widehat{B}_T}.$$

This is straightforward but needs some care.

## A.6 Proof of Proposition 5

**The law of motion.** Taking the difference between the low of motion of the log of debt and the log of output, i.e.,

$$d\log B_t = \left(r_t + \frac{G_t-T_t}{B_t}\right)dt,$$
$$d\log y_t = \left(g_y - \frac{\sigma_y^2}{2}\right)dt + \sigma_y dZ_t^y - Z_t dJ_t,$$

We obtain the law of motion of the log of debt-to-GDP ratio

$$d\widehat{B}_t = \left(r_t - g_y - n + \alpha_D + \frac{\sigma_y^2}{2} - \beta_D\widehat{B}_t\right)dt - \sigma_y dZ_t^y + Z_t dJ_t.$$

The interest rates are

$$r^s = \rho + \gamma g_y - \frac{\gamma(\gamma+1)}{2}\sigma_y^2 - \lambda(\mathbb{E}_Z e^{\gamma Z} - 1),$$
$$r_t = r^s - \alpha_u + \beta_u \widehat{B}_t.$$

As a result

$$d\widehat{B}_t = \left(\widetilde{\alpha} - \beta \widehat{B}_t\right) dt - \sigma_y dZ_t^y + Z_t dJ_t.$$

where

$$\widetilde{\alpha} = \rho + \gamma g_y - \frac{\gamma(\gamma+1)}{2}\sigma_y^2 - \alpha_u - g_y - n + \alpha_D + \frac{\sigma_y^2}{2} - \lambda(\mathbb{E}_Z e^{\gamma Z} - 1) = \alpha - \lambda(\mathbb{E}_Z e^{\gamma Z} - 1).$$

which is similar to the definition of $\alpha$ before but that takes into account rare disasters.

**Stationary distribution.** Taking into account the fact that $\beta = 0$, we can write the Kolmogorov forward equation for the density function $f = f(\widehat{B}, t)$ in the region $\widehat{B} > \widehat{B}_{min}$ as

$$\frac{\partial f}{\partial t} = -\widetilde{\alpha}\frac{\partial f}{\partial \widehat{B}} + \frac{\sigma_y^2}{2} \cdot \frac{\partial^2 f}{\partial \widehat{B}^2} + \lambda \left\{ \mathbb{E}\left[ f(\widehat{B} - Z, t) \right] - f(\widehat{B}, t) \right\}. \tag{A.16}$$

The stationary distribution satisfied $\partial f / \partial t = 0$. We guess and verify that

$$f(\widehat{B}) = \overline{f} \cdot e^{-\xi \widehat{B}},$$

where $\overline{f}$ and $\widehat{B}$ are constants to be determined. Plugging this guess in equation A.16 and taking into account that $\partial f / \partial t = 0$, we obtain the implicit equation that determines the rate parameter $\xi$

$$\widetilde{\alpha}\xi + \frac{\sigma_y^2}{2}\xi^2 = \lambda\left(1 - \mathbb{E}\left[e^{\xi Z}\right]\right). \tag{A.17}$$

This equation has one obvious solution of $\xi = 0$, which immediately implies that $f(\widehat{B}) = 0$ for all $\widehat{B} > \widehat{B}_{min}$. This must not be the case for the reflected process. As a result, we discard this solution. It is easy to see by plotting the left- and the right-hand sides of equation A.17 as functions of $\xi$, that the remaining solution of the equation is positive when $\widetilde{\alpha} + \lambda\mathbb{E}[Z] < 0$ and it is negative in the opposite case of $\widetilde{\alpha} + \lambda\mathbb{E}[Z] > 0$. In the knife edge case of $\lambda\mathbb{E}[Z] = -\widetilde{\alpha}$, there is only one solution of $\xi = 0$. The stationary distribution exists only for negative $\xi$.

Constant $\overline{f}$ is determined by requiring that

$$\int_{\widehat{B}_{min}}^{\infty} f(\widehat{B}) d\widehat{B} = 1.$$

As a result,

$$\overline{f} = \xi e^{\xi \widehat{B}_{min}},$$

and

$$f(\widehat{B}) = \xi e^{-\xi\left(\widehat{B} - \widehat{B}_{min}\right)}.$$

**Transversality condition.** To prove that the transversality condition holds if and only if $\alpha_u - \alpha_D > 0$, we consider two separate cases.

First, consider the case when the stationary distribution does not exist. Specifically, suppose that $\widetilde{\alpha} + \lambda \mathbb{E}[Z] > 0$. In this case, the debt-to-GDP ratio increases unboundedly on average. As a result, we can ignore the influence of the lower reflecting barrier. As a result, the expectation in the transversality condition

$$\mathbb{E}_t e^{nT} \xi_T b_T = \frac{1}{(1-\gamma_G)^2} \mathbb{E}_t e^{-(\rho-n)T} e^{\widehat{B}_T + (1-\gamma)\log y_T}.$$

Next, express $\widehat{B}_T + (1-\gamma)\log y_T$ as

$$\begin{aligned} d\widehat{B}_t + (1-\gamma) d\log y_t &= \widetilde{\alpha} dt - \sigma_y dZ_t^y + Z_t dJ_t + (1-\gamma)\left[\left(g_y - \frac{\sigma_y^2}{2}\right) dt + \sigma_y dZ_t^y - Z_t dJ_t\right] \\ &= \left[\widetilde{\alpha} dt + (1-\gamma)\left(g_y - \frac{\sigma_y^2}{2}\right)\right] dt - \gamma\sigma_y dZ_t^y + \gamma Z_t dJ_t \\ &= \left[\rho - \frac{\gamma^2}{2}\sigma_y^2 - \alpha_u - n + \alpha_D - \lambda(\mathbb{E}_Z e^{\gamma Z} - 1)\right] dt - \gamma\sigma_y dZ_t^y + \gamma Z_t dJ_t. \end{aligned}$$

Integrate the last equation

$$\begin{aligned} \widehat{B}_T + (1-\gamma)\log y_T - [\widehat{B}_t + (1-\gamma)\log y_t] &= \left[\rho - \frac{\gamma^2}{2}\sigma_y^2 - \alpha_u - n + \alpha_D - \lambda(\mathbb{E}_Z e^{\gamma Z} - 1)\right](T-t) \\ &\quad - \gamma\sigma_y\left(Z_T^y - Z_t^y\right) + \gamma \sum_{k=1}^{n_{t,T}} Z_{t_k}, \end{aligned}$$

where $n_{t,T}$ is a (random) number of Poisson event arrivals between $t$ and $T$. As a result,

$$\begin{aligned} &\mathbb{E}_t e^{-(\rho-n)T} e^{\widehat{B}_T + (1-\gamma)\log y_T} \\ =& e^{-(\rho-n)t + \widehat{B}_t + (1-\gamma)\log y_t} \mathbb{E}_t e^{[-\gamma^2\sigma_y^2/2 - \alpha_u + \alpha_D - \lambda(\mathbb{E}_Z e^{\gamma Z} - 1)](T-t) - \gamma\sigma_y\left(Z_T^y - Z_t^y\right) + \gamma\sum_{k=1}^{n_{t,T}} Z_{t_k}} \\ =& e^{-(\rho-n)t + \widehat{B}_t + (1-\gamma)\log y_t} \mathbb{E}_t e^{\left[-\frac{\gamma^2}{2}\sigma_y^2 - \alpha_u + \alpha_D - \lambda(\mathbb{E}_Z e^{\gamma Z} - 1)\right](T-t) + \frac{\gamma^2\sigma_y^2}{2}(T-t) + (T-t)\lambda\mathbb{E}_t[e^{\lambda Z_t} - 1]} \\ =& e^{-(\rho-n)t + \widehat{B}_t + (1-\gamma)\log y_t} \mathbb{E}_t e^{[-\alpha_u + \alpha_D](T-t)}. \end{aligned}$$

It is clear from the last expression that the transversality condition holds if and only if $-\alpha_u + \alpha_D < 0$.

Consider now the second case in which the stationary distribution exists, that is, $\widetilde{\alpha} + \lambda\mathbb{E}[Z] < 0$. For simplicity of exposition, we consider the case of $\gamma = 1$, which allows us to write

$$\begin{aligned} \mathbb{E}_t e^{nT} \xi_T b_T &= \frac{1}{(1-\gamma_G)^2} \mathbb{E}_t e^{-(\rho-n)T} e^{\widehat{B}_T} \\ &= \frac{1}{(1-\gamma_G)^2} e^{-(\rho-n)T} \int_{\widehat{B}_{\min}}^{\infty} e^{\widehat{B}} \xi e^{-\xi\widehat{B}} d\widehat{B} \\ &= \frac{1}{(1-\gamma_G)^2 (1-\xi)} e^{-(\rho-n)T} \xi \int_{\widehat{B}_{\min}}^{\infty} e^{(1-\xi)\widehat{B}} d(1-\xi)\widehat{B} \\ &= \frac{1}{(1-\gamma_G)^2 (\xi-1)} e^{-(\rho-n)T} \xi e^{(1-\xi)\widehat{B}_{\min}} \end{aligned}$$

where the last inequality is only valid under $\xi > 1$. As a result, the TVC holds if $\xi > 1$ and $\rho > n$.

## A.7 Derivation of Asset Returns with Default

We lay out a heuristic derivation here.

**Safe interest rate.** The discount factor is

$$m_{t,t+dt} = e^{-\rho dt} \frac{c_{t+dt}^{-\gamma}}{c_t^{-\gamma}} = \begin{cases} e^{-\rho dt} e^{-\gamma g_y dt}, & \text{no disaster,} \\ e^{-\rho dt} e^{-\gamma g_y dt} e^{-\gamma(-Z)}, & \text{disaster,} \end{cases}$$
$$= \begin{cases} e^{-(\rho+\gamma g_y)dt}, & \text{no disaster,} \\ e^{-(\rho+\gamma g_y)dt+\gamma Z}, & \text{disaster,} \end{cases}$$

the return is

$$R_{t,t+dt} = e^{r_t^s dt}.$$

The Euler equation is

$$1 = \mathbb{E}_t \left( m_{t,t+dt} R_{t,t+dt} \right).$$

Use the values of the SDF and the return

$$1 = (1 - \lambda dt) e^{-(\rho+\gamma g_y)dt} e^{r_t^s dt} + \lambda dt \mathbb{E}_Z e^{-(\rho+\gamma g_y)dt+\gamma Z} e^{r_t^s dt},$$

and simplify

$$e^{(\rho+\gamma g_y - r_t^s)dt} = 1 + \lambda dt \left( \mathbb{E}_Z e^{\gamma Z} - 1 \right),$$
$$r_t^s = \rho + \gamma g_y - \lambda \left( \mathbb{E}_Z e^{\gamma Z} - 1 \right).$$

Note that this last formula is just a special case of a more general formula in Proposition 8 in Appendix B.

**Defaultable interest rate.** The return is

$$R_{t,t+dt} = \begin{cases} e^{r_t dt}, & \text{no default,} \\ 0 & \text{default.} \end{cases}$$

The Euler equation is

$$1 = \mathbb{E}_t \left( m_{t,t+dt} R_{t,t+dt} \right).$$

Use the values of the SDF and the return

$$1 = (1 - \lambda dt) e^{-(\rho+\gamma g_y)dt} e^{r_t dt} + \lambda dt e^{-(\rho+\gamma g_y)dt} e^{r_t dt} \mathbb{P} \left( Z < \log \left( \frac{\mathcal{B}_{FL}^{nr}}{B_t / Y_t} \right) \right) \mathbb{E} \left[ e^{\gamma Z} | Z < \log \left( \frac{\mathcal{B}_{FL}^{nr}}{B_t / Y_t} \right) \right],$$

and simplify

$$
\begin{aligned}
e^{(\rho+\gamma g_y-r_t)dt} =&(1-\lambda dt)+\lambda dt\mathbb{P}\left(Z<\log\left(\frac{\mathcal{B}_{FL}^{nr}}{B_t/Y_t}\right)\right)\mathbb{E}\left[e^{\gamma Z}|Z<\log\left(\frac{\mathcal{B}_{FL}^{nr}}{B_t/Y_t}\right)\right],\\
(\rho+\gamma g_y-r_t) =&\lambda\left\{\mathbb{P}\left(Z<\log\left(\frac{\mathcal{B}_{FL}^{nr}}{B_t/Y_t}\right)\right)\mathbb{E}_{Z|Z<\log\left(\frac{\mathcal{B}_{FL}^{nr}}{B_t/Y_t}\right)}[e^{\gamma Z}]-1\right\},\\
r_t =&\rho+\gamma g_y-\lambda\left\{\mathbb{P}\left(Z<\log\left(\frac{\mathcal{B}_{FL}^{nr}}{B_t/Y_t}\right)\right)\mathbb{E}_{Z|Z<\log\left(\frac{\mathcal{B}_{FL}^{nr}}{B_t/Y_t}\right)}[e^{\gamma Z}]-1\right\}.
\end{aligned}
$$

In the case of the exponential distribution of random variable $Z$, i.e., a change in log output, with the probability distribution function

$$
f_Z(z)=\begin{cases}\xi e^{-\xi z}, & z\geq 0,\\ 0, & z<0,\end{cases}
$$

we have

$$
\begin{aligned}
r_t &= \rho+\gamma g_y-\lambda\left[\int_0^{\log\frac{\mathcal{B}_{FL}^{nr}}{B_t/Y_t}} e^{\gamma z}dF_Z(z)-1\right]\\
&= \rho+\gamma g_y-\lambda\left[\xi\int_0^{\log\frac{\mathcal{B}_{FL}^{nr}}{B_t/Y_t}} e^{(\gamma-\xi)z}dz-1\right]\\
&= \rho+\gamma g_y-\lambda\left[\frac{\xi}{\gamma-\xi}e^{(\gamma-\xi)z}\Big|_0^{\log\frac{\mathcal{B}_{FL}^{nr}}{B_t/Y_t}} -1\right]\\
&= \rho+\gamma g_y-\lambda\left\{\frac{\xi}{\xi-\gamma}\left[1-\left(\frac{\mathcal{B}_{FL}^{nr}}{B_t/Y_t}\right)^{\gamma-\xi}\right]-1\right\}.
\end{aligned}
$$

## A.8 Proof of Proposition 6

To see why the points to the right of $\mathcal{B}_{FL}^{nr}$ fail the backward induction argument, start with the debt-to-GDP level of $\mathcal{B}_{FL}^{nr}$ and the situation when uncertainty has not been resolved yet. In the next instant, the debt-to-GDP will exceed $\mathcal{B}_{FL}^{nr}$ because the debt servicing cost is larger than the surplus. Since the government defaults after disaster of any size in this region, the interest rate on public debt is $r^d=\rho+\gamma g_y+\lambda$. However, the highest level of debt-to-GDP that the government can sustain at this interest rate is $\overline{s}/(r^d-g_y-n)$, which is smaller than $\mathcal{B}_{FL}^{nr}$.[28] As a result, the households refuse to purchase government bonds leading to immediate default. However, predictable defaults with capital losses are not possible in our model where agents have rational expectations. This means that $\mathcal{B}_{FL}^{nr}$ could not be an equilibrium level of debt-to-GDP in the first place. Continuing this logic all the way from $\mathcal{B}_{FL}^{nr}$ to $\mathcal{B}_{FL}^{r}$, we deduce that when risk is still present, all point to the right of $\mathcal{B}_{FL}^{r}$ are not an equilibrium.

[28]Formally, $\mathcal{B}_{FL}^{nr}=\overline{s}/(\rho+\gamma g_y-g_y-n)>\overline{s}/(\rho+\gamma g_y+\lambda-g_y-n)$.

## A.9 Change in Distribution of Disasters

In this section, we study how a rightward shift in the distribution of disaster sizes affects fiscal limits. Formally, we consider a change from distribution $F(z)$ to $\widetilde{F}(z)$, where the latter first-order stochastically dominated the former. Recall that sense. If the distribution $\widetilde{F}(z)$ first-order stochastically dominates the distribution $F(z)$, then, for any increasing function, including $D(z) = e^{\gamma z}$, we have that $\mathbb{E}_F g(z) < \mathbb{E}_{\widetilde{F}} g(z)$.

This change has two opposing effects on the debt servicing cost curve. First, it reduces the safe interest rate in equation (23) via the household's stochastic discount factor. Second, it increases the probability of default conditional on a disaster and, as a result, raises the sovereign default premium. Each of these two forces can dominate at any level of debt - a contrast to the previous case where a change in $\lambda$ had an unambiguous effects on the interest rate for debt-to-GDP ratios above or below $\mathcal{B}^*$.

To illustrate this point, consider a special case when the distribution $F$ is exponential with the mean of $\bar{z}$ that satisfies $\bar{z}\gamma < 1$. Without this parameters restriction, which states that either the risk aversion is small enough or the average disaster is not too large, the demand for safe assets is infinite. Figure A.1 presents three examples in which the net effect of the above two forces is either negative for all levels of debt, positive for all levels of debt, or negative for some and positive for the other levels of debt. The left panel presents the case in which the households are risk neutral, i.e., $\gamma = 0$. In this case, the effect of larger disasters on the safe interest rate is absent and, hence, a higher probability of default conditional on a disaster dominates. As can be seen from Figure A.1, for all levels of the debt-to-GDP ratio between 0 and the fiscal limit without risk, the interest rate $\widetilde{r}_t$ under larger disasters rises relative to $r_t$ . By contrast, the right panel of Figure A.1 draws the change in the interest rate in the case when the risk aversion is relatively high, such as $\gamma = 1$. In this case, a safe interest rate decline is the dominant force that pushes down the public debt interest rate for all debt-to-GDP levels considered. Next, the middle panel of Figure A.1, presents an intermediate case with the coefficient of relative risk aversion $\gamma = 0.5$. In this scenario, there is a cutoff value of the debt-to-GDP ratio below which the decline in the safe interest rate dominates and above which a higher probability of disasters dominates. Finally, we note that this interest rate behavior directly translates into the behavior of the debt-servicing costs. As a result, in general, it is impossible to determine the direction of a change in the fiscal limit without risk and flipping points.

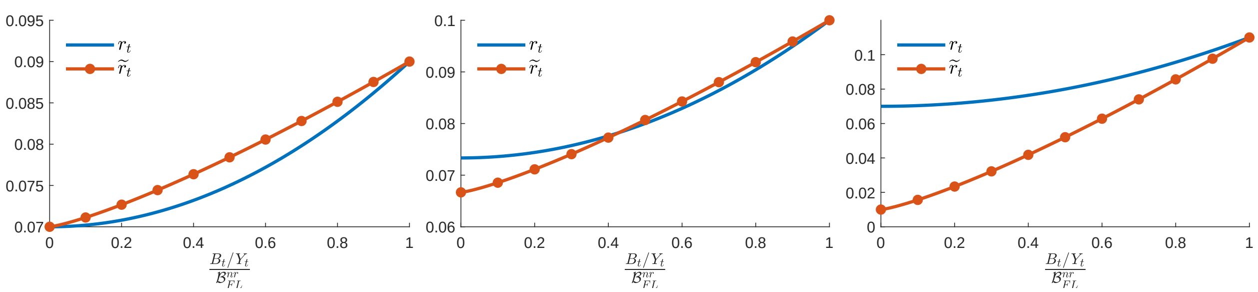


**Figure A.1:** The three plots of this figure illustrate how the public bonds interest rate changes from $r_t$ to $\widetilde{r}_t$ when the distribution of disasters, represented by the exponential distribution with the density of $f(z) = \bar{z}^{-1} \exp(-z/\bar{z})$, changes its mean from $\bar{z}$ to $\widetilde{\bar{z}} > \bar{z}$. The left panel presents the case of $\gamma = 0$. The middle panel shows the case for $\gamma = 0.5$. The right panel plots the interest rate for $\gamma = 1$. All the other parameters are kept constant.

# B A Model with EZW Preferences

This section of the Appendix provides the details omitted in Section 3.7 and 4.4 by first describing the recursive preferences and then by stating some results that we prove in an online appendix C.

## B.1 A no-Disaster Case

In a model without disasters, a typical household maximizes the following preferences

$$W_t = V_t + \mathbb{E}_t \int_t^{\infty} \pi_{t,s} y_s u\left(\frac{b_s}{y_s}\right) ds, \tag{B.1}$$

where

$$\begin{aligned} V_t &= \mathbb{E}_t \int_t^{\infty} f(c_s, V_s) ds, \\ f(c_s, V_s) &= \frac{[(1-\gamma)V_s]^{\frac{\theta-\gamma}{1-\gamma}}}{1-\theta} \{c_s^{1-\theta} - (\rho - n)[(1-\gamma)V_s]^{\frac{1-\theta}{1-\gamma}}\}, \\ \pi_{t,s} &= e^{\left\{\frac{\theta-\gamma}{1-\gamma}\left[\rho-n-(1-\theta)\left(g-\frac{\gamma\sigma_y^2}{2}\right)\right]-(\rho-n)(1-\gamma)\right\}\frac{s-t}{1-\theta}} \left[\rho - n - (1-\theta)\left(g_y - \frac{\gamma\sigma_y^2}{2}\right)\right]^{-\frac{\theta-\gamma}{1-\theta}} C_s^{-\gamma}. \end{aligned} \tag{B.2}$$

Formally, the utility function (3) consists of two terms that capture the utility from consumption and utility from holding government bonds. We assume that the utility from consumption is represented by the Epstein-Zin-Weil preferences with subjective discount factor $\rho$, the coefficient of relative risk aversion $\gamma$, and the intertemporal elasticity of substitution $1/\theta$. One advantage of using these preferences is that they allow for separation of the coefficient of relative risk aversion (CRRA) and the intertemporal elasticity of substitution (IES) that will be convenient in our calibration. We use the continuous-time formulation of these preferences introduced by Duffie and Epstein (1992). When $\gamma = \theta$, the preferences in (B.1) reduce to the preferences we used in the main text and that are given by equation (3).

With the process (B.2) entering the preferences for public debt (3), the demand for liquid bonds does not depend on current consumption of the household in equilibrium, i.e., the wealth effect on demand for government bonds is zero in equilibrium.

By repeating steps is the proof of Lemma 2, which can be found in the Appendix A.2, we can write the following law of motion for the log of debt-to-GDP ratio.

$$d\widehat{B}_t = \left(r_t - g_y - n + \alpha_D + \frac{\sigma_y^2 - \sigma_B^2}{2} - \beta_D \widehat{B}_t\right) dt + \sigma_{\hat{B}} dZ_t^{\hat{B}},$$

where $dZ_t^{\hat{B}} \equiv (\sigma_B/\sigma_{\widehat{B}})dZ_t^B - (\sigma_y/\sigma_{\widehat{B}})dZ_t^y$ and $\sigma_{\hat{B}}^2 \equiv \sigma_B^2 + \sigma_y^2$. Note that we added disasters in this expression.

Asset market clearing conditions combined with optimal choices by households gives the asset pricing equations summarized in the next proposition.

**Proposition 8.** *In equilibrium, the interest rate on safe assets and liquid government bonds are:*

$$r^s = \rho + \theta g_y - \frac{\gamma(\theta+1)}{2}\sigma_y^2,$$
$$r_t = r^s - \alpha_u + \beta_u + \beta_u \widehat{B}_t,$$

*and the drift and diffusion terms for the return on the risky asset are given by:*

$$\mu_t = r^s + \gamma\sigma_y^2,$$
$$\sigma_t = \sigma_y.$$

The proof is in Appendix C. Proposition 8 states that the only difference in the asset pricing in this extended model compared to the model in Section 3 is the explicit presence of the IES parameter in the safe interest rate

## B.2 A Case with Disasters

Adding disasters is straightforward. Equation B.2 has to be modified to take into account the fact that the household faces not only Brownian but also disaster risk. The law of motion of debt to GDP becomes

$$d\widehat{B}_t = \left(r_t - g_y - n + \alpha_D + \frac{\sigma_y^2 - \sigma_B^2}{2} - \beta_D \widehat{B}_t\right) dt + \sigma_{\hat{B}} dZ_t^{\hat{B}} + Z_t dJ_t, \tag{B.3}$$

We present the extension of Proposition 8 to the disaster case.

**Proposition 9.** *In equilibrium, the interest rate on safe assets and liquid government bonds are:*

$$r^s = \rho + \theta g_y - \frac{\gamma(\theta+1)}{2}\sigma_y^2 + \lambda \mathbb{E}\left[\frac{\theta-\gamma}{1-\gamma}(e^{-(1-\gamma)Z} - 1) - (e^{\gamma Z} - 1)\right],$$
$$r_t = r^s - \alpha_u + \beta_u + \beta_u \widehat{B}_t,$$

*and the drift and diffusion terms for the return on the risky asset are given by:*

$$\mu_t = r^s + \gamma\sigma_y^2 + \lambda \mathbb{E}_Z\left[(e^{\gamma Z} - 1)(1 - e^{-Z})\right],$$
$$\sigma_t = \sigma_y.$$

It is straightforward to extend the proof of Proposition 8 to the case with disasters by following, for example, Tsai and Wachter (2015).

As a result, the law of motion of the log of public debt-to-GDP ratio is

$$d\widehat{B}_t = \left(\overline{\alpha} - \beta \widehat{B}_t\right) dt + \sigma_{\hat{B}} dZ_t^{\hat{B}} + Z_t dJ_t.$$

where

$$\overline{\alpha} \equiv \rho + \gamma g_y - \{\sigma_B^2 + [\gamma(\theta+1) - 1]\,\sigma_y^2\}/2 - \alpha_u - g_y - n + \alpha_D + \lambda \mathbb{E}_Z\left[\frac{\theta-\gamma}{1-\gamma}\left(e^{-(1-\gamma)Z} - 1\right) - (e^{\gamma Z} - 1)\right].$$

which is similar to the definition of $\widetilde{\alpha}$ in Proposition 5 but that takes into account the fact that the IES and CRRA are not equal each other.

Note that the stationary distribution of $\widehat{B}_t$ when there is a lower reflecting barrier $\widehat{B}_{\min}$ and $\beta = 0$ is again exponential with the rate parameter that solves

$$\overline{\alpha}\xi + \frac{\sigma_{\hat{B}}^2}{2}\xi^2 = \lambda(1 - \mathbb{E}_Z[e^{\xi Z}]).$$

Assume that government defaults when the debt jumps over the debt limit as in Section 4. The household needs to be compensated for this risk. The next proposition presents the interest rate paid on government debt absent default.

**Proposition 10.** *Conditional on no default, public debt pays*

$$r_t = r^s + \lambda\mathbb{E}\left[e^{\gamma Z_t}\mathbb{I}\left(Z > \log\left(\frac{\mathcal{B}_{FL}}{B_t/Y_t}\right)\right)\right].$$

The proof of this result uses Proposition 1 from Tsai and Wachter (2015). As a result,

$$r_t = \rho + \theta g_y - \frac{\gamma\left(\theta+1\right)}{2}\sigma_y^2 + \lambda\frac{\theta-\gamma}{1-\gamma}\mathbb{E}\left[e^{-(1-\gamma)Z} - 1\right] - \lambda\mathbb{E}\left[e^{\gamma Z}\mathbb{I}\left(Z < \log\left(\frac{\mathcal{B}_{FL}}{B_t/Y_t}\right)\right) - 1\right].$$

When we assume that $Z$ has an exponential distribution with the pdf $f_Z(z) = \overline{z}^{-1}e^{-z/\overline{z}}$ for $z \geq 0$, we get

$$r_t = \rho + \theta g_y - \frac{\gamma\left(\theta+1\right)}{2}\sigma_y^2 - \lambda\frac{\theta-\gamma}{1+\overline{z}-\overline{z}\gamma}\overline{z} - \lambda\left(\frac{1}{1-\overline{z}\gamma}\left(1-\left(\frac{B_t/Y_t}{\mathcal{B}_{FL}}\right)^{\frac{1-\overline{z}\gamma}{\overline{z}}}\right) - 1\right).$$

where I used the fact that $\lim_{z\to\infty} e^{(\gamma-1/\overline{z}-1)z} = \lim_{z\to\infty} e^{(\gamma-1/\overline{z})z} = 0$, which can only happen when $\gamma < \xi < \xi + 1$, where the second inequality holds automatically. Moreover, the equity premium is

$$\mu_t - r^s = \gamma\sigma_y^2 + \lambda\frac{1+\overline{z}+1-\overline{z}\gamma}{\left(1-\gamma\overline{z}\right)\left(1+\overline{z}\right)\left(1-\gamma\overline{z}+\overline{z}\right)}\gamma\overline{z}^2.$$

and the safe rate is

$$r^s = \rho + \theta g_y - \frac{\gamma\left(\theta+1\right)}{2}\sigma_y^2 - \lambda\overline{z}\left(\frac{\theta-\gamma}{1+\overline{z}-\overline{z}\gamma} + \frac{\gamma}{1-\overline{z}\gamma}\right).$$

# C Proof of Proposition 8

**Step 0: preliminaries.** First, the partial derivatives of function

$$f(c,V) = \frac{\left[(1-\gamma)V\right]^{\frac{\theta-\gamma}{1-\gamma}}}{1-\theta}\left[c_t^{1-\theta} - (\rho-n)\left((1-\gamma)V\right)^{\frac{1-\theta}{1-\gamma}}\right]$$

are

$$f_1(c,V) = c_t^{-\theta}\left[(1-\gamma)V_t\right]^{\frac{\theta-\gamma}{1-\gamma}},$$
$$f_2(c,V) = \frac{\theta-\gamma}{1-\gamma}\cdot\frac{f_t}{V_t} - \rho + n.$$

**Second**, we now evaluate the value of $V_t$ when the consumption follows a geometric Brownian motion process. Formally, we solve the follow system of equations

$$V_t = \mathbb{E}_t \left[ \int_t^\infty f(c_u, V_u) du \right],$$
$$f(c, V) = \frac{[(1-\gamma)V]^{\frac{\theta-\gamma}{1-\gamma}}}{1-\theta} \left[ c^{1-\theta} - (\rho - n) \left( (1-\gamma)V \right)^{\frac{1-\theta}{1-\gamma}} \right],$$
$$\frac{dc_t}{c_t} = g_y dt + \sigma_y dZ_t^y.$$

We guess the solution of the form

$$V_t = v c_t^{1-\gamma},$$

where $v$ is a positive constant. We plug this guess

$$\begin{aligned} f(c_t, V_t) &= \frac{\left[(1-\gamma) v c_t^{1-\gamma}\right]^{\frac{\theta-\gamma}{1-\gamma}}}{1-\theta} \left[ c_t^{1-\theta} - (\rho - n) \left( (1-\gamma) v c_t^{1-\gamma} \right)^{\frac{1-\theta}{1-\gamma}} \right] \\ &= \frac{[(1-\gamma)v]^{\frac{\theta-\gamma}{1-\gamma}}}{1-\theta} \left[ 1 - (\rho - n) \left( (1-\gamma) v \right)^{\frac{1-\theta}{1-\gamma}} \right] c_t^{1-\gamma}. \end{aligned}$$

As a result

$$\begin{aligned} V_t &= \mathbb{E}_t \left[ \int_t^T f(c_u, V_u) du + V_T \right] \\ &= \frac{[(1-\gamma)v]^{\frac{\theta-\gamma}{1-\gamma}}}{1-\theta} \left[ 1 - (\rho - n) \left( (1-\gamma) v \right)^{\frac{1-\theta}{1-\gamma}} \right] \mathbb{E}_t \int_t^T c_u^{1-\gamma} du + v \mathbb{E}_t c_T^{1-\gamma}. \end{aligned}$$

To compute the last expectations note that

$$d \log c_t = \frac{dc_t}{c_t} - \frac{1}{2} \left( \frac{dc_t}{c_t} \right)^2 = \left( g_y - \frac{\sigma_y^2}{2} \right) dt + \sigma_y dZ_t^y.$$

As a result,

$$
\begin{aligned}
d\log c_t^{1-\gamma} &= (1-\gamma)\left(g_y - \frac{\sigma_y^2}{2}\right)dt + (1-\gamma)\sigma_y dZ_t^y,\\
\log c_u^{1-\gamma} - \log c_t^{1-\gamma} &= (1-\gamma)\left(g_y - \frac{\sigma_y^2}{2}\right)(u-t) + (1-\gamma)\sigma_y Z_{u-t}^y,\\
c_u^{1-\gamma} &= c_t^{1-\gamma}\exp\left\{(1-\gamma)\left(g_y - \frac{\sigma_y^2}{2}\right)(u-t) + (1-\gamma)\sigma_y Z_{u-t}^y\right\},\\
\mathbb{E}_t c_u^{1-\gamma} &= c_t^{1-\gamma} e^{(1-\gamma)\left(g-\frac{\gamma\sigma_y^2}{2}\right)(u-t)},\\
\mathbb{E}_t\int_t^T c_u^{1-\gamma}du &= \int_t^T \mathbb{E}_t c_u^{1-\gamma}du\\
&= \frac{c_t^{1-\gamma}}{(1-\gamma)\left(g-\frac{\gamma\sigma_y^2}{2}\right)}\left[e^{(1-\gamma)\left(g_y-\frac{\gamma\sigma_y^2}{2}\right)(T-t)} - 1\right].
\end{aligned}
$$

This implies

$$
\begin{aligned}
V_t =& \frac{[(1-\gamma)v]^{\frac{\theta-\gamma}{1-\gamma}}}{1-\theta}\left[1-(\rho-n)\left((1-\gamma)v\right)^{\frac{1-\theta}{1-\gamma}}\right]\mathbb{E}_t\int_t^T c_u^{1-\gamma}du + v\mathbb{E}_t c_T^{1-\gamma}\\
=& e^{(1-\gamma)\left(g_y-\frac{\gamma\sigma_y^2}{2}\right)(T-t)} c_t^{1-\gamma}\left\{\frac{\frac{[(1-\gamma)v]^{\frac{\theta-\gamma}{1-\gamma}}}{1-\theta}\left[1-(\rho-n)\left((1-\gamma)v\right)^{\frac{1-\theta}{1-\gamma}}\right]}{(1-\gamma)\left(g_y-\frac{\gamma\sigma_y^2}{2}\right)} + v\right\}\\
&- \frac{[(1-\gamma)v]^{\frac{\theta-\gamma}{1-\gamma}}}{1-\theta}\left[1-(\rho-n)\left((1-\gamma)v\right)^{\frac{1-\theta}{1-\gamma}}\right]\frac{c_t^{1-\gamma}}{(1-\gamma)\left(g_y-\frac{\gamma\sigma_y^2}{2}\right)}.
\end{aligned}
$$

The term with $T-t$ must be equal to zero for the conjecture to be correct

$$
\begin{aligned}
\frac{[(1-\gamma)v]^{\frac{\theta-\gamma}{1-\gamma}}}{1-\theta} &= (\rho-n)\frac{(1-\gamma)v}{1-\theta} - v(1-\gamma)\left(g_y-\frac{\gamma\sigma_y^2}{2}\right),\\
v &= \frac{1}{1-\gamma}\left[\rho-n-(1-\theta)g_y+(1-\theta)\frac{\gamma\sigma_y^2}{2}\right]^{-\frac{1-\gamma}{1-\theta}}.
\end{aligned}
$$

As a result, we obtain

$$
V_t = \frac{c_t^{1-\gamma}}{1-\gamma}\left[\rho-n-(1-\theta)\left(g_y-\frac{\gamma\sigma_y^2}{2}\right)\right]^{-\frac{1-\gamma}{1-\theta}}.
$$

**Third**, we will later show that the discount factor in this economy is given by

$$
\frac{\xi_s}{\xi_t} = e^{-n(s-t)}e^{\int_0^s f_2(c_\tau,V_\tau)d\tau}f_1(c_s,V_s).
$$

We next compute the equilibrium stochastic discount factor multiplied by population increase and show that it equals $\pi_{t,s}$.

$$
\frac{\xi_s}{\xi_0}e^{ns} = e^{\int_0^s f_2(c_\tau,V_\tau)d\tau}f_1(c_s,V_s) = e^{\int_0^s\left[\frac{\theta-\gamma}{1-\gamma}\cdot\frac{f_t}{V_t}-\rho+n\right]d\tau}c_s^{-\theta}\left[(1-\gamma)V_s\right]^{\frac{\theta-\gamma}{1-\gamma}}
$$

Consider the argument of the exponent first

$$e^{\int_0^s f_2(c_\tau,V_\tau)d\tau} = e^{\int_0^s \left[\frac{\theta-\gamma}{1-\gamma}\cdot\frac{1}{1-\theta}[(1-\gamma)V_u]^{-\frac{1-\theta}{1-\gamma}}c_u^{1-\theta}-(\rho-n)\frac{1-\gamma}{1-\theta}\right]du}$$
$$= e^{\int_0^s \left[\frac{\theta-\gamma}{1-\gamma}\cdot\frac{1}{1-\theta}\left[(1-\gamma)vc_u^{1-\gamma}\right]^{-\frac{1-\theta}{1-\gamma}}c_u^{1-\theta}-(\rho-n)\frac{1-\gamma}{1-\theta}\right]du}$$
$$= e^{\left\{\frac{\theta-\gamma}{1-\gamma}\left[\rho-n-(1-\theta)\left(g_y-\frac{\gamma\sigma_y^2}{2}\right)\right]-(\rho-n)(1-\gamma)\right\}\frac{s}{1-\theta}}.$$

As a result,

$$\frac{\xi_s}{\xi_0}e^{ns} = e^{\left\{\frac{\theta-\gamma}{1-\gamma}\left[\rho-n-(1-\theta)\left(g_y-\frac{\gamma\sigma_y^2}{2}\right)\right]-(\rho-n)(1-\gamma)\right\}\frac{s}{1-\theta}}c_s^{-\theta}\left[(1-\gamma)vc_s^{1-\gamma}\right]^{\frac{\theta-\gamma}{1-\gamma}}$$
$$= e^{\left\{\frac{\theta-\gamma}{1-\gamma}\left[\rho-n-(1-\theta)\left(g_y-\frac{\gamma\sigma_y^2}{2}\right)\right]-(\rho-n)(1-\gamma)\right\}\frac{s}{1-\theta}}c_s^{-\gamma}\left[\rho-n-(1-\theta)\left(g_y-\frac{\gamma\sigma_y^2}{2}\right)\right]^{-\frac{\theta-\gamma}{1-\theta}}.$$

The last expression equals $\pi_{0,t}$ introduced in the text.

Note that in the case of the CRRA utility, $\pi_s$ has the following familiar look

$$\pi_{0,s} = e^{-(\rho-n)s}c_s^{-\gamma}.$$

**Step 1: Intertemporal Budget Constraint.**

$$\max_{\{c_t,w_t,,x_t,b_t,s_t\}} W_0,$$
$$s.t.: dw_t = (r_t^s s_t + r_t b_t - c_t - T_t - nw_t)dt + w_t x_t dr_t^x,$$
$$s_t + b_t + x_t w_t = w_t,$$

Rewrite the problem by substituting out $dr_t^x$ and $s_t$ as follows

$$\max_{c_t,w_t,\phi_t,x_t,b_t} W_0,$$
$$s.t.: \frac{d\left(e^{nt}w_t\right)+e^{nt}\left[c_t+T_t+\left(r_t^s-r_t\right)b_t\right]dt}{e^{nt}w_t} = \left[r_t^s + x_t(\mu_t - r_t^s)\right]dt + x_t\sigma_t dZ_t^y + \phi_t\sigma_t^\phi dZ_t^B,$$

Let the discount factor be $\xi_t$, which exists and is unique under the complete markets assumption, and must satisfy

$$\frac{d\xi_t}{\xi_t} = -r_t^s dt - \kappa_t^x dZ_t^y. \tag{C.1}$$

where $\kappa_t \equiv (\mu_t - r_t^s)/\sigma_t$. Note that $\xi_t$ is the per member of the household discount factor. Under such interpretation, optimally invested wealth must satisfy

$$w_t = \mathbb{E}_t\int_t^\infty [c_s + T_s + (r_s^s - r_s^B)b_s]e^{n(s-t)}\frac{\xi_s}{\xi_t}ds.$$

As a result, the household problem is

$$\max_{c_t,b_t} W_0(a_0;\widehat{B}_0),$$
$$s.t.:\, w_0 = \mathbb{E}_0\int_0^\infty [c_t + T_t + (r_t^s - r_t)b_t]\,\frac{e^{nt}\xi_t}{\xi_0}dt,$$

where we omitted $x_t$ and $w_t$ from maximization arguments because we assume that the wealth is optimally allocated across assets safe and risky assets. The Lagrangian of this problem is

$$\mathcal{L}_0 = W_0 - \kappa\left[\mathbb{E}_0\int_0^\infty [c_t + T_t + (r_t^s - r_t)b_t]\,\frac{e^{nt}\xi_t}{\xi_0}dt - w_0\right].$$

Note that $\mathcal{L}_0$ is a functional such that $\mathcal{L}_0 : \mathbb{L}\times\mathbb{L}\to\mathbb{R}$, where $\mathbb{L}$ is a space of square integrable progressively measurable processes with values in $\mathbb{R}$.

**Step 2: First Order Conditions.** The first order conditions for this optimization take the following form, where we use notation of Duffie and Skiadas (1994),

$$\nabla\mathcal{L}_0(c,\widetilde{c}) = 0,\ \forall\widetilde{c},$$
$$\nabla\mathcal{L}_0(b,\widetilde{b}) = 0,\ \forall\widetilde{b}.$$

The last two equations state that *the Gateaux derivative* of the Lagrangian with respect to consumption and bond holdings processes are zeros in any direction $\widetilde{c}$ (in case of consumption) and $\widetilde{b}$ (in case of liquid bonds). We next compute these derivatives explicitly. We start with $\nabla V_0(c,\widetilde{c})$ and $\nabla V_0(b,\widetilde{b})$.

$$\nabla W_0(c,\widetilde{c}) = \mathbb{E}_0\int_0^\infty e^{\int_0^s f_2(c_\tau,V_\tau)d\tau} f_1(c_s,V_s)\widetilde{c}_s ds,$$
$$\nabla W_0(b,\widetilde{b}) = \mathbb{E}_t\int_t^\infty (1-\theta)y_s^{1-\theta}\frac{\widetilde{b}_s}{y_s}u'\left(\frac{b_s}{y_s}\right)ds,$$

As a result, the derivative of the Lagrangian with respect to consumption process is

$$\begin{aligned}
0 &= \nabla\mathcal{L}_0(c,\widetilde{c})\\
&= \mathbb{E}_0\int_0^\infty e^{\int_0^s f_2(c_\tau,V_\tau)d\tau} f_1(c_s,V_s)\widetilde{c}_s ds - \kappa\mathbb{E}_0\int_0^\infty \widetilde{c}_t\frac{e^{nt}\xi_t}{\xi_0}dt\\
&= \mathbb{E}_0\int_0^\infty \left(e^{\int_0^s f_2(c_\tau,V_\tau)d\tau} f_1(c_s,V_s) - \frac{\kappa e^{ns}\xi_s}{\xi_0}\right)\widetilde{c}_s ds.
\end{aligned}$$

Because, the last equation has to hold for any $\widetilde{c}$, we must have that

$$e^{\int_0^s f_2(c_\tau,V_\tau)d\tau} f_1(c_s,V_s) = \frac{\kappa e^{ns}\xi_s}{\xi_0}.$$

Taking the ratio of this equation at times $t$ and $s$ and using explicit expression for partial derivative $f_1$, we obtain

$$e^{\int_s^t f_2(c_\tau,V_\tau)d\tau}\left(\frac{c_t}{c_s}\right)^{-\theta}\left(\frac{V_t}{V_s}\right)^{\frac{\theta-\gamma}{1-\gamma}} = e^{n(t-s)}\frac{\xi_t}{\xi_s}. \tag{C.2}$$

Analogously, the optimality wrt to liquid debt is

$$\pi_{t,s} u'\left(\frac{b_s}{y_s}\right) = \left(r_s^s - r_s^b\right) \frac{\kappa e^{ns} \xi_s}{\xi_0}. \tag{C.3}$$

Diving the last two equations, we obtain

$$r_s^b = r_s^s - \frac{\pi_{t,s} u'\left(\frac{b_s}{y_s}\right)}{e^{\int_t^s f_2(c_\tau, V_\tau) d\tau} f_1(c_s, V_s)}.$$

In equilibrium, we have

$$\begin{aligned} r_s^b &= r_s^s - \frac{\pi_{t,s} u'\left(\frac{b_s}{y_s}\right)}{e^{\int_t^s f_2(c_\tau, V_\tau) d\tau} f_1(c_s, V_s)} \\ &= r_t^s - u'\left(\frac{b_t}{y_t}\right) \end{aligned} \tag{C.4}$$

Finally, when households optimize, it must be true that

$$V_a = f_1. \tag{C.5}$$

**Step 3: Stochastic Discount Factor.** First, we want to compute the law of motion of $\xi_t$ (here we assume $\kappa \equiv \kappa/\xi_0$). We will apply the Ito's lemma to the FOC wrt to $c$. To do it, we separately compute several stochastic differentials

$$\begin{aligned} df_1(c_t, V_t) =& d\left\{\omega c_t^{-\theta} \left[(1-\gamma)V_t\right]^{\frac{\theta-\gamma}{1-\gamma}}\right\} \\ =& \omega\left(\left[(1-\gamma)V_t\right]^{\frac{\theta-\gamma}{1-\gamma}} d\left\{c_t^{-\theta}\right\} + c_t^{-\theta} d\left\{\left[(1-\gamma)V_t\right]^{\frac{\theta-\gamma}{1-\gamma}}\right\} + d\left\{c_t^{-\theta}\right\} d\left[(1-\gamma)V_t\right]^{\frac{\theta-\gamma}{1-\gamma}}\right) \\ =& \omega\Bigg(\left[(1-\gamma)V_t\right]^{\frac{\theta-\gamma}{1-\gamma}} \left(-\theta c_t^{-\theta}\right)\left[\frac{dc_t}{c_t} - \frac{1+\theta}{2} \cdot \frac{dc_t^2}{c_t^2}\right] \\ &+ c_t^{-\theta}(\theta-\gamma)\left[(1-\gamma)V_t\right]^{\frac{\theta-\gamma}{1-\gamma}} \left[\frac{dV_t}{(1-\gamma)V_t} + \frac{1}{2}(\theta-1)\frac{dV_t^2}{\left[(1-\gamma)V_t\right]^2}\right] \\ &+ \left(-\theta c_t^{-\theta}\right)\left[\frac{dc_t}{c_t} - \frac{1+\theta}{2} \cdot \frac{dc_t^2}{c_t^2}\right](\theta-\gamma)\left[(1-\gamma)V_t\right]^{\frac{\theta-\gamma}{1-\gamma}} \left[\frac{dV_t}{(1-\gamma)V_t} + \frac{1}{2}(\theta-1)\frac{dV_t^2}{\left[(1-\gamma)V_t\right]^2}\right]\Bigg) \\ =& f_1(c_t, b_t, V_t)\Bigg(-\theta\left[\frac{dc_t}{c_t} - \frac{1+\theta}{2} \cdot \frac{dc_t^2}{c_t^2}\right] + \frac{\theta-\gamma}{1-\gamma}\left[\frac{d\left[(1-\gamma)V_t\right]}{(1-\gamma)V_t} + \frac{1}{2} \cdot \frac{\theta-1}{1-\gamma} \cdot \frac{\left(d\left[(1-\gamma)V_t\right]\right)^2}{\left[(1-\gamma)V_t\right]^2}\right] \\ &- \theta\frac{\theta-\gamma}{1-\gamma} \cdot \frac{dc_t}{c_t} \cdot \frac{d(1-\gamma)V_t}{(1-\gamma)V_t}\Bigg). \end{aligned}$$

Note that the preferences have the following differential representation

$$\frac{dV_t}{V_t} = -\frac{f(c_t, V_t)}{V_t} dt + \sigma_{V,y} dZ_t^y, \tag{C.6}$$

where $\sigma_V$ can be time varying. As a result, (also taking into account that $c_t = (1-\gamma_G)y_t$)

$$\begin{aligned}
\frac{df_1}{f_1} =& -\theta\left[\frac{dc_t}{c_t} - \frac{1+\theta}{2}\cdot\frac{dc_t^2}{c_t^2}\right] + \frac{\theta-\gamma}{1-\gamma}\left[\frac{d\left[(1-\gamma)V_t\right]}{(1-\gamma)V_t} + \frac{1}{2}\cdot\frac{\theta-1}{1-\gamma}\cdot\frac{\left(d\left[(1-\gamma)V_t\right]\right)^2}{\left[(1-\gamma)V_t\right]^2}\right] - \theta\frac{\theta-\gamma}{1-\gamma}\cdot\frac{dc_t}{c_t}\cdot\frac{d\left[(1-\gamma)V_t\right]}{(1-\gamma)V_t} \\
=& \left[-\theta\left(g_y - \frac{1+\theta}{2}\sigma_y^2 + \frac{\theta-\gamma}{1-\gamma}\sigma_y\sigma_{V,y}\right) + \frac{\theta-\gamma}{1-\gamma}\left(-\frac{f_t}{V_t} + \frac{\sigma_{V,y}^2}{2}\cdot\frac{\theta-1}{1-\gamma}\right)\right]dt + \left(\frac{\theta-\gamma}{1-\gamma}\sigma_{V,y} - \theta\sigma_y\right)dZ_t^y.
\end{aligned}$$

Next

$$\begin{aligned}
de^{\int_0^t f_2(c_\tau,V_\tau)d\tau} &= e^{\int_0^t f_3(c_\tau,V_\tau)d\tau} d\int_0^t f_2(c_\tau,V_\tau)d\tau + \frac{1}{2}e^{\int_0^t f_2(c_\tau,V_\tau)d\tau}\left[d\int_0^t f_2(c_\tau,V_\tau)d\tau\right]^2 \\
&= e^{\int_0^t f_2(c_\tau,V_\tau)d\tau} f_2(c_t,V_t)dt.
\end{aligned}$$

Note that the last expression implies that $de^{\int_0^t f_2(c_\tau,V_\tau)d\tau}df_1(c_t,V_t) = 0$. As a result,

$$\begin{aligned}
e^{nt}\kappa d\xi_t + \kappa\xi_t e^{nt} n dt &= de^{\int_0^t f_2(c_\tau,V_\tau)d\tau} f_1(c_t,V_t) + e^{\int_0^t f_2(c_\tau,V_\tau)d\tau} df_1(c_t,V_t) + de^{\int_0^t f_2(c_\tau,V_\tau)d\tau} df_1(c_t,V_t) \\
&= \kappa\xi_t e^{nt}\left[f_2(c_t,V_t)dt + \frac{df_1(c_t,V_t)}{f_1(c_t,V_t)}\right].
\end{aligned}$$

Collecting previous results, we obtain

$$\begin{aligned}
\frac{d\xi_t}{\xi_t} =& -ndt + f_1(c_t,V_t)dt + \frac{df_1(c_t,V_t)}{f_1(c_t,V_t)} \\
=& -\left[\rho + \theta g_y - \theta\frac{1+\theta}{2}\sigma_y^2 + \frac{\theta-\gamma}{1-\gamma}\left(\theta\sigma_y\sigma_{V,y} - \frac{\theta-1}{1-\gamma}\cdot\frac{\sigma_{V,y}^2 + \sigma_{V,B}^2}{2}\right)\right]dt \\
& -\left(\theta\sigma_y - \frac{\theta-\gamma}{1-\gamma}\sigma_{V,y}\right)dZ_t^y. \qquad \text{(C.7)}
\end{aligned}$$

**Step 4: Riskless Rate.** No arbitrage implies that the price $p_t$ of any security that pays dividends $d_s$ to its holder equals

$$p_t = \frac{1}{\xi_t}\mathbb{E}_t\int_t^\infty \xi_s d_s ds. \qquad \text{(C.8)}$$

The differential version of this equation is

$$0 = \xi_t d_t dt + \mathbb{E}_t\left[d(\xi_t p_t)\right]. \qquad \text{(C.9)}$$

The safe bond is a security with the price of 1 and the dividend $r_t^s$. As a result,

$$\begin{aligned}
&0 = \xi_t r_t^s dt + \mathbb{E}_t d\xi_t, \\
&r_t^s = -\frac{1}{dt}\mathbb{E}_t\left(\frac{d\xi_t}{\xi_t}\right) = \rho + \theta g_y - \frac{\theta(\theta+1)}{2}\sigma_y^2 + \frac{\theta-\gamma}{1-\gamma}\left(\theta\sigma_y\sigma_{V,y} - \frac{\theta-1}{1-\gamma}\cdot\frac{\sigma_{V,y}^2+\sigma_{V,B}^2}{2}\right).
\end{aligned}$$

First, by guessing a verifying that the value function in equilibrium is a power function of total wealth, we show that

$$\sigma_{V,y} = \frac{1-\gamma}{\gamma}\kappa_t. \qquad \text{(C.10)}$$

Second, we have two expressions for $d\xi_t$ in equations (C.1) and (C.7)

$$-\kappa_t dZ_t^y = -\left(\theta\sigma_y - \frac{\theta-\gamma}{1-\gamma}\sigma_{V,y}\right) dZ_t^y,$$

Again, because the last expression has to hold for all realizations of shocks, we obtain

$$\kappa_t = \theta\sigma_y - \frac{\theta-\gamma}{1-\gamma}\sigma_{V,y}, \tag{C.11}$$

Equations (C.10) and (C.11) lead to

$$\sigma_{V,y} = (1-\gamma)\sigma_y.$$

As a result, the riskless rate is

$$\begin{aligned} r_t^s &= \rho + \theta g_y - \frac{\theta(\theta+1)}{2}\sigma_y^2 + \frac{\theta-\gamma}{1-\gamma}\left(\theta\frac{\sigma_y}{\sigma_{V,y}} - \frac{\theta-1}{2(1-\gamma)}\right)\sigma_{V,y}^2 \\ &= \rho + \theta g_y - \gamma\frac{(\theta+1)}{2}\sigma_y^2. \end{aligned}$$

**Step 5: the risky asset price.**

$$\begin{aligned} q_t =& \mathbb{E}_t \int_t^\infty \frac{\xi_s}{\xi_t} y_s ds \\ =& \mathbb{E}_t \int_t^\infty \frac{\overline{\pi}_s e^{-ns}}{\overline{\pi}_t e^{-nt}} y_s ds \\ =& c_t^\gamma \mathbb{E}_t \int_t^\infty e^{\left\{\frac{\theta-\gamma}{1-\gamma}\left[\rho-n-(1-\theta)\left(g_y-\frac{\gamma\sigma_y^2}{2}\right)\right]-(\rho-n)(1-\gamma)\right\}\frac{s-t}{1-\theta}} c_s^{-\gamma}\left[\rho-n-(1-\theta)\left(g_y-\frac{\gamma\sigma_y^2}{2}\right)\right]^{-\frac{\theta-\gamma}{1-\theta}} e^{-n(s-t)} y_s ds \\ =& y_t\left[\rho-n-(1-\theta)\left(g_y-\frac{\gamma\sigma_y^2}{2}\right)\right]^{-\frac{\theta-\gamma}{1-\theta}} \\ & \cdot \int_t^\infty e^{\left\{\frac{\theta-\gamma}{(1-\theta)(1-\gamma)}\left[\rho-n-(1-\theta)\left(g_y-\frac{\gamma\sigma_y^2}{2}\right)\right]-\frac{(\rho-n)(1-\gamma)}{1-\theta}-n+(1-\gamma)\left(g_y-\frac{\gamma\sigma_y^2}{2}\right)\right\}(s-t)} ds. \end{aligned}$$

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# Main Text Tables and Figures

| | 17 Advanced Countries | | | United States | | |
|---|---|---|---|---|---|---|
| | Median | 25th perc. | 75th perc. | Median | 25th perc. | 75th perc. |
| Long-term nominal interest rate | 4.61 | 3.62 | 6.38 | 3.92 | 3.32 | 5.48 |
| Inflation rate | 2.14 | 0.11 | 4.39 | 1.75 | 0.00 | 3.51 |
| Real interest rate | 2.71 | 1.17 | 4.82 | 2.66 | 1.52 | 4.29 |
| Real GDP per capita growth | 2.01 | 0.28 | 3.82 | 1.89 | -0.45 | 3.75 |
| Population growth | 0.80 | 0.44 | 1.17 | 1.39 | 0.97 | 1.91 |
| Debt to GDP ratio | 44.2 | 24.3 | 68.6 | 36.4 | 15.1 | 59.0 |
| No. of observations | 2145 | | | 134 | | |

Real interest rate is the long-term nominal interest rate less a three-year moving average of inflation rates. All variables expressed as percentage points. Statistics based on data set after observations with fiscal cost more than 10 percent or less than -10 percent are dropped.

**Table 1:** Moments of macro variables

| | 17 Advanced Countries | | | United States | | |
|---|---|---|---|---|---|---|
| | 1870-2016 | 1870-1914, 1946-2016 | 1946-2016 | 1870-2016 | 1870-1914, 1946-2016 | 1946-2016 |
| r - g (percent) | | | | | | |
| 25th percentile | -2.7 | -1.8 | -3.0 | -2.2 | -2.2 | -2.2 |
| Median | 0.1 | 0.2 | -0.8 | -0.3 | -0.4 | -1.0 |
| 75th percentile | 2.5 | 2.4 | 1.4 | 1.7 | 1.7 | 0.9 |
| Fraction < 0 | 49 | 47 | 58 | 52 | 55 | 71 |
| Fraction < -2% | 30 | 24 | 34 | 31 | 27 | 36 |
| No. of observations | 491 | 373 | 238 | 29 | 22 | 14 |

Real interest rate is the long-term nominal interest rate less a three-year moving average of inflation rates. "Fraction < 0" is the fraction of years experssed in percent with negative debt servicing cost. "Fraction < -2%" is the fraction of years with the debt servicing cost of less than negative two percent. Statistics are based on the dataset after observations with the fiscal cost more than 10 percent or less than -10 percent are winsorized at thresholds.

**Table 2:** Moments of the debt servicing cost.

| Parameter | Description | Value | Source/target |
|---|---|---|---|
| $g_y$ | Productivity growth rate | 0.02 | US data |
| $\sigma_y$ | Output per capita std | 0.025 | US data |
| $n$ | Population growth rate | 0.0115 | US data |
| $1/\theta$ | Intertemporal elasticity of substitution | 0.75 | Standard in macro lit. |
| $\alpha_u$ | Liquidity yield independent of debt | 0.0052 | AAA bonds & gov. yield |
| $\beta_u$ | Semi elasticity of liquidity yield | 0.0028 | AAA bonds & gov. yield |
| $\sigma_B$ | Public bonds growth rate shocks std | 0.45 | $cor(dy_t/y_t, r_t) = -0.056$ |

**Table 3:** Parameters common across the three calibrations.

| Parameter | Description | Cal. 1 | Cal. 2 | Cal. 3 | Source/target |
|---|---|---|---|---|---|
| $\gamma$ | CRRA | 83 | 83 | 3.47 | Equity premium |
| $\rho$ | Subjective disc. factor | 0.07 | 0.07 | 0.06 | Mean 10-year Treasury yield |
| $\beta_d$ | Fiscal rule parameter | 0.16 | $-\beta_u$ | $-\beta_u$ | Debt-to-GDP distribution |
| $\alpha_d$ | Fiscal rule parameter | -0.04 | -0.02 | -0.03 | Debt-to-GDP distribution |
| $\min(B_t/Y_t)$ | Reflecting boundary | - | 0.18 | 0.18 | Debt-to-GDP distribution |
| $\lambda$ | Disaster's arrival rate | - | - | 0.02 | Barro (2006) |
| $\overline{z}$ | Mean log disaster size | - | - | 0.23 | Barro (2006) |

**Table 4:** Calibration-specific parameters. Columns Cal. 1, Cal. 2, and Cal. 3 refer to our calibrations 1-3.

| Variable | Description | Baseline | **Ex. 1**: $n\downarrow$ | **Ex. 2**: $g\downarrow$ | **Ex. 3**: $\mathbb{E}[\mu_t]-r^s\uparrow$ |
|---|---|---|---|---|---|
| | | **Panel A: calibration 1** | | | |
| $\mathbb{E}[B_t/Y_t]$ | Mean of debt-to-GDP | 0.55 | 0.57 | 0.54 | 0.48 |
| $std(B_t/Y_t)$ | Std of debt-to-GDP | 0.53 | 0.54 | 0.51 | 0.45 |
| $r^s$ | Safe rate | 0.032 | 0.032 | 0.014 | 0.009 |
| $\mathbb{E}[r_t]$ | Average liquid rate | 0.024 | 0.025 | 0.006 | 0.001 |
| $\mathbb{E}[\mu_t]-r^s$ | Equity risk premium | 0.052 | 0.052 | 0.052 | 0.072 |
| | | **Panel B: calibration 2** | | | |
| $\mathbb{E}[B_t/Y_t]$ | Mean of debt-to-GDP | 0.92 | 1.08 | 0.80 | 0.56 |
| $std(B_t/Y_t)$ | Std of debt-to-GDP | - | - | - | - |
| $r^s$ | Safe rate | 0.032 | 0.032 | 0.014 | 0.009 |
| $\mathbb{E}[r_t]$ | Average liquid rate | 0.024 | 0.025 | 0.006 | 0.001 |
| $\mathbb{E}[\mu_t]-r^s$ | Equity risk premium | 0.052 | 0.052 | 0.052 | 0.072 |
| | | **Panel C: calibration 3** | | | |
| $\mathbb{E}[B_t/Y_t]$ | Mean of debt-to-GDP | 0.92 | 1.08 | 0.80 | 0.59 |
| $std(B_t/Y_t)$ | Std of debt-to-GDP | - | - | - | - |
| $r^s$ | Safe rate | 0.032 | 0.032 | 0.014 | 0.009 |
| $\mathbb{E}[r_t]$ | Average liquid rate | 0.024 | 0.025 | 0.006 | 0.001 |
| $\mathbb{E}[\mu_t]-r^s$ | Equity risk premium | 0.052 | 0.052 | 0.052 | 0.072 |

**Table 5: Moments generated under Calibrations 1-3**. Column "Baseline" shows the moments under the baseline version of a calibration, while columns "Ex. 1," "Ex. 2," and "Ex. 3" shows the moments for the three experiments that correspond to a decline in the population growth rate (Ex. 1), a decline in productivity growth rate (Ex. 2), and an increase in equity premium (Ex. 3).

| Variable | **Baseline** | **Ex. 1**: $n\uparrow$ | **Ex. 2**: $g\uparrow$ | **Ex. 3**: $\mathbb{E}[\mu_t]-r^s\downarrow$ |
|---|---|---|---|---|
| | | **Panel A:** $\bar{s}=0.05$ | | |
| $\mathcal{B}_{FP}$ | 1.06 | 1.59 | 1.01 | 1.04 |
| $\mathcal{B}_{FL}^{r}$ | 1.44 | 1.98 | 1.39 | 1.45 |
| $\mathcal{B}_{FL}^{nr}$ | 2.22 | 3.12 | 2.14 | 2.22 |
| | | **Panel B:** $\bar{s}=0.1$ | | |
| $\mathcal{B}_{FP}$ | 1.18 | 1.59 | 1.13 | 1.14 |
| $\mathcal{B}_{FL}^{r}$ | 1.95 | 2.36 | 1.91 | 1.96 |
| $\mathcal{B}_{FL}^{nr}$ | 2.90 | 3.58 | 2.83 | 2.90 |

**Table 6: Thresholds**. Column "Baseline" shows the cutoffs under the baseline version of a calibration, while columns "Ex. 1," "Ex. 2," and "Ex. 3" show the cutoffs for the three experiments that correspond to an increase in the population growth rate (Ex. 1), an increase in the productivity growth rate (Ex. 2), and a decline in equity premium through lower probability of disasters $\lambda$ (Ex. 3). The values of subjective discount factor $\rho$ are 0.022 in Panel A, and 0.034 in Panel B. The variables $\mathcal{B}_{FP}$, $\mathcal{B}_{FL}^{r}$, and $\mathcal{B}_{FL}^{nr}$ denote the flipping point, the fiscal limits with and without risk, respectively.

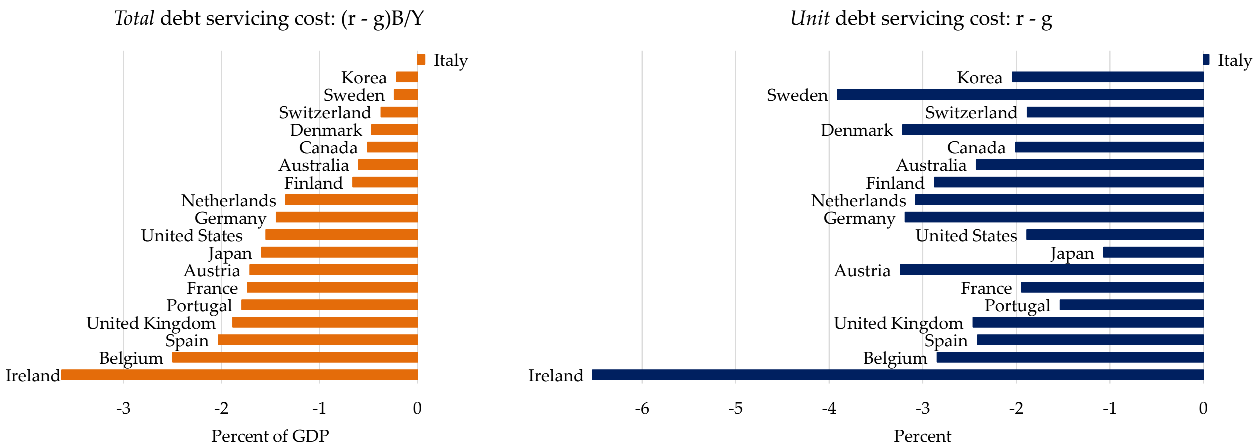


**Figure 1:** The left panel shows the total public debt servicing costs, while the right panel presents the unit costs. Both panels present average values over 2016-19 period.

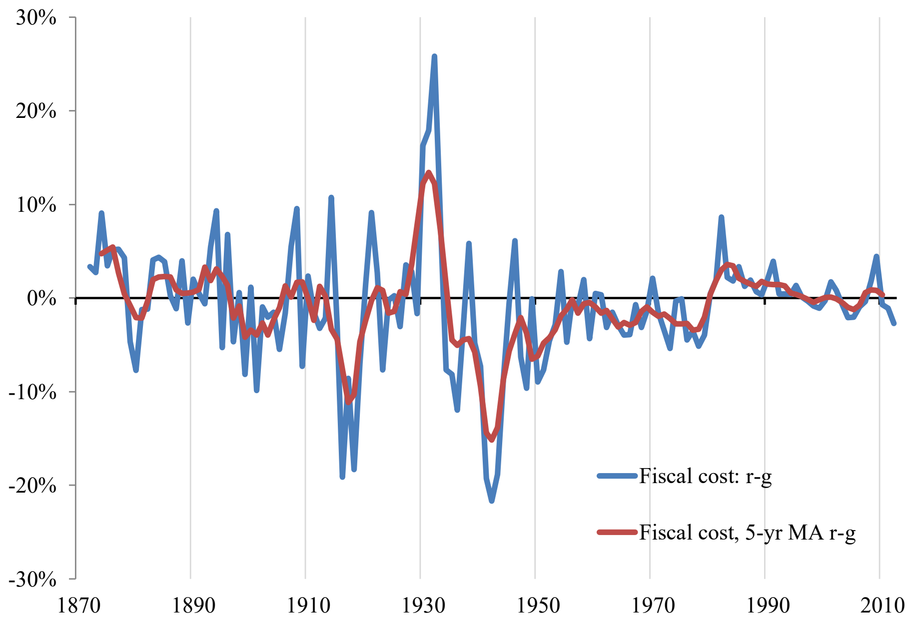


**Figure 2:** US (unit) debt servicing cost $r - g$.

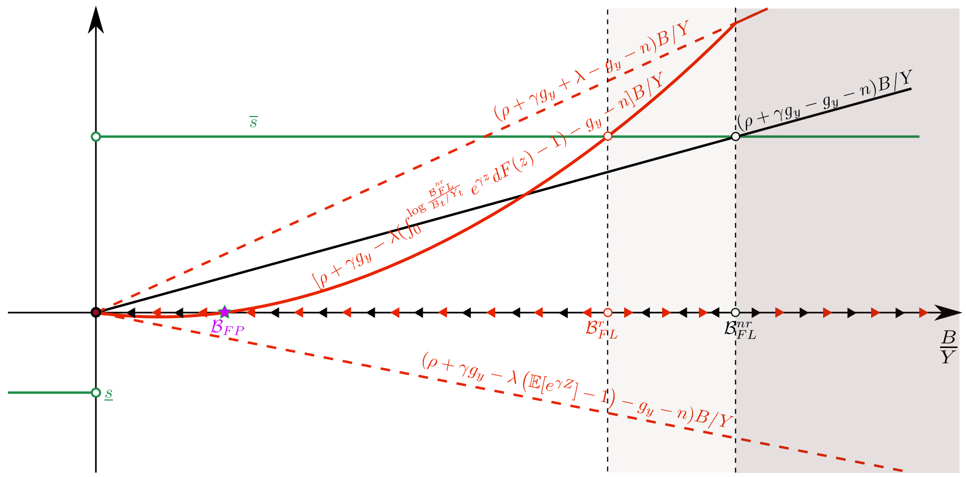


**Figure 3:** Debt-to-GDP dynamics before (red arrows) and after (black arrows) resolution of uncertainty. The shaded areas represent the regions where government defaults absent uncertainty (darker gray region) and with uncertainty (lighter gray region).

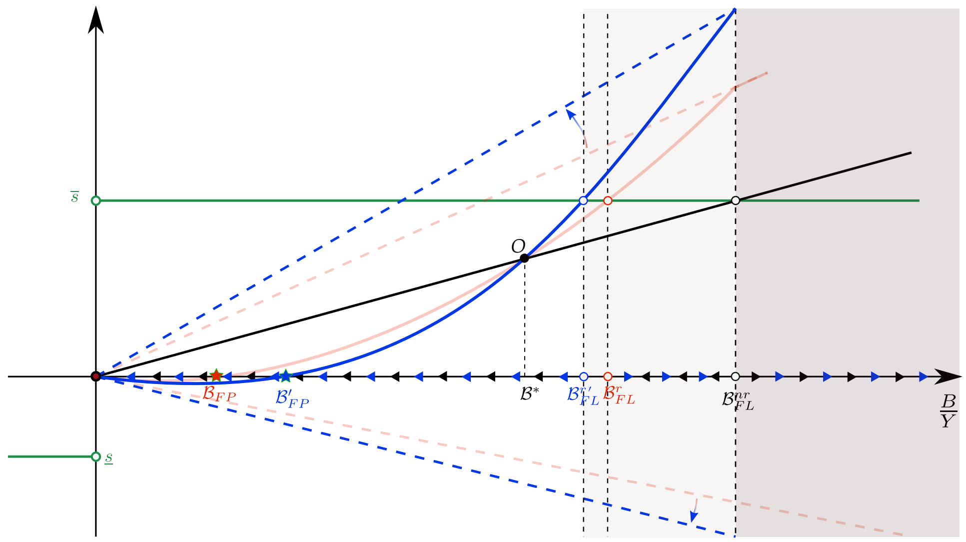


**Figure 4:** Debt-to-GDP dynamics before the resolution of uncertainty under different arrival rate of disasters $\lambda$. The light red solid and dashed lines correspond to a low value of $\lambda$, while the blue solid and dashed lines correspond to high levels of $\lambda$. To avoid cluttering the diagram, we omitted explicit labels that are similar to those in Figure 3.