Strategic Investment under Uncertainty with First- and Second-mover Advantages*

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Abstract

We analyze a duopoly entry game where firms trade off the first-mover advantage (of earning monopoly rents) against the second-mover advantage (of paying a lower entry cost) in the classic real-option framework. The equilibrium solution features five regions. There are two waiting regions: a new waiting-to-be-the-second-mover region and the standard option-value-of-waiting region. For sufficiently high market demand, there is no first-mover advantage in equilibrium as Follower immediately enters after Leader. Therefore, firms play mixed strategies and become Leader with a rate increasing in market demand, giving rise to a probabilistic entry region. For intermediate levels of market demand, firms rush to enter, giving rise to the first-mover-advantage-induced "rent-equalization" region (Fudenberg and Tirole, 1985; Grenadier, 1996). Finally, a second probabilistic entry region emerges to connect the rent-equalization region and the waiting-to-be-the-second-mover region. Quantitatively, the second-mover advantage can cause firms to significantly delay entry and substantially erode firm value.

Keywords: wars of attrition, real option games, mixed strategies, Stackelberg game, duopoly, separation principle, first-mover advantage, second-mover advantage

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1 Introduction

Corporate decisions, e.g., entry into a new product market and R&Ds, typically involve significant upfront fixed costs and are costly to reverse. These decisions in essence are American-style real-option-exercising problems, which are widely taught in business schools and used by management consultants and practitioners.[1] However, a key limiting assumption in standard real-options models is that a firm has an exclusive permanent access to the investment project and thus in effect solves a monopolist's real-option exercising problem. In reality, firms routinely compete against each other for investment opportunities. Despite the real-world importance of strategic interactions, research at the intersection between real options and game theory has been quite limited with a few exceptions, e.g., Fudenberg and Tirole (1985), Grenadier (1996, 2002), and Back and Paulsen (2009).

The existing real-option duopoly models assume that firms incur the same cost when making entry or investment decisions. However, in many industries, the second mover often has a more efficient production technology and/or pays a lower entry cost than the first mover. The ulcer-relief drug Zantac is a well known case study of a successful second mover (Berndt, Pindyck and Azoulay, 2003).[2] While the pioneer pays a steep price in creating the product category, the later entrant can learn from the experience of the pioneer, enjoying lower costs and making fewer mistakes as a result.[3]

Anticipating the second-mover advantage, in addition to preserve the option value in standard real-option models, firms also have incentives to wait so as to lower its entry cost.

[1] McDonald and Siegel (1986) is the pioneering contribution to the real-options literature and Dixit and Pindyck (1994) is the standard reference of this literature. Abel, Dixit, Eberly and Pindyck (1996) make the connection between the real options approach and the q theory of investment (see, e.g., Hayashi (1982) and Abel and Eberly (1994, 1996)). Grenadier and Malenko (2010) develop a Bayesian real-options approach and Orlov, Skrzypacz and Zryumov (2020) study Bayesian persuasion in a real-options environment. Applications of real-options models include natural resources (Brennan and Schwartz, 1985); real estate (Titman (1985) and Grenadier (1996)); corporate default (Leland, 1994); mergers (Lambrecht, 2004); takeovers (Morellec and Zhdanov, 2008); and external equity financing (Hugonnier, Malamud and Morellec, 2015), among others. Grenadier and Malenko (2011) analyze real-option signaling games. Economic applications involving discrete-choice models are also real-option models.

[2] Berndt, Pindyck and Azoulay (2003) show that for the anti-ulcer drug market there are both brand-level and product-level network externalities. When the product-level externality is stronger than the brand-level externality, it is better to go second and let the first mover incur various costs of resolving the uncertainties (e.g., about the likelihood of successful FDA testing and the size of the potential market).

[3] This is paraphrased from Northwestern Kellogg's Insight at `https://insight.kellogg.northwestern.edu/article/the_second_mover_advantage`, which is based on Shankar, Carpenter and Krishnamurthi (1998).

We show that this new waiting motive fundamentally alters the economics of duopoly entry games. Depending on the level of the total market demand, we show that there are up to five mutually exclusive regions coexisting in equilibrium. First, the two distinct waiting motives induce two equilibrium *waiting* regions where it is optimal for firms not to enter. Second, there exists a *first-mover advantage* region where firms compete for entry. This is because after the first mover enters, the second mover faces a smaller market and rationally waits, enabling Leader to collect monopoly rents before the total market demand rises to an even higher level triggering Follower to voluntarily enter as in Fudenberg and Tirole (1985) and Grenadier (1996).

Third, there are two additional regions where firms prefer to enter as Follower. However, waiting is costly as firms forgo operating profits. in equilibrium both firms use mixed strategies to enter probabilistically. Why do we have two mixed-strategy regions? In one region where the total market demand is not very high, Leader collects monopoly rents in equilibrium as it is suboptimal for Follower to immediately enter. In the other mixed-strategy region, Follower immediately enters after Leader does because the total market demand is sufficiently high so that it is optimal for both firms to be in the market.

Now we sketch out our duopoly model. Two *ex ante* identical firms, Alice's and Bob's, compete to enter a new product market. To ease exposition, we assume that the total profit of the industry is exogenous and stochastic. We assume that Leader, the firm that enters first, has a monopoly power over the market demand until Follower enters. As soon as it enters, Follower takes away a half of the total market demand from Leader.

Entering this new market is exercising an entry option, which can be quite costly for a firm. As we expect, there are various upfront fixed costs that a firm must incur when entering a new market. For example, a firm that opens new factories and sells its products overseas must learn how to work with local governments, familiarize itself with local business and legal environments, and learn about customer preferences, just to name a few. Entering the market as the first mover (Leader) can be particularly costly as the firm has to start pretty much everything from scratch and pay various kinds of setup and learning costs.

In contrast, Follower (the second mover) incurs a smaller upfront entry cost than Leader. For example, by observing Leader and learning from its success and failure experiences in the new market, Follower can come up with a more efficient entry strategy and economize its cost structure. That is, Leader's entry generates a positive externality on Follower by lowering Follower's entry cost. When this positive externality of reducing Follower's entry cost is sufficiently large, firms *en ante* then have incentives to be the second mover. To be precise, Leader's value is lower than Follower's value. We show that this second-mover advantage in our duopoly model drives key predictions of our model.

We analyze both mixed-strategy and pure-strategy equilibria for our duopoly setting. For both types of equilibria, we obtain closed-form solutions for value functions and optimal entry strategies. Finally, we quantify our model's predictions and find substantial option value erosion and socially inefficient entry delay.

Our first and most important contribution is to characterize the mixed-strategy equilibrium, which is symmetric in that the two firms' strategies are the same. They both wait with probability one when the total profit in the industry x is below an endogenously determined cutoff threshold $\overline{x}$. When the industry profit is sufficiently high, i.e., $x \geq \overline{x}$, both firms enter probabilistically at an equilibrium rate of $\lambda^*(x)$. Once one firm enters, the other immediately follows, which means there is no monopoly profit for Leader.

As firms prefer to be Follower, then why are they willing to enter probabilistically? This is because the other alternatives, entering for sure as Leader and never entering, are worse. A firm that never exercises its entry option is worth zero. Stochastic entry is thus a compromised outcome between the two firms. How do we determine a firm's entry strategy in the mixed-strategy equilibrium, characterized by the equilibrium entry rate $\lambda^*(x)$?

On one hand, because of the second-mover advantage, each firm wants to free ride on the other's entry by being Follower in order to save its entry cost. This encourages firms to wait. On the other hand, it is also costly for firms to wait as it forgoes the opportunity of collecting the current profits. Each firm balances the benefit of waiting, which preserves the option value of becoming the second mover (hence winning the attrition game) and the opportunity cost of missing the current period's profit.

In equilibrium, both firms must be indifferent between entering and waiting for another period. To make them indifferent between the two options, the entry rate $\lambda^*(x)$ must equal the ratio between (1.) a firm's net income, the difference between operating profits and the interest payment of the entry cost, and (2.) the wedge between Leader's and Follower's entry costs (the reward for the attrition game's winner). This is because the competitor's entry rate, which equals $\lambda^*(x)$ in equilibrium, is the rate at which a firm wins the attrition game. To the best of our knowledge, our paper is among the first to characterize the mixed-strategy equilibrium in strategic real-option exercising games. We obtain closed-form solutions for both the entry strategy and value functions. The equilibrium in Grenadier (1996) is of pure strategy, because there is a first-mover advantage in his model. Firms prefer to be Leader and thus have no incentive to randomize their entry decisions.

Our second key contribution is to develop a new solution method for the mixed-strategy equilibrium, which we refer to as the separation principle. This principle allows us to decompose the mixed-strategy equilibrium solution into two subproblems. First, we solve a single firm's optimal stopping problem (ignoring strategic interactions between the two firms). Second, we derive a generalized war-of-attrition formula for the equilibrium entry rate in dynamic entry games. Technically, we extend the standard war-of-attrition result to settings with stochastic investment opportunities and endogenous entry. We show that the interactions between the war-of-attrition force and the option value of waiting significantly enrich our duopoly game analysis. We emphasize that the separation principle holds broadly for duopoly competition models with second-mover advantages.

The separation principle offers at least three advantages. First, solving a single firm's real-option problem is much easier than analyzing a dynamic duopoly entry game. Second, we show that the war-of-attrition part of our analysis boils down to a straightforward calculation for the equilibrium entry rate $\lambda^*(x)$ as if firms were behaving myopically by only taking the current net income and the reward of being winner (saved entry cost by being Leader) into account. Finally, we show how to combine the real-options analysis and the war-of-attrition analysis to obtain the equilibrium outcome. Interestingly, these two forces interact in an economically intuitive and analytically tractable way. In sum, this two-step procedure significantly deepens our understanding of the duopoly game's solution and mechanism.

Our third key contribution is to solve for the pure-strategy equilibria and provide a tight connection between the mixed-strategy and pure-strategy equilibria. As in the mixed-strategy equilibrium, Leader exercises its entry option later than the socially optimal level in our pure-strategy equilibria. This is because Leader takes into account the immediate entry by Follower. Follower's incentives to grab one-half of the market share from Leader and free ride on Leader's entry cost cause Leader to inefficiently delay its entry. That is, viewed from the lens of the war-of-attrition game, the loser of the game (Leader) waits too long before entry. This inefficiency result differs from standard war-of-attrition examples, where the pure-strategy equilibria are efficient as the loser immediately drops out (Levin, 2004). The loser's inefficient delay is due to the option value of waiting in our model. Again, this result highlights the rich predictions generated by the interaction between the real options force and the second-mover advantage in a stochastic entry game.

We further show that Leader's pre-entry value in a pure-strategy equilibrium equals a firm's pre-entry value in the mixed-strategy equilibrium. An implication of this result is that the threshold above which both firms adopt mixed strategies equals Leader's optimal entry threshold in a pure-strategy equilibrium. Despite the two types of equilibria have the same entry region where $x \geq \overline{x}$, entry is further delayed in the mixed-strategy equilibrium than in the pure-strategy equilibria path by path. This is because firm entry occurs instantly in the pure-strategy equilibria but only probabilistically in the mixed-strategy equilibrium.

Finally, we show that the quantitative effects of competition and the second-mover advantage on firm value and equilibrium entry strategies are quite large. We then characterize the distributions of entry time using tractable partial differential equations with economically intuitive boundary conditions for both pure-strategy and mixed-strategy equilibria. Using these tractable formulas, we show that the quantitative effects of competition and the second-mover advantage on the distributions of entry time are very large. Compared with the socially efficient outcome, a firm significantly delays its entry timing as it prefers to be the second mover and only has one half of the market share. This result that entry is inefficiently delayed holds for both pure-strategy and mixed-strategy equilibria.

Moreover, the mixed-strategy equilibrium is even more inefficient than the pure-strategy equilibria and quantitatively the predictions are quite different for the two types of equilibria. Although the two types of equilibria have the same entry region $x \geq \overline{x}$, Leader is determined endogenously and probabilistically in the mixed-strategy equilibrium while Leader enters with probability one in the pure-strategy equilibria whenever $x \geq \overline{x}$. Therefore, the realized entry time is often much later in the mixed-strategy equilibrium than in the pure-strategy equilibria. It is this further entry delay in the mixed-strategy equilibrium that makes the total market capitalization of the industry lower in the mixed-strategy equilibrium than that in the pure-strategy equilibria.

While the main focus of our paper is the second-mover advantage, we can generalize our main model so that the first-mover and second-mover advantages coexist in equilibrium. Which advantage dominates depends on market demand and the entry-cost wedge for Leader and Follower. In Section 8, we provide an example with state-contingent advantages.

Related literature. Our paper is naturally related to the real-options, strategic competition, and war-of-attrition literature. As we have noted, our paper is closely related to Fudenberg and Tirole (1985) and Grenadier (1996).[4] Unlike these papers, we study settings with second-mover advantages. As a result, the key differences in terms of results include (*a.*) both pure-strategy and mixed-strategy equilibria exist in our model and only the pure-strategy equilibrium exists in Grenadier (1996); (*b.*) the key driving force for both pure-strategy and mixed-strategy equilibria is the incentive to free ride on Leader in our model while the key driving force in Grenadier (1996) is a firm's incentive to make a preemptive entry move and the equilibrium rent equalization force emphasized in Fudenberg and Tirole (1985); (*c.*) our model predicts excessively delayed entry while Grenadier (1996) predicts socially inefficient rushed real-option exercising.[5]

Our paper is also closely related to Grenadier (2002) and Back and Paulsen (2009), who study oligopoly games where incumbents make irreversible incremental capital accumulation. In contrast, we analyze dynamic entry games. Mathematically, their game-theoretic analyses build on an individual firm's optimal singular control while our duopoly game builds on stopping-time models. As a result, both the mathematical structure and economic predictions of our model are quite different from Grenadier (2002) and Back and Paulsen (2009).

Our paper is also closely related to Lambrecht (2001), who develops a duopoly exit model in a standard real-option setting with a second-mover advantage. Different from Lambrecht (2001), the second-mover advantage is about entry in our model and the mixed-strategy equilibrium analysis, which is a focus of our paper, is new. Lambrecht and Perraudin (2003) introduce incomplete information into an equilibrium real-option exercising model.[6]

[4] Fudenberg, Gilbert, Stiglitz and Tirole (1983) model preemption games (e.g., patent races) in deterministic settings. Smets (1991) studies irreversible investment in a duopoly setting and analyzes an asymmetric leader-follower equilibrium. Murto (2004) studies a duopoly exit game and focuses on pure strategies.

[5] The monopolist's real-option model is based on McDonald and Siegel (1986) and Dixit and Pindyck (1994). The cooperative duopoly model against which we calculate social surplus loss is related to a similar benchmark in Weeds (2002).

[6] Anderson, Friedman and Oprea (2010) generalize Lambrecht and Perraudin (2003) to settings with

Weeds (2002) integrates a real-options model with strategic interactions by incorporating technological uncertainty into models along the lines of Grenadier (1996). There is no war-of-attrition force and the equilibria are of the pure-strategy type in Weeds (2002).

War-of-attrition models are widely used in economics.[7] We build on and generalize classic deterministic war-of-attrition-style duopoly exit models, e.g., Ghemawat and Nalebuff (1985, 1990), Fudenberg and Tirole (1986), and Hendricks, Weiss and Wilson (1988) to incorporate stochastic payoffs and the real option value of waiting. Unlike these papers, we study endogenous entry in a duopoly game where attrition means entering the market and letting the other firm to free ride on entry cost reduction. It is worth emphasizing that the payoff from becoming Leader (upon entering) in our entry game is endogenous and can only be obtained via backward induction. This is a major difference between our entry game and standard war-of-attrition exit games.

Importantly, the interactions between the real-option value of waiting and the war-of-attrition considerations generate novel predictions in both mixed-strategy and pure-strategy equilibria. For example, the equilibrium entry rate $\lambda^*(x)$ is state dependent and the pure-strategy equilibria are socially inefficient unlike in standard war-of-attrition games.

There is also a growing literature that integrates industrial organization considerations into asset pricing models. For example, Dou, Ji and Wu (2021) extend the standard Lucas-tree asset pricing model to allow for endogenous strategic competition. Chen, Dou, Guo and Ji (2022) study how strategic competition and financial distress dynamically interact.

2 Model

In this section, we set up an entry game in which two *ex ante* identical firms (entrants) decide their optimal timing to enter a new market with stochastic profits.

2.1 Market Demand and Industry Structure

As in McDonald and Siegel (1986), Dixit and Pindyck (1994), and Grenadier (1996), we assume that the total market profit is governed by a stochastic process, $\{X_t; t \geq 0\}$, which

multiple firms.

[7]Section 8.1 of Tirole (1988) and Levin (2004) offer introductions to the war-of-attrition literature.

follows a geometric Brownian motion:

$$dX_t = \mu X_t dt + \sigma X_t d\mathcal{Z}_t \,, \tag{1}$$

where μ is the expected growth rate of X, $\sigma > 0$ is the constant volatility for the growth rate of X, $\{\mathcal{Z}_t; t \geq 0\}$ is a one-dimensional standard Brownian motion, and the initial value of X is known: $X_0 = x_0 > 0$.[8]

Let τ_L denote the stochastic time when Leader enters the market and let τ_F denote the stochastic time when Follower enters. By definition, $\tau_F \geq \tau_L$. Let $K_1 > 0$ and $K_2 > 0$ denote the one-time upfront fixed entry cost that Leader and Follower have to pay at their respective entry time τ_L and τ_F. More broadly, we interpret Leader's upfront entry cost K_1 as the present value of all expenses that Leader incurs and similarly K_2 as the present value of all expenses that Follower incurs.[9] It is plausible that Leader incurs larger costs than Follower does as Leader may have to pay additional innovation costs, learn about a new product market, and work with local governments in the new markets. Follower can save some of the costs by observing Leader's actions, learning from Leader's experiences and mistakes, and even possibly imitating Leader's success and copying Leader's strategies.

The industry structure has three phases. First, before either firm enters ($t < \tau_L$), the market is inactive and neither firm receives any cash flow. Which firm becomes Leader is endogenous and stochastic. Second, after Leader enters at τ_L and before Follower enters at τ_F, Leader receives a monopoly profit at a rate of $\{X_s; s \in [\tau_L, \tau_F)\}$. Third, after Follower enters at τ_F, the economy permanently switches from a monopoly to a duopoly setting in which Follower and Leader equally split the total market profit and both receive profits indefinitely at a rate of $\{X_s/2; s \geq \tau_F\}$.

As a key goal of our paper is to study the implications of a second-mover advantage on firm entry and duopoly equilibrium, we assume that Follower's entry cost K_2 is lower than a half of Leader's entry cost K_1: $K_2 \leq K_1/2$. This assumption about entry costs is closely related to the other key assumption that Leader loses one half of its monopoly market share to Follower upon the latter's entry. With this pair of assumptions, we can show that there is a second-mover advantage and therefore a firm prefers to be Follower rather than Leader.[10]

[8]Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ denote the probability space. We assume that the process $\{\mathcal{Z}_t; t \geq 0\}$ is progressively measurable with respect to $\{\mathcal{F}_t\}_{t\geq 0}$.

[9]We can generalize our model by incorporating ongoing operating costs that may be different for Follower and Leader. For brevity, we leave this extension out. Our key results are robust to this extension.

[10]For a lower entry-cost wedge $\Delta K = K_1 - K_2$, we show that the first-mover and second-mover advantages

In sum, two *ex ante* identical firms, firm a (Alice's) and firm b (Bob's), maximize their values by taking the total market profit $\{X_s; s \geq 0\}$ process and the industry structure described above as given. Let τ_a and τ_b denote firm a's and b's stochastic entry time, respectively. Both firms are risk-neutral and discount profits at the constant interest rate r. As in the standard real-option models, we require $r > \mu$ and $r > 0$, which ensure that firm value is finite.[11] Below we summarize these assumptions, which apply throughout our analysis:

$$Assumptions: \quad r > \mu, \quad r > 0, \quad K_1 \geq 2K_2 > 0 \ . \tag{2}$$

For brevity we do not refer to (2) for the remainder of our paper.

2.2 Leader's Post-entry and Follower's Pre-entry Values: $L(x), F(x)$

Definitions. Follower's pre-entry value, i.e., for any $t \geq \tau_L$, is given by:

$$F(x) = \max_{\tau_F \geq t} \mathbb{E}_t^x \left[\int_{\tau_F}^{\infty} e^{-r(s-t)} \frac{X_s}{2} ds - e^{-r(\tau_F - t)} K_2 \right] , \tag{3}$$

where $X_t = x > 0$ and $\mathbb{E}_t^x[\cdot] = \mathbb{E}_t[\cdot | X_t = x]$ as our model is Markovian.[12] Let τ_F^* denote the optimal stopping time for (3). Taking τ_F^* and $F(x)$ as given, we define Leader's post-entry value function, $L(x)$, for any $t \in [\tau_L, \tau_F^*]$ as follows:

$$L(x) = \mathbb{E}_t^x \left[\int_t^{\tau_F^*} e^{-r(s-t)} X_s ds + \int_{\tau_F^*}^{\infty} e^{-r(s-t)} \frac{X_s}{2} ds \right] , \tag{4}$$

where the first term in (4) gives Leader's time-t value for its post-entry stochastic monopoly period and the second term gives the value of being a duopoly after Follower enters at τ_F^*. Note that $F(x)$ includes Follower's entry cost K_2 but $L(x)$ does not include Leader's entry cost K_1. We define $F(x)$ and $L(x)$ this way to ease exposition.

As we show later, both pure-strategy and mixed-strategy equilibria exist in our model. The pure-strategy equilibria are asymmetric and the mixed-strategy equilibrium is symmetric between the two firms. We analyze both types of equilibria. First, we study the economically more interesting symmetric mixed-strategy equilibrium.

can coexist. Which advantage dominates depends on the value of x. In Section 8, we use an example to show state-contingent advantages.

[11] We can equivalently interpret our optimization problems under the risk-neutral measure (i.e., risk adjusted). In this case, μ is the drift under the risk-neutral measure. Introducing risk premia via a stochastic discount factor allows us to study the asset pricing applications of competition (Duffie, 2001).

[12] For $t = 0$, we write $\mathbb{E}_0^x[\cdot]$ as $\mathbb{E}^x[\cdot]$.

2.3 Entry Equilibrium

For a given pair of entry times (τ_a, τ_b), Firm i's value function at time t is given by[13]

$$\mathbb{E}_t^x\Big[e^{-r(\tau_i \wedge \tau_{-i} - t)}\left[\mathbf{1}_{\tau_i<\tau_{-i}}(L(X_{\tau_i}) - K_1) + \mathbf{1}_{\tau_i>\tau_{-i}}F(X_{\tau_{-i}})\right]\Big], \quad i = a,\, b, \tag{5}$$

where $X_t = x > 0$ and $\mathbf{1}_A$ is an indicator function that equals one if event A occurs and zero otherwise. The first term in (5) captures the event where firm i is Leader and the second term captures the event where firm i is Follower. As the event $\tau_i = \tau_{-i}$ has zero probability almost surely for mixed strategies, we exclude this possibility in (5) to ease exposition.

Here, we focus on the mixed strategies when firms make their entry decisions. We characterize the Markov perfect mixed-strategy equilibrium by using the firms' stochastic entry rate processes. Let $\lambda_i(X_t)$ denote this controlled stochastic entry rate process at which firm i exercises its investment option. For any $t < \tau_L$, the probability that firm i becomes Leader over a small time interval $[t, t+dt]$ is $\lambda_i(X_t)dt$. Firm i's entry time τ_i is a doubly stochastic process as the associated intensity process $\{\lambda_i(X_t)\}_{t\geq 0}$ is also stochastic.[14] Leader's entry time τ_L is then given by[15]

$$\tau_L = \min\{\tau_a, \tau_b\} \tag{6}$$

and is also doubly stochastic but with an intensity process of $\{\lambda_a(X_t) + \lambda_b(X_t)\}_{t\geq 0}$. Next, we define feasible mixed strategies and the Markov perfect mixed-strategy equilibrium.

Definition 1 An entry rate λ_i is a measurable function from $\mathbb{R}_+$ to $\mathbb{R}_+$. A pair of strategy (λ_a, λ_b) is feasible if and only if for any $t > 0$, $\int_0^t \lambda_i(X_s)ds < \infty$ almost surely. Let Φ denote the set of all feasible mixed strategies.

[13] We show later that there exist a symmetric mixed-strategy equilibrium and asymmetric pure-strategy equilibria. For both cases, we can ignore the event where $\tau_a = \tau_b$ almost surely. For brevity, we thus leave out the $\tau_a = \tau_b$ scenario in our definition of value functions.

[14] Stopping time τ is doubly stochastic if the underlying counting process $\{\mathcal{N}_t\}_{t\geq 0}$ whose first jump time τ is doubly stochastic. A counting process $\{\mathcal{N}_t\}_{t\geq 0}$ is doubly stochastic if its associated intensity process $\{\lambda_t\}_{t\geq 0}$ is $\{\mathcal{F}_t\}_{t\geq 0}$-predictable and for all t and $s > t$, conditional on the σ-algebra generated by $\{\mathcal{N}_u\}_{u\in[0,t]}$ and $\mathcal{F}_s$, the random variable $(\mathcal{N}_s - \mathcal{N}_t)$ has a Poisson distribution with parameter $\int_t^s \lambda_u du$. Now we apply these definitions to our model. Let $\{\mathcal{G}_t\}_{t\geq 0}$ be the σ-algebra generated by $\{\mathcal{F}_t\}_{t\geq 0}$ and $\{\mathcal{N}_t^i\}_{t\geq 0}$, where $i = a, b$. For any $t \geq 0$ and $s > t$, conditional on the σ-algebra generated by $\mathcal{G}_t \bigcup \mathcal{F}_s$, the counting processes $\{\mathcal{N}_u^a - \mathcal{N}_t^a\}_{u\in[t,s]}$ and $\{\mathcal{N}_u^b - \mathcal{N}_t^b\}_{u\in[t,s]}$ are independent and the random variable $(\mathcal{N}_s^i - \mathcal{N}_t^i)$ has a Poisson distribution with parameter $\int_t^s \lambda_i(X_u)du$ for $i = a, b$. Firm i's entry time τ_i is thus doubly stochastic with the underlying counting process $\{\mathcal{N}_t^i\}_{t\geq 0}$ and the associated intensity process $\{\lambda_i(X_t)\}_{t\geq 0}$. See Lando (1998) and Duffie (2005) among others for applications of doubly stochastic processes to affine credit-risk models.

[15] Technically, $\tau_L = \min\{t \geq 0 : \mathcal{N}_t^a + \mathcal{N}_t^b = 1\}$.

Definition 2 Let $J_i(x; \lambda_a, \lambda_b)$ denote firm i's value at time t defined in (5) for a given $X_t = x > 0$ and a feasible Markov mixed strategy pair (λ_a, λ_b). A feasible strategy pair $(\lambda_a^*, \lambda_b^*)$ is a *Markov perfect mixed-strategy equilibrium* if for any $x > 0$, the following conditions hold:

$$J_a(x; \lambda_a^*, \lambda_b^*) \geq J_a(x; \lambda_a, \lambda_b^*), \quad \forall\, (\lambda_a, \lambda_b^*) \in \Phi, \tag{7}$$

$$J_b(x; \lambda_a^*, \lambda_b^*) \geq J_b(x; \lambda_a^*, \lambda_b), \quad \forall\, (\lambda_a^*, \lambda_b) \in \Phi. \tag{8}$$

Let $V_i(x)$ denote firm i's equilibrium value function: $V_i(x) = J_i(x; \lambda_a^*, \lambda_b^*)$.

Before analyzing duopoly competition, we summarize the solutions for two benchmarks: a monopoly and a planner's problem for the cooperative duopoly setting.

3 Monopoly and Cooperative Duopoly

We first summarize the solution for the standard single firm's real-option model, which we also refer to as the monopoly problem, and later use it as a benchmark with which we compare our duopoly competition model solution. Additionally, when summarizing the monopoly solution we introduce a few functions that are helpful for our duopoly analysis.

Monopoly. A stand-alone firm chooses its entry time, τ_M, to solve the following problem:

$$M^*(x) \;=\; \max_{\tau_M \geq t} \quad \mathbb{E}_t^x \left[e^{-r(\tau_M - t)} \left(\int_{\tau_M}^{\infty} e^{-r(s-\tau_M)} X_s ds - K_1 \right) \right], \tag{9}$$

where $X_t = x > 0$ and $M^*(x)$ is the optimal value function. The firm's value after exercising its option ($t \geq \tau_M$) is given by the standard Gordon growth model:

$$\Pi(x) = \mathbb{E}_t^x \left[\int_t^{\infty} e^{-r(s-t)} X_s ds \right] = \frac{x}{r - \mu}. \tag{10}$$

The $\Pi(x)$ function is the (gross) payoff value for the firm.

The optimal investment policy for the standard real-option problem (9) takes the form of an endogenous threshold which we denote by x_M. That is, the monopolist enters the first moment τ_M^* when $\{X_s\}$ exceeds x_M to be reported later: $\tau_M^* = \inf\{s \geq t : X_s \geq x_M\}$.

The standard approach to solving (9) is using the widely used smooth-pasting condition as in McDonald and Siegel (1986) and Dixit and Pindyck (1994). Here, we solve the real-option problem as a monopolist's value-maximizing problem. Doing so has an additional benefit of allowing us to derive an intermediate result useful for our duopoly analysis.[16]

[16] Of course, the monopoly solution method (less used but also known in the literature) and the standard

First, we calculate the firm's option value associated with an exogenously given investment threshold $\widehat{x}$. The value for a firm that invests at the first moment X_s exceeds $\widehat{x}$: $\tau_M = \inf\{s \geq t : X_s \geq \widehat{x}\}$, denoted by $M(x;\widehat{x})$, is given by

$$M(x;\widehat{x}) = \left(\frac{x}{\widehat{x}}\right)^{\beta} \left(\Pi(\widehat{x}) - K_1\right), \quad x < \widehat{x}\,, \tag{11}$$

$$M(x;\widehat{x}) = \Pi(x) - K_1, \quad x \geq \widehat{x}\,, \tag{12}$$

where $\Pi(x)$ is given by (10) and $\beta > 1$ is the optimality parameter given by[17]

$$\beta = \frac{-(\mu - \frac{1}{2}\sigma^2) + \sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2r\sigma^2}}{\sigma^2}\,. \tag{13}$$

In the $x < \widehat{x}$ region the firm waits and in the $x \geq \widehat{x}$ region the firm invests.

Second, the firm chooses its threshold $\widehat{x}$ to maximize (11), which is effectively a (static) monopolist's problem. A higher value of $\widehat{x}$ increases the *quantity* $(\Pi(\widehat{x}) - K_1)$, the net payoff upon investing at τ_M, but decreases the *price* (time-t value of a dollar paid at τ_M): $\mathbb{E}_t^x[e^{-r(\tau_M - t)}] = (x/\widehat{x})^{\beta}$. The firm chooses $\widehat{x}$ to maximize its value $M(x;\widehat{x})$, the product of $(\Pi(\widehat{x}) - K_1)$ and $(x/\widehat{x})^{\beta}$. We obtain the following closed-form solution for $\widehat{x}^* = x_M$:

$$x_M = \frac{\beta}{\beta - 1}(r - \mu)K_1\,. \tag{14}$$

For any given $x \in (0, x_M)$, we can show that $M(x;\widehat{x})$ is increasing in $\widehat{x}$ for $\widehat{x} \in [x, x_M]$ and decreasing in $\widehat{x}$ for $\widehat{x} > x_M$.[18] Therefore, x_M is the optimal entry threshold for (11): $\widehat{x}^* = x_M$ and the firm's value function is $M^*(x) = M(x; x_M)$. We next summarize the above main results below.

Proposition 1 *The optimal entry threshold x_M is given in (14) and the monopolist's value function is given by*

$$M^*(x) = M(x; x_M)\,, \tag{15}$$

where $M(x;\widehat{x})$, firm value for a given entry threshold $\widehat{x}$, is given by (11)-(12). For a given $x \in (0, x_M)$, $M(x;\widehat{x})$ is increasing in the threshold $\widehat{x}$ for $\widehat{x} \in [x, x_M]$, which implies

$$M(x;\widehat{x}) \geq M(x;x) = \Pi(x) - K_1 \text{ for } \widehat{x} \in [x, x_M]\,. \tag{16}$$

smooth-pasting-condition-based approach are mathematically equivalent.

[17]That is, β is the larger root of the fundamental quadratic equation, $\sigma^2 z(z-1)/2 + \mu z - r = 0$, associated with the GBM X process (1) in standard real option models.

[18]For any $x \in (0, \widehat{x})$, $\dfrac{\partial M(x;\widehat{x})}{\partial \widehat{x}} = \dfrac{x^{\beta}}{\widehat{x}^{\beta+1}}(\beta - 1)\left[\dfrac{\beta}{\beta - 1}K_1 - \Pi(\widehat{x})\right] = \dfrac{x^{\beta}}{\widehat{x}^{\beta+1}}\dfrac{\beta - 1}{r - \mu}(x_M - \widehat{x})$. As $\beta > 1$, $\frac{\partial M(x;\widehat{x})}{\partial \widehat{x}} > 0$ for $\widehat{x} \in (x, x_M)$ and $\frac{\partial M(x;\widehat{x})}{\partial \widehat{x}} < 0$ for $\widehat{x} > x_M$. Also by definition, $\frac{\partial M(x;\widehat{x})}{\partial \widehat{x}} = 0$ for $\widehat{x} \in (0, x)$.

The last result in Proposition 1 implies that the longer the firm waits before $\tau_M^* = \inf\{s \geq t : X_s \geq x_M\}$, the higher its value $M(x;\widehat{x})$. We establish the second-mover advantage using this result in the next section. Next, we solve a planner's total market capitalization maximization problem, which we refer to this case as a cooperative duopoly.

Cooperative Duopoly. A planner who maximizes the total market capitalization of the two firms chooses Leader's entry time $\tau_L \geq t$ and Follower's entry time $\tau_F \geq \tau_L$ by solving:

$$\mathbb{E}_t^x \left[\int_{\tau_L}^{\infty} e^{-r(s-t)} X_s ds - K_1 e^{-r(\tau_L - t)} - K_2 e^{-r(\tau_F - t)} \right]. \tag{17}$$

Let $W(x)$ denote the planner's value function and let $(\widetilde{\tau}_L, \widetilde{\tau}_F)$ denote the pair of Leader's and Follower's optimal entry timing strategies. By definition, $\widetilde{\tau}_L \leq \widetilde{\tau}_F$. Since Follower's value entirely comes from grabbing a half of the industry profits from Leader and moreover it also incurs an upfront fixed entry cost K_2. It is therefore socially optimal to only allow one firm to enter and give it the entire profits. Next, we summarize this monopoly efficiency result.

Proposition 2 *The planner's value $W(x)$ equals a monopolist's value $M^*(x)$. Leader's entry time is the same as the monopolist's: $\widetilde{\tau}_L = \inf\{s \geq t : X_s \geq x_M\}$, where $M^*(x)$ and x_M are given in Proposition 1. Finally, Follower never enters: $\widetilde{\tau}_F = \infty$.*

In our model, permanently granting one firm monopoly rights and excluding the other firm is socially optimal. We purposefully choose this simple setting in order to focus on the effect of the second-mover advantage in our duopoly competition model.

4 Duopoly Competition: Mixed-Strategy Equilibrium

In this section, we solve for the mixed-strategy equilibrium and value functions for our duopoly model. To guide our analysis in this section, in Figure 1 we divide the duopoly game into three periods and then highlight the value functions in each period: $t \geq \tau_F$ (after Follower enters), $t \in [\tau_L, \tau_F)$ (after Leader enters but before Follower enters), and $t < \tau_L$ (before Leader enters). Using backward induction, we first solve Follower's problem.

4.1 Follower's Pre-entry and Leader's Post-entry Values: $F(x), L(x)$

After Leader enters ($t \geq \tau_L$), Follower solves its optimal entry decision problem.

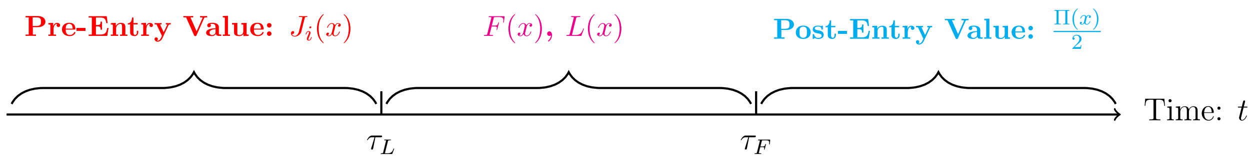

Figure 1: This figure summarizes various value functions for a given pair of entry timing (τ_L, τ_F) in three time periods: $t < \tau_L$ (before Leader's entry); $t \in [\tau_L, \tau_F]$; and $t > \tau_F$ (after Follower's entry).

Follower's Optimal Entry and Pre-entry Value. Follower's problem (3) is the same as a monopolist's problem with K_1 and $\{X_s; s \geq 0\}$ replaced by K_2 and $\{X_s/2; s \geq 0\}$, respectively, in Proposition 1. Follower's pre-entry value $F(x)$ is thus given by:

$$F(x) = \left(\frac{\Pi(x_F)}{2} - K_2\right)\left(\frac{x}{x_F}\right)^{\beta}, \quad x < x_F, \tag{18}$$

$$F(x) = \frac{\Pi(x)}{2} - K_2\,, \quad x \geq x_F, \tag{19}$$

where the optimal entry threshold, x_F, is given by

$$x_F = \frac{2\beta}{\beta - 1}(r - \mu)K_2\,. \tag{20}$$

As in standard real option models, Follower's pre-entry value $F(x)$ is increasing and convex. The higher the volatility σ, the higher the value $F(x)$.

Equations (14) for x_M and (20) for x_F imply that under the assumption $K_1 \geq 2K_2$ a monopoly with an exclusive access to the industry enters later than Follower in our duopoly setting: $x_F \leq x_M$. This result implies second-mover advantage in our model.

Leader's Post-entry Value. Solving (4) for $t \geq \tau_L$, we obtain

$$L(x) = \Pi(x) - \frac{\Pi(x_F)}{2}\left(\frac{x}{x_F}\right)^{\beta}, \quad x < x_F \tag{21}$$

$$L(x) = \frac{\Pi(x)}{2}, \quad x \geq x_F\,. \tag{22}$$

In the $x \geq x_F$ region, both Leader and Follower are active and they equally split the market share, valued at $\Pi(x)/2$. In the $x < x_F$ region, Leader's time-t value $L(x)$ thus equals the difference between the industry's total market capitalization $\Pi(x)$ and $\frac{\Pi(x_F)}{2}\left(\frac{x}{x_F}\right)^{\beta}$. The latter term equals the present value of Leader's lost profits caused by Follower's entry.[19]

[19]The term $\frac{\Pi(x_F)}{2}\left(\frac{x}{x_F}\right)^{\beta}$ equals the value of lost profits $\Pi(x_F)/2$ at τ_F, multiplied by $(x/x_F)^{\beta}$, the time-t value of a dollar paid when $\{X_s\}$ reaches x_F.

Next, we summarize the key results for $L(x)$ and $F(x)$.

Proposition 3 *Follower's optimal entry time is given by* $\tau_F^* = \inf\{s \geq \tau_L : X_s \geq x_F\}$, *where* x_F *is its optimal entry threshold given by (20). In the* $x \geq x_F$ *region, Follower's pre-entry and Leader's post-entry values,* $F(x)$ *and* $L(x)$, *are given by (19) and (22), respectively. In the* $x < x_F$ *region,* $F(x)$ *and* $L(x)$ *are given by (18) and (21), respectively. Finally,*

$$L(x) - K_1 < F(x)\,, \quad x > 0\,. \tag{23}$$

Equation (23) states that a firm is always better off being Follower. That is, our model features a second-mover advantage for all $x > 0$. We discuss the forces behind this key result in two steps. First consider the $x \geq x_F$ region. As Follower pays a lower upfront entry cost than Leader, $(L(x) - K_1) - F(x) = K_2 - K_1 < 0$. Second, in the $x \in (0, x_F)$ region, using (21) for $L(x)$ and (18) for $F(x)$, we obtain:

$$\begin{aligned}(L(x) - K_1) - F(x) &= (\Pi(x) - K_1) - (\Pi(x_F) - K_2)\,(x/x_F)^\beta \\ &< (\Pi(x) - K_1) - (\Pi(x_F) - K_1)\,(x/x_F)^\beta \\ &= M(x; x) - M(x; x_F) \leq 0\,. \end{aligned} \tag{24}$$

The first inequality follows from $K_1 > K_2$. The second inequality follows from (16) by using the property that Follower's entry trigger x_F is lower than the monopolist's entry trigger x_M: $x_F \leq x_M$, implied by $K_1 \geq 2K_2$. In sum, we have shown that our duopoly competition model features a second-mover advantage: $L(x) - K_1 < F(x)$ for any x.[20]

Inequality (23), the definition of $J_i(x; \lambda_a, \lambda_b)$ given in (5), and $V_i(x)$ given in Definition 2 together imply that the equilibrium value function $V_i(x)$ satisfies:

$$L(x) - K_1 \leq V_i(x) \leq F(x)\,, \quad x > 0\,. \tag{25}$$

The inequality on the left holds because a firm can always become Leader immediately. The inequality on the right also holds because the best that a firm can be is Follower: $V_i(x) \leq \mathbb{E}_t^x[e^{-r(\tau_L - t)} F(X_{\tau_L})] \leq F(x)$. In essence, (25) states that it is always advantageous to be the second mover (Follower). Next, we turn to a firm's decision to become Leader.

[20] We can prove a stronger result than $L(x) - K_1 < F(x)$ shown in (24): $L(x) - K_1 < \Pi(x)/2 - K_2$. (see Remark 1 in Appendix). This inequality states that the Leader's net payoff upon entry, $L(x) - K_1$, is always strictly lower than the net payoff value of being Follower: $\Pi(x)/2 - K_2$. The inequality $L(x) - K_1 < \Pi(x)/2 - K_2$ implies $L(x) - K_1 < F(x)$ as an option is always at least worth as much as its net payoff value upon immediate exercising: $F(x) \geq \Pi(x)/2 - K_2$ for all $x > 0$.

4.2 Closed-Form Markov Perfect Mixed-Strategy Equilibrium

First, we solve for firm i's value, $J_i(x; \lambda_a(x), \lambda_b(x))$ for a given mixed strategy pair $(\lambda_a(x), \lambda_b(x))$. The following HJB equation for $J_i(x) = J_i(x; \lambda_a(x), \lambda_b(x))$ holds:

$$rJ_i(x) = \frac{\sigma^2 x^2}{2} J_i''(x) + \mu x J_i'(x) + \lambda_i(x)[L(x) - K_1 - J_i(x)] + \lambda_{-i}(x)[F(x) - J_i(x)]\,, \quad (26)$$

where $L(x)$ is given by (21) and (22), and $F(x)$ is given by (18) and (19). The intuition for the HJB equation (26) is as follows. The first two terms on the right side are standard and capture the effects of diffusion and drift of X on $J_i(x)$. The third term describes the effect of Firm i's own entry strategy on its value. The last term describes the effect of the competitor's mixed entry strategy on firm i's value. If the competitor enters, firm i becomes Follower and its value function jumps from $J_i(x)$ to $F(x)$. The sum of these four terms on the right side equals the annualized firm value $rJ_i(x)$ (Duffie, 2001).

Next, we turn to the symmetric Markov perfect equilibrium. Let $\lambda^*(x) = \lambda_a^*(x) = \lambda_b^*(x)$ denote the symmetric equilibrium Markov perfect mixed strategy for the two firms. Let $V_i(x)$ denote firm i's equilibrium value function: $V_i(x) = J_i(x; \lambda_a^*(x), \lambda_b^*(x))$. There are two scenarios to consider: 1.) $\lambda^*(x) > 0$ and 2.) $\lambda^*(x) = 0$. When $\lambda^*(x) > 0$, the firm must be indifferent between entering the market (becoming Leader) and waiting, which means the value functions from the two strategies are equal:

$$V_i(x) = L(x) - K_1\,, \quad \text{if} \quad \lambda^*(x) > 0\,. \quad (27)$$

Using (26) and (27), we obtain the following HJB equation for $V_i(x)$:

$$rV_i(x) = \frac{\sigma^2 x^2}{2} V_i''(x) + \mu x V_i'(x) + \lambda^*(x)[F(x) - V_i(x)]\,, \quad (28)$$

which hold for both $\lambda^*(x) > 0$ and $\lambda^*(x) = 0$ cases. The key term in (28) is the last one, which captures the expected change of firm i's value due to its competitor's entry. Although the industry demand X is continuous, firm value is discontinuous and jumps when its competitor enters the market.

Re-arranging (28) yields the following expression for $\lambda^*(x)$:[21]

$$\lambda^*(x) = \frac{rV_i(x) - \left[\frac{\sigma^2 x^2}{2} V_i''(x) + \mu x V_i'(x)\right]}{F(x) - V_i(x)}\,. \quad (29)$$

[21] Mathematically, the numerator of (29) is $-\mathcal{A}V_i(x)$, where $\mathcal{A} = \frac{\sigma^2 x^2}{2}\frac{\partial^2}{\partial x^2} + \mu x \frac{\partial}{\partial x} - r$ is the infinitesimal generator.

When $\lambda^*(x) > 0$, substituting $V_i(x) = L(x) - K_1$ given in (27) into (29), we obtain

$$\lambda^*(x) = \frac{rL(x) - \left[\frac{\sigma^2 x^2}{2} L''(x) + \mu x L'(x)\right] - rK_1}{F(x) - (L(x) - K_1)} \,. \tag{30}$$

That is, $\lambda^*(x)$ is fully determined by $L(x)$ and $F(x)$.

We later show that $\lambda^*(x) > 0$ holds for $x \geq \overline{x}$ and $\lambda^*(x) = 0$ holds for $x < \overline{x}$, where the threshold for the mixed strategy, $\overline{x}$, satisfies the following value-matching and smooth-pasting conditions:

$$V_i(\overline{x}) = L(\overline{x}) - K_1, \tag{31}$$

$$V_i'(\overline{x}) = L'(\overline{x}) \,. \tag{32}$$

While these two boundary conditions resemble the standard value-matching and smooth-pasting conditions for a single firm's optimal threshold in the standard models, the economics underpinning (31)-(32) is different from standard real-option models. Mathematically, we generalize the variational-inequality analysis in standard real-option models to our strategic setting in the mixed-strategy equilibrium. In Section 5, we propose a separation principle that links our duopoly competition model to standard real-option problems.

Note that in the $\lambda^*(x) = 0$ region, (28) implies the following HJB equation for $V_i(x)$:

$$rV_i(x) = \frac{\sigma^2 x^2}{2} V_i''(x) + \mu x V_i'(x) \,, \;\; x < \overline{x} \,. \tag{33}$$

We can show that $\overline{x} > x_F$ holds in equilibrium, which implies that it is optimal for Follower to enter immediately after Leader does at τ_L^*. Therefore, using (22), we obtain the following linear payoff function for $L(x)$ at $\overline{x}$: $L(\overline{x}) = \Pi(\overline{x})/2$. Substituting $L(\overline{x}) = \Pi(\overline{x})/2$ into (31)-(33), we obtain the closed-form expression for $V_i(x)$, denoted by $V^*(x)$:

$$V^*(x) = \left(\frac{x}{\overline{x}}\right)^{\beta} \left(\frac{\Pi(\overline{x})}{2} - K_1\right) \,, \quad x < \overline{x} \,, \tag{34}$$

$$V^*(x) = \frac{\Pi(x)}{2} - K_1 \,, \quad x \geq \overline{x} \,, \tag{35}$$

$$\overline{x} = \frac{2\beta}{\beta - 1}(r - \mu)K_1 \,. \tag{36}$$

Equations (34) and (35) resemble the standard value function expressions in the waiting and exercising regions as in McDonald and Siegel (1986), Dixit and Pindyck (1994), and Grenadier (1996). Equation (36) implies that the threshold above which firms enter probabilistically, $\overline{x}$, equals the optimal entry trigger for a (hypothetical) monopoly who has a perpetual option to enter by paying a one-time cost K_1 and afterwards receives $\{X_s/2\}$

infinitely. Because Follower's entry cost is lower than Leader's ($K_2 < K_1$), $x_F < \overline{x}$ which implies that Follower immediately enters after Leader does ($\tau_F^* = \tau_L^*+$). As a result, Leader never enjoys monopoly rents in equilibrium.

It is worth emphasizing that

$$\overline{x} = 2x_M \,. \tag{37}$$

That is, the threshold above which firms stochastically enter, $\overline{x}$, is twice as high as the monopolist's entry x_M, which maximizes the cooperative duopoly's total surplus. Intuitively, competition in our model discourages firms from entering rather than encourages them to make preemptive moves. This is because firms anticipate no monopoly rents and prefer to be the second mover so as to save $K_1 - K_2$ out of the entry cost.

In the $x \geq \overline{x}$ region, both firms optimally randomize their entry decisions. Therefore, $V_i(x)$ equals Leader's net payoff value $\big(\Pi(x)/2 - K_1\big)$, as given in (35). Because both firms wait with probability one in the $x < \overline{x}$ waiting region ($\lambda^*(x) = 0$), firm i's pre-entry value $V_i(x)$ equals the product of (a.) $(x/\overline{x})^\beta$, the present value of a dollar paid at the moment of Leader's entry τ_L^*, and (b.) $\big(\Pi(\overline{x})/2 - K_1\big)$, Leader's value netting of investment cost K_1. Firm i's pre-entry value, $V_i(x)$, is increasing and convex in x.

Using the no-arbitrage asset-pricing equation for $L(x)$ to simplify (30), we obtain[22]

$$\lambda^*(x) = \frac{CF_L(x) - rK_1}{F(x) - (L(x) - K_1)} \,, \tag{38}$$

where $CF_L(x)$ is Leader's equilibrium cash flow. The numerator in (38) is the firm's net income (the net benefit of becoming Leader) per unit of time and the denominator $F(x) - (L(x) - K_1)$ is the forgone value of becoming Leader. The equilibrium symmetric entry rate $\lambda^*(x)$ must equal the ratio given in (38) so that the firm is indifferent between becoming Leader now and waiting to enter at the rate of $\lambda^*(\,\cdot\,)$ the next instant. This result is related to the war-of-attrition argument for exit games (Levin, 2004). Unlike standard war-of-attrition games, ours is an entry game with stochastic and endogenous cash flows and reward payoffs. Additionally, the option value of waiting is crucial in our model.

Since Leader can only capture one half of the market share, we have $CF_L(x) = x/2$ and $F(x) - (L(x) - K_1) = K_1 - K_2$ for $x \geq \overline{x}$. Therefore, in equilibrium, (38) implies the following

[22] The asset-pricing equation for $L(x)$ is: $rL(x) = CF_L(x) + \mu x L'(x) + \frac{\sigma^2 x^2}{2} L''(x)$, which states that the total expected rate of return, including both cash flows and capital gains (the drift and volatility terms), for Leader equals the risk-free rate r.

expression for the equilibrium entry rate $\lambda^*(x)$:

$$\lambda^*(x) = \frac{x/2 - rK_1}{K_1 - K_2} > 0\,, \quad x \geq \overline{x}\,. \tag{39}$$

The entry rate $\lambda^*(x)$ increases linearly with x for $x \geq \overline{x}$ and approaches ∞ as $x \to \infty$. The numerator in (39) equals firm i's *net income* given by the operating profit $x/2$ minus rK_1, the interest expense of financing the upfront investment cost K_1.

While both firms prefer to be Follower, for sufficiently high values of x (in the $x \geq \overline{x}$ region), they are indifferent in equilibrium between 1.) entering and becoming Leader instantly and 2.) waiting for another instant with the hope that the other firm becomes Leader meanwhile and if not both continue playing the mixed strategy. Because Follower has a cost-saving advantage over Leader (second-mover advantage), Follower immediately enters as soon as Leader does. As a result the denominator in (39) equals $F(x) - (L(x) - K_1) = K_1 - K_2 = \Delta K$, the difference between Leader's and Follower's upfront entry cost.

In equilibrium, the entry rate $\lambda^*(x)$ equals the ratio between firm i's current net income $\frac{1}{2}x - rK_1$ and the entry-cost wedge $K_1 - K_2$. This insight is analogous to the war-of-attrition argument for standard exit games Levin (2004). Here, as Follower is clearly better off, neither firm is *ex ante* willing to become Leader voluntarily. Both firms prefer free-riding the other by being Follower and saving ΔK out of its entry cost. The mixed-strategy equilibrium is thus a compromised outcome between the two firms. As a firm waits for the other to enter, it forgoes the opportunity of collecting profits $x/2 - rK_1$, but preserves the option value of being the second mover and saving $\Delta K = K_1 - K_2$. In equilibrium, both firms are indifferent between entering and waiting when λ^* is given in (39). The higher the value of x, the higher the costs of forgoing one-period profit and thus the more likely it enters (e.g., $\lambda^*(x)$ increasing in x.)

Finally, note the discontinuity of $\lambda^*(x)$ as x reaches $\overline{x}$ from the left: $\lambda^*(x)$ is zero in the $x < \overline{x}$ waiting region, jumps to $\lambda^*(\overline{x}) = \left(\frac{\beta}{\beta-1}\frac{r-\mu}{r} - 1\right)\frac{rK_1}{K_1-K_2} > 0$ at $x = \overline{x}$.

We summarize the equilibrium solution in Figure 2. While a duopoly entry game generally features three stochastic time periods as illustrated in Figure 1, in equilibrium $\tau_F^* = \tau_L^*+$ and there are only two time periods in our model. For $t < \tau_F^* = \tau_L^*+$, $V_a(x) = V_b(x)$. For $t > \tau_F^* = \tau_L^*+$, each firm receives a half of the market share and is valued at $\Pi(x)/2$. Regarding entry, one firm randomly becomes Leader paying K_1 at τ_L^* and the other firm immediately enters at $\tau_F^* = \tau_L^*+$ paying K_2 as Follower. The probability that firm a ends up

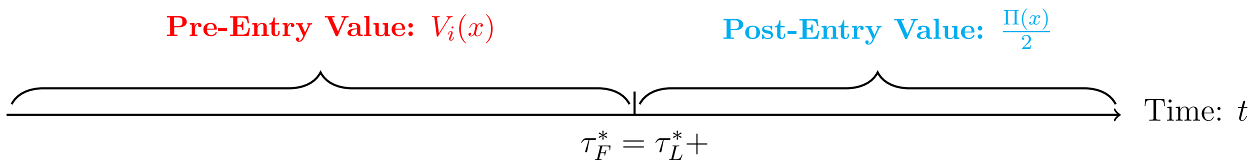

Figure 2: This figure summarizes the mixed-strategy equilibrium solution. For $t < \tau_F^* = \tau_L^* +$, $V_a(x) = V_b(x)$. For $t > \tau_F^* = \tau_L^* +$, each firm receives one half of the market share and is valued at $\Pi(x)/2$. Regarding entry, one firm randomly becomes Leader paying K_1 at τ_L^* and the other firm immediately enters at $\tau_F^* = \tau_L^* +$ paying K_2 as Follower.

being the winner (Follower) is one half. Next we summarize our duopoly model solution.

Theorem 1 *Firm i's value function is given by* (34)-(35)*. The symmetric Markov perfect equilibrium strategy is given by* $(\lambda_a^*(x), \lambda_b^*(x)) = (\lambda^*(x), \lambda^*(x))$*. In the* $x < \overline{x}$ *region, where* $\overline{x}$ *is the threshold for the mixed trategy given by* (36)*, both firms wait:* $\lambda^*(x) = 0$*. In the* $x \geq \overline{x}$ *region, both firms enter stochastically at the rate of* $\lambda^*(x) > 0$ *given in* (39)*. As soon as one firm enters, the other also enters immediately:* $\tau_F^* = \tau_L^* +$*.*

Next, we provide an alternative solution method for the duopoly mixed-strategy equilibrium, which we refer to as the separation principle. This principle helps us understand the mechanism for the mixed-strategy equilibrium.[23]

5 Separation Principle and Application to Our Model

Before introducing the separation principle, it is helpful to first define the following real-option problem. A single firm chooses its optimal entry time τ to receive a gross payoff value of $L(x)$ given in (21)-(22) by paying a fixed cost K_1. Mathematically, the firm solves the following optimal stopping problem:

$$H(x) := \max_{\tau \geq t} \mathbb{E}_t^x \left[e^{-r(\tau - t)} (L(X_\tau) - K_1) \right] . \tag{40}$$

We can show that the value function $H(x)$ equals firm i's value function $V_i(x)$ for the mixed-strategy equilibrium, which leads to the separation principle.

[23]Our separation principle is different from the separation principle in the incomplete information optimal control literature Liptser and Shiryaev (1977), which states that the optimization problem with incomplete symmetric information can be decomposed into two steps: first estimate the state variables using filtering techniques and then solve dynamic programming problems using the filtered state variables.

5.1 Separation Principle

Next we state the separation principle and then discuss the intuition for this principle.

Theorem 2 *The value function $H(x)$ for a single firm's real-option problem (40) equals firm i's value function $V_i(x)$ for the mixed-strategy equilibrium. Therefore, we can equivalently obtain the mixed-strategy equilibrium solution in two steps. First, we solve a single firm's real-option problem (40) to obtain the value function in our duopoly setting: $V_i(x) = H(x)$. Second, we obtain the equilibrium entry rate $\lambda^*(x)$ in the region where $H(x) = L(x) - K_1$ by using a war-of-attrition argument given in (30).*

The separation principle allows us to decompose the mixed-strategy equilibrium solution into two subproblems. First, we solve a single firm's optimal stopping problem (which ignores the strategic interaction between the two firms). Second, we obtain the equilibrium entry rate using a war-of-attrition argument. This decomposition result holds for a general duopoly competition model with a second-mover advantage.

We derive the separation principle in two steps. First, we show that firm i's value function in the mixed-strategy equilibrium satisfies the following variational inequality:

$$\max\left\{\frac{\sigma^2 x^2}{2} V_i''(x) + \mu x V_i'(x) - rV_i(x), (L(x) - K_1) - V_i(x)\right\} = 0\,, \tag{41}$$

which is the same variational inequality for $H(x)$, the value function of a single firm's entry problem (40) (Øksendal, 2013). As the variational inequality (41) admits a unique solution Friedman (1982), $V_i(x) = H(x)$.[24]

Second, we calculate the equilibrium entry rate $\lambda^*(x) = \frac{CF_L(x) - rK_1}{F(x) - H(x)} > 0$ using (38) in the region where $H(x) = L(x) - K_1$. This formula allows us to interpret the stochastic entry game as a generalized war-of-attrition game where the game payoffs and cash flows are

[24] We derive the variational inequality (41) as follows. The HJB equation (28) and the inequality $V_i(x) \leq F(x)$ given in (25) together imply

$$\frac{\sigma^2 x^2}{2} V_i''(x) + \mu x V_i'(x) - rV_i(x) \leq 0\,.$$

The other inequality $L(x) - K_1 \leq V_i(x)$ given in (25) and (27) together imply $\lambda^*(x) = 0$ if $L(x) - K_1 < V_i(x)$. Substituting this result into (28), we obtain

$$\frac{\sigma^2 x^2}{2} V_i''(x) + \mu x V_i'(x) - rV_i(x) = 0 \quad \text{if} \quad L(x) - K_1 < V_i(x).$$

Combining above with $L(x) - K_1 \leq V_i(x)$, we obtain the variational inequality (41).

endogenous and the winner of the game is Follower.

In order to make firm i indifferent between quitting the attrition game (by entering) and continuing the game for another period, its competitor (firm $-i$) must set its entry rate $\lambda_{-i}(x)$ to $\lambda^*(x)$ by solving the following equation in the $H(x) = L(x) - K_1$ region:

$$CF_L(x) - rK_1 = \lambda^*(x)\left[F(x) - H(x)\right] . \tag{42}$$

The (flow) cost of waiting, $CF_L(x) - rK_1$ on the left side of (42), equals the (flow) benefit of waiting, which equals the reward of being Follower (the attrition game's winner) multiplied by $\lambda^*(x)$, the equilibrium rate at which firm i wins the attrition game.

While (42) may at first appear to be a myopic analysis as the flow benefit on the left side seems to ignore the dynamics of the state variable x, it is optimal and time consistent. This seemingly myopic strategy is optimal because the firm's entry region is already optimized (from the first step) and a version of the envelope condition is at work.

As our preceding analysis does not depend on specific assumptions of our duopoly model, the separation principle thus applies broadly to duopoly games with second-mover advantages. Next, we apply the separation principle to our mixed-strategy equilibrium.

5.2 Application of Separation Principle to Our Model

First, using the standard value-matching and smooth-pasting conditions to solve the variational inequality (41), we obtain the following closed-form solutions for the value function associated with the single firm's real-option problem (40):[25]

$$H(x) = V^*(x), \tag{43}$$

where $V^*(x)$ is given by (34)-(35). It is helpful to emphasize $H(x) = L(x) - K_1$ in the $x \geq \overline{x}$ region, where $\overline{x}$ is given by (36).

Second, substituting $CF_L(x) = x/2$ and $F(x) - (L(x) - K_1) = K_1 - K_2$ into (38), we obtain the equilibrium entry rate $\lambda^*(x)$ given in (39). As discussed earlier, this result follows from the war-of-attrition argument adapted to our duopoly setting. In order to make firm i indifferent between quitting the attrition game (by entering) and continuing the game for

[25] We show that only the linear part of the payoff $L(x) - K_1$ in the $x \geq x_F$ region, given by (22), is used to solve for $H(x)$, which allows us to derive the same closed-form solutions.

another period, its competitor (firm $-i$) must set its entry rate $\lambda_{-i}(x)$ to $\lambda^*(x)$ by solving:

$$x/2 - rK_1 = \lambda^*(x)(K_1 - K_2)\,, \quad \text{for} \quad \text{all} \quad x \geq \overline{x}\,. \tag{44}$$

The (flow) cost of waiting, $x/2 - rK_1$ on the left side of (44), equals the (flow) benefit of waiting, which equals the saved entry cost $K_1 - K_2 = \Delta K$ by being Follower multiplied by $\lambda^*(x)$, the equilibrium rate at which firm i wins the attrition game.

In sum, we can obtain the mixed-strategy equilibrium of our duopoly entry model as follows. First, we solve a single firm's optimal entry problem (40) to obtain firm value $V_i(x) = V^*(x)$ given in (34)-(35) and a threshold $\overline{x}$ given in (36). Second, we use this threshold to define the stochastic entry region $x \geq \overline{x}$, where $V_i(x) = L(x) - K_1$ and $\lambda^*(x) > 0$. Additionally, we pin down $\lambda^*(x)$ using (44) based on a generalized war-of-attrition argument as discussed above.

6 Pure-strategy Equilibria

In this section, we analyze pure-strategy equilibria.

Pure-strategy Equilibrium Definition. Let $\mathcal{E}_i \subset (0, \infty)$ denote a closed set associated with firm i's entry strategy: firm i enters at t if and only if $X_t \in \mathcal{E}_i$. Let Φ denote the set of all feasible entry strategies for firms a and b: $(\mathcal{E}_a, \mathcal{E}_b)$. Then for each $(\mathcal{E}_a, \mathcal{E}_b) \in \Phi$, firm i's time-t value is given by

$$\mathbb{E}_t^x \left[e^{-r(\tau_L - t)} \left(\mathbf{1}_{\tau_i < \tau_{-i}} (L(X_{\tau_i}) - K_1) + \mathbf{1}_{\tau_i > \tau_{-i}} F(X_{\tau_{-i}}) + \mathbf{1}_{\tau_i = \tau_{-i}} \frac{L(X_{\tau_i}) - K_1 + F(X_{\tau_i})}{2} \right) \right], \tag{45}$$

where $\tau_L = \tau_i \wedge \tau_{-i}$ and $\tau_i = \inf\{s \geq t : X_s \in \mathcal{E}_i\}$ is the first time firm i enters $\mathcal{E}_i$. The first term in (45) captures the event where firm i is Leader and the second term captures the event where firm i is Follower. The last term in (45) accounts for the possibility that the two firms enter at the same time. Next, we define pure-strategy equilibria.

Definition 3 A pair of entry strategy $(\mathcal{E}_a^*, \mathcal{E}_b^*)$ is a *pure-strategy equilibrium* if for any $x > 0$ the following conditions hold:

$$J_a(x; \mathcal{E}_a^*, \mathcal{E}_b^*) \geq J_a(x; \mathcal{E}_a, \mathcal{E}_b^*), \quad \forall\, (\mathcal{E}_a, \mathcal{E}_b^*) \in \Phi, \tag{46}$$

$$J_b(x; \mathcal{E}_a^*, \mathcal{E}_b^*) \geq J_b(x; \mathcal{E}_a^*, \mathcal{E}_b), \quad \forall\, (\mathcal{E}_a^*, \mathcal{E}_b) \in \Phi. \tag{47}$$

In equilibrium, $J_a(x; \mathcal{E}_a^*, \mathcal{E}_b^*)$ and $J_b(x; \mathcal{E}_a^*, \mathcal{E}_b^*)$ are the value functions for firms a and b.

Consider an asymmetric pure-strategy equilibrium where firm b never becomes Leader and firm a becomes Leader at τ_L.[26] In this equilibrium, firm a solves the real-option problem (40). Let $P_L(x)$ and x_L denote firm a's value function and its optimal trigger, respectively. Then, $P_L(x) = H(x) = V^*(x) = V_i(x)$ and $x_L = \overline{x}$, where $\overline{x}$ is given in (36). Hence, firm a's optimal entry time is $\tau_L^* = \inf\{s \geq t : X_s \geq \overline{x}\}$. That is, Leader's value function $P_L(x)$ in the pure-strategy equilibria equals firm value $V_i(x)$ in the mixed-strategy equilibrium and the optimal trigger x_L equals the threshold for the mixed-strategy entry region $\overline{x}$.

Second, after firm a enters at τ_L^*, firm b optimally enters at $\tau_F^* = \inf\{s \geq \tau_L^* : X_s \geq x_F\}$, where x_F is given in (20). Because $\overline{x} > x_F$, $\tau_F^* = \tau_L^* +$. Therefore, Follower's value is

$$P_F(x) = \mathbb{E}_t^x \left[e^{-r(\tau_L^* - t)} (\Pi(X_{\tau_L^*})/2 - K_2) \right], \tag{48}$$

where $\tau_L^* = \inf\{s \geq t : X_s \geq \overline{x}\}$. Solving (48), we obtain the following closed-form solutions:

$$P_F(x) = F(x) = \Pi(x)/2 - K_2, \quad x \geq \overline{x}, \tag{49}$$

$$P_F(x) = (x/\overline{x})^\beta F(\overline{x}) = (x/\overline{x})^\beta (\Pi(\overline{x})/2 - K_2), \quad x < \overline{x}. \tag{50}$$

Theorem 3 *In an asymmetric pure-strategy equilibrium, Leader enters at $\tau_L^* = \inf\{s \geq t : X_s \geq x_L\}$, where the threshold x_L equals $\overline{x}$ as given in (36), and its value function $P_L(x)$ equals $V^*(x)$ as given in (34)-(35). Follower enters at $\tau_F^* = \inf\{s \geq \tau_L^* : X_s \geq x_F\}$, where x_F is given in (20). Because $x_L = \overline{x} > x_F$, Follower enters immediately after Leader ($\tau_F^* = \tau_L^*+$) and its value function $P_F(x)$ is given by (49)-(50). Mathematically, $\mathcal{E}_a^* = [x_L, \infty)$ and $\mathcal{E}_b^* = \emptyset$ form an asymmetric pure-strategy entry equilibrium.*

Next, we compare the total market capitalization for our mixed-strategy and pure-strategy equilibria.

Corollary 1 *The asymmetric pure-strategy equilibrium yields a higher total value than the symmetric mixed-strategy equilibrium: $2V_i(x) \leq P_L(x) + P_F(x)$ for all $x > 0$.*

The above result follows from $V_i(x) = P_L(x) \leq P_F(x)$ for the two types of equilibria. Note that in our pure-strategy equilibria, Leader still exercises its entry option later than the socially optimal level. This is because Leader anticipates that its competitor will immediately follow it to enter. Follower's plan to grab one half of the market share from Leader causes Leader to inefficiently delay its entry. This result differs from simple war-of-attrition

[26] Naturally, by switching firm a's role with b's, we obtain another asymmetric pure-strategy equilibrium.

examples, where the pure-strategy equilibria are socially efficient as one firm immediately drops out Levin (2004). Why are our pure-strategy equilibria socially inefficient? This is because Leader (the loser in the attrition game) also has the real option value. This result highlights the rich predictions generated by the interaction between the real-option value and the second-mover advantage in our stochastic entry game.

We have focused our analysis on the three equilibria: the symmetric mixed-strategy equilibrium and two asymmetric pure-strategy equilibria (one with firm a being Leader and the other with firm a being Follower). We point out that there are other equilibria which heuristically speaking involve a combination of mixed-strategy and pure-strategy equilibria solutions. For brevity, we leave the details of these equilibria out of the paper.

7 Model Implications and Quantitative Analysis

In this section, we further study model implications and provide a quantitative analysis.

Parameter Choices. Our model is parsimonious with only five parameters in total. As in Grenadier (1996), we set the annual risk-free rate to $r = 0.04$, the expected growth rate (drift) of the profit process X to $\mu = 0.02$, and the volatility of the growth rate of X to $\sigma = 0.1$ per annum. We normalize Leader's fixed entry cost to $K_1 = 1$. Since Follower's entry cuts Leader's profits by half at all time, we set Follower's entry cost to half of Leader's, $K_2 = 0.5$, to keep the cost-benefit (profit) ratio the same for Leader and Follower.

First, we discuss our model's implications in the symmetric mixed-strategy equilibrium.

7.1 Value Functions and Equilibrium Entry Strategies

We first analyze the mixed-strategy equilibrium and then the pure-strategy equilibria.

7.1.1 Mixed-strategy Equilibrium

In Panel A of Figure 3, we plot value functions for the mixed-strategy equilibrium. Before either firm enters the market ($t < \tau_L^*$). the two firms are symmetric and their value functions are equal: $V_a(x) = V_b(x)$. There are two regions to consider. For sufficiently low demand ($x < \overline{x} = 0.097$), the dominant strategy for both firms is to wait ($\lambda^*(x) = 0$). The solid

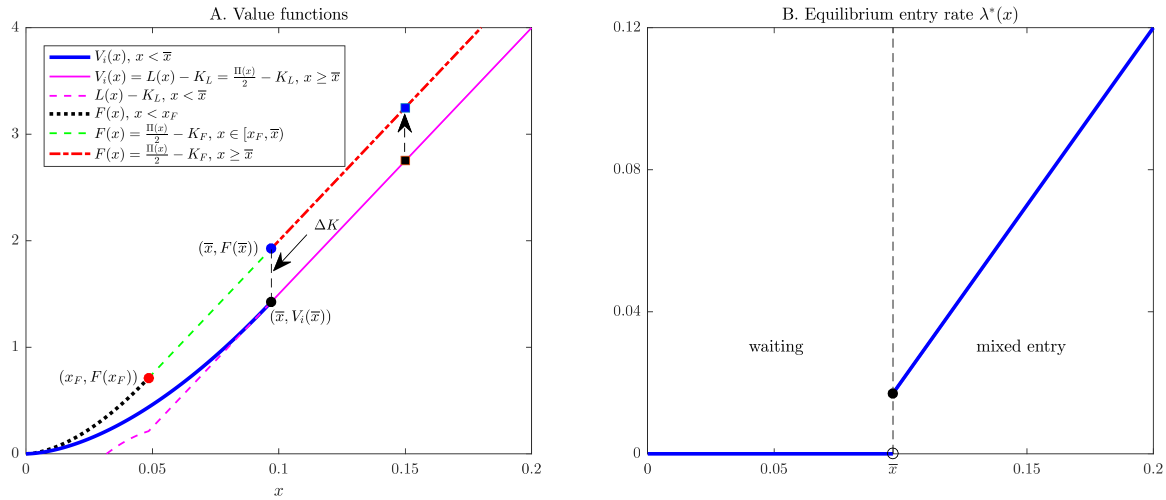

Figure 3: Value functions and entry rate in the mixed-strategy equilibrium. Both firms probabilistically enter at the rate of $\lambda^*(x) > 0$ for all $x \geq \overline{x} = 0.097$. Follower immediately enters at τ_F^* after Leader enters at τ_L^*: $\tau_F^* = \tau_L^*+$. For $t < \tau_L^*$, $V_a(x) = V_b(x)$.

blue line depicts the corresponding firm value $V_i(x)$, which is increasing and convex for $i = a, b$. For sufficiently high demand ($x \geq \overline{x} = 0.097$), both firms are willing to enter but only probabilistically. As they are using mixed strategies, they must be indifferent between becoming Leader and waiting for another period, which means $V_a(x) = V_b(x) = L(x) - K_1$.

Why do firms choose mixed strategies when x is sufficiently high? On one hand firms are willing to pay the entry cost K_1 to become Leader as the payoffs from entering the market are sufficiently large. But on the other hand, firms prefer to be Follower as its entry cost is lower than Leader's by $\Delta K = K_1 - K_2$. These two considerations make firms settle for mixed strategies, a compromise between waiting and entering with probability one.

The cutoff threshold above which firms adopt the mixed strategy, $\overline{x}$, is determined by the smooth-pasting condition linking $V_i(x)$ with firm i's net payoff value function from being Leader, $L(x) - K_1$ (the magenta line), at $x = \overline{x} = 0.097$.

Next, we turn to Follower's pre-entry problem, which is a standard real-option entry problem as in McDonald and Siegel (1986), Dixit and Pindyck (1994). Follower behaving as a monopolist receives a profit flow at the rate of $X_s/2$ after entering the market. Therefore, Follower's optimal entry threshold equals x_F given in (20). Follower's pre-entry value $F(x)$ has two segments: the convex option value in the $x \leq x_F$ region (the black dotted line) and the linear net payoff value $\Pi(x)/2 - K_2$ in the $x > x_F$ region (the green dashed and red dash-dotted lines).

Note that the threshold above which a firm stochastically becomes Leader, $\overline{x}$, is larger than Follower's entry threshold x_F:

$$\overline{x} = \frac{2\beta}{\beta - 1}(r-\mu)K_1 > \frac{2\beta}{\beta - 1}(r-\mu)K_2 = x_F \,.$$

As a result, as soon as one firm becomes Leader at τ_L^*, the other firm immediately enters at $\tau_F^* = \tau_L^*+$ as Follower.[27]

Next, we use Panel A of Figure 3 to illustrate this Leader/Follower entry dynamics. Suppose by playing mixed strategies, firm i stochastically becomes Leader at $\tau_L^* = \tau_i$ when $X_{\tau_i} = 0.15 > \overline{x}$ (the black square on the magenta solid straight line $V_i(x) = \Pi(x)/2 - K_1$). Immediately, its competitor (firm $-i$) exercises its entry option as Follower and its value jumps from the same black square by $\Delta K = K_1 - K_2 = 0.5$ to the blue square on the dash dotted red line $F(x) = \Pi(x)/2 - K_2$. The payoff lines for Leader and Follower are linear and parallel with a slope of $1/(2(r-\mu))$.[28]

Panel B of Figure 3 plots the equilibrium mixed-strategy intensity $\lambda^*(x)$. For $x < \overline{x} = 0.097$, both firms wait with probability one. For $x \geq \overline{x} = 0.097$, both firms probabilistically enter as Leader at the rate of $\lambda^*(x)$, which increases linearly with x. Note the discontinuous jump as we reach $\overline{x}$ from the left of $\overline{x} = 0.097$. The entry rate $\lambda^*(x)$ equals the ratio of (a) the (flow) benefit of being Leader and (b.) the (stock) cost of being Leader: $\lambda^*(x) = (x/2 - rK_1)/\Delta K$. The (flow) benefit of being Leader equals the difference between duopoly profit and the interest expense of the entry cost: $x/2 - rK_1$. The (stock) cost of being Leader equals the entry-cost wedge: $\Delta K = K_1 - K_2$.

Intuitively, in the $x \geq \overline{x}$ region, neither firm is willing to become Leader with probability one due to the "free-rider" problem (second-mover cost-saving advantage): As Leader, Follower receives the same post-entry payoff $\Pi(x)/2$, but with a lower entry cost K_2. Thus, the only way to determine Leader in this region is for both firms to randomize their entry decisions at the equilibrium rate $\lambda^*(x)$.

This is exactly the war-of-attrition argument. But unlike the standard wars of attrition

[27]This is because (a.) Leader's entry cost is larger than Follower's: $K_1 > K_2$ and (b.) both firms receive the same post-entry payoffs in equilibrium: $\Pi(x)/2$. Therefore, Follower is more willing to exercise its entry option than Leader, which means $x_F < \overline{x}$. By definition $\tau_F^* \geq \tau_L^*$, therefore in equilibrium as soon as one firm enters the market, the other immediately follows.

[28]The vertical distance between the two lines equals $\Delta K = K_1 - K_2 = 0.5$ for all $x \geq \overline{x}$.

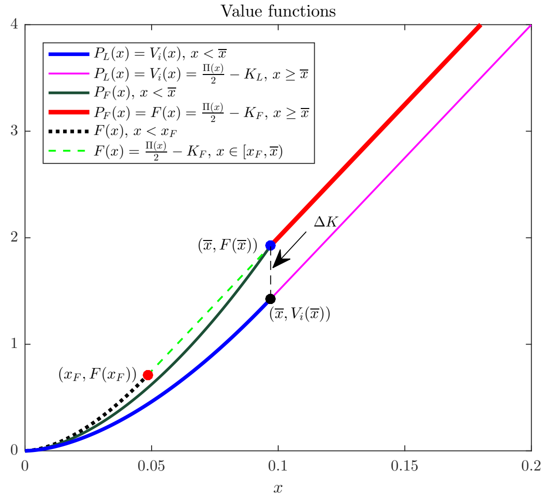

Figure 4: VALUE FUNCTIONS FOR PURE-STRATEGY EQUILIBRIA. Leader's value equals the one in the mixed-strategy equilibrium: $P_L(x) = V_a(x) = V_b(x)$ and Follower's value $P_F(x)$ is higher than Leader's value: $P_F(x) > P_L(x)$. The optimal Leader's entry threshold x_L equals the threshold, $\overline{x}$, for stochastic entry in the mixed-strategy equilibrium: $x_L = \overline{x} = 0.097$.

in graduate micro theory lecture note (Levin, 2004), our duopoly model is an entry rather than an exit game and moreover it blends the insights from both the real-option theory and the war-of-attrition literature. Importantly, we show that the interaction of these two forces generates new predictions. While competition erodes a firm's option value, it does so not by speeding up entry but rather by delaying entry. Next, we analyze pure-strategy equilibria.

7.1.2 Pure-strategy Equilibria

In Figure 4, we plot Leader's and Follower's value functions, $P_L(x)$ and $P_F(x)$, for the asymmetric pure-strategy equilibria, and then compare them with the value function $V_i(x)$ for the symmetric mixed-strategy equilibrium.

In a pure-strategy equilibrium, firms are pre-assigned to be Leader or Follower (e.g., firm a is Leader and b is Follower). The solid lines depict the equilibrium pre-entry Leader's value $P_L(x)$ where the blue segment is increasing and convex in x in the waiting region ($x < \overline{x}$) and the magenta line is Leader's net linear payoff function $\Pi(x)/2 - K_1$ in the entry region ($x \geq \overline{x}$). The solid red line gives Follower's net linear payoff function $P_F(x) = F(x) = \Pi(x)/2 - K_2$ at τ_F^* in the region where Follower enters ($x \geq \overline{x}$).

Also, Follower's pre-entry value function $P_F(x)$ in the waiting region ($x < \overline{x}$) is increasing and convex (the solid green line.) Because Follower can only enter when x exceeds $\overline{x}$, which is higher than Follower's unconstrained entry threshold x_F given in (20), Follower's value function is lower than $F(x)$, i.e., $P_F(x) < F(x)$ in our pure-strategy equilibrium. The black dotted and green dashed line segments for $F(x)$ in Figure 4 aid our understanding of the model's mechanism and solutions but are off-the-equilibrium path.[29]

7.1.3 Comparing Mixed-strategy with Pure-strategy Equilibria

Now we link the symmetric mixed-strategy equilibrium with the asymmetric pure-strategy equilibria. First, in both mixed-strategy and pure-strategy equilibria, the dominant strategy for both firms is to wait in the $x \le \overline{x}$ region with probability one. This is because second-mover advantages prevail in both types of equilibria: Neither firm has incentives to become Leader as the competitor will immediately enter by paying a lower entry cost K_2 and taking a half of the total market share. Therefore, there are no monopoly profits for Leader in equilibrium.

Second, Leader in a pure-strategy equilibrium sets its entry threshold x_L at $\overline{x}$ as its problem is equivalent to a real-option problem with an entry cost of K_1 and a payoff that is one-half of the market share as we show in Theorem 3. The pure-strategy equilibrium solution for Leader corresponds to a single firm's problem in McDonald and Siegel (1986) with a properly chosen payoff function. Then using our separation principle for the mixed-strategy equilibrium, we conclude that (1.) the entry threshold must also equal $\overline{x}$ and (2.) $V_i(x)$ equals Leader's value in a pure-strategy equilibrium $P_L(x)$:

$$V_a(x) = V_b(x) = P_L(x)\,.$$

Third, as $\overline{x} = 2x_M > x_F$, Follower enters immediately after Leader does in both types of equilibria. As $K_1 > K_2$, Follower's value in the pure-strategy equilibria is larger than in the mixed-strategy equilibrium: $P_F(x) > V_i(x)$. The industry's total market capitalization in a pure-strategy equilibrium is thus larger than in the mixed-strategy equilibrium for all $x > 0$: $P_L(x) + P_F(x) - [V_a(x) + V_b(x)] = P_F(x) - V_i(x) = P_F(x) - P_L(x) > 0$, as $P_L(x) = V_a(x) = V_b(x)$ and $P_F(x) > P_L(x)$ (implied by the second-mover advantage).

Let $\Psi(x)$ denote the fractional loss of the industry's total market capitalization as we

[29]To ease exposition, we use solid lines to draw all the on-the-equilibrium-path value functions.

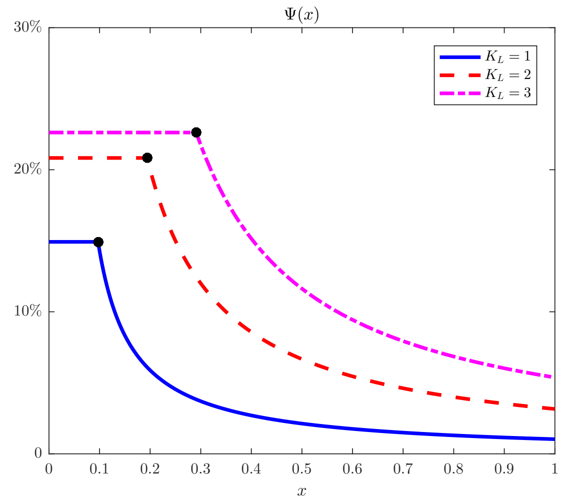

Figure 5: Fractional loss of the industry's total market capitalization $\Psi(x)$. This figure plots $\Psi(x)$ as we move from a pure-strategy equilibrium to the mixed-strategy equilibrium and shows that the mixed-strategy equilibrium is less efficient than the pure-strategy equilibria. Quantitatively, these effects are significant especially in the $x \leq \overline{x}$ waiting region. For the three cases: $K_1 = 1, 2, 3$, $\overline{x} = 0.097, 0.194, 0.291$ (the black dots).

move from a pure-strategy equilibrium to the mixed-strategy equilibrium:

$$\Psi(x) = 1 - \frac{V_a(x) + V_b(x)}{P_L(x) + P_F(x)} = \frac{P_F(x) - P_L(x)}{P_F(x) + P_L(x)} > 0\,, \quad x > 0\,.$$

In Figure 5, we plot $\Psi(x)$ for three levels of Leader's entry cost: $K_1 = 1, 2, 3$. The higher the entry cost K_1, the larger the total market capitalization differences between the two types of equilibria $\Psi(x)$. The black dots depict the relation: $\overline{x}(K_1) = \frac{2\beta}{\beta-1}(r-\mu)K_1$.

In the $x \in (0, \overline{x})$ region, both firms wait and the fractional loss $\Psi(x)$ as we move from a pure-strategy equilibrium to the mixed-strategy equilibrium is constant: $\Psi(x) = \frac{K_1 - K_2}{\frac{\beta+1}{\beta-1}K_1 - K_2}$. In Figure 5, we demonstrate that the value loss is large and crucially depends on K_1. As we increase the entry cost from $K_1 = 1$ to $K_1 = 3$, the threshold $\overline{x}$ increases from 0.097 to 0.291, and the fractional loss $\Psi(x)$ increases from 14.9% to 22.62% in the $x \in (0, \overline{x})$ region.

In the $x \geq \overline{x}$ region, $\Psi(x) = \frac{K_1 - K_2}{\Pi(x) - K_1 - K_2}$, which decreases with x. Intuitively, the higher the value of x the more likely firms enter in the mixed-strategy equilibrium. The inefficiency of the mixed-strategy equilibrium relative to pure-strategy equilibria decreases.

Having compared mixed-strategy and pure-strategy equilibrium solutions for a given economy, next we study the effect of competition on welfare by comparing our duopoly competition model solution to the cooperative duopoly solution.

7.2 Competition and Option Value Erosion

We measure inefficiency by comparing the total market capitalization of the competitive duopoly industry with the cooperative duopoly setting. We first analyze the mixed-strategy equilibrium and then turn to the pure-strategy equilibria.

7.2.1 Mixed-strategy Equilibrium

Let $\Delta(x)$ denote the fractional value loss of the industry due to duopoly competition:

$$\Delta(x) = 1 - \frac{V_a(x) + V_b(x)}{W(x)}\,, \tag{51}$$

where $V_a(x) + V_b(x)$ is the industry's total market capitalization in the mixed-strategy equilibrium and $W(x) = M^*(x)$ is the cooperative duopoly (also monopoly) value given by (15).

Recall that a monopolist enters whenever x exceeds the threshold $x_M = \beta/(\beta-1)(r-\mu)K_1$ and in contrast firms in the mixed-strategy equilibrium enter probabilistically when $x \geq \overline{x} = 2x_M$. Note that $\overline{x}$ is twice as high as the monopolist's threshold x_M indicating substantial inefficient delay. We divide the entire $x > 0$ in three regions to ease our discussion of $\Delta(x)$.

Using closed-form expressions, we can show that in the $x \leq x_M$ region, both firms wait with probability one and the fractional value loss equals $\Delta(x) = 1 - (1/2)^{\beta-1}$, which is independent of Leader's entry cost K_1. This independence result is reflected by the three squares on the horizontal line at the top of panel A. In our example, $\beta = 1.70$ and $\Delta(x) = 38.5\%$ in the $x \leq x_M = 0.0485K_1$ region. This almost 40% substantial value loss comes from anticipated significant entry delay in the future.

In the intermediate region where $x \in (x_M, \overline{x}) = (x_M, 2x_M) = (0.0485K_1, 0.097K_1)$, firms in our duopoly model still wait even though it is socially efficient to enter. The fractional value loss is given by

$$\Delta(x) = 1 - \frac{(x/\overline{x})^{\beta}\,(\Pi(\overline{x}) - 2K_1)}{\Pi(x) - K_1}. \tag{52}$$

The numerator in the second term is the total market capitalization while waiting and the denominator equals the monopolist's value (by exercising the entry option). Panel A of Figure 6 shows that $\Delta(x)$ decreases with x and reaches the same value $\Delta(\overline{x}) = (\beta-1)/(\beta+1)$ regardless of Leader's entry cost K_1 at $x = \overline{x}$. The three black dots on the dashed black line reflect this result. In our example, this fractional value loss is substantial: $\Delta(\overline{x}) = 25.97\%$.

Finally, in the $x \geq \overline{x}$ region where both firms stochastically enter, the fractional value

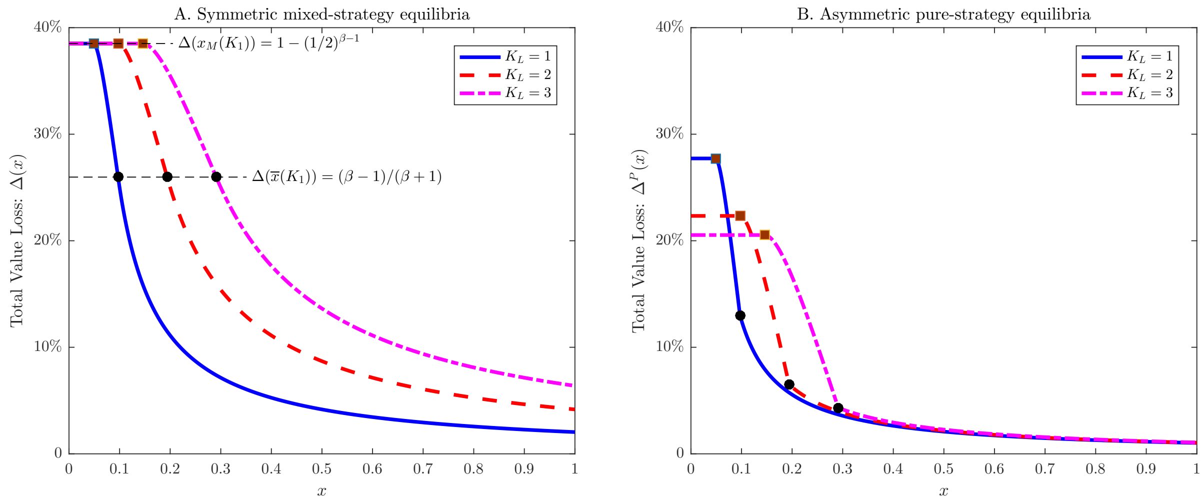

Figure 6: Total Value Losses (as a Fraction of Cooperative Duopoly Value $W(x)$). Panels A and B plot the value loss, $\Delta(x)$ for the mixed-strategy equilibrium and $\Delta^P(x)$ for the pure-strategy equilibria, respectively. Quantitatively, the mixed-strategy equilibrium is significantly more inefficient than the pure-strategy equilibria.

loss equals $\Delta(x) = \frac{K_1}{\Pi(x)-K_1}$ as both firms probabilistically enter without coordinating. Note that $\Delta(x)$ is independent of Follower's entry cost K_2 for all $x > 0$. This is because Follower immediately enters after Leader does.

Unlike in Grenadier (1996) where firms in equilibrium make preemptive moves under competition and hence enter sooner, in our model firms enter later than the socially optimal level as they try to capture the second-mover advantage. The higher Leader's entry cost K_1, the stronger incentives firms have to delay their entry decisions and the higher the fractional value loss $\Delta(x)$. Next, we analyze the pure-strategy equilibria.

7.2.2 Pure-strategy Equilibria

Let $\Delta^P(x)$ denote the fractional value loss of the industry due to duopoly competition:

$$\Delta^P(x) = 1 - \frac{P_L(x) + P_F(x)}{W(x)}, \tag{53}$$

where $P_L(x) + P_F(x)$ is the industry's total market capitalization in a pure-strategy equilibrium and $W(x) = M^*(x)$ is the cooperative duopoly (also monopoly) value given by (15).

As for the mixed-strategy equilibrium, we also divide the entire $x > 0$ range into three regions to ease our discussion of $\Delta^P(x)$. In the $x \leq x_M$ region, both firms wait with probability one and the fractional value loss is given by $\Delta^P(x) = 1 - \frac{1}{2^{\beta-1}}\left(1 + (\beta-1)\frac{K_1-K_2}{2K_1}\right)$, which is constant and lower than the corresponding constant fractional loss $\Delta(x) = 1 - \frac{1}{2^{\beta-1}}$ in the same $x \le x_M$ region for the mixed-strategy equilibrium. In our example with $K_1 = 1$ and $K_2 = 0.5$, $\Delta^P(x) = 27.72\%$ in the $x \le x_M = 0.0485$ region.

In the intermediate region where $x \in (x_M, \overline{x}) = (x_M, 2x_M) = (0.0485K_1, 0.097K_1)$, firms in our duopoly model continue to wait even though it is socially efficient to enter. Then,

$$\Delta^P(x) = 1 - \frac{(x/\overline{x})^\beta \left(\Pi(\overline{x}) - K_1 - K_2\right)}{\Pi(x) - K_1} < \Delta(x) \tag{54}$$

in this region. Finally, in the $x \ge \overline{x}$ region, Leader enters with probability one and the fractional value loss equals $\Delta^P(x) = \frac{K_2}{\Pi(x)-K_1}$, which is again lower than $\Delta(x)$ in the mixed-strategy equilibrium. This is because there is no more inefficient delay once $\{X_s\}$ reaches $\overline{x}$ in a pure-strategy equilibrium. In contrast, firms continue to play a war-of-attrition game in the mixed-strategy equilibrium even when x is very large.

Panel B of Figure 6 plots $\Delta^P(x)$ for three levels of Leader's entry cost: $K_1 = 1, 2, 3$ in a pure-strategy equilibrium. As panel A, panel B confirms our preceding qualitative analysis for a pure-strategy equilibrium.

Quantitatively, the competition effect of firm value in a pure-strategy equilibrium is also large. And importantly, the differences between the fractional value loss $\Delta(x)$ for the mixed-strategy equilibrium and $\Delta^P(x)$ for the pure-strategy equilibrium are also large.

Comparing the two panels in Figure 6 makes it clear that the mixed-strategy equilibrium while more natural to us (say due to its symmetric treatment of the two firms) is much more inefficient than the pure-strategy equilibria. This is because firms enter probabilistically with the hope that the other firm becomes Leader in the mixed-strategy equilibrium. In contrast, the pre-assigned Leader has no incentives to further delay once the threshold $\overline{x}$ is reached or exceeded, as Leader anticipates the immediate entry by Follower.

Next, we analyze our model-implied distributions of time to entry.

7.3 Distributions of Time to Entry $\tau_L^* - t$

Definitions. Fix a calendar date T and let $X_t = x$ for any $t \le T$. Let $\underline{G}(t, x; T)$ denote the time-t cumulative distribution function (CDF) that Leader enters before T in the mixed-strategy equilibrium. Similarly, let $\overline{G}(t, x; T)$ denote the time-t CDF for the same event in the pure-strategy equilibria. Mathematically, for any $x > 0$ and time $t \in [0, T]$:

$$\underline{G}(t,x) = \mathbb{P}_t^x(\tau_L^{\text{mixed}} - t \le T - t) \quad \text{and} \quad \overline{G}(t,x) = \mathbb{P}_t^x(\tau_L^{\text{pure}} - t \le T - t)\,. \tag{55}$$

In (55), we use superscripts, mixed and pure, to indicate that Leader's entry time τ_L^* in the mixed-strategy equilibrium (characterized in Theorem 1) and the pure-strategy equilibria (characterized in Theorem 3) , respectively.

It is worth noting that for every sample path, entry in the pure-strategy equilibria is sooner than in the mixed-strategy equilibrium. This is because firms follow trigger strategies with the same entry region $x \ge \overline{x}$ for both mixed-strategy and pure-strategy equilibria. However, firms enter with probability one in the entry region for the pure-strategy equilibria but only stochastically in the mixed-strategy equilibrium. This path-by-path dominance result implies that the CDF $\underline{G}(t,x)$ for time to entry $\tau_L^* - t$ in the mixed-strategy equilibria also first-order stochastically dominates the CDF $\overline{G}(t,x)$ for $\tau_L^* - t$ in the pure-strategy equilibrium: $\underline{G}(t,x) \le \overline{G}(t,x)$ for any $x > 0$ and $t \in [0, T]$.

CDF for the Mixed-strategy Equilibrium: $\underline{G}(t,x;T)$**.** The CDF for time to entry $\tau_L^* - t$ satisfies the following partial differential equation (PDE) for $t < T$ and all $x > 0$:

$$\underline{G}_t(t,x) + \mu x \underline{G}_x(t,x) + \frac{1}{2}\sigma^2 x^2 \underline{G}_{xx}(t,x) + 2\lambda^*(x)(1 - \underline{G}(t,x)) = 0 \tag{56}$$

subject to economically intuitive boundary conditions: $\underline{G}(t,0) = 0$ and $\lim_{x\to\infty} \underline{G}(t,x) = 1$ for $t \in [0,T)$ and $\underline{G}(T,x) = 0$ for $x \in (0,\infty)$. The first three terms in the PDE (56) are the standard terms describing the calendar time effect, the drift effect of x, and the volatility effect of x on the CDF. The last term captures the "jump" effect of stochastic entry, which is only present for the mixed-strategy equilibrium. Because both firms stochastically become Leader at the rate of $\lambda^*(x)$, $\underline{G}(t,x)$ increases to one at the rate of $2\lambda^*(x)$ and therefore the expected change of the CDF $\underline{G}(t,x)$ equals $2\lambda^*(x)(1 - \underline{G}(t,x))$.

CDF for the Pure-strategy Equilibria: $\overline{G}(t,x;T)$**.** The CDF for $\tau_L^* - t$ in the pure-strategy equilibria, $\overline{G}(t,x)$, satisfies the following PDE for $t < T$ and $x \in [0, \overline{x})$:

$$\overline{G}_t(t,x) + \mu x \overline{G}_x(t,x) + \frac{1}{2}\sigma^2 x^2 \overline{G}_{xx}(t,x) = 0\,, \quad x \in [0, \overline{x})\,, \tag{57}$$

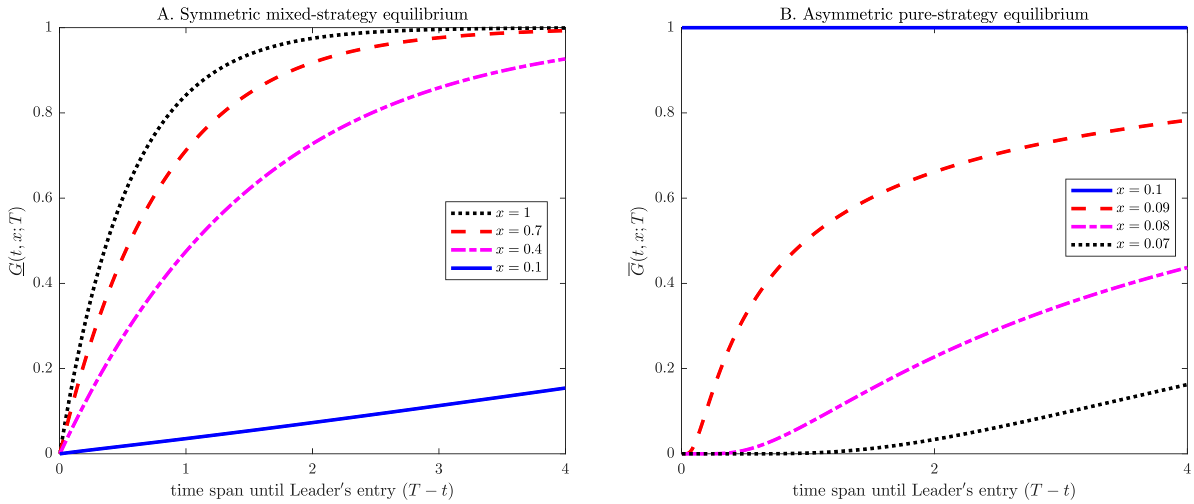

Figure 7: CDF OF TIME TO ENTRY $\tau_L^* - t$ IN PURE-STRATEGY AND MIXED-STRATEGY EQUILIBRIA. Panel A plots the CDF of $\tau_L^* - t$ in the mixed-strategy equilibrium for four levels of x: $0.1, 0.4, 0.7, 1$. Panel B plots the CDF of $\tau_L^* - t$ in the pure-strategy equilibrium for four levels of x: $0.07, 0.08, 0.09, 0.1$.

subject to intuitive boundary conditions: $\overline{G}(t,\overline{x}) = 1$ and $\overline{G}(t,0) = 0$ for $t \in [0,T)$ and $\overline{G}(T,x) = 0$ for $x \in [0,\overline{x})$. The CDF $\overline{G}(t,x)$ has the following closed-form solution:

$$\overline{G}(t,x) = \Phi(d_2) + (x/\overline{x})^{(1-2\mu/\sigma^2)}\,\Phi(d_1), \tag{58}$$

where $\Phi(\,\cdot\,)$ is the CDF for the standard normal distribution and the pair (d_1, d_2) is given by

$$d_1 = d_2 - \left(2\mu/\sigma^2 - 1\right)\sigma\sqrt{T-t}\,, \tag{59}$$

$$d_2 = \frac{\ln(x/\overline{x}) + (\mu - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\,. \tag{60}$$

The first term $\Phi(d_2)$ in (58) equals the time-t probability for the event $X_T \geq \overline{x}$.[30] The second term gives the probability for all the events where $X_T < \overline{x}$ but $\{X_s; s \in (t,T)\}$ exceeds $\overline{x}$ at least once at some $s \in (t,T)$.

Comparing CDFs for Mixed-strategy and Pure-strategy Equilibria. The CDFs of time to entry $\tau_L^* - t$ for the two types of equilibria are dramatically different qualitatively and quantitatively. Panel A in Figure 7 plots the CDFs $\underline{G}(t,x;T)$ of $\tau_L^* - t$ in the mixed-strategy equilibrium for four levels of x: $0.1, 0.4, 0.7, 1$. When $X_t = x = 0.1$, firms enter within one

[30] The first term is analogous to the conditional (risk-neutral) probability that the option holder receives a strictly positive payoff at the option maturity date in the Black-Scholes option pricing formula.

year with a small probability (3.57%). Even within four years, firms only enter with 15.4% probability. In contrast, in a pure-strategy equilibrium, as $X_t = x = 0.1 > \overline{x} = 0.097$, entry occurs with probability one. This comparison of CDFs for the mixed-strategy and pure-strategy equilibria shows that quantitative predictions of the model are quite different depending on which equilibrium we choose. To us, the mixed-strategy equilibrium is more natural and robust as it is symmetric between the two firms.[31]

In the mixed-strategy equilibrium, entry can take significantly much longer time. For example, even when $X_t = x = 1$, there is still $16\% = 1 - \overline{G}(t, 1; t + 1)$ probability that firms have not entered within one year. This is in sharp contrast with the prediction in a pure-strategy equilibrium where entry is immediate provided that $x \geq \overline{x} = 0.097$ as we discussed earlier.

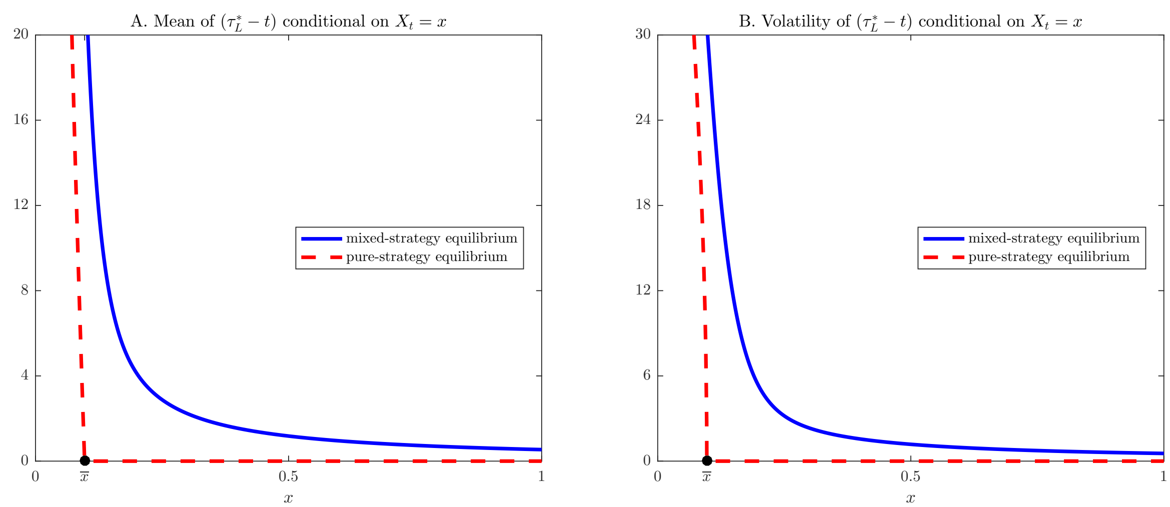

Figure 8: Mean and volatility of time to entry $\tau_L^* - t$ conditional on $X_t = x$. The entry threshold is $\overline{x} = 0.097$.

Finally, Panel A of Figure 8 compares the conditional mean of time to entry $\tau_L^* - t$ in mixed-strategy and pure-strategy equilibria.[32] Again, we see that for $x \geq \overline{x}$, it can take much time for firms to enter in mixed-strategy equilibria while entry is immediate in pure-strategy equilibria (as it's above the entry threshold). Panel B of Figure 8 compares the

[31] Additionally, as known in the war of attrition literature, the mixed-strategy equilibrium is the unique one in settings with incomplete information about the competitor's type.

[32] For the pure-strategy equilibria, in the $x \in (0, \overline{x})$ waiting region, the conditional mean of time to entry equals $\mathbb{E}_t^x[\tau_L^{\text{pure}} - t] = \frac{\log(\overline{x}/x)}{\mu - \sigma^2/2}$, and the conditional variance of time to entry equals $\text{var}_t^x[\tau_L^{\text{pure}} - t] = \sigma^2 \frac{\log(\overline{x}/x)}{(\mu - \sigma^2/2)^3}$, for any $\mu > \sigma^2/2$.

conditional volatility of time to entry $\tau_L^* - t$ for the two types of equilibria and shows that the conditional volatility in the mixed-strategy equilibrium is quite large even when x is significantly larger than the threshold $\overline{x}$. The key takeaway from this figure is that the mixed-strategy equilibrium can be much more inefficient than the pure-strategy equilibria.

8 State-contingent Advantages

So far we have focused on how the second-mover advantage alters strategic real-option exercising, e.g., entry, decisions. In this section, we show that both the first-mover and second-mover advantages endogenously arise in equilibrium for a small positive entry-cost wedge $\Delta K = K_1 - K_2$. For this case, we can show that the first-mover advantage as emphasized in Fudenberg and Tirole (1985) and Grenadier (1996) dominates and causes firms to enter for intermediate values of x. For sufficiently large values of x, the second-mover advantage still dominates as in our baseline model of Section 2. In sum, for a small entry-cost wedge ΔK, the equilibrium solution features state-contingent advantages where firms play pure entry strategies for intermediate values of x due to the first-mover advantage but play a war-of-attrition game for sufficiently large values of x.

We can offer a complete characterization of our model solutions for any positive pair of (K_1, K_2). Due to space considerations, we provide an example illustrating the key mechanism of the case with state-contingent advantages.

In Figure 9, we plot the equilibrium value functions $V_a(x) = V_b(x)$ using the solid line, which divides the positive real line into four line segments, for the new $K_1 = 0.6$ case which implies $\Delta K = 0.1$. Compared with the baseline example analyzed in Section 7 where $K_1 = 1$ and $\Delta K = 0.5$, the entry-cost wedge for the new case is much lower.

A key implication of reducing the entry-cost wedge ΔK to 0.1 from 0.5 is that the first-mover advantage endogenously arises in the $(\widehat{x}_L, \widehat{x}_F) = (0.0221, 0.0414)$ region, as Leader's net payoff $L(x) - K_1$ is above Follower's value $F(x)$: the dashed red line segment is above the dash-dotted magenta line segment. It is the dominant strategy for both firms to enter and become Leader in this region. Which firm becomes Leader is purely random with one half probability. As a result, their pre-entry value functions in this region equal $V_a(x) = V_b(x) = (L(x) - K_1 + F(x))/2$. In sum, in this region, a firm becomes Leader by realizing its first-mover advantage. Its (unlucky) competitor will enter only when $\{X_s\}$ reaches the

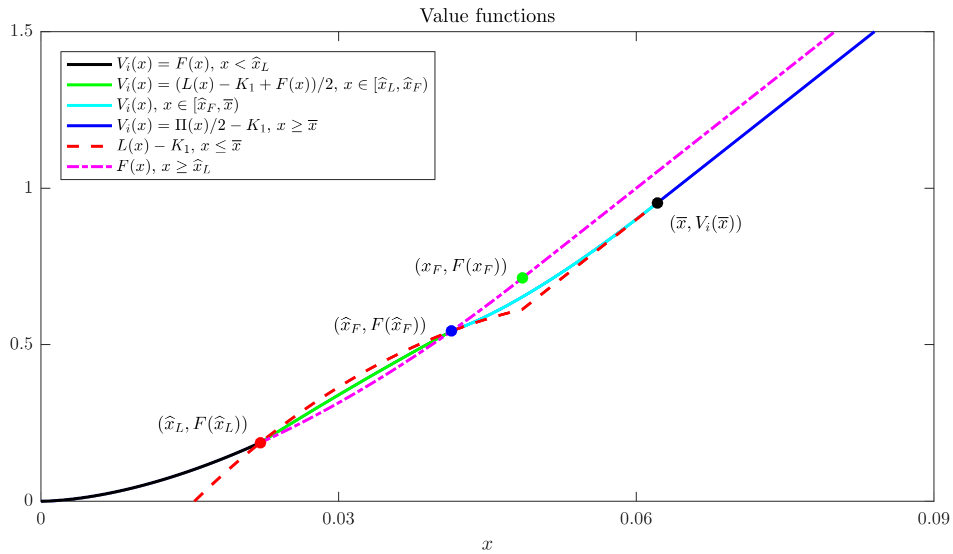

Figure 9: EQUILIBRIUM VALUE FUNCTIONS $V_a(x) = V_b(x)$. In a symmetric equilibrium, in the $x \in [\widehat{x}_L, \widehat{x}_F] = [0.0221, 0.0414]$ (first-mover advantage) region, firms compete to enter: one wins the competition as Leader and the other enters as Follower only when $\{X_s\}$ reaches $x_F = 0.0485$. In the $x \geq \overline{x} = 0.0621$ region, firms probabilistically enter at the rate of $\lambda^*(x)$. After one firm enters, the other also immediately enters. For all other values of x, both firms wait with probability one. The solid line depicts the value function $V_i(x)$. All parameter values are the same as in Section 7 other than Leader's entry cost: $K_1 = 0.6$.

endogenous threshold $x_F = 0.0485$ (the green dot). Unlike in our baseline model, here firms enter sequentially with an economically significant time gap $\tau_F^* - \tau_L^* > 0$.

For $x > \widehat{x}_F = 0.0414$, the second-mover advantage dominates: $F(x) > L(x) - K_1$: the dash-dotted magenta line for $F(x)$ is above the dashed red line for Leader's net payoff $L(x) - K_1$ in Figure 9. But firms enter *probabilistically* when x is sufficiently large: $x \geq \overline{x} = 0.062$. This is because further delaying entry with probability one is simply too costly. In equilibrium, the two firms compromise using the mixed strategies to enter.

Then what happens in the region $x \in (\widehat{x}_F, \overline{x})$, which lies between the first-mover-advantage entry region $(\widehat{x}_L, \widehat{x}_F)$ and the stochastic entry region $(\overline{x}, \infty)$? There is a second-mover advantage as $L(x) - K_1 < F(x)$ when $x \in (\widehat{x}_F, \overline{x})$. Therefore, firms have incentives to wait. Moreover, because the cost of waiting (forgone profit $x/2 - rK_1$) is also not high, firms choose to wait with probability one in this region. As a result, $(\widehat{x}_F, \overline{x})$ is a waiting region that divides the two entry regions.

The $x < \widehat{x}_L = 0.0221$ region is the other waiting region where it is optimal for both firms

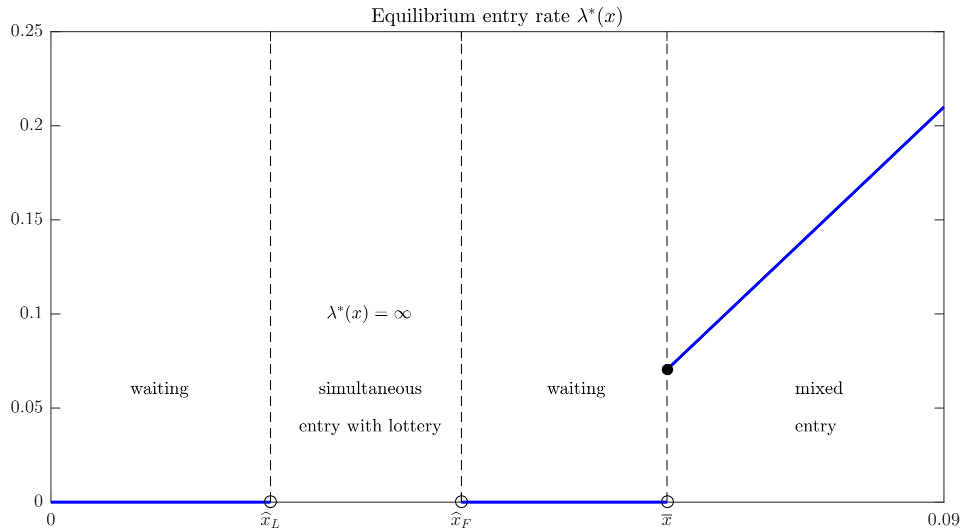

Figure 10: Entry rates in the symmetric equilibrium. Both firms probabilistically enter at the rate of $\lambda^*(x) = (x/2 - rK_1)/\Delta K = 5x - 0.24 > 0$ in the $x \geq \overline{x} = 0.0621$ region. In the $x \in [\widehat{x}_L, \widehat{x}_F] = [0.0221, 0.0414]$ region, both firms are willing to enter immediately. In equilibrium, one firm is randomly chosen to be Leader with probability 1/2 and the other firm enters as Follower in the future when $\{X_s\}$ reaches $x_F = 0.0485$. All parameter values are the same as in Section 7 other than Leader's entry cost: $K_1 = 0.6$.

to preserve the option value by delaying entry. In this region, firm value is increasing and convex: $V_a(x) = V_b(x) = F(x)$ (the solid black line segment).

The first-mover and second-mover advantages not only co-exist but also are interconnected. The triggers ($\widehat{x}_F$, $\widehat{x}_L$) that define first-mover and second-mover advantages are endogenous and are part of the interdependent value functions $L(x)$ and $F(x)$. Figure 9 shows that as x increases from zero to ∞, a firm finds itself in one of the four mutually exclusive regions: 1.) the (first) waiting region (to preserve option value); 2.) the first-mover-advantage region where the two firms compete to enter; 3.) the (second) waiting region; and 4.) the mixed-strategy probabilistic entry region. In Figure 10, we plot the corresponding equilibrium mixed-strategy entry rates $\lambda^*(x)$. In the two disjoint waiting regions ($x < \widehat{x}_L$ and $(\widehat{x}_F, \overline{x})$), $\lambda^*(x) = 0$ by definition. In the $[\widehat{x}_L, \widehat{x}_F]$ region, one firm (lucky Leader decided by a lottery outcome) enters immediately ($\lambda^*(x) = \infty$) and the other waits until $\{X_s\}$ reaches x_F to enter as Follower. Finally, in the $(\overline{x}, \infty)$ region, both firms enter probabilistically at the rate of $\lambda^*(x) = (x/2 - rK_1)/\Delta K$ and once one firm enters the other immediately follows.

9 Conclusion

We present a tractable model of duopoly competition where firms make their irreversible market entry timing decisions. Firms endogenously arise as Leader and Follower in equilibrium. A key property of the duopoly industry structure that we analyze is the second-mover advantage in that Leader's net payoff upon entry is lower than Follower's pre-entry value. In equilibrium, firms prefer to be Follower rather than Leader.

We derive closed-form solutions for both mixed-strategy and pure-strategy equilibria. We develop and prove a separation principle which allows us to derive the mixed-strategy equilibrium solution using a two-step procedure: first we obtain the solution for a monopolist' real-option problem and then use a generalized war-of-attrition argument to derive firms' equilibrium entry rates in the symmetric mixed-strategy equilibrium. Finally, we conduct quantitative analysis and find that (*a.*) the welfare losses due to competition and second-mover advantage are substantial and (*b.*) entry in our duopoly model is much further delayed in the mixed-strategy equilibrium, which we consider a more natural equilibrium concept, than in the pure-strategy equilibria.

To derive our results in a most parsimonious setting we have made some simplifying assumptions. While we mainly focus on the case with only the second-mover advantage, in Section 8 we also highlight the key result in a setting where the first-mover advantage, the key driving force in Fudenberg and Tirole (1985) and Grenadier (1996), and the second-mover advantage coexist and interact with each other, creating state-contingent advantages.[33] We have also assumed that a firm has complete information about its competitor's cost structure and type. We plan to study the effects of reputation as in Kreps and Wilson (1982), Milgrom and Roberts (1982), and Abreu and Gul (2000) on equilibrium real-option exercising strategies.

[33] Due to space considerations, we have omitted the details for the complete analysis of the first-mover and second-mover advantages in our duopoly setting.

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A Proofs

Let $\mathcal{A}V(x)$ denote the infinitesimal generator operating on a function $V(x)$:

$$\mathcal{A}V(x) = \frac{\sigma^2}{2}x^2V''(x) + \mu xV'(x) - rV(x) \tag{61}$$

First we introduce the following lemma.

Lemma 1 *The optionality parameter β given in (13) satisfies $\beta > 1$ and $\frac{\beta}{\beta-1}\frac{r-\mu}{r} > 1$.*

Proof of Lemma 1: As in McDonald and Siegel (1986), β is the positive root of the (fundamental) quadratic equation: $g(z) = 0$ where $g(z) = \frac{1}{2}\sigma^2 z(z-1) + \mu z - r$. Using $g(1) = \mu - r < 0$ and $g(z) \to \infty$ as $z \to \infty$, we obtain $\beta > 1$, a standard result. The inequality $\frac{\beta}{\beta-1}\frac{r-\mu}{r} > 1$ is the same as $r > \beta\mu$. This result apparently holds when $\mu \leq 0$. When $\mu > 0$, $g(r/\mu) = \frac{1}{2}\sigma^2(r/\mu)(r/\mu - 1) > 0$ and $g(\beta) = 0$ imply $r > \beta\mu$. □

Proof of Proposition 1: First, recall that in footnote 18, we have shown the monotonicity of $M(x;\widehat{x})$ in $\widehat{x}$ for $\widehat{x} \in [x, x_M]$. Next, we verify that $M(x;x_M)$ satisfies the following variational inequality (Øksendal, 2013):

$$\max\{\mathcal{A}M^*(x), (\Pi(x) - K_1) - M^*(x)\} = 0. \tag{62}$$

Equation (16) implies $M(x;x_M) \geq \Pi(x) - K_1$ for $x < x_M$. Since $\mathcal{A}M(x;x_M) = 0$ for $x < x_M$ and $M(x;x_M) = \Pi(x) - K_1$ for $x \geq x_M$, it suffices to show $\mathcal{A}(\Pi(x) - K_1) \leq 0$ for $x > x_M$.[34] We can show that for $x > x_M$:

$$\mathcal{A}(\Pi(x) - K_1) = rK_1 - x \leq rK_1 - x_M = rK_1\left(1 - \frac{\beta}{\beta-1}\frac{r-\mu}{r}\right) < 0\,, \tag{63}$$

where the last inequality follows from $\frac{\beta}{\beta-1}\frac{r-\mu}{r} > 1$ in Lemma 1. □

Proof of Proposition 2: See the discussion preceding this proposition. □

Proof of Proposition 3: See the discussion preceding and following this proposition. □

Before proving Theorems 1-3, we introduce the following lemma.

[34] At $x = x_M$, $M''(x;x_M)$ does not exist but $\mathcal{A}M(x_M;x_M) \geq 0$ holds in a generalized sense; see, e.g., Friedman (1982).

Lemma 2 *Let $V^*(x)$ and $\overline{x}$ be given in (34)-(36). We have*

$$\mathcal{A}V^*(x) \leq 0, \quad x > 0, \tag{64}$$

$$V^*(x) \geq L(x) - K_1, \quad x > 0\,. \tag{65}$$

Proof of Lemma 2: Proposition 1 implies that $V^*(x)$ and $\overline{x}$ given in (34)-(36) are the value function and optimal threshold for a monopolist's real-option entry problem where the entry cost is K_1 and the post-entry profit equals $\{X_s/2\}$:

$$V^*(x) \;=\; \max_{\tau_M \geq t} \quad \mathbb{E}_t^x \left[\int_{\tau_M}^{\infty} e^{-r(s-t)} \frac{X_s}{2} ds - e^{-r(\tau_M - t)} K_1 \right]. \tag{66}$$

Using the relation between an optimal stopping problem and its variational inequality, we immediately obtain (64).

By the definitions of $F(x)$ and $\Pi(x)$ given in (3) and (10), we have

$$F(x) - \left(\frac{\Pi(x)}{2} - K_2 \right) = \mathbb{E}_t^x \left[K_2 (1 - e^{-r(\tau_F^* - t)}) - \int_t^{\tau_F^*} e^{-r(s-t)} \frac{X_s}{2} ds \right], \tag{67}$$

where $\tau_F^* = \inf\{s \geq t : X_s \geq x_F\}$. (66) implies

$$V^*(x) \geq \mathbb{E}_t^x \left[\int_{\tau_F^*}^{\infty} e^{-r(s-t)} \frac{X_s}{2} ds - e^{-r(\tau_F^* - t)} K_1 \right]. \tag{68}$$

Combining (4) with (68), we obtain

$$\begin{aligned} V^*(x) - (L(x) - K_1) \;&\geq\; \mathbb{E}_t^x \left[K_1 (1 - e^{-r(\tau_F^* - t)}) - \int_t^{\tau_F^*} e^{-r(s-t)} X_s ds \right] \\ &\geq\; \mathbb{E}_t^x \left[2K_2 (1 - e^{-r(\tau_F^* - t)}) - \int_t^{\tau_F^*} e^{-r(s-t)} X_s ds \right] \\ &=\; 2 \left[F(x) - \left(\frac{\Pi(x)}{2} - K_2 \right) \right] \geq 0, \end{aligned} \tag{69}$$

where the second inequality follows from $K_1 \geq 2K_2$ and the equality follows from (67). Therefore, (65) holds. □

Remark 1 One implication of the preceding proof is that Leader's net payoff upon entry, $L(x) - K_1$, is always strictly lower than Follower's net payoff upon entry, $\frac{\Pi(x)}{2} - K_2$:

$$L(x) - K_1 < \frac{\Pi(x)}{2} - K_2\,, \quad \text{for any} \;\; x > 0. \tag{70}$$

Indeed, for any $x > 0$, using $L(x)$ and $\Pi(x)$ defined in (4) and (10), respectively, we have

$$L(x) - \frac{\Pi(x)}{2} = \mathbb{E}_t^x \left[\int_t^{\tau_F^*} e^{-r(s-t)} \frac{X_s}{2} ds \right] \leq \mathbb{E}_t^x \left[K_2 (1 - e^{-r(\tau_F^* - t)}) \right] < K_2 \leq K_1 - K_2\,,$$

where the first inequality follows from (67) and $F(x) \geq \frac{\Pi(x)}{2} - K_2$ for any $x > 0$. Finally, we can derive (23) by using (70) (see footnote 20).

Next we prove the theorems.

Proof of Theorem 1: Let $f(x) := \mathcal{A}V^*(x)$ for $x \geq 0$ where $V^*(x)$ is given in (34)-(35). We can verify

$$f(x) := \mathcal{A}V^*(x) = \lambda^*(x)[L(x) - K_1 - F(x)], \quad x > 0, \tag{71}$$

where $\lambda^*(x)$ is given by (39) for $x \geq \overline{x}$ and $\lambda^*(x) = 0$ for $x < \overline{x}$. Using $V^*(x)$ as in (34)-(35), we can show $f(x) = 0$ for any $x < \overline{x}$ and $f(x) = rK_1 - x/2 < 0$ for any $x > \overline{x}$. As $\lambda^*(x) \geq \lambda^*(\overline{x}) = \left(\frac{\beta}{\beta-1}\frac{r-\mu}{r} - 1\right)\frac{rK_1}{K_1-K_2} > 0$ for any $x > \overline{x}$, we can conclude

$$e^{-\int_t^\infty \lambda^*(X_s)ds} = 0, \quad \text{almost surely.} \tag{72}$$

Next, we complete the proof in two steps.

Step 1: We prove $V^*(x) \geq J^a(x; \lambda_a, \lambda^*)$ where $(\lambda_a, \lambda^*) \in \Phi$.

Let τ_a and τ_b be firm a's and b's stochastic entry time associated with λ_a and $\lambda_b = \lambda^*$, respectively, and let $\tau := \min\{\tau_a, \tau_b\}$. Note that $V^*(x) \in \mathcal{C}^2(\mathbb{R}_+ \setminus \{\overline{x}\}) \cap \mathcal{C}^1(\mathbb{R}_+)$. Applying Itô's Lemma to $e^{-rs}V^*(X_s)$ for $s \in [t, \tau]$, we obtain

$$V^*(x) = \mathbb{E}_t^x[e^{-r(\tau-t)}V^*(X_\tau)] - \mathbb{E}_t^x\left[\int_t^\tau e^{-r(s-t)}\mathcal{A}V^*(X_s)ds\right]. \tag{73}$$

Substituting (65) into (73), we obtain

$$V^*(x) \geq \mathbb{E}_t^x[e^{-r(\tau-t)}(L(X_\tau) - K_1)] - \mathbb{E}_t^x[\int_t^\tau e^{-r(s-t)}\mathcal{A}V^*(X_s)ds]. \tag{74}$$

Note that

$$\begin{aligned} J^a(x; \lambda_a, \lambda^*) =&\mathbb{E}_t^x\left[e^{-r(\tau-t)}\left[\mathbf{1}_{\tau_a<\tau_b}(L(X_\tau) - K_1) + \mathbf{1}_{\tau_a>\tau_b}F(X_\tau)\right]\right] \\ =&\mathbb{E}_t^x[e^{-r(\tau-t)}(L(X_\tau) - K_1)] - \mathbb{E}_t^x\left[\mathbf{1}_{\tau_a>\tau_b}e^{-r(\tau-t)}(L(X_\tau) - K_1 - F(X_\tau))\right], \end{aligned} \tag{75}$$

where the second equality follows from the property: $\mathbf{1}_{\tau_a=\tau_b} = 0$ almost surely. Using (75) and (74), we obtain

$$\begin{aligned} &J^a(x; \lambda_a, \lambda^*) \\ \leq &V^*(x) + \mathbb{E}_t^x\left[\int_t^\tau e^{-r(s-t)}\mathcal{A}V^*(X_s)ds - \mathbf{1}_{\tau_a>\tau_b}e^{-r(\tau-t)}(L(X_\tau) - K_1 - F(X_\tau))\right]. \end{aligned} \tag{76}$$

We can show

$$
\begin{aligned}
&\mathbb{E}_t^x\left[\int_t^{\tau} e^{-r(s-t)}\mathcal{A}V^*(X_s)ds\right]\\
=&\mathbb{E}_t^x\left[\int_t^{\tau} e^{-r(s-t)}f(X_s)ds\right]\\
=&\mathbb{E}_t^x\left[\int_t^{\infty}\int_t^{\tau_a\wedge z} e^{-r(s-t)}f(X_s)\lambda^*(X_z)e^{-\int_t^z\lambda^*(X_u)du}dsdz\right]\\
=&\mathbb{E}_t^x\left[\int_t^{\tau_a}\int_s^{\infty}\lambda^*(X_z)e^{-\int_t^z\lambda^*(X_u)du}dze^{-r(s-t)}f(X_s)ds\right]\\
=&\mathbb{E}_t^x\left[\int_t^{\tau_a} e^{-\int_t^s(r+\lambda^*(X_u))du}f(X_s)ds\right]\\
=&\mathbb{E}_t^x\left[\int_t^{\tau_a} e^{-\int_t^s(r+\lambda^*(X_u))du}\lambda^*(X_s)[L(X_s)-K_1-F(X_s)]ds\right]\\
=&\mathbb{E}_t^x\left[\mathbf{1}_{\tau_a>\tau_b}e^{-r(\tau_b-t)}[L(X_{\tau_b})-K_1-F(X_{\tau_b})]\right] \qquad (77)
\end{aligned}
$$

using (71), Tonelli's Theorem (to interchange the integration order in the third equality as $f(x)\le 0$ and $\lambda^*(x)\ge 0$ for any $x>0$), integration by parts, and (72). Combining (76) and (77) yields $J^a(x;\lambda_a,\lambda^*)\le V^*(x)$.

Step 2: We prove $V^*(x)=J^a(x;\lambda^*,\lambda^*)$.

Let τ_a^* and τ_b^* be firm a's and b's stochastic entry time, respectively, associated with strategy $(\lambda_a(x),\lambda_b(x))=(\lambda^*(x),\lambda^*(x))$, and let $\tau^*:=\min\{\tau_a^*,\tau_b^*\}$. Because $\lambda^*(x)=0$ for any $x<\overline{x}$, we have $X_{\tau^*}\ge\overline{x}$, which implies $V^*(X_{\tau^*})=L(X_{\tau^*})-K_1$. Therefore, we can see that (74)-(76) hold with equality if λ_a, τ_a, τ_b and τ therein are set to λ^*, τ_a^*, τ_b^* and τ^*, respectively. We have thus shown $V^*(x)=J^a(x;\lambda^*,\lambda^*)$.

In sum, combining our analyses in Steps 1 and 2, we obtain $J_a(x;\lambda^*,\lambda^*)\ge J_a(x;\lambda_a,\lambda^*)$. By symmetry, we also have $J_b(x;\lambda^*,\lambda^*)\ge J_b(x;\lambda^*,\lambda_b)$ for $(\lambda^*,\lambda_b)\in\Phi$. □

Proof of Theorem 2: It suffices to show that $V_i(x)$ satisfies the variational inequality (41). See footnote 24. □

Proof of Theorem 3: We first prove (43). Equation (66) implies

$$
V^*(x) \;=\; \mathbb{E}_t^x\left[e^{-r(\tau_L^*-t)}\left(\frac{\Pi(X_{\tau_L^*})}{2}-K_1\right)\right]=\mathbb{E}_t^x\left[e^{-r(\tau_L^*-t)}(L(X_{\tau_L^*})-K_1)\right], \qquad (78)
$$

where $\tau_L^*=\inf\{s\ge t: X_s\ge\overline{x}\}$. We thus only need to show that for stopping time $\tau\ge t$, the following inequality holds

$$V^*(x) \geq \mathbb{E}_t^x \left[e^{-r(\tau - t)} (L(X_\tau) - K_1) \right]. \tag{79}$$

Note that $V^*(x) \in \mathcal{C}^2(\mathbb{R}_+ \setminus \{\overline{x}\}) \cap \mathcal{C}^1(\mathbb{R}_+)$. Applying Itô's Lemma to $e^{-rs}V^*(X_s)$ for $s \in [t, \tau]$, we obtain

$$V^*(x) = \mathbb{E}_t^x[e^{-r(\tau-t)}V^*(X_\tau)] - \mathbb{E}_t^x \left[\int_t^\tau e^{-r(s-t)} \mathcal{A}V^*(X_s) ds \right].$$

Using (64) and (65), we obtain (79).

Next, we prove $J_a(x; \mathcal{E}_a^*, \mathcal{E}_b^*) \geq J_a(x; \mathcal{E}_a, \mathcal{E}_b^*)$, where $(\mathcal{E}_a, \mathcal{E}_b^*) \in \Phi$. Denote $\tau_a = \inf\{s \geq t : X_s \in \mathcal{E}_a\}$. As $\mathcal{E}_b^* = \emptyset$, we have

$$\begin{aligned} J_a(x; \mathcal{E}_a^*, \mathcal{E}_b^*) &= \mathbb{E}_t^x \left[e^{-r(\tau_L^* - t)} (L(X_{\tau_L^*}) - K_1) \right] = V^*(x) \\ &\geq \mathbb{E}_t^x[e^{-r(\tau_a - t)}(L(X_{\tau_a}) - K_1)] = J_a(x; \mathcal{E}_a, \mathcal{E}_b^*), \end{aligned}$$

where the above inequality follows from (79).

Finally, we prove $J_b(x; \mathcal{E}_a^*, \mathcal{E}_b^*) = P_F(x) \geq J_b(x; \mathcal{E}_a^*, \mathcal{E}_b)$, where $(\mathcal{E}_a^*, \mathcal{E}_b) \in \Phi$. By definition we immediately obtain $J_b(x; \mathcal{E}_a^*, \mathcal{E}_b^*) = P_F(x)$. Let $\tau_a^* := \inf\{s \geq t : X_s \geq \overline{x}\}$, $\tau_b := \inf\{s \geq t : X_s \in \mathcal{E}_b\}$, and $\tau := \min\{\tau_a^*, \tau_b\}$. For any $x \geq \overline{x}$, we have

$$J_b(x; \mathcal{E}_a^*, \mathcal{E}_b) \leq \mathbb{E}_t^x[e^{-r(\tau-t)}F(X_\tau)] = F(x) + \mathbb{E}_t^x \left[\int_t^\tau e^{-r(s-t)} \mathcal{A}F(X_s) ds \right] \leq F(x) = P_F(x),$$

where the first and second inequalities follow from $F(x) \geq L(x) - K_1$ and $\mathcal{A}F(x) \leq 0$, respectively. For any $x \in (0, \overline{x})$, applying Itô's Lemma to $e^{-rs}P_F(X_s)$ where $s \in [t, \tau]$, we obtain

$$\begin{aligned} P_F(x) =& \mathbb{E}_t^x[e^{-r(\tau-t)}P_F(X_\tau)] = \mathbb{E}_t^x \left[e^{-r(\tau-t)} \Big(F(X_{\tau_a^*}) \mathbf{1}_{\tau_a^* \leq \tau_b} + P_F(X_\tau) \mathbf{1}_{\tau_a^* > \tau_b} \Big) \right] \\ \geq& \mathbb{E}_t^x \left[e^{-r(\tau-t)} \Big(F(X_{\tau_a^*}) \mathbf{1}_{\tau_a^* \leq \tau_b} + (L(X_\tau) - K_1) \mathbf{1}_{\tau_a^* > \tau_b} \Big) \right] \geq J_b(x; \mathcal{E}_a, \mathcal{E}_b^*), \end{aligned}$$

where the first inequality follows from $X_\tau \leq \overline{x}$ and $P_F(x) \geq L(x) - K_1$ for any $x \leq \overline{x}$,[35] the second equality uses $X_{\tau_a^*} \geq \overline{x}$ and $P_F(x) = F(x)$ for any $x \geq \overline{x}$, and the last inequality uses the result $F(x) \geq L(x) - K_1$ for any $x > 0$. □

[35]For any $x \in (0, \overline{x}]$, $P_F(x) = (x/\overline{x})^\beta F(\overline{x}) \geq (x/\overline{x})^\beta (L(\overline{x}) - K_1) = V_i(x) \geq L(x) - K_1$ (see Lemma 2).