A Modern Perspective on the Classical Approach to the Lender of Last Resort

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Abstract

The classical lender of last resort approach (Thornton 1802 and Bagehot 1873) emphasizes that the lender supports financial markets and not individuals; behave sconsistently with longer run (inflation) objectives; and lends freely "to this man and that man" on good collateral at a high rate. Importantly, the classical approach stresses that a lender of last resort is a monetary—not a credit—operation, where the objective is to get cash into the hands of people that need it. These days the classical prescriptions are seen by some as being outdated and anachronistic—perhaps being relevant back in the 1800's but not in today's complex, modern financial economy. Instead, the lender of last resort should use its balance sheet to pursue credit/interest rate policies that directly affect long term assets and/or rescue large, interconnected and insolvent institutions, things that the classical writers were fearsely opposed to. We use a standard, dynamic monetary model to assess the classical approach and find that "lending freely at a high rate" on good collateral enhances social welfare when the economy is hit by servere liquidity shocks. In fact, the classical approach can be seen in some of the policies pursued by, e.g., the Federal Reserve, in response to aggregate liquidity shortages,

1 Introduction

The architects of the classical approach to the lender of last resort (Thornton 1802 and Bagehot 1873) advocated policies that mitigated losses in economic activity brought about by a significant and unexpected loss in the economy's means of payment. When the economy is hit by a signficant adverse shock that results in a loss of monetary purchasing power, the classical approach emphasized the importance of getting money into the hands of people that spend it—so economic activity does not become severely depressed. As a result they prescribed that the lender of last resort should lend freely "to this man and that man" on good collateral at a high rate. Among other things, the architects did *not* view their policy recommendations: (i) as a bailout to a failing institution/individual;[1] (ii) as some sort of credit or interest rate policy; or (iii) at variance with other objectives of the monetary authority, such as price stability (or low inflation). Bagehot (1873) recommended that the policies of the lender of last resort be widely advertised, known to all. Furthermore, it was emphasized that the lender of last resort should not be viewed as a source of everyday, on-going liquidity needs but rather to be tapped on those occasions when the markets for money are significantly stressed or panicked.

In modern times, the classical prescriptions have been viewed as being outdated and anarchronistic (Freixas, Parigi and Rochet 2004), irrelevant (Goodfriend and King 1988) or completely misinterpreted (Humphrey 2010). Frexias, Parigi and Rochet (2004) believe that asymmetric information and moral hazard, which are prevalent in banking relationships, undermine the classical lender of last resort theory. They and others[2] take a more expansive view of the role of a lender of last resort: one that directly engages in credit policy or undertakes risky asset exchanges for the sake of, e.g., saving large, interconnected financial institutions. Mishkin and White (2014) provide a number of examples spanning over 150 years and multiple jurisdictions where central banks have, in fact, behaved in such an expansive manner. Goodfriend and King (1986) conclude that central banks' current focus on targeting short-term, overnight interest rates—e.g., the federal funds rate in the US—eliminates any distinction between monetary policy—i.e., low inflation—and lender of last resort. And Humphrey (2010), after examining Federal Reserve policy in the aftermath of the Great Financial Crisis, claims that "[t]he Fed has deviated from the classical model in so many ways as to make a mockery of the notion that it

[1] They explicitly recommended against support to insolvent institutions.

[2] Freixas, Giannini, Hoggarth and Soussa (2000), Freixas, Parigi and Rochet (2000), Kahn and Santos (2001), Repullo (2000), Choi, Santos and Torulmazer (2017) among others

is a L[ender] O[f] L[ast] R[esort].

We agree that some aspects of past central banks' policies resemble fiscal policy and are stark departures from classical lender of last resort policies. But we do not think that classical lender of last resort policies are necessarily ineffective. Nor do we think that a central bank's policy of targetting an overnight interest rate is a reason to conclude that the distinction between monetary and lender last resort policies has vanished.[3] We rather think that the classical approach to the Lender of Last Resort is actually very insightful and deserves a rigourous investigation. As a matter of fact, the classical approach can be seen in some of the policies pursued by, e.g., the Federal Reserve, in response to aggregate liquidity shortages.

In this paper we propose a framework to assess the classical approach and analyze whether lending freely at a high rate" on good collateral enhances social welfare when the economy is hit by servere liquidity shocks. Since a flight to liquidity in a financial crisis is really a sudden and massive increase in the demand for money, we think that these questions can be satisfactorily answered only by studying an environment in which liquidity is cash - with its unique role as a medium of exchange - and not some generic asset called money for convenience. Only in a truly monetary model, in our view, it is possible to fully understand the effects of this type of intervention by the central bank.

The model we propose has two types of investors with different financial opportunities. Those that we call *credit investors* can be monitored and can therefore finance investment using unsecured credit. Those that we call *cash investors*, instead, are anonymous and can only acquire investment goods with money. Investors learn their type at the beginning of each period after they made their portfolio decisions. If they find out that they are cash investors they tipically will need some extra cash that they can obtain by selling illiquid bonds in a competitive financial market. Credit investors, instead, do not need the cash to buy investment goods and prefer to exchange their cash for bonds that give a higher return. What makes the model interesting is that the number of credit investors relative to cash investors is a random variable; in some states of the world there may be a large number of cash investors who are willing to sell assets to obtain liquidity and a small number of credit investors that are willing to buy assets and provide liquidity. These states of the world are situations of extreme stress in the market, resembling the fire sales that are often observed in periods of crisis.

[3] If anything, targetting an overnight rate (range) for every conceivable liquidity shock might be interpreted as a rather misguided lender of last resort policy. i.e., a lender of last resort policy that lends freely but *not* at a high rate.

In our model the central bank provides liquidity through a repo standing facility. We focus on this particular type of intervention, but the model is abstract enough to provide insights on the variety of collateralized loan facilities set up by central banks during the recent crises.

We start with a case in which the aggregate shock, i.e. the fraction of credit investors in the market, takes only two values. In the first state there are many credit investors and relative fewer cash investors and cash investors sell all their government bonds in the competitive financial market while credit investors do not exchange all of their real balances. In the other state there are few credit investors relative to cash investors and cash investors do not sell all of their government bonds in the competitive financial market. Credit investors exchange all of their real balances, so that cash investors find themselves in the need for more liquidity but cannot obtain it in the market. We find that the standing facility improves on the insurance against the risk of illiquidity provided by the existence of a financial market. However, by providing liquidity insurance, it induces investors to hold more illiquid assets, discouraging the holding of cash.[4] In states in which liquidity is abundant, this has negative effects on welfare since it limits the ability of cash constrained firms to acquire capital. On the other side, when credit conditions deteriorates and liquidity is scarce, central bank liquidity injections allow firms to relax their cash constraint and lower interest rates. We show however that a repo standing facility has positive welfare effects if the probability that liquidity is scarce in some states is sufficiently high.

We also analyze our model under the assumption that the aggregate shock has a continuous distribution. In this case we can identify three possible regimes that we call "abundant" liquidity, "sufficient" liquidity and "scarce" liquidity. This more general model confirms the results discussed above but, interestingly, we find that the standing facility should be set up in a way that intervention occurs only in situations of severe distress. This supports the view that, from a welfare point of view, the market should provide most of the necessary liquidity and that intervention must be limited to situations in which credit conditions deteriorate significantly.

We then use our general model also to evaluate a recent policy innovation by the Fed that, in 2021, has set up a permanent repo facility. The establishment of this instrument has been preceded by an intense debate, inside the Fed, over the relative merits of temporary facilities - like the ones adopted during the Covid crisis and the Great Financial Crisis - versus permanent

[4] Indeed, as argued by Berentsen et al. (2014) the existence itself of a financial market may induce agents to hold insufficient cash.

facilities. A temporary facility has the advantage of limiting the decrease of liquidity in liquidity abundant states, but is less effective in states of scarce liquidity. The opposite is true for a permanent facility. While a priori it is difficult to decide which one of the two facilities is better from a welfare point of view, our calibration points decisively toward the adoption of a permanent facility.

Our model is also able to account for two important phenonema. The first one is the impact of monetary intervention on asset prices, a fact that has been at the center of the policy debate during the Covid crisis. Our model shows, quite naturally that active purchases of assets through the standing facility leads to higher prices of assets. The second concerns the effects of monetary policy on aggregate economic activity. In our model, a repo operation has positive real effects even though prices are fully flexible.

2 Benchmark model

Cash investors require money to purchase investments. By selling assets or using them as collateral for repo finance, they can get more money and buy more investment goods. *Credit investors* finance investments using unsecured credit. As a result, they can contribute to *market liquidity* by using their idle balances to purchase assets outright or provide repo finance. Market liquidity varies with *credit conditions*, where credit conditions are measured as the fraction of credit investors in the economy. When credit conditions and, hence, market liquidity deteriorate, a central bank can support the economy's liquidity needs by providing repo finance to cash investors. The central bank's repo facility essentially "buys" assets from cash investors *with newly issued cash* with the promise that the investors repurchase them in a near future date. We now turn to the details of the model.

Investors hold a portfolio of one-period government bonds, b, and real money balances, z. A government bond pays one unit of a real consumption good at maturity and money is a fiat, nominal object.[5] The government sells $\bar{b}$ one-period bonds each period. Bond repayments are financed by a lump-sum tax T_b, where $T_b = \bar{b}$ since the measure of investors is normalized to 1. A random fraction σ of investors are cash investors and the remainder are credit investors. For expositional simplicity, we initially assume the *credit shock* σ takes on 2 values, σ_L and σ_H, where $0 < \sigma_L < \sigma_H < 1$. We later extend the analysis to allow for a general distribution. State

[5] For simplicity and without loss of generality, we assume that the non-monetary asset is a one-period real bond. Our results are unaffected if we assume, e.g., the asset is a nominal government bond or a Lucas tree.

H corresponds to a deterioration of credit conditions since $\sigma_H > \sigma_L$. State L (H) occurs with probability π_L ($\pi_H = 1 - \pi_L$), where $0 \leq \pi_L \leq 1$.[6] There is *aggregate risk* when $0 < \pi_L, \pi_H < 1$ and no aggregate risk when $\pi_L, \pi_H = 0, 1$.

Investors are infinitely lived and time is indexed by t. Each time period t has 3 subperiods. The first subperiod is the financial market or *FM subperiod*. Investors enter the FM subperiod with portfolio (b, z) and learn whether they are cash or credit investors. Investors can adjust their liquidity/asset positions in a competitive financial market and with the central bank. Cash investors can either sell bonds or get repo finance—using their bonds as collateral—in the financial market. We assume that investors incur a very small transactions cost when selling or buying assets in the FM subperiod, which implies that cash investors use repo arrangements to obtain additional real balances.[7] A repo contract in state $i = L, H$ specifies two prices, p_i^{FM} and p_i^R, where p_i^{FM} is the competitive repo price per unit of government bond (or collateral) measured in terms of real balances and $p_i^R \geq p_i^{FM}$ is the price at which cash investors repurchase their collateral in a subsequent (third) period. Hence, a cash investor can exchange collateral $b_i^c \leq b$ for $p_i^{FM} b_i^c$ real balances. A credit investor provides $z_i^n \leq z$ real balance in repo finance which is secured by $z_i^n / p_i^{FM} \equiv b^n$ collateral, where the subscript n denotes that the investor is *not* a cash investor. Demand for market liquidity—or, equivalently, aggregate demand for real balances—is $\sigma_i p_i^{FM} b_i^c$ and supply of market liquidity is $(1 - \sigma_i) z_i^n$. The central bank provides liquidity in the FM subperiod through a repo facility. At the beginning of each period, before the state is revealed, the central bank repo facility posts two non-state contingent prices, p^{CB} and p^R, and stands ready to purchase government bonds in any amount from investors at repo price p^{CB}. The repo arrangement requires that investors repurchase their bonds in a subsequent (third) subperiod at the repurchase price $p^R \geq p^{CB}$. The FM subperiod competitive *market repo rate* in state $i = L, H$ is defined as $r_i^{FM} \equiv (p_i^R - p_i^{FM})/p_i^{FM}$ and the *central bank repo rate* is $r^{CB} \equiv (p^R - p^{CB})/p^{CB}$. If a cash investor repo finances collateral equal to b_i^c in the competitive financial market and b_i^{CB} at the central bank's repo facility, then his real balance holdings increase by $p_i^{FM} b_i^c + p^{CB} b_i^{CB}$ and his (implicit) bond holdings decrease by $b_i^c + b_i^{CB}$. Feasibility requires $b_i^c + b_i^{CB} \leq b$. Cash investors exit the FM subperiod holding portfolio

[6] We examine a more general specification in Section 5 and Appendix D, where σ identically and independently is distributed over $[0, 1]$.

[7] More generally, we assume that the transactions cost associated with repo finance are less than those associated with selling and buying assets, a condition that holds in practice. If the transactions cost associated with repo finance and selling and buying assets are equal, then investors would be indifferent between repo transactions and buying and selling assets.

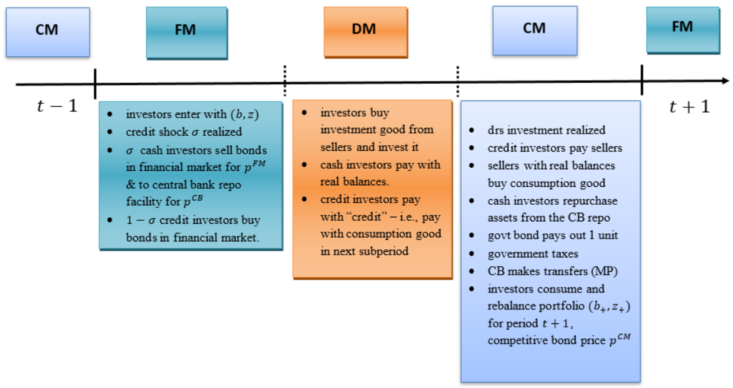


Figure 1: Timing of events

$(b - p_i^R b_i^c - p^R b_i^{CB}, z + p_i^{FM} b_i^c + p^{CB} b_i^{CB})$ while credit investors hold $(b + p_i^R z_i^n / p_i^{FM}, z - z_i^n)$.[8]

In the second subperiod, cash and credit investors bargain with *sellers* over the amount, y_i, and total price, p_i^j, of investment goods to exchange in state $i = L, H$, where p_i^j is measured in real balances and $j = c, n$. In this decentralized investment market or *DM subperiod*, the seller receives p_i^c real balances from the cash investor while the credit investor pays the seller p_i^n consumption goods in the *subsequent* subperiod.[9] The latter is a credit arrangement since the seller extends a loan, y_i, to be repaid, p_i^n, in a later subperiod. Sellers are infinitely lived and their measure is at least equal to 1.[10] Sellers, and only sellers, can produce perishable investment goods using their labor in a linear technology, where a unit of labor produces a unit of the investment good. The seller's disutility of y labor is $c(y)$, where $c(0) = 0$, $c' > 0$ and $c'' \geq 0$. The investment good is used in technology f that is only available to investors. The technology generates $f(y)$ units of a consumption good in the next subperiod, where $f(0) = 0$, $f' > 0$, $f'' < 0$ and $f'(0) > c'(0)$. The market structure in the DM subperiod has each investor being matched with a seller. In each match, investors and sellers bargain over the terms of trade (p_i^j, y_i^j), $j = c, n$. We assume that (p_i^j, y_i^j) is determined by the Kalai bargaining solution. Intuitively, if the investor has bargaining power θ and S represents total surplus generated by the investor-seller match, p_i^j and y_i^j are set so that the investor and seller receive θS and $(1-\theta)S$, respectively,

[8] It is convenient to represent a repo transaction as a change in real money balances in the FM subperiod along with the change in the bond holding adjusting for the repurchase price, p_i^R or p^R. For example, in the case of the cash investor, his real balances increase by $p_i^{FM} b_i^c + p^{CB} b^{CB}$ in the FM subperiod, but in order to get this increase in real balances, he must repurchase the equivalent of $p_i^R b_i^c + p^R b^{CB}$ bonds in the subsequent CM subperiod from his repo counterparty.

[9] Real balances are measured in terms of the consumption good.

[10] Without loss of generality, we assume that sellers do not participate in the FM subperiod financial markets. We elaborate on this below.

of the surplus. Feasibility for the cash investor requires that $p_i^c \leq z + p_i^{FM} b_i^c + p^{CB} b_i^{CB}$, where the right side is the total real balances held by the buyer. Cash investors exit the DM subperiod holding the net portfolio $(b - p_i^R b_i^c - p^R b_i^{CB}, z + p_i^{FM} b_i^c + p^{CB} b_i^{CB} - p_i^c)$ while credit investors exit with the same portfolio of money and bonds they entered the DM with, $(b + p_i^R z_i^n / p_i^{FM}, z - z_i^n)$ along with the credit obligation p_i^n. Matched sellers exit holding either p_i^c real balances or p_i^n worth of real credit.

In the third and final subperiod, which we call the competitive market or *CM subperiod*, there exists a competitive market where investors and sellers trade money for the consumption good, investors pay off their debts—to sellers, the central bank and other investors—and rebalance their portfolio of assets they intend to take into the next period. The government levies the lump-sum tax, T_b, and pays off its one-period debt obligations. Each investor receives a payoff from his investment equal to $f(y_i^j)$ units of the consumption good, $j = c, n$, at the very beginning of the CM subperiod. Investors that entered into a repo contract with either the central bank or other investors in the previous FM repurchase their collateral at the stated price—p^R for central bank repo and p_i^R for competitive repo—with real balances. If ϕ represents the amount the consumption goods that 1 unit of fiat money can buy in the CM subperiod competitive market, then $1/\phi$ is the price of the consumption good measured in terms of money, e.g., dollars. Hence, if an agent holds m units of nominal balances, then real balance holdings are simply $z = \phi m$. The price of the newly issued one-period government bonds in the CM subperiod is denoted by p^{CM}, measured in terms of the consumption good (or real balances). Sellers and investors have linear, one-to-one preferences over the consumption good. The representative investor exits the CM subperiod of period t and enters period $t+1$ with portfolio (b_+, z_+).[11] See Figure 1 for a summary of the timing of events in a typical period t.

In practice, repo finance is designed to provide the lender with some insurance against default. This is typically accomplished by having the repo lender provide a cash loan that is less than the market value of the collateral, and is referred to as applying a "haircut" to the collateral. In the event of a default, the haircut helps the repo lender recover the total loan repayment, principle and interest, in most circumstances.[12] Because there is no uncertainty regarding the payoff of

[11] Owing to the specification of preferences, the CM subperiod asset price does not depend on the state i and all investors exit the CM subperiod of period t holding the same portfolio, independent of whether the investor was a cash or credit investor in period t. Since sellers do not participate in the FM subperiod financial markets, their demands for real balances and real assets in the CM subperiod are zero.

[12] The repo lender will not recover the total loan repayment if, for example, the market value of the collateral experiences a significant decline.

the one period government bond in the CM—each bond pays one unit—any haircut yields the same equilibrium since the repo lender possesses the asset and receives a one unit payoff in the repo borrower, for some reason, does not repurchase the asset. Therefore, without of generality we assume there is no haircut and that $p^R = p_i^R = 1$. Since the repurchase price is essentially "predetermined," the competitive and central bank repo contracts are fully described by the FM subperiod repo prices, p_i^{FM} and p^{CB}, respectively. Notice that the standing repo facility will be *inactive* in state i whenever $p^{CB} < p_i^{FM}$ since cash investors strictly prefer getting repo finance in the competitive financial market.

In addition to operating the standing repo facility, the central bank sets an inflation target, π^*. Define M_t to be the aggregate stock of nominal money at the beginning of period t. Then $M_{t+1} = \mu_t M_t$, where μ_t represents the gross growth rate in aggregate nominal money balances between periods t and $t+1$. Since, in practice, central banks are reluctant to pursue a deflationary policy and do not have taxing authority, we have $\mu_t \geq 1$. New money is injected into the economy by lump-sum transfers T_t^M to investors at the beginning of the CM subperiod of period t. If ϕ_t is price of money in period t, then the value of aggregate real money balances in the CM subperiod of period t is $\phi_t(M_t + T_t^M)$.

In the language of Holmstrom and Tirole (1998), credit investors are able to *pledge* all of investment income, $f(y_i)$, while cash investors cannot pledge any of it.[13] Credit investors are able to pledge all of their investment income because they can be monitored in both the DM and CM subperiods of period t. If a credit investor defaults on his obligation p_i^n in the CM subperiod, then he is banished from the economy forever. We assume that this punishment is severe enough to ensure that credit investors never default. Cash investors cannot be monitored in either the DM and CM subperiods of period t and are, therefore, anonymous. Anonymity implies that sellers will not extend credit to cash investors since the cash investors can (and will) costlessly default. Hence, none of their investment income is pledgeable. We assume that ownership of the government bond is digitally stored at a repository that can only be accessed when financial markets operate, which is in the FM and CM subperiods. Since government bond ownership cannot be verified and transferred in the DM subperiod, bonds cannot serve as a medium of exchange in the DM subperiod. This implies that any transfer of investment goods from sellers to cash investors must be settled in cash in the DM subperiod.

[13] That credit investors can pledge all of their investment income implies that $p_i^n \leq f(y_i)$, which is always the case in equilibrium.

3 An always inactive central bank repo facility

Here we assume that the competitive financial market is the sole source of liquidity in the FM subperiod. This occurs, for example, if $p^{CB} < \min\{p_H^{FM}, p_L^{FM}\}$, which implies that (in equilibrium) the central bank repo facility is always inactive.[14] We parameterize σ_L and σ_H in a way that highlights the intuition that underlies the potential costs and benefits associated with an active central bank standing repo facility.[15] Our parameterization results in an equilibrium that is characterized by:

- σ_L cash investors repo financing all their collateral (i.e., government bond holdings) in the FM subperiod competitive financial market, $b_L^c = b$, and $1 - \sigma_L$ credit investors not using all of their real balances, $z_L^n < z$, for repo finance when the state is $i = L$; and

- σ_H cash investors not repo financing all of their collateral in the FM subperiod competitive financial market, $b_H^c < b$, and $1-\sigma_H$ credit investors using all of their real balances, $z_H^n = z$, for repo finance when the state is $i = H$.

Intuitively, when $\sigma = \sigma_L$ credit conditions are "loose" and market liquidity is *abundant* in the sense that: (i) there is cash equal to $(1 - \sigma_L)(z - z_L^n)$ sitting on the "sidelines," i.e., stays in the portfolios of credit investors and (ii) cash investors are able to repo finance all of their collateral. When $\sigma = \sigma_H$ credit conditions are "tight" and market liquidity is *scarce* in the sense that: (i) there is no cash sitting on the sidelines and (ii) cash investors hold some government bonds in their portfolio. Such an equilibrium configuration arises when σ_L is sufficiently small and σ_H is sufficiently large. We now provide some of the details of investors' and sellers' decision making so that we can fully characterize the equilibrium.

The value function for an investor at the beginning of the CM subperiod, $W(b, z, y, d)$, is given by

$$W\left(b,z,y,d\right) = \max_{x,b',z'} \left\{x + \beta \mathbb{E}_i J\left(b', z', \sigma_i\right)\right\},$$
$$\text{s.t. } x + p^{CM} b' + \phi/\phi' z' + d = b + z + \phi T^M + f(y) - T_b$$

[14] For example, by setting $p^{CB} = 0$ the central bank can ensure that its repo facility is always inactive.

[15] We emphasize that our parameterization is in no way "contrived." A more general model, presented and analyzed in Section 5 and Appendix D, has three kinds of equilibria that emerge in the FM subperiod, two of which are described by our parameterizations of σ_L and σ_H. The third equilibrium configuration is basically a mixture of the two that are presented in this section.

where b and z are the amounts of government bonds (minus repo repayments) and real balances brought into the CM subperiod, y is the amount of the investment good invested in DM subperiod, d is the real credit obligation to a seller (which equals p^n for a credit investor and zero for a cash investor), x is the amount consumed of the real consumption good, b' and z' are government bonds and real balances, respectively, brought into the next period, ϕ' is the price of one unit of fiat money in the next period, T^M is the central bank's lump sum *nominal* monetary transfer, T_b is the government's *real* lump-sum tax and $J(b', z', \sigma_i)$ is the investor's value function at the beginning of the subsequent FM subperiod when the state of the world is $i = H, L$. The expectation is taken with respect to the state in the next period, either $i = H$ or L. We can eliminate x from the CM subperiod value function using the budget constraint to get

$$W(b, z, y, d) = f(y) + b + z + \phi T^M - d - T_b + \max_{b', z'}[-p^{CM} b' - \phi/\phi' z' + \beta \mathbb{E}_i J(b', z', \sigma_i)].$$

The first-order (Euler) conditions, assuming interior solutions, are

$$b' \quad : \quad p^{CM} = \beta \mathbb{E}_i J_1(b', z', \sigma_i), \tag{1}$$

$$z' \quad : \quad \frac{\phi}{\phi'} = \beta \mathbb{E}_i J_2(b', z', \sigma_i), \tag{2}$$

and the envelope conditions are

$$W_1(b, z, y, d) = W_2(b, z, y, d) = 1, W_3(b, z, y, d) = f'(y) \text{ and } W_4(b, z, y, d) = -1.$$

Intuitively, owing to the linearity in preferences over the consumption good, investors value both an additional unit of a government bond that matures in that CM subperiod and an additional unit of real balances at one, the amount of the consumption good that they can purchase. Linearity also implies

$$W(b, z, y, d) = f(y) - d + W(b, z, 0, 0). \tag{3}$$

The value function at the beginning of the FM subperiod in state $i = L, H$, $J(b, z, \sigma_i)$, is given by

$$J(b, z, \sigma_i) = \sigma_i J^c(b, z, \sigma_i) + (1 - \sigma_i) J^n(b, z, \sigma_i),$$

where $J^c(b, z, \sigma_i)$ is the cash investor's state i value function, $J^n(b, z, \sigma_i)$ is the credit investor's state i value function and b and z represent the bond and real balance holdings at the beginning of the FM subperiod.

In the FM subperiod cash investors use their collateral to obtain additional real balances so they can purchase more investment goods in the DM subperiod. Since the financial market in the FM subperiod is competitive, cash investors take the bond repo price, p_i^{FM}, as given and solve

$$J^c(b, z, \sigma_i) = \max_{b_i^c \leq b} V^c(b - b_i^c, z + z_i^c, \sigma_i) \qquad (4)$$
$$\text{s.t.} \quad p_i^{FM} b_i^c = z_i^c,$$

where V^c is cash investor's value function in the DM subperiod and $b_i^c \leq b$ is the repo collateral constraint, i.e., a cash investor can't repo finance collateral that he does not hold.

We assume that investors get matched with sellers with probability 1 in the DM subperiod. The terms of trade in a match are determined by Kalai bargaining, where the investor has bargaining power θ.[16] If a cash investor enters the DM subperiod with real balances z_i in state i, the quantity of investment good produced y_i^c and real payment p_i^c for those goods are determined by,

$$\max_{y^j, p^j \leq z_i^j} [f(y_i^c) - p_i^c]$$
$$\text{s.t.} \ f(y_i^c) - p_i^c = \theta[f(y_i^c) - c(y_i^c)]$$
$$p_i^c \leq z_i,$$

i.e., the terms of trade maximize the cash investor's surplus subject to the investor getting a fraction θ of the total match surplus and a cash constraint. The amount of investment good produced and payment for it expressed as functions of z_i, $Y(z_i)$ and $P(z_i)$, respectively, are

$$Y(z_i) = \begin{cases} v^{-1}(z_i) & \text{if } z_i = p_i^c < v(y^*) \\ y^* & \text{otherwise} \end{cases}, P(z_i^c) = \begin{cases} z_i & \text{if } z_i = p_i^c < v(y^*) \\ z^* = v(y^*) & \text{otherwise} \end{cases}$$

where $v(y) = (1-\theta) f(y) + \theta c(y)$, y^* is the efficient level of the investment good, i.e., $f'(y^*) = c'(y^*)$.[17] We assume that b, the amount of government bonds that investors hold at the beginning of the FM subperiod, is not "large" in the sense that even if the cash investor repo finances all of his collateral in the FM subperiod market in state $i = L$, total real balance holdings are strictly less than $v(y^*)$, the amount of real balances needed to purchase the first-best level of the investment good, y^*.[18] One can express the cash investor's DM subperiod value function in

[16] The same analysis can be applied to a general trading mechanism as in Gu and Wright (2016).

[17] Notice that if the cash constraint does not bind, then the solution to the above maximization problem is $f'(y^*) = c'(y^*)$ and the cash investor pays $(1-\theta)f(y^*) + \theta c(y^*) = v(y^*)$ real balances for y^* units of the investment good.

[18] In Appendix E we examine the case where b is "large" in the sense that $z + p_L^{FM} b_L^c = z^*$ in the liquidity abundant state $i = L$, where $b_L^c \leq b$. That is, in state $i = L$ the buyer is able to purchase the efficient amount of the investment good y^*. When b is "large," market liquidity will be scarce in state H when σ_H is sufficiently large.

state i as

$$V^c(b_i, z_i, \sigma_i) = W[b_i, z_i - P(z_i), Y(z_i), 0]. \quad (5)$$

Intuitively, the matched investor invests $Y(z_i)$ and pays $P(z_i)$ for it if he has a total of z_i real balances at the beginning of the DM subperiod.

Combining (4) and (5), and using (3) we get

$$\begin{aligned} J^c(b, z, \sigma_i) &= \max_{b_i^c \leq b} \{f[Y(z + z_i^c)] + W(b - b_i^c, z + z_i^c - P(z + z_i^c), 0, 0)\} \\ & \text{s.t. } p_i^{FM} b_i^c = z_i^c, \end{aligned} \quad (6)$$

Exploiting the linearity of W and using the budget constraint to eliminate z_i^c, (6) can be rewritten as,

$$\begin{aligned} J^c(b, z, \sigma_i) &= \max_{b_i^c \leq b} \{f\left[Y(z + p_i^{FM} b_i^c)\right] - P(z + p_i^{FM} b_i^c) \\ & -b_i^c(1 - p_i^{FM}) + W(b, z, 0, 0)\}. \end{aligned} \quad (7)$$

The first-order condition for the right-side of (7) is

$$\lambda(z + p_i^{FM} b_i^c) \begin{cases} = (1 - p_i^{FM})/p_i^{FM} & \text{if } b_i^c < b \\ \geq (1 - p_i^{FM})/p_i^{FM} & \text{if } b_i^c = b \end{cases}, \quad (8)$$

where

$$\lambda(z + p_i^{FM} b_i^c) \equiv \frac{f'\left[Y^c\left(z + p_i^{FM} b_i^c\right)\right]}{v'\left[Y^c\left(z + p_i^{FM} b_i^c\right)\right]} - 1 = \frac{\theta\left\{f'\left[Y^c\left(z + p_i^{FM} b_i^c\right)\right] - c'\left[Y^c\left(z + p_i^{FM} b_i^c\right)\right]\right\}}{v'\left[Y^c\left(z + p_i^{FM} b_i^c\right)\right]}. \quad (9)$$

$\lambda(\cdot)$ can be interpreted as liquidity premium for *real balances* in the FM subperiod and $(1 - p_i^{FM})/p_i^{FM}$ represents the marginal cost of (converting collateral into) real balances.

A credit investor's state i value function in the FM subperiod is given by

$$\begin{aligned} J^n(b, z, \sigma_i) &= \max_{z_i^n \leq z} W(b + b_i^n, z - z_i^n, 0, 0) + f(y^*) - v(y^*), \\ \text{s.t. } p_i^{FM} b_i^n &= z_i^n, \end{aligned}$$

Since credit investors are not constrained by a means of payment in the DM subperiod—their investment income is fully pledgeable—they negotiate an outcome with a seller that maximizes total surplus $f(y) - c(y)$, which implies that $p_i^n \equiv v(y^*)$ and $y = y^*$. Credit investors can use their real balances to provide repo finance in the FM subperiod competitive market. Again, exploiting the linearity of W and using the budget constraint to eliminate b_i^n, $J^n(b, z, \sigma_i)$ can be written as

$$J^n(b, z, \sigma_i) = \max_{z_i^n \leq z} \left(\frac{1}{p_i^{FM}} - 1\right) z_i^n + W(b, z, 0, 0) + f(y^*) - v(y^*) \quad (10)$$

The solution to the right-side of (10) is

$$z_i^n \begin{cases} \in (0, z) & \text{if } p_i^{FM} = 1 \\ = z & \text{if } p_i^{FM} < 1. \end{cases}$$

Intuitively, a credit investor is indifferent between providing and not providing repo finance in the FM subperiod when the FM subperiod repo bond price, p_i^{FM}, equals 1, the payoff of a government bond in the subsequent CM subperiod. However, when p_i^{FM} is less than one, a credit investor supplies all of his real balances for repo finance since the excess payoff per unit of collateral supplied, $1 - p_i^{FM}$, is strictly positive. Notice that $p_i^{FM} < 1$ necessarily implies that $\lambda(\cdot) > 0$, see (8). An immediate implication of all this is that, *in equilibrium*, we must have $p_i^{FM} \leq 1$. If $p_i^{FM} > 1$, then the supply of repo finance will be zero and the demand will be strictly positive; hence, the FM subperiod financial market will not clear.

We can now use the (1) and (2) to determine the equilibrium CM subperiod price of the newly issued government bonds and demand for real balances. Since there are only two states, L and H, it will be convenient to express (1) and (2) as

$$p^{CM} = \pi_L \beta[\sigma_L J_1^c(b, z, \sigma_L) + (1 - \sigma_L) J_1^n(b, z, \sigma_L)] \\ + \pi_H \beta[\sigma_H J_1^c(b, z, \sigma_H) + (1 - \sigma_H) J_1^n(b, z, \sigma_H)] \quad (11)$$

and

$$\frac{\phi}{\phi'} = \pi_L \beta[\sigma_L J_2^c(b, z, \sigma_L) + (1 - \sigma_L) J_2^n(b, \sigma_L)] \\ + \pi_H \beta[\sigma_H J_2^c(b, z, \sigma_H) + (1 - \sigma_H) J_2^n(b, z, \sigma_H)], \quad (12)$$

respectively. When $i = L$, credit conditions are loose/market liquidity is abundant and cash investors repo finance all of their collateral in the FM subperiod. The former implies that credit investors are indifferent between holding cash and providing repo finance so $p_L^{FM} = 1$; the latter implies that $b_L^c = b$. Hence, the cash investor's FM subperiod value function (7) in state L is

$$J^c(b, z, \sigma_L) = f \circ Y(z + b) - P(z + b) - b_H^* + W(b, z, 0, 0),$$

and the envelope conditions imply

$$J_1^c(b, z, \sigma_L) = \lambda(z + b) + 1, \quad (13)$$

$$J_2^c(b, z, \sigma_L) = \lambda(z + b) + 1. \quad (14)$$

The credit investor's FM subperiod value function (10) becomes

$$J^n(b,z,\sigma_L) = W(b,z,0,0) + f(y^*) - v(y^*),$$

and we have

$$J_1^n(b,z,\sigma_L) = 1, \tag{15}$$

$$J_2^n(b,z,\sigma_L) = 1. \tag{16}$$

When $i = H$, market credit conditions are tight/market liquidity is scarce. Since cash investors do not repo finance all of their collateral in the FM subperiod financial market, let b_H^* be the amount of collateral that a cash investor repo finances, where b_H^* solves (8) with equality. Hence, the cash investor's FM subperiod value function (7) in state H is

$$J^c(b,z,\sigma_H) = f \circ Y\left(z + p_H^{FM} b_H^*\right) - P(z + p_H^{FM} b_H^*) + W(b,z,0,0),$$

and we have

$$J_1^c(b,z,\sigma_H) = 1, \tag{17}$$

$$J_2^c(b,z,\sigma_H) = \lambda(z + p_H^{FM} b_H^*) + 1. \tag{18}$$

Since $z_H^n = z$, the credit investor's value function (10) when $i = H$ can be written as

$$J^n(b,z,\sigma_H) = \left(\frac{1}{p_H^{FM}} - 1\right) z + W(b,z,0,0) + f(y^*) - v(y^*),$$

and we have

$$J_1^n(b,z,\sigma_H) = 1, \tag{19}$$

$$J_2^n(b,z,\sigma_H) = \frac{1}{p_H^{FM}} = \lambda(z + p_H^{FM} b_H^*) + 1, \tag{20}$$

where the second equality in (20) follows from (8) with equality.

We consider a competitive steady-state equilibrium with rational expectations. The steady state equilibrium requirement means that real variables—such as z_t, p_t^{CM}, $\phi_t(M_t + T_t^M)$, and so on—are unchanging over time. The competitive equilibrium requirement means that supply equal demand and rational expectations means that agents' forecasts are consistent with equilibrium outcomes. Substituting (13)-(20) into (11)-(12), imposing the steady state conditions and market clearing for government bonds in the CM subperiod, $b = \bar{b}$, we get[19]

$$p^{CM} = \beta + \beta \pi_L \sigma_L \lambda(z + \bar{b}), \tag{21}$$

[19] The market for real balances clear if $z'/\phi' = M_t + T_t$, which we fully characterize below.

and

$$\frac{\phi}{\phi'\beta} - 1 = \pi_L \sigma_L \lambda(z + \bar{b}) + \pi_H \lambda(z + p_H^{FM} b_H^*). \tag{22}$$

Since equilibrium in the FM subperiod in state $i = H$ requires $\sigma_H b_H^* p_H^{FM} = (1 - \sigma_H) z$ or $z + p_H^{FM} b_H^* = z/\sigma_H$, we can rewrite (22) as

$$\frac{\phi}{\phi'\beta} - 1 = \pi_L \sigma_L \lambda(z + \bar{b}) + \pi_H \lambda(z/\sigma_H). \tag{23}$$

Three observations are in order. First, the so-called Fisher equation relates the nominal interest rate (on an illiquid one period bond), denoted as ι, to the real interest rate r and inflation rate $\phi/\phi' - 1$. More specifically, the Fisher equation is

$$\iota = \frac{\phi}{\phi'\beta} - 1. \tag{24}$$

Hence, the left sides of (22) and (23) can be interpreted as a nominal interest rate. Second, since the fundamental value for a government bond is β and for real balances is zero,[20] (21) and (23) indicate that asset prices exceed their fundamental values, i.e., assets have a liquidity premium in the CM subperiod. Third, the equilibrium can be solved sequentially. The right side of (23) is decreasing in z and becomes negative as z gets arbitrarily large and approaches $\theta/(1-\theta)$ as $z \to 0$. Hence, (23) solves uniquely for z if θ is not "too small."[21] This value of z can then be plugged into (21) to solve for p^{CM}.

Notice that each asset's CM subperiod price is tightly related to FM subperiod liquidity premia, $\lambda(\cdot)$. Intuitively, an asset price equals its fundamental value—β for the government bond and zero for fiat money—plus any expected liquidity premia that cash and credit investors receive.[22] For the government bond, cash investors receive a liquidity premium only in the state $i = L$ because they repo finance all of their collateral holdings in the FM subperiod, while credit

[20] The discounted value of real bond payments is sometimes called the the bond's fundamental value. A one-period government bond that pays one unit of the consumption good has a fundamental value of $1/(1+r) \equiv \beta$. Since fiat money does not provide any interest or dividends, its discounted stream or fundamental value is zero.

[21] If θ is a very small number, then the benefit of an additional unit of liquidity to an investor is also very small. In this situation, the seller is basically the beneficiary of the additional liquidity because his bargaining power is so high and the investor's marginal benefit of the liquidity does not cover the marginal cost, ι. The investor's bargaining power, θ, has to be sufficiently large to exceed the cost of accumulating any real balances.

[22] Since sellers do not need liquidity in the DM subperiod, they do not attach a liquidity premium to these assets. As a result, sellers have no incentive to buy real balances or real assets in the CM subperiod—which we assumed—since these assets embed a liquidity premium in their CM subperiod prices. If sellers can participate in the FM subperiod, they would have no incentive to purchase the real asset in the CM subperiod since $p_i^{FM} \leq p^{CM}$. Athough the real asset can be purchased "cheap"—with real balances—in the FM subperiod in state $i = H$, it is straightforward to show that sellers will not purchase real balances in the CM subperiod because the cost of holding real balances minus the expected benefit associated with purchasing the real asset in the FM subperiod in state $i = H$ is strictly positive. Therefore, our assumption that sellers do not participate in the FM subperiod is not binding.

investors never receive a liquidity premium. Since only the cash investor receives a liquidity premium and only in state $i = L$, the liquidity premium term $\lambda(z + \bar{b})$ in (21) is multiplied by $\pi_L \sigma_L$. For fiat money, cash investors receive liquidity premia in both states $i = L, H$ while credit investors receive a liquidity premium in the state where the competitive repo price is less than 1, in state $i = H$. Since only the cash investor receives a liquidity premium in the state $i = L$, the liquidity premium term for state $i = L$ in (23), $\lambda(z + \bar{b})$, is multiplied by $\pi_L \sigma_L$; since all investors receive a liquidity premium in state $i = H$, the liquidity premium term for state $i = H$, $\lambda(z/\sigma_H)$, is multiplied by $\pi_H(\sigma_L + \sigma_H) = \pi_H$.

Since (steady state) equilibrium requires $\phi_t(M_t + T_t^M) = \phi_{t+1}(M_{t+1} + T_{t+1}^M)$, we have

$$\frac{\phi_t}{\phi_{t+1}} = \frac{M_{t+1} + T_{t+1}^M}{M_t + T_t^M} = \mu_t \frac{M_t + T_t^M}{M_t + T_t^M} = \mu_t.$$

The central bank must set $\mu_t = \pi^* + 1 \equiv \phi_t/\phi_{t+1}$ for all t to hit its inflation target of π^*.[23] The lump sum transfer in the CM subperiod of period t, T_t^M, required to hit the inflation target π^* is

$$T_t^M = \pi^* M_t. \tag{25}$$

Equations (21) and (23), along with $\phi/\phi' \equiv \pi^* + 1$, can be used to pin down the equilibrium CM government bond price and real balances when the central bank repo facility is always inactive, which we denote as $\tilde{p}^{CM}$ and $\tilde{z}$, respectively. The other equilibrium variables of interest can be calculated as follows: $\tilde{\phi}_t = \tilde{z}/[(\pi^* + 1)^t M_0]$,

$$\begin{aligned} \tilde{p}_H^{FM} &= \frac{1}{\lambda(\tilde{z}/\sigma_H) + 1},\ \tilde{b}_H^c = \frac{(1-\sigma)\,\tilde{z}}{\tilde{p}_H^{FM}\sigma_H},\ \tilde{y}_H^c = Y(\frac{\tilde{z}}{\sigma_H}), \\ \tilde{p}_L^{FM} &= \tilde{p}^{CM},\ \tilde{b}_L^c = \bar{b},\ \tilde{y}_L^c = Y(\tilde{z} + \bar{b}), \end{aligned}$$

where M_0 represents the nominal money stock at the beginning of date 0 and $T_0 = \pi^* M_0$.

This equilibrium is consistent with the stylized facts that as credit conditions tighten, aggregate output falls and liquidity becomes more scarce. To see this, define aggregate output in state i, Q_i, as

$$Q_i \equiv \sigma_i f(\tilde{y}_i^c) + (1 - \sigma_i) f(y^*).$$

Clearly, $Q_H < Q_L$ since $\tilde{y}_H^c < \tilde{y}_L^c < y^*$ and $\sigma_H > \sigma_L$. Since credit conditions tighten when moving from state $i = L$ to state $i = H$, aggregate output declines as market liquidity becomes more scarce. Furthermore, when market liquidity is scarce, government bond prices become

[23] At this monetary growth rate, agents *expect* inflation to equal π^* in future periods.

depressed, $\tilde{p}_H^{FM} < \tilde{p}_L^{FM} = 1$, which is consistent with the notion that when market liquidity "drys up" assets "sell at fire sale" prices, i.e., the repo price is less than 1, the fundamental value of a one-period asset at the end of one period.

4 Active central bank repo facility in state H

Since market liquidity is scarce in state $i = H$, there may be a role for the central bank to play as a liquidity provider. The central bank repo facility will be active in state $i = H$ only if its posted repo price is strictly greater than the *equilibrium* FM subperiod repo price when the facility is always inactive, i.e., if $p^{CB} > \tilde{p}_H^{FM}$. Furthermore, in any equilibrium where the central bank is active in state $i = H$, competitive market repo price must equal the central bank's posted repo price, i.e., $p_H^{FM} = p^{CB}$. If this was not the case then either: (i) $p^{CB} < p_H^{FM} = \tilde{p}_H^{FM}$ which implies that the central bank repo facility would be inactive, a contradiction; or (ii) $p^{CB} > p_H^{FM}$ which implies that cash investors' demand for market (repo) liquidity would be zero and there will be an excess supply of repo finance in the FM subperiod market. For the time being we assume that p^{CB} is strictly greater than but close to $\tilde{p}_H^{FM}$. This assumption, which we will subsequently relax, implies that cash investors demand central bank repo finance in the FM subperiod but do not repo finance all of their collateral, i.e., $b_H^* < b$.

The characterization of equilibrium when the central bank repo facility is active in state $i = H$ is almost identical to that of an always inactive facility except that p_H^{FM} is replaced with p^{CB}. Intuitively, in the previous section investors face prices $\{1, p_L^{FM}, p_H^{FM}, \phi/\phi_+\}$ and now they face $\{1, p_L^{FM}, p^{CB}, \phi/\phi_+\}$ since $p^{CB} = p_H^{FM}$ in equilibrium. The asset pricing equations are

$$p^{CM} = \beta + \beta\pi_L\sigma_L\lambda(z + \bar{b}) \tag{26}$$

and

$$\iota = \pi_L\sigma_L\lambda(z + \bar{b}) + \pi_H\lambda(z + p^{CB}b_H^*). \tag{27}$$

Comparing these with (21) and (23), a notable and important quantitative difference between worlds with and without an active central bank repo facility lies in the cash investor's state $i = H$ real balance holdings. When the standing repo facility is always inactive, the cash investor's real balances at the beginning of the DM subperiod in state $i = H$ are $z + p_H^{FM}b_H^* = z/\sigma_H$; when it is active in state $i = H$, real balance holdings are $z + p^{CB}b_H^* = z + p^{CB}(b_H^c + b_H^{CB}) = z/\sigma_H + p^{CB}b_H^{CB}$. Intuitively, the standing repo facility allows cash investors to augment their real balance holdings beyond what is supplied in private markets. To complete the characterization of the equilibrium, we need the updated version of (8) with equality for $i = H$. In particular, if we replace b_i^c with b_H^* and p_H^{FM} with p^{CB} in (8) we get

$$\lambda(z + p^{CB} b_H^*) = \frac{1 - p^{CB}}{p^{CB}}. \tag{28}$$

The equilibrium can be solved sequentially. First, substitute (28) into (27) to get

$$\iota = \pi_L \sigma_L \lambda(z + \bar{b}) + \pi_H \frac{1 - p^{CB}}{p^{CB}}, \tag{29}$$

which solves the equilibrium z.[24] Second, substitute the equilibrium z into (26) and use this equation to solve for equilibrium p^{CM}.[25] For a given $p^{CB} \geq \tilde{p}_H^{FM}$, denote the equilibrium CM subperiod bond price and real balances as $\hat{p}^{CM}$ and $\hat{z}$, respectively. By construction, $\hat{p}^{CM} = \tilde{p}^{CM}$ when $p^{CB} = \tilde{p}_H^{FM}$ which implies that an increase in p^{CB} from $\tilde{p}_H^{FM}$ necessarily decreases $\hat{z}$ from $\tilde{z}$—see (29)—and increases $\hat{p}^{CM}$ from $\tilde{p}^{CM}$—see (26). Intuitively, an active central bank repo facility makes government bonds more liquid resulting in a higher bond premium, $\hat{p}^{CM} - \beta$, as reflected by its higher CM subperiod price. And because real balances are costly to hold, an increase in the liquidity of bonds induces investors to reduce their real balance holdings.

Since the central bank repo facility is inactive in state $i = L$, the CM subperiod transfer T_t^M given by (25) is consistent with the central bank hitting its inflation target. In state $i = H$, however, (25) is *not* consistent with the central bank hitting its inflation target, π^*. Intuitively, the central bank earns interest income from its repo transaction in state $i = H$: the money balances it receives from cash investors to settle their repo obligations in the CM subperiod exceeds the money balances they provided to the cash investors in the FM subperiod. Therefore, if state H occurs in period t and the transfer is given by (25), then $M_{t+1} < M_t(1 + \pi^*)$. This necessarily implies that if the central bank gives a transfer equal to (25) in all states $i = H, L$, then investors' inflation expectations must necessarily be strictly less than π^* and, as a result, the central bank will not hit its inflation target π^*.[26] In order to validate inflation expectations of π^*, the central bank must *increase* its CM subperiod transfer beyond (25) by an amount $(1 - p^{CB}) b_H^* / \phi_H$ in state $i = H$, which represents the interest income earned by the central

[24] Again, for existence θ cannot be too small.

[25] Recall that $\iota = (1 + \pi^*)/\beta - 1$ can be viewed as being "exogenous." Below, we discuss how the central bank is able to hit the inflation target π^*.

[26] More specifically, the standing repo facility injects $p^{CB} b_H^* / \phi_H$ nominal balances in the FM subperiod and "withdraws" $p^{CM} b_H^* / \phi_H$ nominal balances when buyers repurchase their collateral from the standing repo facility, where $\phi_H = \phi_{t-1}/(1 + \pi^*)$. Since $p^{CB} < p^{CM}$, if the CM subperiod transfer is equal to $\pi^* M_t$, then $M_{t+1} < M_t(1 + \pi^*)$.

bank's repo facility.[27] When the central bank rebates the repo interest income back to the economy, the aggregate money supply growth rate in all states will be equal to π^*. There are two important observations here. First, there is no inconsistency between a central bank achieving its long run inflation target while, at the same time, providing liquidity to financial markets when market liquidity is scarce. And second, the above results and discussion clearly indicate that the central bank's role of lender of last resort—via standing repo—is a purely monetary/liquidity operation that has no fiscal policy implications.

It may seem puzzling at first that an injection of *nominal* money balances in the FM subperiod has *real* effects in an economy where prices are flexible. This puzzle can be resolved when it is recognized that the increased nominal balances are withdrawn later on in the period—in the CM subperiod—when cash investors repurchase their collateral from the central bank repo facility. The net result (along with appropriate CM subperiod transfers T_t^M) is that inflation expectations—equal to π^*—will be unaffected. Hence, a nominal FM subperiod injection of money balances via the repo facility translates into higher DM subperiod *real* balances and, therefore, a higher transfer of investment goods between cash investors and sellers, and higher output in the CM subperiod.

Although the central bank's repo facility enhances the liquidity of government bonds, it is not obvious that investors and sellers are better off. As we show below, although total consumption *increases* in state $i = H$, it *decreases* in state $i = L$[28] and, as a result, the effect on social welfare is ambiguous. We assess the effect that the central bank repo facility has on the economy by appealing to a measure of social welfare that sums the discounted expected utility of agents in the economy, where the planner discounts the future at the same rate as agents. Owing to the linearity of the CM subperiod utility functions, this measure of social welfare simplifies to the difference between the discounted sum of per period expected investment output—which also equals discounted sum of per period expected consumption of investors and sellers in the CM subperiod—and the discounted sum of expected cost associated with producing the investment good by sellers in the DM subperiod. Since we focus on steady-state equilibria, our measure of social welfare is proportional to $W(p^{CB})$ plus a constant, where

$$W(p^{CB}) \equiv \pi_H \sigma_H \left\{ f[y_H(p^{CB})] - c[y_H(p^{CB})] \right\} + \pi_L \sigma_L \{ f[y_L(p^{CB})] - c[y_L(p^{CB})] \}. \tag{30}$$

[27] That is, the total tranfer in state $i = H$ must be $T_H^M = \pi^* M_t + (p^{CB} - p^{CM}) b_H^* / \phi_H$. Notice that by construction, we have $M_{t+1} = M_t + (p^{CB} b_H^* - p^{CM} b_H^* + T_H^M)/\phi_H = (1+\pi^*) M_t$ in state H.

[28] We demonstrate this in Proposition 2.

The function $W(p^{CB})$ captures the "welfare" generated by the cash investors; the welfare generated by credit investors is independent of p^{CB} and hence is a constant.

The model and analysis allows for the possibility of no aggregate risk, when either $\pi_L = 0$ or $\pi_H = 0$. Our first main result provides an important insight about central bank repo facilities when there is no aggregate risk.

Proposition 1 *When there is no aggregate risk in the economy, an active central bank repo facility cannot increase welfare.*

See Appendix A for the proof.

One might think that since an active central bank repo facility provides cash investors with additional cash from their collateral holdings, they will be able to increase their investments and production of the consumption good and, as a result, social welfare increases. This intuition is incorrect. Because real balances are costly to hold, investors reduce their money accumulation, z, in the CM subperiod by the amount of liquidity that is provided by the central bank repo facility. This result is reminiscent of Holmstrom and Tirole (1998), where government provided liquidity cannot improve outcomes in the absence of aggregate risk.

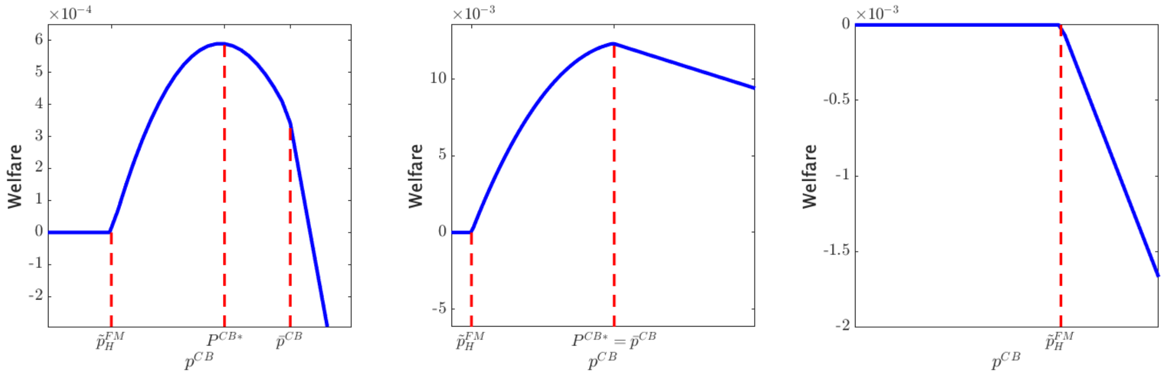
When there is aggregate risk, a central bank repo facility can improve matters for society under the conditions described in the following proposition,

Proposition 2 *(i) An active central bank repo facility increases investment and consumption in state $i = H$ and decreases both in state $i = L$. (ii) If λ/λ' is increasing on $[\tilde{z}/\sigma_H, \infty)$, then an active central bank repo facility can increase social welfare when σ_H is "sufficiently large" in the liquidity scarce state $i = H$.*

See Appendix B for the proof.

Just as in Proposition 1, an active central bank repo facility in state $i = H$ reduces investors' demand for real balances in the CM subperiod. The reduced real balance holdings, z, necessarily implies that investment and consumption for cash investors and sellers fall in state $i = L$ because total liquidity of a cash investors, $z + p_L^{FM} b_L^c = z + \bar{b}$ falls. The benefit associated with the central bank repo facility is the increased investment and consumption that occurs in state $i = H$. The requirement that λ/λ' is increasing on $[\tilde{z}/\sigma_H, \infty)$ has a nice economic interpretation and is not very restrictive.[29] The ratio λ/λ' is more likely to be increasing on $[\tilde{z}/\sigma_H, \infty)$, the more concave

[29] For example, suppose that $f(x) = [(x + \varepsilon)^{1-\gamma} - \varepsilon^{1-\gamma}]/(1 - \gamma)$, $c(x) = x$ and buyers make take-it-or-leave-it



(a) $\sigma_H = 0.87$ (b) $\sigma_H = 0.95$ (c) $\sigma_H = 0.80$

Figure 2: Numerical Examples: Welfare

is $f(\cdot)$ and the more concave is $f(\cdot)$, the faster the marginal value of liquidity increases as liquidity becomes scarce.[30] Hence, Proposition 2(ii) essentially says that an active central bank repo facility will be welfare enhancing if the marginal value of liquidity is significantly higher in the liquidity scarce state $i = H$ than the liquidity abundant state $i = L$ or, equivalently, if σ_H is sufficiently large.[31]

We now provide some numerical examples that illustrate what is meant by σ_H being "sufficiently large." The various panels in Figure 2 show how social welfare changes with the central bank's repo bond price p^{CB} for different values of σ_H, where $\sigma_H = 0.87, 0.95$ and 0.80 in the left, middle and right panels, respectively. In all cases, state $i = H$ is characterized by scarce market liquidity when the central bank repo facility is always inactive.[32] We plot the percentage change in social welfare compared to the case when the central bank standing repo is always

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offers. Then

$$\frac{d[\lambda(x)/\lambda'(x)]}{dx} = \frac{1}{\gamma}[(\gamma+1)(x+\varepsilon)^{\gamma} - 1] = \frac{(x+\varepsilon)^{\gamma}}{\gamma}[(\gamma+1) - (x+\varepsilon)^{-\gamma}]. \tag{31}$$

In state $i = H$, $x \equiv \tilde{z} + \tilde{p}^{FM} b^* = \tilde{z}/\sigma_H$ meaning that λ/λ' is increasing on $[\tilde{z}/\sigma_H, \infty)$ if $(\tilde{z}/\sigma_H + \varepsilon)^{-\gamma} - 1 < \gamma$. Because $(\tilde{z}/\sigma_H + \varepsilon)^{-\gamma} - 1 = \lambda(\tilde{z}/\sigma_H)$, we can use (28) to deduce that $(\tilde{z}/\sigma_H + \varepsilon)^{-\gamma} - 1 < \gamma$ iff

$$\tilde{r}_H^{FM} < \gamma,$$

where $\tilde{r}_H^{FM} = \tilde{p}^{CM}/\tilde{p}_H^{FM} - 1$ is the FM subperiod competitive repo rate when the central bank does not operate a standing repo facility. If γ is extremely low, say 0.1, a standing repo facility will be welfare improving if $\tilde{r}_H^{FM} < 10.0\%$. Even in this example the real rate has to be unrealistically high in order for λ/λ' to be *decreasing* on $[\tilde{z}/\sigma_H, \infty)$.

[30] Notice that an increasing λ/λ' implies that $\lambda/|\lambda'|$ is decreasing because $\lambda' < 0$. The latter means that as liquidity increases, the marginal value of liquidity normalized by $|\lambda'|$ decreases. Because $|\lambda'|$ is normally decreasing, this restriction requires that the marginal value of liquidity decreases sufficiently fast.

[31] Alternative interpretation is a semi-elasticity, how much real balances z change for a one percentage change in the cost of liquidity.

[32] The values for the other parameters in the examples are: $\beta = 0.98$, $\sigma_L = 0.03$, $\theta = 0.7$, $\gamma = 0.7$, $\varepsilon = 0.05$, $\iota = 0.04$, $\bar{b} = 0.2$, $\pi_L = 0.8$ and $\pi_H = 0.2$.

inactive. In the left panel of Figure 2, social welfare, at least initially, smoothly increases and then decreases with p^{CB}. But as p^{CB} continues to increase, the social welfare function kinks at $p^{CB} = \bar{p}^{CB}$, where the qualitative nature of the equilibrium changes. At and beyond the kink, the equilibrium is characterized by cash investors using all their collateral for repo finance *and* credit investors providing all their real balances for repo finance in the FM subperiod.[33] Notice that the social welfare maximizing repo price p^{CB} lies in between $\tilde{p}_H^{FM}$ and $\bar{p}^{CB}$ in the left panel, which implies that cash investors do not repo finance all their collateral in the FM subperiod, i.e., $b_H^b + b_H^{CB} < b$. Hence, total liquidity remains scarce in state $i = H$ at the social welfare maximizing central bank repo price. In contrast, social welfare reaches its maximum at $\bar{p}^{CB}$ in the middle panel. Here, market liquidity is so scarce—because $\sigma_H = 0.95$—that buyers want to repo finance all of their collateral at the social welfare maximizing repo price, $\bar{p}^{CB}$. When $\tilde{p}_H^{FM} < p^{CB} < \bar{p}^{CB}$, however, buyers do not repo finance all of their collateral and total liquidity is scarce in state $i = H$. The left and middle panels provide examples of σ_H being "sufficiently large" and, as a result, an active central bank repo facility in state $i = H$ is welfare enhancing.

The right panel in Figure 2 provides an example where σ_H is not "sufficiently large." When $\sigma_H = 0.80$, market liquidity is scarce in state $i = H$ in the equilibrium when the central bank repo facility is always inactive. But unlike the two other examples, σ_H is not sufficiently large. When the central bank's standing facility becomes active in state $i = H$ (by having p^{CB} increase above $\tilde{p}_H^{FM}$), social welfare immediately declines—see the right panel. In this example, *expected* market liquidity is impaired by an active central bank repo facility because the decrease in market liquidity in state $i = L$ dominates the increase in total liquidity in state $i = H$.

These examples indicate that a central bank repo facility is beneficial only when market liquidity is "really" scarce in state $i = H$ and that scarce market liquidity need not imply that an active central bank repo facility is welfare improving. Our two-state model succinctly identifies the costs and benefits associated with an active standing repo facility when state $i = L$ is characterized by an abundance of market liquidity and $i = H$ by scarcity. Although the two-state model is both simple and illustrative, it does not capture all of the potential equilibrium configurations that can arise; and the equilibria that do arise seem to depend critically on how

[33] This equilibrium configuration can exist (when the central bank is always inactive) if the value of σ_H greater than σ_L, as we have assumed, but not too large. Such an equilibrium configuration, when the central bank is always inactive, is characterized by scarcity in state $i - L$ and by having cash investors sell all of their assets and credit investors use all of their real balances to purchase them in state $i = H$. In our more general model, in Section 5 and Appendix D, we fully characterize this FM subperiod equilibrium outcome.

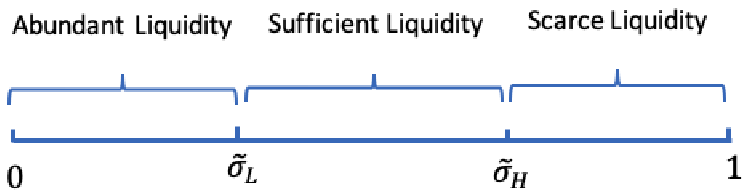


Figure 3: Market liquidity

we choose the parameter values for σ_L and σ_H.[34] To remedy these issues we expand the number of possible states from 2 to $[0, 1]$. When $\sigma \in [0, 1]$, all possible configurations will arise in equilibrium.

5 A General Model

We now assume that σ is continuous and independently and identically distributed over $[0, 1]$ with distribution $F(\sigma)$. Three distinct equilibrium configurations can arise in the FM subperiod when the central bank repo facility is always inactive: when σ is "small," the equilibrium is characterized by abundant market liquidity; when σ is "large," it is characterized by scarce market liquidity; and when σ is neither small nor large, it is characterized by *sufficient* market liquidity, see figure 3.[35]

Critical cutoffs values, $\tilde{\sigma}_L$ and $\tilde{\sigma}_H$, that separate the three equilibrium regions in Figure 3 are determined as follows.[36] Market liquidity is abundant for all σ that satisfy $\sigma\bar{b} < (1-\sigma)z$. The critical cutoff $\tilde{\sigma}_L$ occurs when total market liquidity just becomes sufficient when the FM subperiod bond price is equal to 1, i.e., when $\tilde{\sigma}_L\bar{b} = (1-\tilde{\sigma}_L)z$ or

$$\tilde{\sigma}_L = \frac{z}{z+\bar{b}}. \tag{32}$$

The critical cutoff $\tilde{\sigma}_H$ occurs when market liquidity is sufficient but becomes scarce when σ is

[34] For example, a two-state model cannot have FM subperiod equilibria that are characterized by market liquidity that is scarce, abundant *and* sufficient. By construction, only two of the three possible configurations can arise.

[35] We continue to assume that $\bar{b}$ is "not large" in the body of the paper. In Appendix E, we precisely characterize what it means for $\bar{b}$ to be "large" and "small," as well the equilibria with and without an active standing repo when $\bar{b}$ is "large."

[36] The critical cutoff $\tilde{\sigma}_L$ is unique. The critical cutoff $\tilde{\sigma}_H$ need not be unique. We are, however, unable to generate any examples where $\tilde{\sigma}_H$ is not unique. We assume that the critical cutoff $\tilde{\sigma}_H$ is unique in the main body of the paper and provide conditions for a unique cutoff in Appendix C.

increased from $\tilde{\sigma}_H$, i.e., when $\lambda(z/\tilde{\sigma}_H) + 1 = 1/p^{FM}_{\tilde{\sigma}_H} = \tilde{\sigma}_H \bar{b}/\left(1 - \tilde{\sigma}_H\right) z$ or

$$\tilde{\sigma}_H = \frac{z[\lambda(z/\tilde{\sigma}_H) + 1]}{z[\lambda(z/\tilde{\sigma}_H) + 1] + \bar{b}}.^{37} \tag{33}$$

When σ is continuous, both the bond pricing and money demand equilibrium equations are straightforward generalizations of (21) and (23), respectively. In particular, asset prices are simply weighted averages of the liquidity premia that arise across the various states σ.[38] For example, regarding real balances, cash investors have strictly positive liquidity premia in all states σ while credit investors have strictly positive premia only in states where market liquidity is not abundant, i.e., in states $\sigma > \tilde{\sigma}_L$. And regarding government bonds, cash investors have strictly positive liquidity premia in states where they repo finance all of their collateral, i.e., in states $\sigma \in (0, \tilde{\sigma}_H)$, while credit investors never receive a strictly positive liquidity premium.

We now show by way of numerical examples that the main results and insights from our two-state environment generally carry over to a continuous distribution world.[39] When the central bank repo facility is always inactive the equilibrium CM subperiod asset price and real balance holdings for our parameterization are $\tilde{p}^{CM} = 0.9990$ and $\tilde{z} = 0.9498$, respectively.

Figure 4(a) illustrates how the equilibrium FM subperiod repo bond price, $\tilde{p}^{FM}_{\sigma}$, varies with σ while figure 4(b) illustrates the equilibrium quantity of collateral repo finance for the representative cash investor in the FM subperiod, $\tilde{b}^{c}_{\sigma}$. At lower values of σ, market liquidity is abundant and the competitive repo price $\tilde{p}^{FM}_{\sigma}$ equals 1, which is the payoff to the government bond in the subsequent CM subperiod. In these states, cash investors repo all of their collateral. Nevertheless, the demand for market liquidity *relative to total potential supply of market liquidity* is small because the total amount of collateral that is repo financed, $\sigma\bar{b}$, is relatively small. (The total supply of collateral that is repo financed is relatively small because σ is small.) The

[37] To understand this equality, notice from (8) we have that $\lambda(z/\tilde{\sigma}_L) + 1 > 1/p^{FM}_{\tilde{\sigma}_L} = 1$ since $z + \bar{b} = z/\tilde{\sigma}_L$ and $p^{FM}_{\tilde{\sigma}_L} = 1$. As σ increases from $\tilde{\sigma}_L$, we have, by continuity,

$$\lambda(z/\sigma) + 1 > 1/p^{FM}_{\sigma} = \bar{b}\sigma/[(1-\sigma)z]$$

for σ "close" to but greater than $\tilde{\sigma}_L$ because $\sigma p^{FM}_{\sigma}\bar{b} = (1-\sigma)z$ implies $p^{FM}_{\sigma} = (1-\sigma)z/(\sigma\bar{a})$. Since

$$\lambda(z/\sigma) + 1 < \bar{b}\sigma/[(1-\sigma)z]$$

as $\sigma \to 1$, by continuity, there exists a $\tilde{\sigma}_H \in (\tilde{\sigma}_L, 1)$ such that $\lambda(z/\tilde{\sigma}_H) + 1 = \bar{b}\tilde{\sigma}_H/[(1-\tilde{\sigma}_H)z]$, which can be rearranged to (33).

[38] The asset pricing equations are derived in Appendix D.

[39] We parametrize the model as follows: F is the standard uniform distribution and support concentrated on $[0, 1]$, $c\,(y) = y$ and

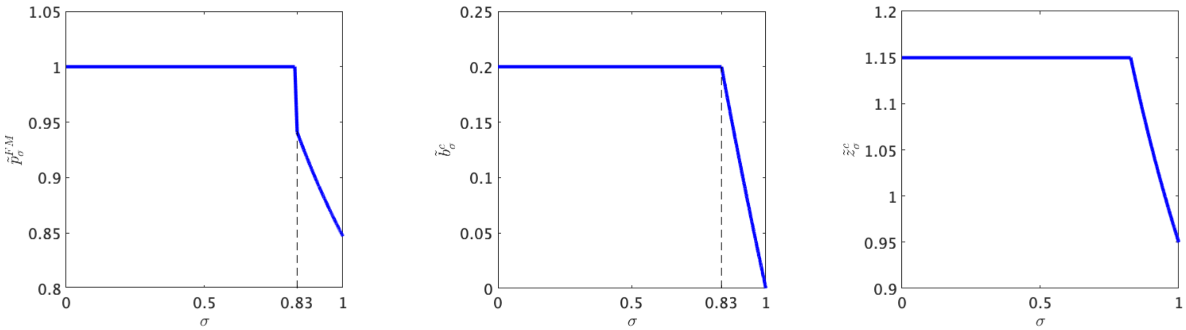
$$f\,(y) = \frac{(y+\varepsilon)^{1-\gamma} - \varepsilon^{1-\gamma}}{1-\gamma}.$$

And we set the parameter values as: $\bar{b} = 0.2$, $\varepsilon = 0.05$, $\gamma = 0.7$, $\delta = 0.01$, $\beta = 0.98$, $\iota = 0.04$ and $\theta = 0.7$.

equilibrium has credit investors providing only a fraction of their money holdings in these states σ for repo finance. As σ increases, we enter the sufficient market liquidity region, $\sigma \in (\tilde{\sigma}_L, \tilde{\sigma}_H)$, where we see, in figure 4(a), that the FM subperiod repo bond price $\tilde{p}_\sigma^{FM}$ drops sharply from 1 with increases in σ. In this region, cash investors repo finance all of their collateral and credit investors use all of their real balances to finance repo. The sharp decline in the FM subperiod bond price $\tilde{p}_\sigma^{FM}$ as σ increases can be explained by the combination of a decline in total market liquidity—since $1 - \sigma$ falls—and an increase in total collateral supply—since σ increases and $\tilde{b}_\sigma^c = \bar{b}$. Finally, when $\sigma \geq \tilde{\sigma}_H$, the FM subperiod repo bond price $\tilde{p}_\sigma^{FM}$ continues to decline with increases in σ but not as rapidly, as shown in figure 4(a). The critical value $\tilde{\sigma}_H$ occurs at the "second kink" in figure 4(a) in the repo bond price—where the slope changes from very steep to less steep—and at the kink in figure 4(b). In our example the sufficient market liquidity region is very small; in particular, $(\tilde{\sigma}_L, \tilde{\sigma}_H) = (0.826, 0.835)$. For σ's beyond $\sigma = \tilde{\sigma}_H$, market liquidity is scarce and cash investors do not repo finance all of their collateral and repo less collateral as σ increases, while credit investors always supply all of their real balances for repo finance. This combination mitigates the decline in $\tilde{p}_\sigma^{FM}$ compared to the previous region, as illustrated in figure 4(a). When market liquidity is scarce, the FM subperiod repo bond price is "too low" to induce cash investors to repo finance all of their collateral.

Figure 4(b) shows that the quantity of bonds that the typical cash investor repo finances is constant over the entire region of σ where $\tilde{p}_\sigma^{FM} = 1$ and market liquidity is abundant, as well as for the small region where $\tilde{p}_\sigma^{FM} < 1$ and market liquidity is sufficient. Figure 4(c) illustrates the total money holdings of the cash investor after the FM subperiod transactions, $\tilde{z}_\sigma^c \equiv \tilde{z} + \tilde{p}_\sigma^{FM} \tilde{b}_\sigma^c$. Money holdings are constant for all σ where $\tilde{p}_\sigma^{FM} = 1$, when market liquidity is abundant. When the FM subperiod repo bond price falls below 1—this happens when $\sigma > \tilde{\sigma}_L = 0.826$—the cash investor's money holdings $\tilde{z}_\sigma^c$ decline with σ. But unlike the equilibrium FM subperiod repo bond price, $\tilde{p}_\sigma^{FM}$, which initially declines rapidly and then tails off, the rate of decline in money holding appears to be more or less constant. This is because when $\tilde{p}_\sigma^{FM}$ drops rapidly, cash investors continue to repo finance all of their collateral (when market liquidity is sufficient) and when the decline in $\tilde{p}_\sigma^{FM}$ moderates, buyers are only repo financing a fraction of their collateral (when market liquidity is scarce). These effects work to smooth the decline in real balance holdings over the two regions where market liquidity is sufficient and scarce and $\tilde{p}_\sigma^{FM} < 1$.

We now examine a sometimes active central bank repo facility. Define $\bar{\sigma}$ as the critical state such that for all $\sigma > \bar{\sigma}$ cash investors receive repo finance from both the competitive financial



(a) p_σ^{FM} (b) $\tilde{b}_\sigma^c$ (c) DM z

Figure 4: Numerical Examples: Beta Distribution for σ

market and the central bank repo facility; for all $\sigma < \bar{\sigma}$ cash investors receive repo finance only from the competitive market, i.e., the central bank repo facility is inactive. As above, we denote real balances, bond prices and other critical values when the central bank repo facility is active in some states with a "hat," i.e., $\hat{z}$, $\hat{p}^{CM}$, $\hat{p}_\sigma^{FM}$, $\hat{\sigma}$ and so on. The precise relationship between $\bar{\sigma}$ and the central bank's repo price, p^{CB}, depends on whether market liquidity is either scarce or sufficient at $\bar{\sigma}$. If market liquidity is scarce at $\bar{\sigma}$, then $\bar{\sigma} > \hat{\sigma}_H$. This occurs if p^{CB} is not too big. Then from (28) we have

$$\lambda(\frac{z}{\bar{\sigma}}) = \frac{1 - p^{CB}}{p^{CB}}. \tag{34}$$

In this siutation cash investors repo finance $b_{\bar{\sigma}}^*$ collateral in total using both financial markets *and* the central bank repo facility in all states $\sigma > \bar{\sigma}$. In these states, market and central bank liquidity is scarce.[40]

If market liquidity is sufficient at $\bar{\sigma}$, then the critical value $\hat{\sigma}_H$ does not exist. This situation occurs if p^{CB} is sufficiently big: cash investors will repo finance all of their collateral in states $\sigma > \bar{\sigma}$. In fact, since $\bar{\sigma} > \hat{\sigma}_L$ the cash investor repo finances all of his collateral in *all* states of the world. The critical value $\bar{\sigma}$ is determined by the equality of economy wide value of repo finance by cash investors, who each repo $\bar{b}$ collateral at repo price p^{CB}, with market liquidity, i.e., $\bar{\sigma} p^{CB} \bar{b} = (1 - \bar{\sigma}) z$ or

$$\bar{\sigma} = \frac{z}{z + p^{CB} \bar{b}}.$$

We now return to our numerical example to illustrate equilibrium outcomes and optimal cen-

[40] Market and central bank liquidity is scarce in the sense that cash investors do not sell all of their bond holdings in the FM subperiod and credit investors use all of their money balances to purchase bonds in the FM subperiod.

tral bank repo policy of the central bank. Figure 5(a) illustrates the CM subperiod equilibrium bond price, $\hat{p}^{CM}$, and real balance holdings, $\hat{z}$, as a function of the central bank repo price p^{CB}. When p^{CB} is strictly less than the FM subperiod bond price at $\sigma = 1$ the central bank repo facility is always inactive, i.e., when $p^{CB} < \tilde{p}_1^{FM} = 0.847$, and an increase in p^{CB} has no effect, at least initially, on the equilibrium outcomes. This is illustrated in Figure 5(a) by the horizontal blue and red lines for $p^{CB} < 0.847$. As p^{CB} increases beyond 0.847, cash investors will access the central bank's repo facility in an increasing number of σ-states to obtain additional liquidity.[41] Because collateral (government bonds) can generate additional liquidity for cash investors, investors' demand for government bonds in the CM subperiod increases. Hence, an increase in p^{CB} increases the equilibrium bond price, $\hat{p}^{CM}$, and decreases real balance holdings, $\hat{z}$, as illustrated by the blue and red lines, respectively, in Figure 5(a). The "kink" in the (red) money demand curve, which occurs at $p^{CB} = 0.923$, has a special significance. For $p^{CB} < 0.923$, $\hat{\sigma}_H > \bar{\sigma}$ which means that states $\sigma > \bar{\sigma}$ are characterized by market and central bank liquidity scarcity.[42] For $p^{CB} > 0.923$, $\hat{\sigma}_H$ does not exist, which means that for all $\sigma \geq \bar{\sigma}$ cash investors repo finance all of their collateral in the FM subperiod.[43]

Figure 5(b) shows the percentage change in welfare associated for various values of p^{CB}. Welfare is maximized when the central bank sets $p^{CB} = 0.923$. Interestingly, this is the value of p^{CB} at the kink of the money demand function in figure 5(a).[44] The optimal central bank intervention sets the repo price so that cash investors repo finance all of their collateral when the repo facility is active. The optimal central bank intervention increases welfare by about 5.93 basis points compared to an always inactive standing repo facility and real balance holdings decline by about 3.5%, (more specifically, $\hat{z} = 0.9171$ versus $\tilde{z} = 0.9498$, respectively).

Figure 6 compares financial market outcomes for an optimal central bank repo facility—in red—and an always inactive repo facility—in blue.[45] As can be seen in Figure 6(a), the optimal central bank repo *rate* provides a cap on market *rates* in high σ-states of the world.[46] When

[41] When the central bank repo price initially increases beyond 0.847, the relevant bond price and money demand equations are given by (48) and (49), respectively, in Appendix D.

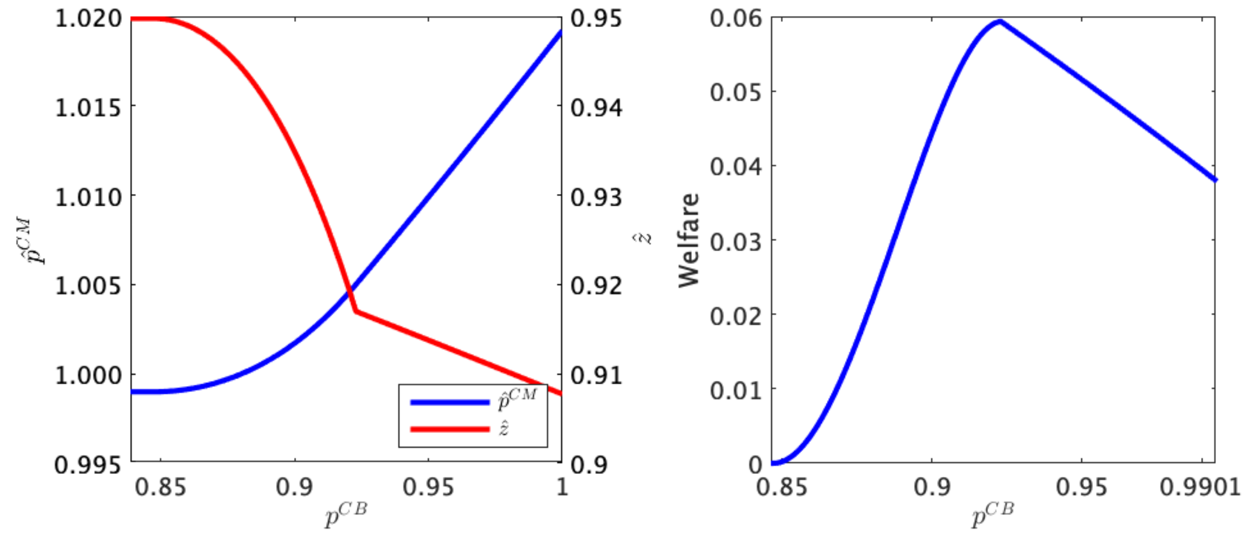
[42] In this case, the equilibrium bond and money demand equations are given by (48) and (49), respectively, in Appendix D.

[43] In this situation, the equilibrium bond and money demand function are given by (50) and (51), respectively, in Appendix D. In terms of our example, p^{CB} is "large enough" when $p^{CB} \geq 9.23$ and, as a result, $\hat{\sigma}_H$ does not exist.

[44] We parameterize the example in the subsequent section so that $\hat{\sigma}_H > \bar{\sigma}$. We were unable to generate any examples where the pdf is symmetric on $[0, 1]$ and $\hat{\sigma}_H > \bar{\sigma}$. We are able to generate an example characterzed by $\hat{\sigma}_H > \bar{\sigma}$—in the next section—by assuming a non-symmetric pdf.

[45] We discuss the yellow lines below.

[46] Using market and central bank repo rates instead of FM subperiod repo bond prices, help facilitate the



(a) Equilibrium (b) Welfare

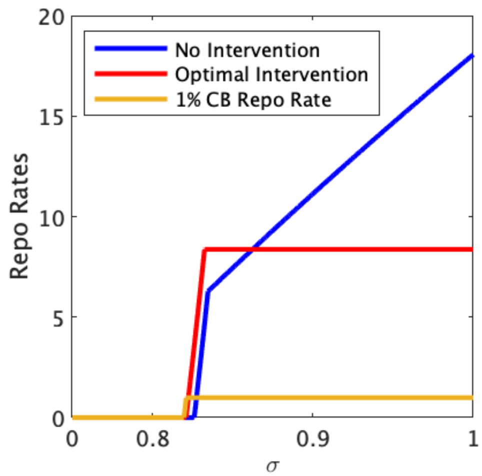
Figure 5: CB Intervention: Uniform Distribution for σ

the central bank repo facility is always inactive, scarcity of market liquidity becomes more acute in higher σ-states and as σ increases, higher market repo rates follow. The market repo rate initially and significantly spikes from zero—for both the optimal and the always inactive central bank repo facilities—when market liquidity is no longer abundant since cash investors continue to repo finance all of their collateral even when FM subperiod bond prices fall. For an always inactive central bank repo facility, the increase in the repo rate is moderated as σ increases since cash investors choose to repo finance smaller fractions of their collateral holdings (owing to lower market repo bond prices). At the point where market liquidity is no longer abundant—and repo rates exceed zero—notice that the market repo rate for an always inactive facility is, at least initially, *less than* the market repo rate for an optimal repo facility, i.e., the red line lies above the blue.[47] But as σ continues to increase at some point—where the blue and red lines cross in figure 6(a)—the repo rate associated with the optimal repo facility is always less than the market repo rate for an always inactive facility. These observations highlight the role played by a central bank repo facility: the facility essentially insures investors against aggregate liquidity risk by equalizing marginal liquidity costs across very high-σ states of the world.

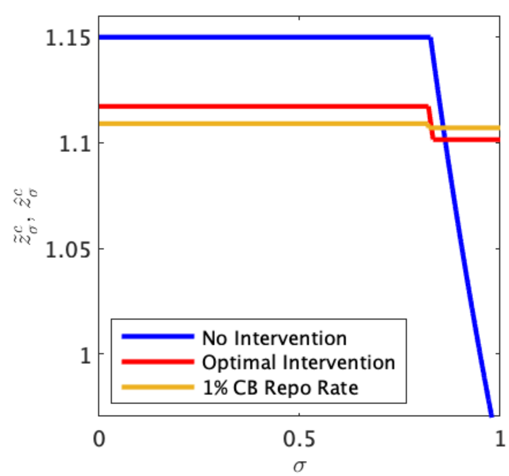
If the central bank repo facility is always inactive, then market liquidity will be scarce in

comparison between an active and always inactive central bank repo facility.

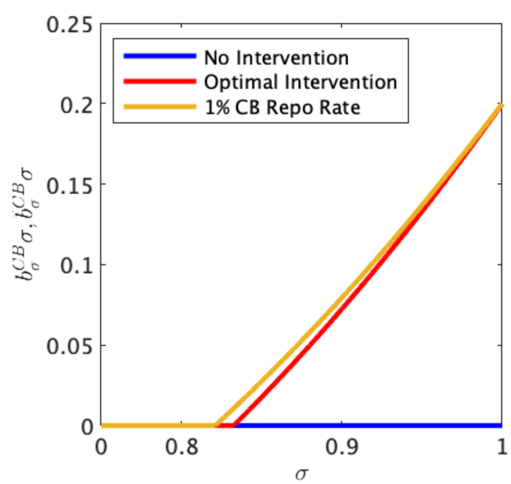
[47] This result should be anticipated since $\tilde{z} > \hat{z}$.



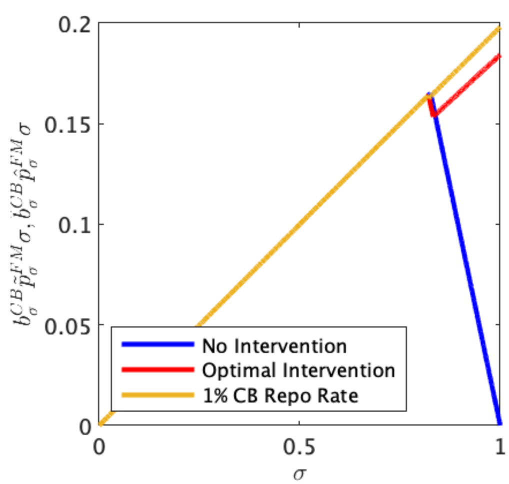
(a) Repo Rates



(b) DM z



(c) Asset Purchased by CB



(d) FM Value of Traded b

Figure 6: Financial Market with CB Intervention

high-σ states. In this situation as σ increases, FM subperiod repo bond prices continue to fall (and market repo rates continue to increase); cash investors respond to these lower repo bond prices (higher market repo rates) by repo financing less collateral. As a result, total liquidity for a representative cash investor, $\tilde{z}_\sigma = \tilde{z} + \tilde{p}_\sigma^{FM} \tilde{b}_\sigma^c$, strictly decreases as σ increases, as illustrated by the blue line in Figure 6(b). If, instead, the central bank repo rate is optimally set, the facility will become active when market liquidity is sufficient.[48] Hence, the collateral that the representative cash investor repo finances is constant over all σ and equal to $\bar{b}$. As a result, total real balance holdings for the representative cash investor entering the DM subperiod, $\hat{z}_\sigma = \hat{z} + \hat{p}_\sigma^{FM} \bar{b}$, are constant across all the high σ-states, where $\sigma \geq \bar{\sigma}$, as illustrated by the red line in Figure 6(b). Although cash investors repo finance all of their collateral in the FM subperiod, total liquidity in higher $\sigma \geq \bar{\sigma}$ states is less than in lower σ states because the FM subperiod repo bond prices are lower. When comparing optimal and always inactive standing repo facilities, there exists a trade-off in the investor's CM subperiod real balance holdings. An investor's DM subperiod real balance holdings are higher when the facility is always inactive compared to the optimal repo facility. When the facility is always inactive the investor essentially "self insures" against high σ's by accumulating more real balances in the CM subperiod compared to an investor that has the optimal standing repo facility insuring against scarcity. This difference in real balance holdings can be seen in figure 6(b) and is measured by the vertical distance between the blue and red lines when market liquidity is abundant, i.e., at lower values of σ. This observation is rather important when assessing the potential welfare gains associated with an active central bank repo facility. In particular, investment and consumption will be lower in low σ-states and higher in high σ-states with the optimal central bank repo facility compared to one that is always inactive.

When the central bank repo rate is optimally set, the central bank repo finances an ever increasing amount of collateral when σ is high and increasing as illustrated by the red line in figure 6(c). The total value of repo finance in the FM subperiod, including central bank repo finance, is illustrated in Figure 6(d). Here the red line shows that the total value of repo finance declines as σ increases when market liquidity is sufficient and the central bank repo facility remains inactive. Over this range of σ although more cash investors are repo financing $\bar{b}$ collateral as σ increases, the decline in repo bond prices results in a reduction in the total value of market value of repo finance. When the central bank repo facility becomes active at

[48] We emphasize that for this particular example is characterized by sufficient market liquidity at $\sigma = \bar{\sigma}$.

$\sigma = \bar{\sigma}$, further increases in σ results in higher total value of repo finance since the central bank's repo facility stabilizes the repo bond price at p^{CB}. When the standing repo facility is always inactive—given by the blue line in Figure 6(d)—once market liquidity ceases to be abundant, i.e., $\sigma > \hat{\sigma}_L$, the total value of repo finance declines to zero as σ increases.

In our example, the optimal repo rate for central bank repo facility is 8.37% which implies that 82.08% of the time market liquidity is abundant with an associated market repo rate equal to 0.0%. In an *ex ante* sense, the central bank's repo rate can be interpreted as a being a *penalty* rate since the vast majority of the time it exceeds the market rate. Cash investors use the standing repo facility only when $\sigma \geq 0.8323$, which occurs only 16.77% of the time: hence, the vast majority of the time, the standing repo facility is not used. When the central bank repo facility is always inactive the market repo rate can be as high as 18.07%. Nevertheless, even in this environment the market repo rate will be zero most (82.60%) of the time. These numbers imply that the *ex ante* welfare gain associated with a central bank repo facility will be very small since the facility is used only a small fraction of the time: the welfare gain associated with an optimal central bank repo facility is about 6 basis points. However, and importantly, when cash investors do use the facility, the *state contingent* welfare gain associated can be quite large. For example, when $\sigma = 0.9$, the welfare gain is 0.57% and when $\sigma = 0.95$, it is 1.58%.

The yellow lines in Figure 6 describe a situation where the central bank's repo rate is set at a non-optimal and very low rate, equal to 1%. Figures 6(b) and 6(d) indicate that setting such a low rate generates rather "stable" outcomes across states. In particular, the liquidity that a representative cash investor brings into the DM subperiod is (almost) invariant to the value of σ—see figure 6(b)—and the total value of repo finance is (essentially) an increasing function of σ—see figure 6(d). Figure 6(c) illustrates that central bank repo finance is always higher when the repo rate is set lower than the optimal setting, and that the facility will be used at lower values of σ. Although these outcomes are stable across states, consumption for cash investors will be less in most states compared to an optimally set central bank repo rate, see Figure 6(b). Since investors know they have access to "inexpensive" liquidity in the FM subperiod, they accumulate less real balances in the CM subperiod, consume less in most states of the world, resulting in lower welfare, compared to the optimal setting. In fact, for this example, the 1% central bank repo rate results in a level of welfare that is lower than an always inactive central bank repo facility.

An important insight from our benchmark, two-state model is that market liquidity has to be very scarce if a central bank standing repo facility is to be beneficial. This insight carries over to the continuous state case. When $\sigma \in (0,1)$, by construction, there always exist states of the world where market liquidity is very scarce. As a result a central bank repo facility can always be welfare improving. But providing central bank repo liquidity "too generously"—i.e., when liquidity is scarce but not very scarce, as in the yellow lines in Figure 6—can actually cause welfare to fall. When the repo rate is optimally set, central bank repo finance will be a low probability event, meaning that even though a central bank repo facility is beneficial and state contingent gains may be large, the *ex ante* welfare gain will typically be very small. Baghot's prescription of "lend freely on good capital but at a high rate" is captured in our example. Setting a central bank repo rate that is "too low" creates a moral hazard problem—investors' accumulate too little private liquidity since they can relie on the central bank for cheap liquidity—which results in a decrease in social welfare.

6 'Standing' v. 'Emergency' Central Bank Repo Facilities

The Federal Reserve established a *standing* repo facility in July 2021. This facility is available every business day to accredited counterparties and the facility's repo rate (price) and other policy parameters are both known and clearly specified in advance. We will use the term 'standing facility' to describe Bagehot's policy prescription that a central bank's liquidity lending rates, as well as other important parameters, be clearly articulated and known to all. Prior to July 2021, the Federal Reserve supplied liquidity to financial markets in the aftermaths of major liquidity events on an "ad hoc" basis, through emergency or temporary repo operations and facilities. For example, after the onset of the 2007-08 financial crisis, the Fed established a variety of emergency repo-type facilities and operations, such as the Primary Dealer Credit Facility.[49] More recently, when a shortfall in central bank reserves significantly and suddenly elevated overnight rates in September 2019, the Fed offered up to $75 billion in overnight repo funding using an auction mechanism with minimum reserve bid. And in March 2020, in response to a huge increase in demand for US dollars resulting from the COVID-19 pandemic, which resulted in dramatic spikes in market repo rates, the Federal Reserve deployed a number of overnight repo operations and facilities, including a Primary Dealer Credit Facility, to supply liquidity to domestic and international market participants. We will use the term 'emergency facility' to describe earlier Federal Reserve ad hoc facilities.

[49] The Primary Dealer Credit Facility provided overnight secured (repo) loans to primary dealers.

In this section we ask whether a standing repo facility—as described by, say, the Federal Reserve's current standing repo facility—provides superior outcomes to an emergency repo facility—as described by, say, past Federal Reserve emergency interventions where policies arise *after* a major liquidity shock has been realized. Even though a standing repo liquidity facility can insure all investors against extreme market liquidity scarcity events, it is not obvious that it is better than an emergency repo facility. There is a tradeoff. Although a standing repo facility insures all investors against adverse shocks, it also reduces their incentive to accumulate CM subperiod real balance holdings—and, hence, reduce market liquidity—compared to an emergency facility. This implies that investment and consumption levels associated with a standing repo facility will be less than those associated with an emergency facility in states of the world where investors do not use any central bank repo facility (since market liquidity is abundant and/or sufficient). The emergency repo facility will generate higher consumption and investment, compared to a standing facility, the vast majority of the time since the central bank's repo facility is not accessed the vast majority of the time.

We view the timing in Figure 1 as one that describes a standing repo facility because the facility is always available to *all* investors at clearly announced FM subperiod terms of trade. We interpret an emergency facility as one where investors understand that the central bank will intervene in the event of severe scarcity of market liquidity but only after it observes this scarcity. But to observe scarce market liquidity necessarily implies that (some) investors trade in the FM subperiod competitive financial market without the support of the central bank. If the competitive financial market reveals significant scarcity of market liquidity, then the central bank reacts by providing liquidity via a repo facility to remaining investors. We model the notion of an emergency repo facility by subdividing the FM subperiod into two parts—"early" and "late"—and assigning a fraction of investors to visit the FM subperiod early and the remainder late. More specifically, at the beginning of the period when investors learn whether they are cash or credit investors, they also learn whether they participate in the early or late FM subperiod. A fraction α of investors enter and trade in the "early" FM subperiod—without the aid of any central bank repo facility—and then exit. The central bank and the remaining $1 - \alpha$ investors observe the market repo price/rate generated by the α investors in the early FM subperiod. The central bank establishes an emergency repo facility only if the early FM subperiod repo price falls below a predetermined FM subperiod bond price p_{α}^{CB}; a facility is not established if the early FM subperiod repo price exceeds p_{α}^{CB}. After the central bank's decision (to establish

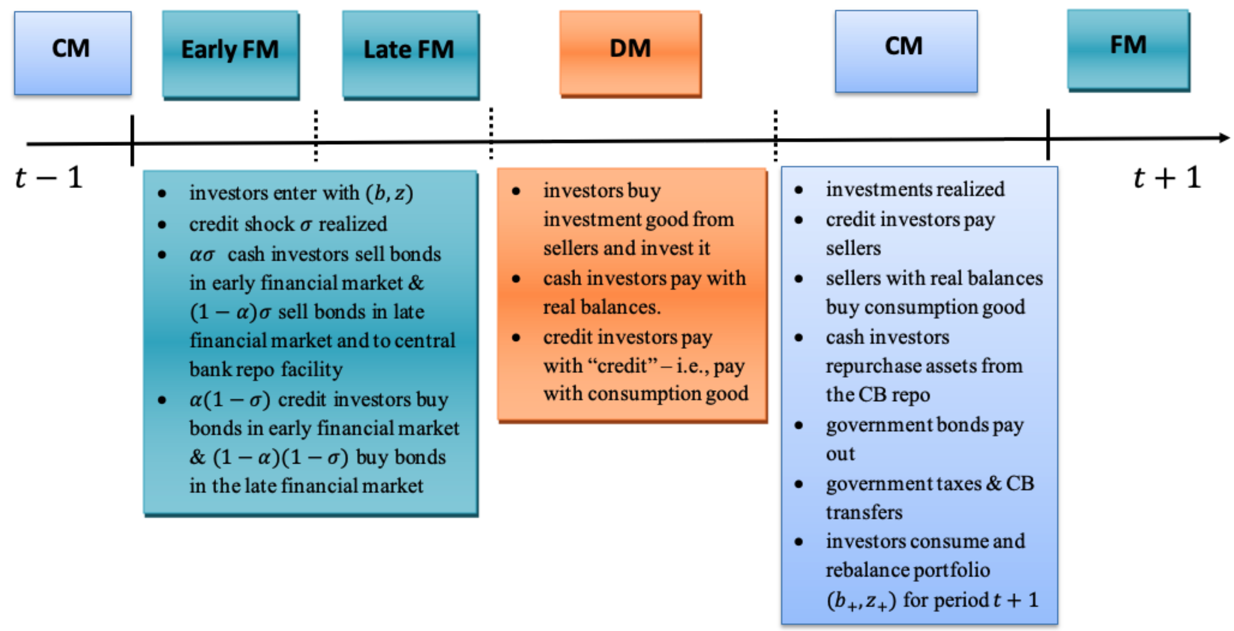


Figure 7: Timing of events for emergency repo facility

an emergency facility), the remaining $1 - \alpha$ investors enter the late FM subperiod and trade in a late FM subperiod competitive financial market and with the central bank, if an emergency facility was established. Intuitively, this timing, which is illustrated in Figure 7, operationalizes the idea that of an ad hoc central bank repo facility, i.e., a repo facility that is established when significant scarcity in market liquidity is observed.

As above, the repo price p^{CB}_{α} will be associated with a state $\bar{\sigma}_a$ such that a central bank emergency facility with repo price p^{CB}_{α} emerges in all states $\sigma > \bar{\sigma}$.[50] Just as in the previous section, both the equilibrium bond pricing and money demand equations when an emergency repo facility may emerge are functions of the expected liquidity premia they generate.[51] The money demand and bond pricing equations not only reflect the existence of an emergency repo facility in states $\sigma > \bar{\sigma}_{\alpha}$ but also that α cash investors—the early investors—cannot use a central bank repo facility in the FM subperiod in those states. And, again, just as in the previous section, the precise form of the bond pricing and money demand equations depend, in part, on whether or not the critical value $\hat{\sigma}^{\alpha}_{H}$ exists.

The equilibrium bond pricing and money demand equations are functions of α, the measure

[50] In all states $\sigma > \bar{\sigma}$, the early market repo price is strictly less than p^{CB}_{α}. When there is an emergecy facility, critical parameters will be indexed by α, e.g., $\bar{\sigma}_{\alpha}$, $\hat{\sigma}^{\alpha}_{L}$, $\hat{\sigma}^{\alpha}_{H}$ and so on,

[51] The precise form of the equilibrium bond pricing and money demand equations for the emergency repo facility are derived and can be found in Appendix D.

of investors that arrive in the early FM subperiod. Increasing α results in higher consumption in all states $\sigma < \bar{\sigma}$. Intuitively, increasing α increases the probability that cash investors will *not* be able to access central bank liquidity when an emergency facility is established. Because of this, investors effectively self-insure against this possibility by increasing their real balance holdings. As a result, consumption increases in those states where market liquidity is not significantly scarce, $\sigma < \bar{\sigma}$. Holding all else constant, this increases welfare. However, increasing α decreases consumption for a greater measure of early cash investors in states $\sigma > \bar{\sigma}$ and, holding all else constant, this decreases welfare. Theoretically, the total effect on welfare from a change in α is ambiguous. To understand how change in α affects welfare we undertake the following quantitative exercise.

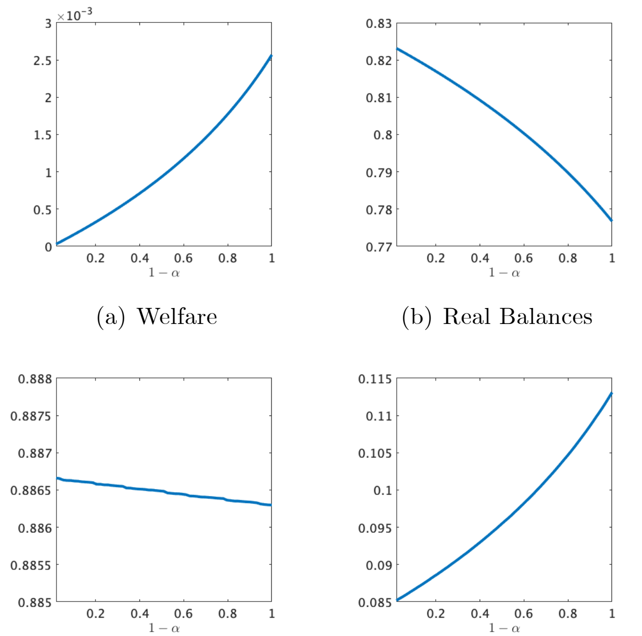
For our quantitative exercise, we set $f(y) = y^{1-\gamma}$, $c(y) = y$ and F to a beta distribution with parameters a_1 and a_2. We need to provide values for parameters a_1, a_2, $\bar{b}$, β, γ, μ and θ. Some of these parameter values can be determined directly from data by while other values can come from predictions of our model that are disciplined by or match the data. We consider an annual model so we set $\beta = 0.98$. We set $1 - \gamma = 0.4$ which can be thought as matching the capital share of 40% found in the data under the assumption that labor supply is inelastic. We set $\mu = 0.02$ to generate a 2% annual inflation and $\theta = 0.5$ so that the investors and the investment good producers have equal bargaining power as a benchmark. It is difficult to pick a_1 and a_2. As a benchmark, we set $a_1 = 2$ and $a_2 = 4$, which implies that the pdf is skewed to the left. One potential benefit of this choice of a_1 and a_2 is that extreme scarcity in market liquidity can be interpreted as a tail event.[52] Given these parameters, we set $\bar{b}$ such that the average repo rate from the model matches the average Broad General Collateral Rate (BGCR) of 2018 and 2019 , which is 2.08%. The derived value of $\bar{b}$ is consistent with the small asset case scenario we have analyzed throughout, where $\bar{b}$ is small in the sense that cash investors are unable to purchase the efficient amount of investment good when market liquidity is abundant. The parameter values for the quantitative exercise can be found in Table 1.

We now use the model to assess whether a standing or ad hoc central bank repo facility is better. We vary α betweeb 0 to 1 and for each α calculate welfare changes for the optimal central bank repo price, p_{α}^{CB}. Recall that α represents the fraction of early investors under an

[52]For the examples in Section 5 we assume a uniform distribution for σ. Since any fixed interval on $[0, 1]$ has the same probability of occuring, extreme scarcity of market liquidity would not be viewed as a tail event. We have also experimented several other distributions. The *qualitative* features described Figure 8 are unchanged by alternative assumptions for the distribution function.

Parameter	Description	Value	Target
β	Discount Factor	0.98	
γ	Curvature of f	0.6	Capital Share 40%
μ	Money Growth	0.02	Inflation of 2%
θ	DM Bargaining Power	0.5	Equal Bargaining Power
a_1	Beta Distribution Par	2	Uniform Distribution
a_2	Beta Distribution Par	4	Uniform Distribution
$\bar{b}$	Bond Quantity	0.681	2.08% Average BGCR

Table 1: Benchmark Parameters for the Quantitative Analysis



(a) Welfare (b) Real Balances

(c) Optimal p^{CB} (d) Intervention Probability

Figure 8: Emergency vs Permanent Facility: Benchmark

emergency repo standing facility scenario that do not have access to central bank repo services. If welfare is maximized at $\alpha = 0$, then we can conclude that a standing central bank repo facility is best; if welfare is maximized at some interior point, then we can conclude that an ad hoc repo facility is best.

Figure 8 illustrates the results from the baseline parameters. Figure 8(a) shows welfare as a function of $1 - \alpha$, normalizing the value at $\alpha = 1$ to 0. As $1 - \alpha$ increases, welfare increases and is maximized at $\alpha = 0$. This implies that a standing central bank repo facility does better than an ad hoc one when the central bank chooses its repo asset price/borrowing rate optimally.[53]

[53] Below, we investigate this relationship if, for some reason, the central bank does not choose the repo price/rate

Figure 8(b) shows a negative relationship between investors' accumulation of CM subperiod real balances and $1-\alpha$, the fraction of investors that can access the central bank's emergency repo facility. When $1-\alpha$ increases, cash investors will have easier access to the emergency repo facility. Hence, investors have less incentive to accumulate real balances in the CM subperiod since they have a higher probability to access the emergency standing facility for repo finance. An implication is that the CM subperiod asset price, p_α^{CM}, increases.

Figure 8(c) shows that the optimal central bank repo price, p_a^{CB}, is a decreasing function of "access" to the emergency repo facility, $1-\alpha$. As $1-\alpha$ increases, more cash investors are able to access the emergency repo facility in states where market liquidity is significantly scarce. As a result, if the central bank keeps its repo price fixed, investors will be induced to accumulate less real balances in the CM subperiod. An implication of this outcome is that when market liquidity is *not* significantly scarce, DM subperiod investment and CM subperiod consumption will be reduced because investors' real balances are smaller. To mitigate this effect, the central bank (optimally) decreases its optimal repo asset price, p_α^{CB}, when $1-\alpha$ increases: a lower central bank repo price reduces FM subperiod liquidity in significantly scarce market liquidity states and investors' will adjust for this loss by accumulating more real balances in the CM subperiod. Hence, there is a negative relationship between p_α^{CB} and $1-\alpha$.

Notice that there are two opposing effects on real balance accumulation in the CM subperiod when $1-\alpha$ is increased. First, holding p_α^{CB} fixed, an increase in access to the central bank's emergency repo facility reduces investors' real balance accumulation in the CM subperiod. Second, decreasing the repo asset price, p_α^{CB}, while holding access $1-\alpha$ constant, induces an increase in real balance accumulation in the CM subperiod. Figure 8(b) illustrates that the first effect dominates the second.

Figure 8(d) shows that the probability that the central bank's emergency repo facility is active is an increasing function of access, $1-\alpha$. Here, as in the case of investors' real balance accumulation in the CM subperiod, there are two countervailing forces at work. First, an increase in access, $1-\alpha$, decreases the central bank repo price p_α^{CB} and, holding all else constant, a decrease in p_α^{CB} increases $\bar{\sigma}_\alpha$ and, hence, the decreases the probability that the emergency facility is used. Second, an increase in access decreases real balance accumulation in the CM subperiod and, holding all else constant, decreasing market liquidity increases the probability that the emergency facility will be used. Figure 8(d) shows that the second effect dominates the

optimally.

first. Interestingly both figures 8(b) and 8(d) have the feature that the "quantity effect," real balances z, dominate the "price effect," p_α^{CB}.

Our numerical example indicates that a standing repo delivers better outcomes than an emergency facility, where the latter arises only after significant scarcity in market liquidity is observed in the early FM subperiod. This result depends critically on central bank pursing the best—welfare maximizing—strategy. The opitmal strategy has the central bank posting a very high repo rate, equal to about 12.8%. This rate is "high" in the sense that when liquidity is abundant, the market repo rate is equal to zero. If, for some reason, the central bank does not pursue such a policy, then, depending upon model parameters, an emergency facility may actually outperform a standing repo facility. In terms of our example, suppose that the central bank cannot commit to such a high repo rate and that the maximum repo rate that it will post is 6.3% (for either the standing or emergency facility). For this repo rate, setting $\alpha = 0.47$ maximizes welfare for the emergency repo facility. If the central bank opens an emergency facility after 47% of investors have traded and posts a repo rate of 6.3%, then the emergency repo facility will deliver higher welfare than a standing repo facility that posts a 6.3% repo rate. This implies that, for the parameters in our example and for a central bank repo rate equal to 6.3%, replacing an emergency repo facility with standing facility would reduce welfare. In fact, for central bank repo rates lower than 6.3%, the emergency facility always does better than a standing facility. Furthermore, if the central bank's repo rate is set sufficiently low, then it is optimal to set $\alpha = 1$ for the "emergency facility," i.e., never open the emergency facility. Creating either partial or no access to the central bank facility by using an emergency facility, compared to a standing facility, is beneficial in these examples because it the increase in market liquidity—by increasing investors' CM real balance holdings z—more than compensates for the loss in "insurance" associated with the standing repo facility.

Although the distribution of σ does not qualitatively affect our results, it does have interesting quantitative implications. In particular, if the distribution has a thicker right tail, then scarce market liquidity is more likely to occur. In this situation, it is optimal for the central bank to intervene more often and more "aggressively" with a lower central bank repo rate. As the right tail thickens, for a fixed α for the emergency facility, the welfare maximizing central bank repo rate falls. And, for a fixed central bank repo rate, a lower α maximizes welfare as the tail thickens. This implies that a permanent repo facility can be optimal if the distribution σ has a thick tail, while an emergency facility can be optimal if the tail is thin. (**Question: When we**

make this final comment, are we thinking about a world where the CB does not set the repo rate opitimally, too low?)

7 Conclusion

To be completed

Appendix

A Proof to Proposition 1

Suppose that $\pi_H = 1$. Because $\iota > 0$, three types of equilibria can occur.

Case 1: Liquidity is scarce. Then from (26) and (27), we have

$$\begin{aligned} p^{CM} &= \beta, \\ \iota &= \lambda(z + p^{CB} b^*). \end{aligned}$$

The first equation implies that the asset is priced at its fundamental value. The second equation implies that the total liquidity holding of a cash investor in the DM subperiod is independent of p^{CB}, i.e., if p^{CB} is increased (decreased), then agents accumulate less (more) real balances, z, and/or sell less (more) assets, b^*, to completely offset the price effect. As a result welfare is independent of p^{CB} and a standing repo facility cannot increase welfare.

Case 2: Liquidity is sufficient. Then the equilibrium z solves

$$\iota = \sigma_H \lambda \left(z + p^{CB}\bar{b}\right) + (1 - \sigma_H)\left(\frac{1}{p^{CB}} - 1\right).$$

This equation implies $z + p^{CB}\bar{b}$ is decreasing in p^{CB}. As a result, a higher p^{CB} leads to lower investment by the cash investor and thus lower welfare.

Case 3: Liquidity is abundant. Then the equilibrium has $p^{FM} = 1$ without the central bank intervention. Since the central bank cannot incur a loss in its repo operations, it cannot offer a price $p^{CB} > 1$. And if it offers $p^{CB} < 1$, the standing facility is inactive. Hence a central bank repo facility cannot affect welfare.

The above argument shows that an increase in p^{CB} weakly reduces welfare and hence it is optimal to reduce p^{CB} so that the standing facility is not active. This proves the proposition.

B Proof to Proposition 2

To see how investment, consumption and welfare are affected by an active repo facility, we set p^{CB} equal to $\tilde{p}_H^{FM}$ and ask what happens when p^{CB} increases. To understand what happens to welfare, differentiate (30) to get

$$\begin{aligned} W'(p^{CB}) \;\approx\; & \pi_L \sigma_L \frac{f' \circ Y\left(z + \bar{b}\right) - c' \circ Y\left(z + \bar{b}\right)}{v' \circ Y\left(z + p^{CM} b\right)} \frac{dz}{dp^{CB}} + \\ & \pi_H \sigma_H \frac{f' \circ Y(z + p^{CB} b_H^*) - c' \circ Y(z + p^{CB} b_H^*)}{v' \circ Y(z + p^{CB} b_H^*)} \frac{d(z + p^{CB} b_H^*)}{dp^{CB}} \end{aligned}$$

where $\approx$ means equal in sign. Using (9), we can rewrite this equation as

$$\begin{aligned} W'(p^{CB}) &\approx \pi_L\sigma_L\lambda\left(z+\bar{b}\right)\frac{dz}{dp^{CB}} \\ &\quad +\pi_H\sigma_H\lambda(z+p^{CB}b_H^*)\frac{d(z+p^{CB}b_H^*)}{dp^{CB}}. \end{aligned} \tag{35}$$

To evaluate the derivatives on the right side of (35), take the derivatives of (29) and (28) with respect to p^{CB} and rearrange to obtain

$$\frac{dz}{dp^{CB}} = \frac{\pi_H}{\sigma_L\pi_L\lambda'(z+\bar{b})\left(p^{CB}\right)^2} < 0, \tag{36}$$

$$\frac{d(z+p^{CB}b_H^*)}{dp^{CB}} = -\frac{1}{\lambda'(z+p^{CB}b_H^*)\left(p^{CB}\right)^2} > 0. \tag{37}$$

Since investment in the DM subperiod is monotone in liquidity, we have established part (i) of this proposition.

To establish part (ii), substitute (36) and (37) into (35) to get

$$W'(p^{CB}) \approx \frac{\pi_H}{\left(p^{CB}\right)^2}\left\{\frac{\lambda\left(z+\bar{b}\right)}{\lambda'\left(z+\bar{b}\right)} - \sigma_H\frac{\lambda(z+p^{CB}b_H^*)}{\lambda'(z+p^{CB}b_H^*)}\right\}.$$

And since $b_H^c + b_H^b = b_H^* < \bar{b}$, $z + p^{CB}b_H^* = z/\sigma_H$, we have

$$\begin{aligned} W'(\tilde{p}_H^{FM}) &\approx \frac{\pi_H}{\left(\tilde{p}_H^{FM}\right)^2}\left\{\frac{\lambda\left(z+\bar{b}\right)}{\lambda'\left(z+\bar{b}\right)} - \sigma_H\frac{\lambda(z+p^{CB}b_H^*)}{\lambda'(z+p^{CB}b_H^*)}\right\} \\ &\approx \left\{\frac{\lambda\left(\tilde{z}+\bar{b}\right)}{\lambda'\left(\tilde{z}+\bar{b}\right)} - \frac{\lambda(\frac{\tilde{z}}{\sigma_H})}{\lambda'(\frac{\tilde{z}}{\sigma_H})}\right\} + (1-\sigma_H)\frac{\lambda(\frac{\tilde{z}}{\sigma_H})}{\lambda'(\frac{\tilde{z}}{\sigma_H})}. \end{aligned} \tag{38}$$

The first bracketed term in (38) is positive because $\tilde{z}+\bar{b} > \tilde{z}/\sigma_H$ and the second term is negative because $\lambda' < 0$. Therefore, $W'(\tilde{p}_H^{FM}) > 0$ if σ_H is sufficiently close to 1.

C Uniqueness of $\tilde{\sigma}_H$

The following is a sufficient but not necessary condition for $\tilde{\sigma}_H$ to be unique.

Condition 3 $1 + \lambda(z) + \lambda'(z)\,z > 0$ *for all* $z \in [0, z^*]$.

This condition is always satisfied if $\theta = 1$ and $\gamma < 1$. By continuity, it holds for θ not too small. Indeed, we have checked that in our numerical examples, this condition hold even though $\theta = 0.5$.

We can then prove

Lemma 4 *Assume condition 3 holds. Then in the interval* $[z/z^*, 1]$, $\tilde{\sigma}_H$ *is unique.*

Proof. Recall that $\tilde{\sigma}_H$ is implicitly defined by

$$\frac{[1+\lambda(z/\sigma)]\,z}{[1+\lambda(z/\sigma)]\,z+\bar{b}}=\sigma.$$

Rearranging, this equation can be written as

$$(1-\sigma)\left[1+\lambda\left(\frac{z}{\sigma}\right)\right]\frac{z}{\sigma}-\bar{b}=0. \tag{39}$$

Differentiate the left-hand side with respect to σ and obtain

$$-(1-\sigma)\,\lambda'\left(\frac{z}{\sigma}\right)\frac{z}{\sigma^2}\frac{z}{\sigma}-\left[1+\lambda\left(\frac{z}{\sigma}\right)\right]\frac{z}{\sigma^2}.$$

This is equal in sign to

$$\begin{aligned}&-(1-\sigma)\,\lambda'\left(\frac{z}{\sigma}\right)\frac{z}{\sigma}-\left[1+\lambda\left(\frac{z}{\sigma}\right)\right]\\ \leq\;&-\lambda'\left(\frac{z}{\sigma}\right)\frac{z}{\sigma}-\left[1+\lambda\left(\frac{z}{\sigma}\right)\right]<0.\end{aligned}$$

The first inequality holds because $\lambda' < 0$ and $\sigma < 1$. The last inequality holds because by assumption $1+\lambda(z)+\lambda'(z)\,z>0$ for all $z\in[0,z^*]$. As a result, there is at most one solution to (39). ■

D General Model

When σ is continuous, the bond pricing and money demand equations, (1) and (2), respectively, for an always inactive central bank repo facility can now be expressed as

$$p^{CM}=\beta\int_{\sigma}[\sigma J_1^c(b,z,\sigma)+(1-\sigma)J_1^n(b,z,\sigma)]dF(\sigma) \tag{40}$$

and

$$\frac{\phi}{\phi'}=\beta\int_{\sigma}[\sigma J_2^c(b,z,\sigma)+(1-\sigma)J_2^n(b,z,\sigma)]dF(\sigma). \tag{41}$$

We already have expressions for $J_1^c(b,z,\sigma)$, $J_2^c(b,z,\sigma)$, $J_1^n(b,z,\sigma)$ and $J_2^n(b,z,\sigma)$ when market liquidity is characterized by either scarcity or abundance. When market liquidity is characterized by sufficiency, i.e., when $\sigma=\sigma_M\in(\tilde{\sigma}_L,\tilde{\sigma}_H)$, the cash investor's FM value function, (7), becomes

$$J^c(b,z,\sigma_M)=f\circ Y(z+p_M^{FM}b)-P(z+p_M^{FM}b)-b(1-p_M^{FM})+W(b,z,0,0),$$

and we have

$$J_1^c(b, z, \sigma_M) = p_M^{FM}[\lambda(z + p_M^{FM} b) + 1], \quad (42)$$

$$J_2^c(b, z, \sigma_M) = \lambda(z + p_M^{FM} b) + 1. \quad (43)$$

When $\sigma = \sigma_M \in (\tilde{\sigma}_L, \tilde{\sigma}_H)$, the credit investor's FM value function, (10), becomes

$$J^n(b, z, \sigma_M) = \left(\frac{1}{p_M^{FM}} - 1 \right) z + W(b, z, 0, 0),$$

and we have

$$J_1^n(b, z, \sigma_M) = 1, \quad (44)$$

$$J_2^n(b, z, \sigma_M) = \frac{1}{p_M^{FM}}. \quad (45)$$

In equilibrium, when market liquidity is sufficient we have $z + p_M^{FM} \bar{b} = z/\sigma_M$, which implies that $p_M^{FM} = (1 - \sigma_M)z/(\sigma_M \bar{b})$. Using (13)-(20) and (42)-(45), the bond pricing and money demand equations (40) and (41) can be simplified to

$$p^{CM} = \beta + \beta \int_0^{\tilde{\sigma}_L} \sigma \lambda(z + \bar{b}) dF(\sigma) + \beta \int_{\tilde{\sigma}_L}^{\tilde{\sigma}_H} \sigma \left\{ \frac{(1 - \sigma)z}{\sigma \bar{b}} \left[\lambda \frac{z}{\sigma} + 1 \right] - 1 \right\} dF(\sigma) \quad (46)$$

and

$$\iota = \int_0^{\tilde{\sigma}_L} \sigma \lambda(z + \bar{b}) dF(\sigma) + \int_{\tilde{\sigma}_L}^{\tilde{\sigma}_H} \left\{ \sigma \lambda \left(\frac{z}{\sigma} \right) + (1 - \sigma) \left[\frac{\sigma \bar{b}}{(1 - \sigma)z} - 1 \right] \right\} dF(\sigma) + \int_{\tilde{\sigma}_H}^1 \lambda \left(\frac{z}{\sigma} \right) dF(\sigma). \quad (47)$$

Intuitively, cash investors repo finance all of their collateral in the FM subperiod in states $\sigma \in (0, \tilde{\sigma}_H)$ and, therefore, get a liquidity benefit from having an additional unit of the bond in any of these states. Cash investors get a liquidity benefit from having an additional unit of real balances in all states. Credit investors, however, get a liquidity benefit only when $p_\sigma^{FM} < 1$ which occurs in states $\sigma \in (\tilde{\sigma}_L, 1)$. Just as in the two-state case, the steady state equilibrium is determined, in part, by two equations in p^{CM} and z, (46) and (47), which can be solved sequentially. First, the equilibrium z can be solved from (47): because the right side is decreasing in z, the equilibrium exists and is unique if θ is not too small. The equilibrium value of p^{CM} is determined by substituting the equilibrium z into (46).

We now assume that the central bank sets its repo price p^{CB} so that its repo facility is active in all states $\sigma > \bar{\sigma}$. We start with the situation where $\hat{\sigma}_H$ exists and $\bar{\sigma} > \hat{\sigma}_H$.[54] The equilibrium

[54] The critical value $\hat{\sigma}_H$ exists if p^{CB} is larger than but sufficiently close to $\tilde{p}_1^{FM} = 1/[1 + \lambda(\tilde{z})]$.

one-period government bond price in the CM subperiod, p^{CM}, is then given by

$$p^{CM} = \beta + \beta \int_0^{\hat{\sigma}_L} \sigma\lambda(z+\bar{b})dF(\sigma) + \beta \int_{\hat{\sigma}_L}^{\hat{\sigma}_H} \sigma \left\{ \frac{(1-\sigma)z}{\sigma\bar{b}} \left[\lambda(\frac{z}{\sigma})+1\right] - 1 \right\} dF(\sigma). \quad (48)$$

Notice that, except for the critical value labels, this expression is identical to the bond price equation when the central bank repo facility is always inactive, (46). This should not surprise since an additional unit of the bond confers no additional benefit to investors beyond the fundamental value when market liquidity is scarce. The money demand equation is

$$\iota = \int_0^{\hat{\sigma}_L} \sigma\lambda(z+\bar{b})dF(\sigma) + \int_{\hat{\sigma}_L}^{\hat{\sigma}_H} \left\{ \sigma\lambda\left(\frac{z}{\sigma}\right) + (1-\sigma)\left[\frac{\sigma\bar{b}}{(1-\sigma)z} - 1\right] \right\} dF(\sigma)$$
$$+ \int_{\hat{\sigma}_H}^{\bar{\sigma}} \lambda\left(\frac{z}{\sigma}\right) dF(\sigma) + \frac{1-p^{CB}}{p^{CB}}[1-F(\bar{\sigma})]. \quad (49)$$

This expression, again except for the critical value labels, is identical to (47) for all states $\sigma < \bar{\sigma}$. When $\sigma > \bar{\sigma}$, the repo facility is active and the equilibrium FM subperiod bond, p_σ^{FM}, is equal to p^{CB}. In this equilibrium cash investors do not access the central bank repo facility when $\sigma < \bar{\sigma}$ since $p_\sigma^{FM} > p^{CB}$ and when $\sigma > \bar{\sigma}$, $b_\sigma^c + b_\sigma^{CB} = b_{\bar{\sigma}}^* < \bar{b}$ for all σ. Intuitively, one should think of this case arising when p^{CB} is not "too big" in the sense that it is "not much" larger than $\tilde{p}_1^{FM} = 1/\left[1-\lambda(\tilde{z})\right]$, the FM subperiod asset price when $\sigma = 1$ and the standing facility is always inactive.

When the central bank chooses p^{CB} "sufficiently large," then $\hat{\sigma}_H$ does not exist. In this case, market and central bank liquidity will be sufficient for all $\sigma > \bar{\sigma}$, meaning that cash investors will repo finance all of their collateral in the FM subperiod in all $\sigma > \bar{\sigma}$. Since $\bar{\sigma} > \hat{\sigma}_L$, market plus central bank liquidity will be sufficient for all states $\sigma > \hat{\sigma}_L$. When $\hat{\sigma}_H$ does not exist, the critical value $\bar{\sigma}$ is determined by the equality of the value of cash investors' repo financing all of their collateral at p^{CB} per unit collateral, $\bar{\sigma}p^{CB}\bar{b}$, with the value of market liquidity, $(1-\bar{\sigma})z$, i.e.,

$$\bar{\sigma} = \frac{z}{z + p^{CB}\bar{b}}.$$

In this situation, the equilibrium government bond price in the CM subperiod, p^{CM}, is

$$p^{CM} = \beta + \beta \int_0^{\hat{\sigma}_L} \sigma\lambda(z+\bar{b})dF(\sigma) + \beta \int_{\hat{\sigma}_L}^{\bar{\sigma}} \sigma \left\{ \frac{(1-\sigma)z}{\sigma\bar{b}} \left[\lambda(\frac{z}{\sigma})+1\right] - 1 \right\} dF(\sigma)$$
$$+ \beta \int_{\bar{\sigma}}^{1} \sigma \left\{ p^{CB} \left[\lambda(\frac{z}{\bar{\sigma}})+1\right] - 1 \right\} dF(\sigma). \quad (50)$$

The first line of this expression, where $\sigma < \bar{\sigma}$, is identical to (46). The second line reflects the fact that cash investors continue to repo finance all of their collateral in all states $\sigma > \bar{\sigma}$ at price

$p_\sigma^{FM} = p^{CB}$. (Recall that in the absence of an active repo facility cash investors do not repo finance all of their collateral in states $\sigma > \tilde{\sigma}_H$.) The money demand equation is

$$\iota = \int_0^{\hat{\sigma}_L} \sigma\lambda(z+\bar{b})dF(\sigma) + \int_{\hat{\sigma}_L}^{\bar{\sigma}} \left\{\sigma\lambda\left(\frac{z}{\sigma}\right) + (1-\sigma)\left[\frac{\sigma\bar{b}}{(1-\sigma)z} - 1\right]\right\} dF(\sigma)$$
$$+ \int_{\bar{\sigma}}^1 \{\sigma\lambda\left(\frac{z}{\bar{\sigma}}\right) + (1-\sigma)\frac{1-p^{CB}}{p^{CB}}\}dF(\sigma), \quad (51)$$

where this expression is identical to (47) for $\sigma < \bar{\sigma}$. In states $\sigma > \bar{\sigma}$, which is described in the second line in the above expression, cash investors repo finance all of their collateral and the equilibrium FM subperiod bond prices are equal to p^{CB}.

Up to this point we have analyzed a standing central bank repo facility. We now discuss and analyze an ad hoc repo facility, where the central bank institutes an emergency facility only if market liquidity is sufficiently scarce (as described in Section 6). With a little abuse of notation, let $\hat{\sigma}_L$ and $\hat{\sigma}_H$ be the cutoffs where liquidity becomes just sufficient and scarce in the early FM subperiod. The critical value $\bar{\sigma}$ describes the set of states where the central bank operates an emergency facility in the late FM subperiod, i.e., when $\sigma > \bar{\sigma}$. Since $\bar{\sigma} > \hat{\sigma}_L$, $\hat{\sigma}_L$ is a relevant statistic for both early and late FM subperiod investors and $\hat{\sigma}_H$ is a relevant statistic for both the early and late FM subperiod investors if $\bar{\sigma} > \hat{\sigma}_H$. If $\hat{\sigma}_H > \bar{\sigma}$, then $\hat{\sigma}_H$ is only relevant for the early FM subperiod investors as they do not have access to the central bank's emergency repo facility. First we examine the case where the central bank's choice of p^{CB} implies $\bar{\sigma} > \hat{\sigma}_H$. When $\sigma > \bar{\sigma}$, the $1-\alpha$ cash investors in the late FM period access the central bank repo facility and repo finance a fraction of their bond holdings, $\bar{b}$. The bond pricing equation is identical to that associated with a standing repo facility, (48). This should not be surprising since an additional unit of bond does not provide value beyond its fundamental in the FM subperiod in all states $\sigma > \hat{\sigma}_H$ and we have $\bar{\sigma} > \hat{\sigma}_H$. The money demand equation is

$$\iota = \int_0^{\hat{\sigma}_L} \sigma\lambda(z+\bar{b})dF(\sigma) + \int_{\hat{\sigma}_L}^{\hat{\sigma}_H} \left\{\sigma\lambda\left(\frac{z}{\sigma}\right) + (1-\sigma)\left[\frac{\sigma\bar{b}}{(1-\sigma)z} - 1\right]\right\} dF(\sigma)$$
$$+ \int_{\hat{\sigma}_H}^{\bar{\sigma}} \lambda\left(\frac{z}{\sigma}\right) dF(\sigma) + (1-\alpha)\left(\frac{1-p^{CB}}{p^{CB}}\right)[1-F(\bar{\sigma})] + \alpha\int_{\bar{\sigma}}^1 \lambda(\frac{z}{\sigma})dF(\sigma). \quad (52)$$

The only significant difference between this equation and that for the standing repo facility, (49), is in the states where the central bank repo facility is active for the late investors, in states $\sigma > \bar{\sigma}$. The liquidity premium for real balances with a standing repo facility is equal to $1/p^{CB} - 1$ in all states $\sigma > \bar{\sigma}$ for all investors. With an ad hoc emergency facility, the liquidity premium is equal to $1/p^{CB} - 1$ in all states $\sigma > \bar{\sigma}$ but for only those $1-\alpha$ late investors; for the early investors the liquidity premium is given by $\lambda(z/\sigma)$, which is increasing in $\sigma > \bar{\sigma}$. Since $\lambda(z/\sigma) > 1/p^{CB} - 1$ for all $\sigma > \bar{\sigma}$, (52) implies that an increase in α, holding p^{CB} constant, necessarily increases z and $\bar{\sigma}$.

We now examine the case where the central bank's choice of p^{CB} implies that $\bar{\sigma} < \hat{\sigma}_H$. In this case, late cash investors repo finance all of their bond $\bar{b}$ in the FM market and/or the repo facility for all σ. The bond pricing equation is

$$\begin{aligned} p^{CM} = \beta + \beta \int_0^{\hat{\sigma}_L} \sigma\lambda(z+\bar{b})dF(\sigma) + \beta \int_{\hat{\sigma}_L}^{\bar{\sigma}} \sigma \left\{ \frac{(1-\sigma)z}{\sigma\bar{b}} \left[\lambda\left(\frac{z}{\sigma}\right)+1\right] - 1 \right\} dF(\sigma) \\ + (1-\alpha) \int_{\bar{\sigma}}^{1} \sigma p^{CB} \left\{ \left[\lambda\left(\frac{z}{\bar{\sigma}}\right)+1\right] - 1 \right\} dF(\sigma) \\ + \beta\alpha \int_{\bar{\sigma}}^{\hat{\sigma}_H} \sigma \left\{ \frac{(1-\sigma)z}{\sigma\bar{b}} \left[\lambda\left(\frac{z}{\sigma}\right)+1\right] - 1 \right\}. \quad (53) \end{aligned}$$

This equation is almost identical to the bond pricing equation for the standing repo facility: the first two lines are (almost) identical to (50) except the second line in (53) is multiplied by the fraction, $(1-\alpha)$, of cash investors that have access to the repo facility in the late FM subperiod. The third line in (53), which is different and reflects the fact that the α early cash investors do not have access to a central bank repo facility. As a result, these investors repo finance all of their bond in the early FM subperiod in states $\sigma \in (\bar{\sigma}, \hat{\sigma}_H)$ and repo finance less than $\bar{b}$ bond in state $\sigma > \hat{\sigma}_H$. The money demand equation is

$$\begin{aligned} \iota = \int_0^{\hat{\sigma}_L} \sigma\lambda(z+\bar{b})dF(\sigma) + \int_{\hat{\sigma}_L}^{\bar{\sigma}} \left\{ \sigma\lambda\left(\frac{z}{\sigma}\right) + (1-\sigma)\left[\frac{\sigma\bar{b}}{(1-\sigma)z} - 1\right]\right\} dF(\sigma) \\ + (1-\alpha)\{\int_{\bar{\sigma}}^{1} \sigma\lambda\left(\frac{z}{\bar{\sigma}}\right) + (1-\sigma)\frac{1-p^{CB}}{p^{CB}}\}dF(\sigma) \\ + \alpha \int_{\bar{\sigma}}^{\hat{\sigma}_H} \left\{ \sigma\lambda\left(\frac{z}{\sigma}\right) + (1-\sigma)\left[\frac{\sigma\bar{b}}{(1-\sigma)z} - 1\right]\right\} dF(\sigma) + \alpha \int_{\hat{\sigma}_H}^{1} \lambda\left(\frac{z}{\sigma}\right) dF(\sigma), \quad (54) \end{aligned}$$

which, again, is almost identical to the money demand equation for the standing repo facility (51). Just as above, the last line of (54) does not have a counterpart in (51) and reflects the fact that α investors do not have access to a standing repo facility in states $\sigma > \bar{\sigma}$.

E Large supply of government bonds

We have so far analyzed the case where the supply of government bonds, $\bar{b}$, is "small." Here we characterize the threshold of government bond supply that distinguishes the small and large supply cases, and characterize the equilibrium when the central bank repo facility is always inactive and sometime active.

The supply of assets is large when cash investors do not sell all their assets and credit investors do not use all their liquidity to buy assets in the FM. We will show below that a precise definition of the region in which the supply of assets is "large" is given by

$$b \geq z^* - \tilde{z}^c \equiv \bar{b}^c. \tag{55}$$

where $\tilde{z}^c$ solves (59) below.

When cash is abundant, i.e. $p_\sigma^{FM} = 1$, and cash investors can sell enough assets to purchase the efficient level of DM investment goods, we have $(1-\sigma) z > \sigma (z^* - z)$. This condition that can be rewritten as

$$z/z^* > \sigma \tag{56}$$

Since cash investors are not constrained by their asset holdings, we must also have

$$z + \bar{b} \geq z^*. \tag{57}$$

In this case the first-order condition (8) holds as an equality. Substituting $p_\sigma^{FM} = 1$ into (8) we obtain

$$\lambda (z + b_\sigma^c) = 0,$$

which implies that cash investors get enough cash to achieve first best consumption, i.e., $z + \bar{b} \geq z^*$. Using the envelope theorem, we obtain

$$\begin{aligned} J_1^c (b, z, \sigma) &= 1, \;\; J_2^c (b, z, \sigma) = 0, \\ J_1^n (b, z, \sigma) &= 1, \;\; J_2^n (b, z, \sigma) = 0. \end{aligned}$$

If conditions (56) and (57) are not satisfied market liquidity is scarce and the envelope conditions applied to the value functions of cash and credit investors are given by (17), (18), (19) and (20). Given the envelope conditions, we can now write the Euler equations (1) and (2) for the "large" asset case as

$$p^{CM} = \beta \tag{58}$$

$$\iota = \int_{z/z^*}^{1} \lambda (z/\sigma) \, dF(\sigma), \tag{59}$$

which implies that the asset is priced fundamentally. Substituting in (57) we obtain (55), i.e. the threshold above which assets are "large".

E.1 The general model with a standing repo facility

Assume that $b \geq \bar{b}^c$. Given p^{CM} and z, p_σ^{FM} is decreasing in σ. Hence, there exists a cutoff $\bar{\sigma}$ such that if $\sigma < \bar{\sigma}$, the standing facility is not used and if $\sigma > \bar{\sigma}$, $p_\sigma^{FM} = p^{CB}$, where $\bar{\sigma}$ solves

$$p^{CB} = \frac{1}{1+\lambda\left(z/\bar{\sigma}\right)}.$$

Because $p^{CB} \leq 1$, $z/z^* < \bar{\sigma}$. Therefore, the Euler equation for money is

$$\iota = \int_{z/z^*}^{\bar{\sigma}} \lambda\left(\frac{z}{\sigma}\right) dF\left(\sigma\right) + \int_{\bar{\sigma}}^{1} \lambda\left(z + p^{CB}b^*\right) dF\left(\sigma\right),$$

where b^* solves $p^{CB}\left[\lambda\left(z+p^{CB}b^*\right)+1\right] = 1$. The Euler condition for the bond is again given by (58)

$$p^{CM} = \beta.$$

QUESTION: CAN WE SHOW DIRECTLY THAT A STANDING FACILITY INCREASES WELFARE?

E.2 Welfare Implications of a Standing Facility in a Two-State Example

Lastly, we show in a two-state example that a standing repo facility can increase welfare in the large asset case. Suppose σ can take value σ_L with probability π_L and value σ_H with probability $\pi_H = 1 - \pi_L$. Suppose the parametrization is such that $\sigma_L < \tilde{z}/z^*$ and $\sigma_H > \tilde{z}/z^*$, where $\tilde{z}$ is the equilibrium real balances that solves

$$\iota = \pi_H \lambda\left(z/\sigma_H\right). \tag{60}$$

The FM price in the high state is $\tilde{p}_H^{FM} = 1/\left[\lambda\left(\tilde{z}/\sigma_H\right)+1\right]$ and that in the low state is $\tilde{p}_L^{FM} = 1$. Moreover, $\tilde{p}^{CM} = \beta$ is the equilibrium bond price because cash investors are not asset constrained in all states.

Now we introduce a repo facility with a price p^{CB}, which is slightly higher than $\tilde{p}_H^{FM}$. Then cash investors are not asset constrained in the high state. Therefore, $\left[\lambda\left(z+p^{CB}b_H^*\right)+1\right]p^{CB} = 1$. Moreover, if $p^{CB} = \tilde{p}_H^{FM}$, cash investors are not constrained in the DM. As a result, $\lambda\left(z_L\right) = 0$ where z_L is the real balances held by cash investors in the DM. Therefore,

$$\begin{aligned}
\left.\frac{\partial W\left(p^{CB}\right)}{\partial p^{CB}}\right|_{p^{CB}=\tilde{p}_H^{FM}} &= \frac{1}{\theta}\pi_H\sigma_H\lambda\left(z+p^{CB}b^*\right)\frac{\partial\left(z+p^{CB}b^*\right)}{\partial p^{CB}} + \frac{1}{\theta}\pi_L\sigma_L\lambda\left(z_L\right)\frac{\partial z_L}{\partial p^{CB}} \\
&= \frac{1}{\theta}\pi_H\sigma_H\lambda\left(z+p^{CB}b^*\right)\frac{\partial\left(z+p^{CB}b^*\right)}{\partial p^{CB}} \\
&= -\frac{1}{\theta}\frac{\pi_H\sigma_H\lambda\left(z+p^{CB}b^*\right)}{\lambda'\left(z+p^{CB}b^*\right)}\frac{1}{\left(p^{CB}\right)^2} > 0.
\end{aligned}$$

Welfare is increasing with p^{CB} if it is close to $\tilde{p}_H^{FM}$. Intuitively, a higher p^{CB} improves investment in the high state but lowers investment in the low state. Because cash investors are making efficient investment in the low state, the improvement in the high state is first order while lower investment in the low state is second order. The net effect on welfare is then positive.

Proposition 5 *When the asset supply is "large," a standing facility can improve welfare in the 2-state case.*