Robust Estimation and Inference in Panels with Interactive Fixed Effects[*]

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Abstract

We consider estimation and inference for a regression coefficient in a panel setting with time and individual specific effects which follow a factor structure. Previous approaches to this model require a "strong factor" assumption, which allows the factors to be consistently estimated, thereby removing omitted variable bias due to the unobserved factors. We propose confidence intervals (CIs) that are robust to failure of this assumption, along with estimators that achieve better rates of convergence than previous methods when factors may be weak. Our approach applies the theory of minimax linear estimation to form a debiased estimate using a nuclear norm bound on the error of an initial estimate of the individual effects. In Monte Carlo experiments, we find a substantial improvement over conventional approaches when factors are weak, with little cost to estimation error when factors are strong.

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1 Introduction

In this paper we consider a linear panel regression model of the form

$$Y_{it} = X_{it}\beta + \sum_{k=1}^{K} Z_{k,it}\delta_k + \Gamma_{it} + U_{it}, \tag{1}$$

where $Y_{it}, X_{it}, Z_{1,it}, \ldots, Z_{K,it} \in \mathbb{R}$ are the observed outcome variable and covariates for units $i = 1, \ldots, N$ and time periods $t = 1, \ldots, T$. The error components $\Gamma_{it} \in \mathbb{R}$ and $U_{it} \in \mathbb{R}$ are unobserved, and the the regression coefficients $\beta, \delta_1, \ldots, \delta_K \in \mathbb{R}$ are unknown. The parameter of interest is $\beta \in \mathbb{R}$, the coefficient on X_{it}. We are interested in "large panels", where both N and T are relatively large.

The error component U_{it} is modelled as a mean-zero random shock that is uncorrelated with the regressors X_{it} and $Z_{k,it}$ and that is at most weakly autocorrelated across i and over t. By contrast, the error component Γ_{it} can be correlated with X_{it} and $Z_{k,it}$ and can also be strongly autocorrelated across i and over t. Of course, further restrictions on Γ_{it} are required to allow estimation and inference on β. For example, the additive fixed effect model imposes that $\Gamma_{it} = \alpha_i + \gamma_t$, where α_i accounts for any omitted variable that is constant over time, and γ_t for any omitted variable that is constant across units. Instead of this additive fixed effect model we will mostly consider the so-called interactive fixed effect model, where

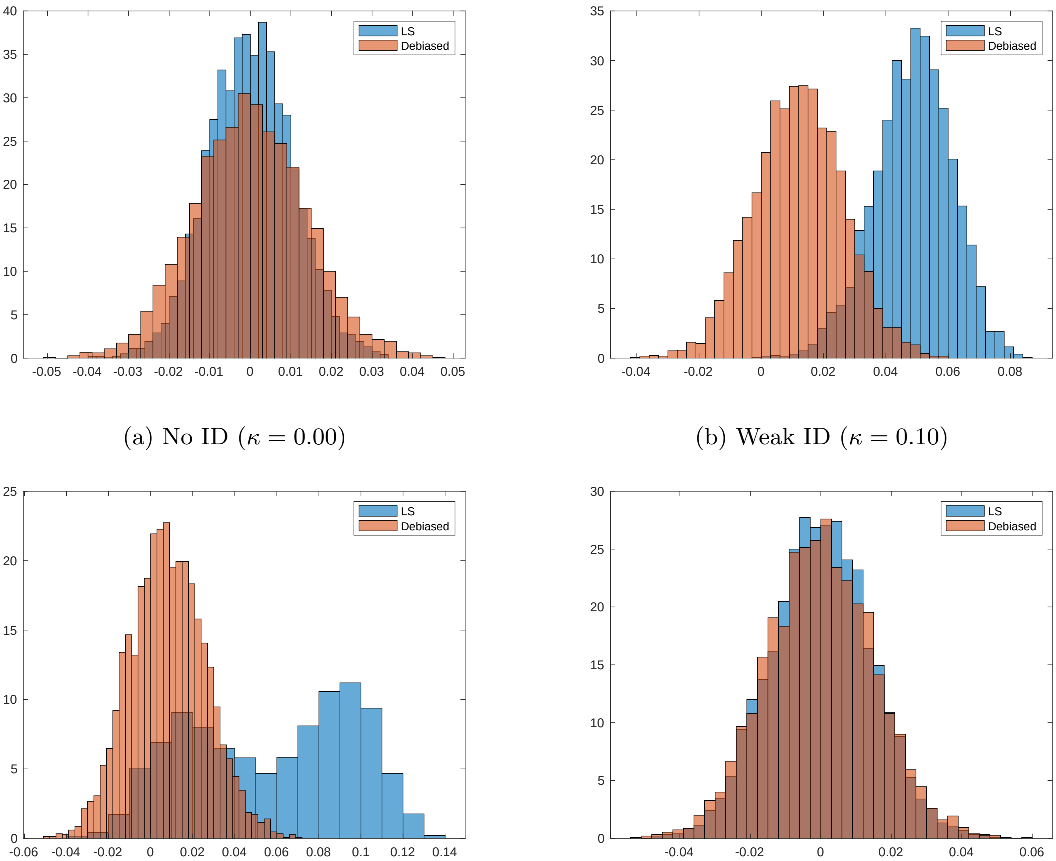
$$\Gamma_{it} = \sum_{r=1}^{R} \lambda_{ir}\, f_{tr} \ . \tag{2}$$

Here, the λ_{ir} and f_{tr} can either be interpreted as unknown parameters or as unobserved shocks. This model for Γ_{it} is also referred to as a factor model with factors loadings λ_{ir} and factors f_{tr}, and we will use the factor and interactive fixed effect terminology synonymously. The number of factors R is unknown, but is assumed to be small relative to N and T. The interactive fixed effect model is attractive, because it introduces enough restrictions to allow estimation and inference on β while still incorporating or approximating a large class of data generating processes (DGPs) for Γ_{it}.

The existing econometrics literature on panel regressions with interactive fixed effect is quite large. Since the seminal work of Pesaran (2006) and Bai (2009), developing tools for estimation and inference on β in model (1)-(2) under large N and large T asymptotics has been a primary focus of this literature. Specifically, Pesaran (2006) introduces the common correlated effects (CCE) estimator, which uses cross-sectional averages of the observed variables as proxies for the unobserved factors. Bai (2009) derives the large N, T properties of the least-squares (LS) estimator that jointly minimizes the sum of squared residuals over the regression coefficients, factors, and factor loadings.[1]

Bai (2009) shows that, under appropriate assumptions, the LS estimator for the regression

[1] This estimator was first introduced by Kiefer (1980).



(a) No ID ($\kappa = 0.00$)

(b) Weak ID ($\kappa = 0.10$)

(c) Weak ID ($\kappa = 0.20$)

(d) Strong ID ($\kappa = 1.00$)

Figure 1: Finite sample distributions of the LS and the debiased estimators, $N = 100$, $T = 50$, $R = 1$

coefficients is $\sqrt{NT}$ consistent and asymptotically normally distributed as both N and T grow to infinity. One of the key assumptions imposed for this result is the so-called "strong factor assumption", which requires all the factor loadings λ_{ir} and factors f_{tr} to have sufficient variation across i and over t, respectively. If the strong factor assumption is violated, then the LS estimator for λ_{ir} and f_{tr} may be unable to pick up the true loadings and factors correctly, because the "weak factors"[2] in Γ_{it} cannot be distinguished from the random noise in U_{it}. This can lead to substantial bias and misleading inference, due to omitted variables bias from Γ_{it} that is not picked up by the estimator.

To illustrate how this can lead to problems with conventional estimates and CIs for β, Figure 1 presents a subset of the results of our Monte Carlo study.[3] When the factors are nonexistent (panel a) or strongly identified (panel d), the distribution of the LS estimator (in blue) is centered at the true parameter value β (equal to 0 in this case). However, when the

[2]See, for example, Onatski (2010, 2012) for a discussion and formalization of the notion of weak factors.
[3]A detailed description of the numerical experiment is provided in Section 5.

factors are present but weak enough that they are difficult to detect (panels b and c), the LS estimator is heavily biased and non-normally distributed. In our Monte Carlo study in Section 5, we show that this indeed leads to severe coverage distortion, with conventional CIs based on the LS estimator having almost zero coverage.

In this paper, we address this issue by developing new tools for estimation and inference on β in the model (1). We develop a debiased estimator along with a bound on the remaining bias, which we use to construct a bias-aware confidence interval. As illustrated in Figure 1, our debiased estimator (shown in red) substantially decreases the bias of the LS estimator when factors are weak, leading to a large improvement in overall estimation error. In addition, this improved performance under weak factors does not come at a substantial cost to performance when factors are strong or nonexistent: our debiased estimator performs similarly to the LS estimator in these cases. Importantly, our CI requires only an upper bound on the number of factors: we show that it is valid uniformly over a large class of DGPs that allows for weak, strong or nonexistent factors up to a specified upper bound on the number of factors. We derive rates of convergence that hold uniformly over this class of DGPs, and we show that our estimator achieves a faster uniform rate of convergence than existing approaches when weak factors are allowed. In the case where N and T grow at the same rate, our estimator achieves the parametric $\sqrt{NT}$ rate.

Our debiasing approach uses a preliminary estimate $\hat{\Gamma}_{\text{pre}}$ of the individual effect matrix Γ along with a bound $\hat{C}$ on the nuclear norm $\|\Gamma - \hat{\Gamma}_{\text{pre}}\|_*$ of its estimation error. Letting $\tilde{\Gamma} := \Gamma - \hat{\Gamma}_{\text{pre}}$, we then consider the augmented outcomes

$$\tilde{Y}_{it} := Y_{it} - \hat{\Gamma}_{\text{pre},it} = X_{it}\beta + \sum_{k=1}^{K} Z_{k,it}\delta_k + \tilde{\Gamma}_{it} + U_{it}.$$

Treating $\tilde{\Gamma}_{it}$ as nuisance parameters satisfying a convex constraint $\|\tilde{\Gamma}\|_* \leq \hat{C}$, we derive linear weights A_{it} such that the estimator $\sum_{i=1}^{N}\sum_{t=1}^{T} A_{it}\tilde{Y}_{it}$ for β optimally uses this constraint, using the theory of minimax linear estimators (see Ibragimov and Khas'minskii, 1985; Donoho, 1994; Armstrong and Kolesár, 2018). In particular, the resulting weights A_{it} control the remaining omitted variables bias $\sum_{i=1}^{N}\sum_{t=1}^{T} A_{it}\tilde{\Gamma}_{it}$ due to possible weak factors in $\tilde{\Gamma} = \Gamma - \hat{\Gamma}_{\text{pre}}$ not picked up by the initial estimate $\hat{\Gamma}_{\text{pre}}$.

A key step in deriving our CI is the contruction of the preliminary estimator $\hat{\Gamma}_{\text{pre}}$ and bound $\hat{C}$ on the nuclear norm of its estimation error. Our CI is bias-aware: it uses the bound $\hat{C}$ to explicitly take into account any remaining bias in the debiased estimator. Our bound is feasible once an upper bound on the number of factors is specified. In our Monte Carlo study, we find that, while our CIs are often conservative, they are about as wide as an "oracle" CI that uses an infeasible critical value to correct the coverage of a CI based on the standard LS estimator.

Although the main focus of this paper is on models with the pure factor structure (2), the proposed approach applies to general interactive fixed effects models as long as it is possible to construct an upper bound on $\|\tilde{\Gamma}\|_*$. In particular, the proposed approach naturally extends to settings with nonseparable interactive unobserved heterogeneity (e.g., Zeleneev, 2019; Fernández-Val, Freeman and Weidner, 2021; Freeman and Weidner, 2021), for which, to the best of our knowledge, no tools for inference were previously available. Our approach also allows relaxing the strong cluster separation assumption in models with grouped unobserved heterogeneity (e.g., Assumption 2(b) in Bonhomme and Manresa, 2015), which is analogous in spirit to the "strong factor assumption". This is practically appealing because, unlike the existing approaches, our method allows overspecifying the true number of clusters.

Finally, rather than imposing the factor model (1), one may wish to impose an a priori bound on the nuclear norm $\|\Gamma\|_*$ of the individual effect matrix directly. In this case, our approach applies with the initial estimate $\hat{\Gamma}_{\rm pre}$ set to zero, which leads to a direct application of minimax linear estimators as in Donoho (1994) and Armstrong and Kolesár (2018). More generally, our approach can be extended to other panel settings with matrix restrictions, such as introducing heterogeneous coefficients β_{it} for each X_{it} and placing rank or nuclear norm restrictions on the matrix of these coefficients (as in Athey, Bayati, Doudchenko, Imbens and Khosravi (2021)). The main requirement is the availability of a convex bound on matrices that enter the regression model, or on the error of an initial estimate of such matrices.

Related literature

There exist various alternative estimation methods for panel regressions with interactive fixed effects. For example, Holtz-Eakin, Newey and Rosen (1988) introduce the the quasi-difference approach, Ahn, Lee and Schmidt (2001, 2013) use generalized method of moments estimation, and Chamberlain and Moreira (2009) explore the decision theoretic approach. All those papers assume fixed T, with only N going to infinity. More recent papers investigating the fixed T large N case include Robertson and Sarafidis (2015), Juodis and Sarafidis (2018), Westerlund, Petrova and Norkute (2019), Higgins (2021), Juodis and Sarafidis (2022). As mentioned before, in the context of large N and large T panels, two seminal papers that have spurred a very large number of follow-up papers are Pesaran (2006) and Bai (2009) — for a review and further references see Bai and Wang (2016). A special case of the violation of the strong factor assumption is when some factor are equal to zero, while all other factors are strong; the inference results of Bai (2009) are usually robust towards that specific violation of the strong factor assumption (Moon and Weidner 2015). This robustness, however, does not carry over to more general weak factors in the DGP of Γ_{it}, as illustrated by Figure 1.

Our approach to using minimax linear estimation to debias an initial estimate mirrors the approach used by Javanmard and Montanari (2014) to debias an initial LASSO estimate in sparse high dimensional regression. Hirshberg and Wager (2020) provide a general discussion and further references for this approach; we refer to this general approach as augmented linear estimation following their terminology. Minimax linear estimation itself goes back at least to Ibragimov and Khas'minskii (1985), with further results on this approach and its optimality properties in Donoho (1994), Armstrong and Kolesár (2018) and Yata (2021), among others.

The particular form of the minimax estimator used for debiasing in our setup follows from a formula given in Armstrong, Kolesár and Kwon (2020).

Requiring Γ_{it} to have the factor structure (2) is equivalent to requiring the matrix of unobserved effects Γ to have rank at most R, i.e., having $\operatorname{rank}(\Gamma) \leq R$. Bounding the nuclear norm of $\tilde{\Gamma}$ or Γ instead can also be seen as a convex relaxation of this requirement. Similar convexifications of the rank constraint have been widely used in the matrix completion completion literature (e.g., Recht, Fazel and Parrilo 2010 and Hastie, Tibshirani and Wainwright 2015 for recent surveys), and for reduced rank regression estimation (e.g., Rohde and Tsybakov 2011). In the econometrics literature, this convex relaxation idea has, for example, been used in by Bai and Ng (2017) to improve estimation in a pure factor models without regressors, by Athey, Bayati, Doudchenko, Imbens and Khosravi (2021) and Fernández-Val, Freeman and Weidner (2021) for treatment effect estimation, by Chernozhukov, Hansen, Liao and Zhu (2018) for the estimation of panel regression models with heterogeneous coefficients, and by Moon and Weidner (2018) to simplify the identification and estimation of panel regression models with homogenous coefficients. However, our paper appears to be the first to obtain valid CIs and improved rates of convergence for β using nuclear norm bounds (or, indeed, using any approach) when Γ may contain weak factors arbitrarily correlated with the variable of interest X.

2 Construction of robust estimates and confidence intervals

2.1 Setup

We consider a panel setting in which we observe a scalar outcome Y_{it}, a scalar covariate X_{it} of interest and additional control covariates $\{Z_{k,it}\}_{k=1}^{K}$ for $i = 1, \ldots, N$, $t = 1, \ldots, T$, which follow the regression model (1). The error term U_{it} is assumed to be mean zero conditional on X, $\{Z_{k,it}\}_{k=1}^{K}$ and Γ, but we allow for heteroskedasticity, which may depend on X_{it} and Γ_{it}, as well as some weak dependence. We write the model in matrix notation as

$$Y = X\beta + Z \cdot \delta + \Gamma + U \,, \qquad\qquad \mathbb{E}[U|X, Z, \Gamma] = 0 \,, \tag{3}$$

where Z denotes the three dimensional array $\{Z_{k,it}\}$ and we define $Z \cdot \delta = \sum_{k=1}^{K} Z_k \delta_k$ where Z_k denotes the matrix with i,tth element $Z_{k,it}$.

We are interested in the coefficient β of X_{it}, which can be interpreted as the effect of a treatment variable X_{it} in a constant treatment effects model. For concreteness, we use panel notation, and we refer to i and t as individuals and time periods respectively. However, we allow for other settings such as network data in which i and t both index individuals in a network. While we will assume a low rank structure on Γ, we allow for arbitrary dependence between the coefficient X_{it} and the individual effect Γ_{it}.

To apply our approach, we require a bound on the nuclear norm of the difference $\Gamma - \hat{\Gamma}$

for some preliminary estimate $\hat{\Gamma}$ of the matrix Γ:

$$\|\widetilde{\Gamma}\|_* \leq \hat{C}\,, \qquad \text{where} \qquad \widetilde{\Gamma} := \Gamma - \hat{\Gamma}\,. \tag{4}$$

Here, $\|\cdot\|_*$ denotes the nuclear norm of the argument matrix, and $\hat{C} \geq 0$ is a known or estimated constant. We focus on two main cases where such bounds are available.

Case 1. $\hat{\Gamma} \neq 0$ and $\hat{C}$ is estimated from the data. This is the case that is practically most relevant in this paper, where $\hat{\Gamma}$ is estimated such that a relatively small value for $\hat{C}$ can be chosen. To obtain $\hat{\Gamma}$ and $\hat{C}$ we will later assume that Γ has a linear factor structure with at most R factors.

Case 2. $\hat{\Gamma} = 0$ and $\hat{C}$ is a known constant. In this case we have $\widetilde{\Gamma} = \Gamma$ and the bound $\hat{C}$ constitutes an a priori bound on the nuclear norm of Γ. While this case is less practically relevant to this paper, it provides for an idealized setting that motivates some of our arguments later in this section.

2.2 (Augmented) linear estimators and CIs

We first define a class of estimators and CIs, indexed by an $N \times T$ matrix A. We then provide a choice of the matrix A, based on finite sample optimality in an idealized setting. Our class of estimators is given in the following definition.

Definition 2.1. *Let $A = A(X, Z)$ be an $N \times T$ matrix of weights $A_{it} \in \mathbb{R}$ that can depend on the matrix X and array Z. Let $\hat{\Gamma}$ be an initial estimate of Γ, and let $\widetilde{Y} = Y - \hat{\Gamma}$. The* augmented linear estimator *with weight matrix A and initial estimate $\hat{\Gamma}$ is given by*

$$\hat{\beta}_A := \sum_{i=1}^{N} \sum_{t=1}^{T} A_{it} \widetilde{Y}_{it} = \langle A, \widetilde{Y} \rangle_F. \tag{5}$$

Here, $\langle \cdot, \cdot \rangle_F$ denotes the entry-wise inner product between between the argument matrices.

Remark 2.1. In Case 2, $\widetilde{Y} = Y$ so that $\hat{\beta}_A$ is a linear estimator in the classical sense: it is linear in the outcomes Y_{it}, with weights depending on the design points $X_{it}, Z_{1,it}, \ldots, Z_{k,it}$. In Case 1, the estimator $\hat{\beta}_A = \langle A, \widetilde{Y} \rangle_F$ applies a linear estimator after an initial estimation step in which the initial estimate $\hat{\Gamma}$ is subtracted from the outcome Y. This mirrors applications of this idea in other settings going back to Javanmard and Montanari (2014); see Hirshberg and Wager (2020) for references (the term "augmented linear estimation" is used in the latter paper).

To analyze this class of estimators, note that subtracting the initial estimate from both sides of the equation (3) gives

$$\widetilde{Y} = X\beta + Z \cdot \delta + \widetilde{\Gamma} + U \tag{6}$$

(recall that $\widetilde{Y} = Y - \hat{\Gamma}$ and $\widetilde{\Gamma} = \Gamma - \hat{\Gamma}$). This gives the decomposition

$$\hat{\beta}_A - \beta = \text{bias}_{\beta,\delta,\widetilde{\Gamma}}(\hat{\beta}_A) + \langle A, U \rangle_F \tag{7}$$

where

$$\text{bias}_{\beta,\delta,\widetilde{\Gamma}}(\hat{\beta}_A) := (\langle A, X \rangle_F - 1)\,\beta + \langle A, Z \cdot \delta \rangle_F + \langle A, \widetilde{\Gamma} \rangle_F. \tag{8}$$

In Case 2, we have $\Gamma = \widetilde{\Gamma}$ and $\text{bias}_{\beta,\delta,\Gamma}(\hat{\beta}_A) = E[\hat{\beta}_A - \beta | X, Z, \Gamma]$ gives the bias of $\hat{\beta}_A$ conditional on X, Z and Γ. In Case 1, $\text{bias}_{\beta,\delta,\widetilde{\Gamma}}$ does not literally give the bias or conditional bias of $\hat{\beta}_A$, since conditioning on $\widetilde{\Gamma} = \Gamma - \hat{\Gamma}$ means conditioning on an information set that depends on Y through the preliminary estimate $\hat{\Gamma}$. We nonetheless refer to $\text{bias}_{\beta,\delta,\Gamma}(\hat{\beta}_A)$ as a bias term, in analogy to Case 2.

Let $\widehat{\text{se}}$ be an estimate of the standard deviation of $\langle A, U \rangle_F = \sum_{i=1}^{N} \sum_{t=1}^{T} A_{it} U_{it}$. For example, to allow for arbitrary heteroskedasticity in U_{it} while imposing independence across i and t, we can use $\widehat{\text{se}} = \sqrt{\sum_{i=1}^{N} \sum_{t=1}^{T} A_{it}^2 \hat{U}_{it}^2}$ where $\hat{U}_{it}$ denotes residuals from an initial regression. If $\text{bias}_{\beta,\delta,\Gamma}(\hat{\beta}_A)$ were zero, then we could form a CI by adding and subtracting a normal critical value times $\widehat{\text{se}}$. To take into account the possibility that $\text{bias}_{\beta,\delta,\Gamma}(\hat{\beta}_A)$ will in general be nonnegligible in our setting, we use the bound (4) to obtain an upper bound on the bias term. In particular, when (4) holds, we have $\left|\text{bias}_{\beta,\delta,\widetilde{\Gamma}}(\hat{\beta}_A)\right| \leq \overline{\text{bias}}_{\hat{C}}(\hat{\beta}_A)$, where for general $C \geq 0$ we define

$$\begin{aligned}
\overline{\text{bias}}_C(\hat{\beta}_A) &:= \sup_{\beta,\delta,\widetilde{\Gamma}:\|\widetilde{\Gamma}\|_* \leq C} \text{bias}_{\beta,\delta,\widetilde{\Gamma}}(\hat{\beta}_A) \\
&= \begin{cases} \sup\limits_{\widetilde{\Gamma}:\|\widetilde{\Gamma}\|_* \leq C} \langle A, \widetilde{\Gamma} \rangle_F & \text{if } \langle A, X \rangle_F = 1, \text{ and } \langle A, Z_k \rangle_F = 0, \text{ for } k = 1, \ldots K, \\ \infty & \text{otherwise} \end{cases} \\
&= \begin{cases} C s_1(A) & \text{if } \langle A, X \rangle_F = 1, \text{ and } \langle A, Z_k \rangle_F = 0, \text{ for } k = 1, \ldots K, \\ \infty & \text{otherwise.} \end{cases}
\end{aligned} \tag{9}$$

Here, for the second equality we used that the supremum over β and δ is unbounded unless $\langle A, X \rangle_F = 1$ and $\langle A, Z_k \rangle_F = 0$, and for the final step we used that the nuclear norm $\|\cdot\|_*$ is dual to the spectral norm, which we denote by $s_1(\cdot)$ since it is equal to the largest singular value of the argument matrix. We refer to $\overline{\text{bias}}_{\hat{C}}(\hat{\beta}_A)$ as the worst-case bias of the estimator $\hat{\beta}_A$ (again, this is only literally true in Case 2, but we use the same terminology in Case 1 by analogy).

Note that, whereas $\text{bias}_{\beta,\delta,\widetilde{\Gamma}}(\hat{\beta}_A)$ depends on the unknown matrix of individual effects Γ through the matrix $\widetilde{\Gamma} = \Gamma - \hat{\Gamma}$, $\overline{\text{bias}}_{\hat{C}}(\hat{\beta}_A)$ is feasible to compute once a bound $\hat{C}$ is given.

Taking into account the possible bias leads to a *bias-aware* CI:

$$\left\{\hat{\beta}_A \pm \left[\overline{\text{bias}}_{\hat{C}}(\hat{\beta}_A) + z_{1-\alpha/2}\widehat{\text{se}}\right]\right\}. \tag{10}$$

To motivate this CI, note that the probability that the lower endpoint is greater than β is

$$P\left(\hat{\beta}_A - \overline{\text{bias}}_{\hat{C}}(\hat{\beta}_A) - z_{1-\alpha/2}\widehat{\text{se}} > \beta\right) = P\left(\sum_{i=1}^{N}\sum_{t=1}^{T} A_{it}U_{it} + \text{bias}_{\beta,\delta,\widetilde{\Gamma}}(\hat{\beta}_A) > \overline{\text{bias}}_{\hat{C}}(\hat{\beta}_A) + z_{1-\alpha/2}\widehat{\text{se}}\right)$$
$$\leq P\left(\sum_{i=1}^{N}\sum_{t=1}^{T} A_{it}U_{it} > z_{1-\alpha/2}\widehat{\text{se}}\right) \approx \alpha/2,$$

where the last step assumes that $\sum_{i=1}^{N}\sum_{t=1}^{T} A_{it}U_{it}$ is approximately normally distributed with zero mean and standard deviation close to $\widehat{\text{se}}$. We provide formal justifications for this later. Combining this with similar calculations for undercoverage in the other direction shows that the coverage is approximately at least $1-\alpha$.

Remark 2.2. In Case 2 where $\widetilde{\Gamma} = \Gamma$ is nonrandom, one can take advantage of the fact that $\text{bias}_{\beta,\delta,\widetilde{\Gamma}}$ is nonrandom, which allows for the shorter CI $\left\{\hat{\beta}_A \pm \text{cv}_\alpha\left(\overline{\text{bias}}_C(\hat{\beta}_A)/\widehat{\text{se}}\right)\cdot\widehat{\text{se}}\right\}$ where $\text{cv}_\alpha(t)$ denotes the $1-\alpha$ quantile of the absolute value of a $N(0,1)$ random variable (see Donoho, 1994; Armstrong and Kolesár, 2018). In order to keep the exposition simple while covering both cases, we focus on the CI given in (10).

2.3 Choice of weights $A = (A_{it})$

As described in the last subsection, one can construct valid confidence intervals for β of the form (10) for any choice of weight matrix A, subject to weak regularity conditions. To get a simple baseline procedure, we compute weights that are optimal in an idealized setting where $U_{it} \overset{iid}{\sim} N(0,\sigma^2)$. In Case 2, $\hat{\beta}_A$ is then normally distributed with variance $\sigma^2\sum_{i=1}^{N}\sum_{t=1}^{T} A_{it}^2 = \sigma^2\|A\|_F^2$ (where $\|\cdot\|_F$ denotes the Frobenius norm), and with bias ranging from $-\overline{\text{bias}}_{\hat{C}}(\hat{\beta}_A)$ to $\overline{\text{bias}}_{\hat{C}}(\hat{\beta}_A)$. Thus, if we choose worst-case MSE under i.i.d. normal errors as our criterion function for the weights, then the optimal weights are obtained by minimizing $\left(\overline{\text{bias}}_{\hat{C}}(\hat{\beta}_A)\right)^2 + \sigma^2\|A\|_F^2$. Plugging in the formula for $\overline{\text{bias}}_{\hat{C}}(\hat{\beta}_A)$ given in (9) gives the following baseline choice of weights, indexed by a tuning parameter b that plays the role of C/σ.

Definition 2.2. *For $b > 0$, define the "optimal" $N \times T$ weight matrix by*

$$A_b^* := \operatorname*{argmin}_{A \in \mathbb{R}^{N\times T}} b^2 s_1(A)^2 + \|A\|_F^2 \qquad \text{s.t.} \quad \langle A, X\rangle_F = 1 \text{ and } \langle A, Z_k \cdot \delta\rangle_F = 0,$$

Here, the constraint $\langle A, Z_k\cdot\delta\rangle_F = 0$ is imposed for all $k \in \{1,\ldots,K\}$.

The weights in Definition 2.2 are optimal in Case 2 when $\hat{C}/\sigma = b$. Heuristically, we also expect that, in Case 1, a good choice of b will correspond to $\hat{C}/\sigma$ such that the bound $\hat{C}$ on the nuclear norm holds with high probability. Conveniently, our nuclear norm bound in the exact factor model in Section 3 scales with the standard deviation σ in the homoskedastic case, which gives us a simple and feasible choice of the tuning parameter b.

We emphasize again that while the definition of A_b^* is motivated by the idealized setting $U_{it} \overset{iid}{\sim} N(0, \sigma^2)$, we do *not* assume that the error terms U_{it} satisfy this strong assumption in this paper. Choosing $A = A_b^*$ to construct the estimator $\hat{\beta}_A$ and the confidence intervals (10) under more general error distributions just means that the resulting estimates and confidence intervals will not be optimal (in finite samples), but we will nevertheless show them to be consistent and valid, respectively.

Remark 2.3. While we have used MSE to motivate our baseline choice of weights A_b^*, one could use other criteria corresponding to different weights on bias and variance. For example, optimizing CI length when $C/\sigma = b$ would give the criterion $bs_1(A) + z_{1-\alpha}\|A\|_F$. If β gives the net welfare gain of an all-or-nothing policy change, then one can target minimax welfare regret as in Ishihara and Kitagawa (2021) and Yata (2021). In our Monte Carlo simulations however, we find that the exact choice of criterion has little effect on performance.

2.4 Practical implementation

The definition of A_b^* is a convex optimization problem that can easily be solved numerically for any given input X, Z, b. Using results from Armstrong, Kolesár and Kwon (2020), it follows that A_b^* can also be computed using the residuals of a nuclear norm regularized regression of X on $Z_1, \ldots, Z_K$ and a matrix of individual effects. In the case with no additional covariates Z, this nuclear norm regularized regression simplifies further: it can be solved by computing the singular value decomposition of X, and then performing soft thresholding on the singular values. The resulting weights A_b^* obtained from the residuals of this regression replace the smallest singular values of X with a constant. We provide details in Appendix B.

In addition to giving alternative methods for computing the weights A_b^*, these results provide some intuition for these weights. The residuals from this nuclear norm regularized regression of X on $Z_1, \ldots, Z_K$ and the individual effects "partial out" potential correlation of X with the estimation error $\widetilde{\Gamma}$, similar to the estimator of Robinson (1988) in the partially linear model. In the case with no additional covariates Z, this amounts to removing the smallest singular values of X and replacing them with a constant.

To summarize, we can compute an estimator $\hat{\beta}_A$ using Definition 2.1 using any matrix of weights A. We can also compute a CI $\left\{\hat{\beta}_A \pm \left[\overline{\text{bias}}_C(\hat{\beta}_A) + z_{1-\alpha/2}\widehat{\text{se}}\right]\right\}$ as in (10), once we have a standard error $\widehat{\text{se}}$ and an upper bound C for the nuclear norm of the error in the initial estimate of Γ. Definition 2.2 gives us a heuristic for computing a reasonable choice of the matrix A, once we have an initial choice of b for the ratio C/σ of the nuclear norm bound to variance of U_{it}.

Thus, to apply our general approach with data $\tilde{Y}, X, Z$ (with $\widetilde{Y}$ computed by subtracting an initial estimate of Γ in Case 1), we need an initial choice b to compute the weights A_b^*

using Definition 2.2. We also need a robust upper bound $\hat{C}$ such that the bound (4) holds with high probability. Finally, we need a robust standard error $\widehat{\text{se}}$. Our CI then takes the form in (10) with $A = A_b^*$ and the given bound $\hat{C}$ standard error $\widehat{\text{se}}$. In Section 3, we give details of these choices, as well as how to compute the initial estimate of Γ, for Case 1, where we impose an exact linear factor structure.

3 Implementation of the panel regression case

In this section we consider the case where $\text{rank}(\Gamma) \leq R$, i.e., $\Gamma_{it} = \sum_{r=1}^{R} \lambda_{ir} f_{tr} = \lambda_i' f_t$. Here R represents an upper bound on the number of factors in the model. As with other methods (e.g. Bai, 2009), our approach requires that this upper bound be specified by the researcher.

Our approach is motivated by bounds on the nuclear norm of an initial estimate of Γ, which we derive formally in Section 4. In particular, we show that (4) holds with $\hat{C} \approx 4Rs_1(U)$ for an initial estimate of Γ based on least squares. Furthermore, the weights A_b^* are designed to be optimal when $U_{it} \overset{iid}{\sim} N(0, \sigma^2)$, which leads to the approximation $s_1(U)/\sigma \approx \sqrt{N} + \sqrt{T}$ (Geman, 1980). We therefore use $b = b^* := 4R(\sqrt{N} + \sqrt{T})$ as our default choice to calibrate $\hat{C}/\sigma$ when computing the weights in Definition 2.2. We then use an upper bound $\hat{C}$ that is valid under heteroskedasticity when computing $\overline{\text{bias}}_{\hat{C}}(\hat{\beta}_{A^*_{b^*}})$ in the construction of the CI.

The following algorithm provides the additional details of our implementation algorithm.

Algorithm 3.1 (Implementation for the factor model)**.**

Input *Data Y, X, Z and R pre-specified by the user, along with tuning parameter ε.*

Output *Estimator and CI for β.*

1. *Compute the least squares (LS) estimator*

$$\left(\hat{\beta}_{\text{LS}}, \hat{\delta}_{\text{LS}}, \hat{\Gamma}_{\text{LS}}\right) = \underset{\{\beta\in\mathbb{R},\delta\in\mathbb{R}^K,G\in\mathbb{R}^{N\times T}\,:\,\text{rank}(G)\leq R\}}{\text{argmin}} \sum_{i=1}^{N}\sum_{t=1}^{T} \left(Y_{it} - X_{it}\beta - Z_{it}'\delta - G_{it}\right)^2 .$$

2. *Compute $\widetilde{Y}_{\text{pre}} = Y - \hat{\Gamma}_{\text{LS}}$ and let $b^* = 4R(\sqrt{N} + \sqrt{T})$. Let*

$$\hat{\beta}_{\text{pre}} = \langle A^*_{b^*}, \widetilde{Y}_{\text{pre}} \rangle.$$

Compute $\hat{\gamma}_{\text{pre}}$ by computing the jth element $\hat{\delta}_{\text{pre},j}$ in the same way as $\hat{\beta}_{\text{pre}}$, but with X and Z_j switched.

3. *Compute $\hat{\Gamma}_{\text{pre}}$ as*

$$\hat{\Gamma}_{\text{pre}} = \underset{\{G\in\mathbb{R}^{N\times T}\,:\,\text{rank}(G)\leq R\}}{\text{argmin}} \sum_{i=1}^{N}\sum_{t=1}^{T} \left(Y_{it} - X_{it}\hat{\beta}_{\text{pre}} - Z_{it}\cdot\hat{\delta}_{\text{pre}} - G_{it}\right)^2 .$$

The solution $\hat{\Gamma}_{\text{pre}}$ to this least squares problem is simply given by the leading R principal components of the residuals $Y_{it} - X_{it}\hat{\beta}_{\text{pre}} - Z_{it} \cdot \hat{\delta}_{\text{pre}}$. Compute $\tilde{Y} = Y - \hat{\Gamma}_{\text{pre}}$.

4. *Compute the final estimate*

$$\hat{\beta} = \hat{\beta}_{A^*_{b^*}} = \langle A^*_{b^*}, \tilde{Y} \rangle_F.$$

To compute the CI, let $\hat{C} = (4+\varepsilon)Rs_1(\hat{U}_{\text{pre}})$ and $\widehat{\text{se}}^2 = \sum_{i=1}^N \sum_{t=1}^T A_{it}^2 \hat{U}^2_{\text{pre},it}$, where

$$\hat{U}_{\text{pre}} = Y - X\hat{\beta}_{\text{pre}} - Z \cdot \hat{\delta}_{\text{pre}} - \hat{\Gamma}_{\text{pre}}.$$

Compute the CI

$$\hat{\beta}_{A^*_{b^*}} \pm \left[\overline{\text{bias}}_{\hat{C}}(\hat{\beta}_{A^*_{b^*}}) + z_{1-\alpha/2}\widehat{\text{se}} \right]$$

where $\overline{\text{bias}}_{\hat{C}}(\hat{\beta}_{A^*_{b^*}}) = \hat{C}s_1(A^*_{b^*})$.

Remark 3.1 (Choice of ε)**.** The quantity $\varepsilon > 0$ is used in the bound $\hat{C} = (4+\varepsilon)Rs_1(\hat{U}_{\text{pre}})$ on $\|\hat{\Gamma}_{\text{pre}} - \Gamma\|_*$ used to compute the CI in the final step. While needed for theoretical results, in our Monte Carlos, we find that we get good coverage when choosing $\epsilon = 0$. As a more principled approach, one could attempt to obtain a sharper bound on the sampling error of $\|\hat{\Gamma}_{\text{pre}} - \Gamma\|_*$, and then choose $\hat{C}$ so that the bound holds with a given probability, and then account for this with a Bonferroni correction with the critical value in the CI. We leave such extensions for future research.

Remark 3.2 (Lindeberg condition)**.** The asymptotic validity of the CI depends on asymptotic normality of the stochastic term $\langle A, U \rangle$ where $A = A^*_{b^*}$ is a nonrandom matrix of weights. This, in turn, depends on a Lindeberg condition on the weights A. To ensure that this holds, we can modify our optimization procedure for computing the weights $A = A^*_{b^*}$ by imposing a bound on the Lindeberg weights

$$\text{Lind}(A) = \frac{\max_{1 \le i \le N, 1 \le t \le T} A_{it}^2}{\sum_{i=1}^N \sum_{t=1}^T A_{it}^2}. \tag{11}$$

A similar approach to showing asymptotic validity is taken in Javanmard and Montanari (2014) in a different setting.

To make this approach practical, we need guidance on what makes $\text{Lind}(A)$ "small enough to use the central limit theorem" in a given sample size. A formal answer to this question is elusive, due to the difficulty of obtaining finite sample bounds on approximation error in the central limit theorem that are practically useful. As a heuristic, we can use comparisons to other settings where the central limit theorem is used. For example, the sample mean $\bar{W} = \frac{1}{n}\sum_{i=1}^n W_i$ with n observations corresponds to an estimator with Lindeberg constant $(1/n)^2/[n \cdot (1/n)^2] = 1/n$. If we are comfortable using the normal approximation in such a setting with, say, $n = 50$, then we can impose a bound $\text{Lind}(A) \leq 1/50$. Noack and Rothe (2019) provide some discussion of these issues in a related setting involving inference in fuzzy regression discontinuity.

In our Monte Carlos, we find that $\text{Lind}(A)$ is very small for the weights used in Algorithm 3.1 once N and T are larger than, say, 20. Thus, imposing a bound on these weights does not appear to be necessary in practice in the data generating processes we have examined.

Remark 3.3 (Standard error)**.** The standard error $\widehat{\text{se}}^2 = \sum_{i=1}^N \sum_{t=1}^T A_{it}^2 \hat{U}_{\text{pre},it}^2$ assumes that U_{it} is uncorrelated across i and t, but allows for heteroskedasticity. Such an assumption will be reasonable if Γ_{it} captures all of the dependence in errors for the outcome. However, incorporating all dependence in Γ_{it} may lead to an unneccessarily conservative choice of C (either directly in Case 2 or through the bound on the number of factors in Case 1). To avoid such conservative bounds on Γ, one can incorporate any dependence that is not directly correlated with X_{it} into the error term U_{it}, and allow for such dependence when constructing the standard error.

4 Asymptotic results

This section gives formal asymptotic results for the estimators and CIs given in Sections 2 and 3. To formally state asymptotic results that allow for weak factors and an unknown error distribution, we introduce some additional notation.

We consider uniform-in-the-underlying distribution asymptotics over a set $\mathcal{P}$ of distributions P for Γ and $X, Z_1, \ldots, Z_K, U$ and a set Θ of parameters $\theta = (\beta, \delta')'$. While we treat $\Gamma, X, Z_1, \ldots, Z_k$ as random variables determined by the unknown probability distribution P for notational purposes, we note that a fixed design setting in which $\Gamma, X, Z_1, \ldots, Z_k$ are non-random (sequences of) matrices can be incorporated by considering a set $\mathcal{P}$ that places a probability one mass on a given value of $\Gamma, X, Z_1, \ldots, Z_k$. We use $\mathbb{P}_{P,\theta}$ to denote probability under the given distribution P and parameters θ. Formally, we consider large N, large T asymptotics in which $N = N_n \to \infty$ and $T = T_n \to \infty$, and we consider sequences of distributions $\mathcal{P} = \mathcal{P}_n$ and parameter spaces $\Theta = \Theta_n$. Asymptotic statements are then taken in the sequence n. However, we supress the dependence on an index sequence n in order to save on notation. For a sequence of vectors or matrices $A_{N,T} = A_{N,T}(\theta, P)$ of fixed dimension (which may depend on θ, P), we use the notation $A_{N,T} = \mathcal{O}_{\Theta,\mathcal{P}}(r_{N,T})$ when, for every $\varepsilon > 0$, there exists C_ε such that

$$\limsup \sup_{P \in \mathcal{P}, \theta \in \Theta} \mathbb{P}_{P,\theta}\left(r_{N,T}^{-1} \|A_{N,T}\| \geq C_\varepsilon \right) \leq \varepsilon,$$

and we use the notation $A_{N,T} = o_{\Theta,\mathcal{P}}(r_{N,T})$ when, for every $\varepsilon > 0$, we have

$$\limsup \sup_{P \in \mathcal{P}, \theta \in \Theta} \mathbb{P}_{P,\theta}\left(r_{N,T}^{-1} \|A_{N,T}\| \geq \varepsilon \right) \to 0.$$

We use the notation $A_{N,T} \asymp_{\Theta,\mathcal{P}} r_{N,T}$ when $A_{N,T} = \mathcal{O}_{\Theta,\mathcal{P}}(r_{N,T})$ and $A_{N,T}^{-1} = \mathcal{O}_{\Theta,\mathcal{P}}(r_{N,T}^{-1})$. We use the notation $A_{N,T} \underset{\Theta,\mathcal{P}}{\overset{d}{\to}} \mathcal{L}$ to denote the statement

$$\limsup \left| \sup_{\theta\in\Theta, P\in\mathcal{P}} \mathbb{P}_{\theta,P}\left(A_{N,T} \leq t\right) - F_{\mathcal{L}}(t) \right| \to 1 \text{ for all } t$$

where $F_{\mathcal{L}}$ denotes the cdf of the probability law $\mathcal{L}$.

We first present results in a general asymptotic setting for a generic initial estimate $\hat{\Gamma}$ and bound C, as in Section 2. We then specialize to the estimator and CI described in Section 3 for the linear factor setting.

4.1 General asymptotic setting

We first show asymptotic validity of the CI (10) under a high level assumption on the weights A_{it} and regression error U_{it}.

Assumption 1.

(i) $\inf_{\theta\in\Theta,P\in\mathcal{P}} \mathbb{P}_{\theta,P}\left(\|\hat{\Gamma} - \Gamma\|_* \leq C\right) \to 1$

(ii) $\frac{\langle A,U\rangle_F}{\widehat{\mathrm{se}}} \underset{\Theta,\mathcal{P}}{\overset{d}{\to}} N(0,1)$

Theorem 1. *Suppose that Assumption 1 holds. Then*

$$\liminf \inf_{\theta\in\Theta,P\in\mathcal{P}} \mathbb{P}_{\theta,P}\left(\beta \in \left\{\hat{\beta}_A \pm \left[\overline{\mathrm{bias}}_C(\hat{\beta}_A) + z_{1-\alpha/2}\widehat{\mathrm{se}}\right]\right\}\right) \geq 1-\alpha.$$

4.2 Asymptotic Results for the Factor Model

We now apply these results to the initial estimate and bound given in Section 3, under the assumption of a linear factor model for Γ. We allow for a side condition on the Lindeberg weights $\mathrm{Lind}(A)$ defined in (11), as described in Remark 3.2. Let $A^*_{b,c}$ be defined in the same way A^*_b, with the modification that we impose the constraint $\mathrm{Lind}(A) \leq c$:

$$\begin{aligned} &\min_A \|A\|_F^2 + b^2 s_1(A)^2, \\ &\text{s.t.} \quad \mathrm{Lind}(A) \leq c, \quad \langle A, X\rangle_F = 1, \quad \langle A, Z_k\rangle_F = 0 \text{ for } k = 1,\ldots,K. \end{aligned} \tag{12}$$

In particular, the weights used in Algorithm 3.1 are given by $A^*_{b^*,\infty} = A^*_{b^*}$, and the weights $A^*_{b^*,c}$ with $c < \infty$ correspond to the modification described in Remark 3.2.

We impose the following conditions.

Assumption 2 (Factor Model)**.** *Suppose that rank*$(\Gamma) \leq R$ *with probability one for all* $P \in \mathcal{P}$ *and the following conditions hold:*

(i) *Write W for $X, Z_1, \ldots, Z_K$ and $W \cdot \gamma = X\beta + \sum_{k=1}^{K} Z_k \delta_k$ where $\gamma = (\beta, \delta')'$. We assume that there exists $b > 0$ such that*

$$\min_{\gamma \in \mathbb{R}^{K+1}: \|\gamma\|=1} \frac{1}{NT} \sum_{r=2R+1}^{\min\{N,T\}} s_r^2(W \cdot \gamma) \geq b$$

with probability approaching 1 uniformly over $P \in \mathcal{P}$.

(ii) $s_1(X) = \mathcal{O}_{\Theta,\mathcal{P}}\left(\sqrt{NT}\right)$, $s_1(Z_k) = \mathcal{O}_{\Theta,\mathcal{P}}\left(\sqrt{NT}\right)$ *for* $k \in \{1, \ldots, K\}$, *and* $s_1(U) \asymp_{\Theta,\mathcal{P}} \max\{\sqrt{N}, \sqrt{T}\}$;

(iii) $\frac{1}{\sqrt{NT}} \langle X, U \rangle_F = \mathcal{O}_{\Theta,\mathcal{P}}(1)$ *and* $\frac{1}{\sqrt{NT}} \langle Z_k, U \rangle_F = \mathcal{O}_{\Theta,\mathcal{P}}(1)$ *for* $k \in \{1, \ldots, K\}$;

(iv) $(s_1(U) - s_r(U)) / s_1(U) = o_{\Theta,\mathcal{P}}(1)$ *for any fixed positive integer r;*

(v) *For any sequence of matrices $A = A_{N,T}(X, Z)$ that is a function of $X, Z_1, \ldots, Z_k$, we have $\langle A, U \rangle_F = \mathcal{O}_{\Theta,\mathcal{P}}(\|A\|_F)$.*

Assumption 2(i) requires that there is sufficient variation in the covariates that is not correlated with the low rank matrix Γ. Assumption 2(ii) places mild bounds on X and Z_k, and places an upper bound on $s_1(U)$ that will hold so long as U_{it} does not exhibit too much dependence over i and t. Assumption 2(iii) will hold so long as U_{it} does not exhibit too much dependence over i and t, and is uncorrelated with X_{it}. Assumption 2(iv) is a high level assumption on the singular values of U. Assumption 2(v) holds so long as U is mean zero given X and Z and satisfies bounds on dependence and second moments.

We also place conditions on the matrix X requiring that there is sufficient variation after controlling for individual effects and the additional covariates Z.

Assumption 3. *For all $P \in \mathcal{P}$, there exists $\pi = \pi_P$ and random matrices H and V such that $X = Z \cdot \pi + H + V$ and the following conditions hold:*

(i) $\|V\|_F \asymp_{\Theta,\mathcal{P}} \sqrt{NT}$, $s_1(V) = \mathcal{O}_{\Theta,\mathcal{P}}(\max\{\sqrt{N}, \sqrt{T}\})$

(ii) $\|H\|_F = \mathcal{O}_{\Theta,\mathcal{P}}(\sqrt{NT})$ *and* $\langle H, V \rangle_F = \mathcal{O}_{\Theta,\mathcal{P}}(\sqrt{NT})$

(iii) $\|Z_k\|_F = \mathcal{O}_{\Theta,\mathcal{P}}(\sqrt{NT})$ *and* $\langle Z_k, V \rangle_F = \mathcal{O}_{\Theta,\mathcal{P}}(\sqrt{NT})$ *for* $k \in \{1, \ldots, K\}$

(iv) $(\mathbf{Z}'\mathbf{Z})^{-1} = \mathcal{O}_{\Theta,\mathcal{P}}\left(\frac{1}{NT}\right)$ *where* $\mathbf{Z} = [\text{vec}(Z_1), \ldots, \text{vec}(Z_K)]$

(v) $\max_{i,t} V_{it}^2 = o_{\Theta,\mathcal{P}}(NTc_{N,T})$ *and* $\max_{i,t} Z_{k,it}^2 = o_{\Theta,\mathcal{P}}\left((NT)^2 c_{N,T}\right)$

Assumption 3 uses a decomposition of X_{it} that depends on an individual effect H_{it} and a random variable V_{it} that is approximately independent and uncorrelated with $Z_{1,it} \ldots, Z_{k,it}$ as well as being approximately uncorrelated with the individual effect H_{it}. Importantly, the individual effect H_{it} can be arbitrarily correlated with Γ_{it} and with the variables $Z_{k,it}$. Note also that we do not place any assumptions on the rank or nuclear norm of the matrix H_{it}.

Part (v) holds under a tail bound on V_{it} and $Z_{k,it}$. For example, if V_{it} are (uniformly) sub-Gaussian then $\max_{i,t} V_{it}^2 = \mathcal{O}_{\Theta,\mathcal{P}}(\log(N+T))$, and the condition $\max_{i,t} V_{it}^2 = o_{\Theta,\mathcal{P}}(NTc_{N,T})$ is satisfied provided that $NTc_{N,T}/\log(N+T) \to \infty$. The only other requirement on $c_{N,T}$ is the requirement that $c_{N,T}\max\{N,T\} \to 0$ in Theorem 3 below. Thus, our results allow for a range of choices of $c_{N,T}$.

Theorem 2. *Let $\hat{\beta} = \hat{\beta}_{A^*_{b^*,c}}$ and $\hat{C} = 4Rs_1(\hat{U}_{\text{pre}})(1+\varepsilon)$ be defined in Algorithm 3.1, with the modification described in Remark 3.2. Suppose that Assumption 2 holds, and that Assumption 3 holds as stated and with Z_k and X interchanged for each $k = 1, \ldots, K$, for the given sequence $c = c_{N,T}$. Then*

$$\hat{\beta} - \beta = \mathcal{O}_{\Theta,\mathcal{P}}(1/\min\{N,T\}).$$

If, in addition, $\langle A^*_{b^*,c}, U\rangle_F/\widehat{\text{se}} \underset{\Theta,\mathcal{P}}{\overset{d}{\to}} N(0,1)$, then

$$\liminf \inf_{\theta\in\Theta, P\in\mathcal{P}} \mathbb{P}_{\theta,P}\left(\beta \in \left\{\hat{\beta} \pm \left[\overline{\text{bias}}_{\hat{C}}(\hat{\beta}) + z_{1-\alpha/2}\widehat{\text{se}}\right]\right\}\right) \geq 1-\alpha.$$

Theorem 2 assumes $\langle A^*_{b^*,c}, U\rangle_F/\widehat{\text{se}} \underset{\Theta,\mathcal{P}}{\overset{d}{\to}} N(0,1)$ as a high level condition. We now give primitive conditions for the case where U_{it} is independent but not necessarily identically distributed, conditional on the covariates $W = (X, Z_1, \ldots, Z_k)$ and the individual effects Γ.

Assumption 4. *There exist constants $\underline{\sigma} > 0$ and $\eta > 0$ such that, for all $P \in \mathcal{P}$, U_{it} is independent over i, t conditional on W, Γ and, for all i, t,*

$$\mathbb{E}_P[U_{it}|W,\Gamma] = 0, \quad \mathbb{E}_P[U_{it}^2|W,\Gamma] > \underline{\sigma}^2, \quad \mathbb{E}_P[U_{it}^4|W,\Gamma] < 1/\eta.$$

Theorem 3. *Let $\hat{\beta} = \hat{\beta}_{A^*_{b^*,c}}$ and $\hat{C} = 4Rs_1(\hat{U}_{\text{pre}})(1+\varepsilon)$ be defined in Algorithm 3.1, with the modification described in Remark 3.2 for $c = c_{N,T}$ with $c_{N,T}\max\{N,T\} \to 0$. Suppose that Assumptions 2(i)-(iv) holds, and that Assumption 3 holds as stated and with Z_k and X interchanged for each $k = 1, \ldots, K$, for the given sequence $c = c_{N,T}$, and that Assumption 4 holds. Let $\widehat{\text{se}}^2 = \sum_{i=1}^N \sum_{t=1}^T A_{it}^2 \hat{U}_{it}^2$ where $A = A^*_{b^*,c}$ and $\hat{U}_{it}$ is the residual from the least squares estimator. Then*

$$\hat{\beta} - \beta = \mathcal{O}_{\Theta,\mathcal{P}}(1/\min\{N,T\})$$

and

$$\liminf \inf_{\theta\in\Theta, P\in\mathcal{P}} \mathbb{P}_{\theta,P}\left(\beta \in \left\{\hat{\beta} \pm \left[\overline{\text{bias}}_{\hat{C}}(\hat{\beta}) + z_{1-\alpha/2}\widehat{\text{se}}\right]\right\}\right) \geq 1-\alpha.$$

5 Simulation evidence

We consider the following design:

$$Y_{it} = X_{it}\beta + \sum_{r=1}^{R} \kappa_r \lambda_{ir} f_{tr} + U_{it},$$
$$X_{it} = \sum_{r=1}^{R} \lambda_{ir} f_{tr} + V_{it},$$

where κ_r controls the identification strength for factor f_{tr}, and R stands for the number of factors. In addition, λ_i, f_t, U_{it} and V_{it} are all mutually independent across both i and t and

$$\lambda_i \sim N(0, I_R) \perp f_t \sim N(0, I_R) \perp \begin{pmatrix} U_{it} \\ V_{it} \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_U^2 & 0 \\ 0 & \sigma_V^2 \end{pmatrix} \right).$$

In the designs considered below, we fix $(\beta, \sigma_U^2, \sigma_V^2) = (0, 1, 1)$ and vary N, T, the number of factors R, and their strengths of identification controlled by κ_r. The number of replications in all of the considered designs is 5000. As before we are interested in estimation of and inference on β.

In Tables 1-3, we report the bias, standard deviation, and rmse for the benchmark LS estimator of Bai (2009) and for the proposed debiased estimator in various designs with 1 and 2 factors. We also report the size of the corresponding tests (with 5% nominal size) and the average length of the confidence intervals (with 95% nominal coverage).

The LS estimator is heavily biased and the associated tests and confidence intervals are heavily size distorted in designs in which at least one factor is weakly identified. At the same time, the proposed estimator effectively reduces the "weak factors" bias without inflating the variance. As a result, the potential efficiency gains from using the debiased estimator can be very large when at least one factor is weakly identified, especially for larger sample sizes (see Section C for additional simulation results). Importantly, even if all the factors are strongly identified, the debiased estimator performs comparably to the LS estimator.

When weak factors are present, the LS confidence intervals can have zero coverage because they are (i) centered around the biased LS estimator and (ii) too short. Hence, the average length of the LS confidence intervals is not a proper benchmark to compare the average length of the debiased confidence intervals with. To provide a relevant comparison, we also construct identification robust confidence intervals by inverting the (absolute value of the) LS based t-statistic using appropriate identification robust critical values (instead of $z_{1-\alpha/2}$). Specifically, for a given design (here, for fixed N, T, and R), we (numerically) compute the least favorable (over κ) critical value for the absolute value of the t-statistic based on the LS estimator. We also construct analogous confidence intervals by inverting the (absolute value of the) t-statistic based on the debiased estimator using the corresponding least favorable critical values. We refer to such intervals as the LS and debiased oracle intervals (because they are based on unknown design specific least favorable critical values) and report their average length denoted by "length*" in the tables below.

Notice that the average length of the LS oracle confidence interval is often comparable to or greater than the actual length of the debiased confidence interval, especially for larger sample sizes (again, see Section C for additional simulation results). The average length of the debiased oracle confidence interval is much smaller than the average length of the LS oracle confidence intervals.

Table 1: $N = 100,\ R = 1$

	LS						debiased					
κ	bias	std	rmse	size	length	length*	bias	std	rmse	size	length	length*
$T = 20$												
0.00	-0.0000	0.0171	0.0171	7.1	0.061	0.299	0.0002	0.0206	0.0206	0.0	0.535	0.137
0.05	0.0242	0.0178	0.0300	37.3	0.062	0.300	0.0095	0.0207	0.0228	0.0	0.535	0.137
0.10	0.0478	0.0200	0.0518	79.3	0.062	0.302	0.0181	0.0215	0.0281	0.0	0.537	0.137
0.15	0.0690	0.0249	0.0734	91.6	0.063	0.308	0.0244	0.0235	0.0339	0.0	0.540	0.138
0.20	0.0792	0.0382	0.0879	85.7	0.067	0.324	0.0250	0.0276	0.0372	0.0	0.544	0.138
0.25	0.0670	0.0531	0.0855	64.8	0.074	0.358	0.0189	0.0306	0.0360	0.0	0.549	0.139
0.50	0.0049	0.0244	0.0248	8.2	0.087	0.425	0.0013	0.0239	0.0240	0.0	0.555	0.139
1.00	0.0004	0.0232	0.0232	5.9	0.088	0.427	0.0001	0.0237	0.0237	0.0	0.555	0.140
$T = 50$												
0.00	-0.0002	0.0103	0.0103	5.9	0.039	0.228	-0.0001	0.0136	0.0136	0.0	0.294	0.079
0.05	0.0244	0.0108	0.0267	67.5	0.039	0.228	0.0064	0.0137	0.0151	0.0	0.295	0.080
0.10	0.0484	0.0124	0.0500	98.2	0.039	0.230	0.0121	0.0143	0.0187	0.0	0.296	0.080
0.15	0.0683	0.0189	0.0709	96.8	0.040	0.237	0.0135	0.0164	0.0213	0.0	0.299	0.080
0.20	0.0580	0.0390	0.0699	72.4	0.046	0.269	0.0084	0.0180	0.0198	0.0	0.301	0.080
0.25	0.0229	0.0306	0.0382	33.5	0.053	0.308	0.0032	0.0164	0.0167	0.0	0.302	0.080
0.50	0.0016	0.0144	0.0145	5.7	0.055	0.324	0.0002	0.0151	0.0151	0.0	0.303	0.080
1.00	0.0001	0.0142	0.0142	5.1	0.055	0.324	-0.0001	0.0151	0.0151	0.0	0.303	0.080
$T = 100$												
0.00	-0.0001	0.0073	0.0073	6.1	0.028	0.183	-0.0001	0.0108	0.0108	0.0	0.203	0.057
0.05	0.0246	0.0077	0.0258	91.0	0.028	0.183	0.0051	0.0109	0.0121	0.0	0.203	0.057
0.10	0.0486	0.0093	0.0495	99.9	0.028	0.185	0.0089	0.0118	0.0148	0.0	0.204	0.057
0.15	0.0619	0.0224	0.0658	92.9	0.030	0.197	0.0068	0.0133	0.0149	0.0	0.206	0.057
0.20	0.0239	0.0267	0.0358	47.4	0.037	0.243	0.0023	0.0125	0.0127	0.0	0.207	0.058
0.25	0.0077	0.0120	0.0143	17.9	0.039	0.256	0.0009	0.0120	0.0120	0.0	0.207	0.058
0.50	0.0009	0.0103	0.0103	5.9	0.039	0.260	0.0001	0.0119	0.0119	0.0	0.207	0.058
1.00	0.0001	0.0102	0.0102	5.4	0.039	0.260	-0.0000	0.0119	0.0119	0.0	0.207	0.058
$T = 300$												
0.00	-0.0000	0.0042	0.0042	5.3	0.016	0.121	0.0000	0.0056	0.0056	0.0	0.138	0.033
0.05	0.0247	0.0046	0.0252	100.0	0.016	0.122	0.0047	0.0056	0.0073	0.0	0.138	0.033
0.10	0.0482	0.0070	0.0487	99.8	0.016	0.123	0.0057	0.0067	0.0088	0.0	0.139	0.033
0.15	0.0178	0.0173	0.0248	60.5	0.021	0.161	0.0015	0.0063	0.0065	0.0	0.140	0.033
0.20	0.0047	0.0064	0.0080	16.4	0.022	0.170	0.0005	0.0061	0.0061	0.0	0.140	0.033
0.25	0.0023	0.0060	0.0064	7.6	0.023	0.171	0.0003	0.0061	0.0061	0.0	0.140	0.033
0.50	0.0003	0.0057	0.0058	4.9	0.023	0.172	0.0001	0.0060	0.0060	0.0	0.140	0.033
1.00	0.0001	0.0057	0.0057	4.9	0.023	0.172	0.0000	0.0060	0.0060	0.0	0.140	0.033

$\mathrm{Lind}(A) \in \{0.0063, 0.0028, 0.0015, 0.0006\}$ for $T \in \{20, 50, 100, 300\}$.

Table 2: $N = 100$, $T = 50$, $R = 2$

κ_1 \ κ_2	LS 0.00	LS 0.05	LS 0.10	LS 0.15	LS 0.20	LS 0.25	LS 0.30	LS 0.40	LS 0.50	LS 1.00	Debiased 0.00	Debiased 0.05	Debiased 0.10	Debiased 0.15	Debiased 0.20	Debiased 0.25	Debiased 0.30	Debiased 0.40	Debiased 0.50	Debiased 1.00
bias																				
0.00	-0.000	0.016	0.031	0.035	0.018	0.008	0.004	0.002	0.001	0.000	-0.000	0.006	0.010	0.010	0.005	0.002	0.001	0.000	0.000	-0.000
0.05	0.016	0.033	0.049	0.060	0.052	0.036	0.030	0.027	0.026	0.025	0.006	0.012	0.017	0.018	0.014	0.010	0.009	0.008	0.007	0.007
0.10	0.031	0.049	0.066	0.080	0.083	0.067	0.057	0.051	0.050	0.049	0.010	0.017	0.022	0.025	0.022	0.018	0.015	0.014	0.014	0.013
0.15	0.035	0.060	0.080	0.097	0.108	0.099	0.082	0.072	0.070	0.068	0.010	0.018	0.025	0.029	0.028	0.024	0.020	0.018	0.017	0.017
0.20	0.018	0.052	0.083	0.108	0.123	0.114	0.089	0.068	0.064	0.060	0.005	0.014	0.022	0.028	0.029	0.024	0.018	0.014	0.013	0.012
0.25	0.008	0.036	0.067	0.099	0.114	0.088	0.054	0.032	0.028	0.025	0.002	0.010	0.018	0.024	0.024	0.017	0.010	0.006	0.005	0.005
0.30	0.004	0.030	0.057	0.082	0.089	0.054	0.027	0.015	0.012	0.010	0.001	0.009	0.015	0.020	0.018	0.010	0.005	0.003	0.002	0.002
0.40	0.002	0.027	0.051	0.072	0.068	0.032	0.015	0.007	0.005	0.004	0.000	0.008	0.014	0.018	0.014	0.006	0.003	0.001	0.001	0.001
0.50	0.001	0.026	0.050	0.070	0.064	0.028	0.012	0.005	0.004	0.002	0.000	0.007	0.014	0.017	0.013	0.005	0.002	0.001	0.001	0.000
1.00	0.000	0.025	0.049	0.068	0.060	0.025	0.010	0.004	0.002	0.000	-0.000	0.007	0.013	0.017	0.012	0.005	0.002	0.001	0.000	-0.000
std																				
0.00	0.009	0.009	0.012	0.019	0.019	0.013	0.011	0.011	0.010	0.010	0.013	0.013	0.014	0.015	0.015	0.014	0.014	0.014	0.014	0.014
0.05	0.009	0.009	0.010	0.014	0.021	0.015	0.012	0.011	0.011	0.011	0.013	0.013	0.013	0.015	0.015	0.014	0.014	0.014	0.014	0.014
0.10	0.012	0.010	0.010	0.012	0.019	0.020	0.014	0.013	0.013	0.013	0.014	0.013	0.014	0.015	0.016	0.015	0.015	0.014	0.014	0.014
0.15	0.019	0.014	0.012	0.012	0.017	0.025	0.021	0.019	0.019	0.020	0.015	0.015	0.015	0.016	0.016	0.017	0.016	0.016	0.016	0.016
0.20	0.019	0.021	0.019	0.017	0.025	0.043	0.045	0.040	0.039	0.039	0.015	0.015	0.016	0.016	0.018	0.020	0.020	0.019	0.019	0.019
0.25	0.013	0.015	0.020	0.025	0.043	0.064	0.055	0.037	0.034	0.032	0.014	0.014	0.015	0.017	0.020	0.023	0.021	0.018	0.018	0.017
0.30	0.011	0.012	0.014	0.021	0.045	0.055	0.036	0.021	0.019	0.019	0.014	0.014	0.015	0.016	0.020	0.021	0.018	0.016	0.016	0.016
0.40	0.011	0.011	0.013	0.019	0.040	0.037	0.021	0.016	0.015	0.015	0.014	0.014	0.014	0.016	0.019	0.018	0.016	0.016	0.016	0.016
0.50	0.010	0.011	0.013	0.019	0.039	0.034	0.019	0.015	0.015	0.015	0.014	0.014	0.014	0.016	0.019	0.018	0.016	0.016	0.015	0.015
1.00	0.010	0.011	0.013	0.020	0.039	0.032	0.019	0.015	0.015	0.015	0.014	0.014	0.014	0.016	0.019	0.017	0.016	0.016	0.015	0.015
rmse																				
0.00	0.009	0.019	0.033	0.039	0.026	0.015	0.012	0.011	0.010	0.010	0.013	0.014	0.017	0.018	0.016	0.014	0.014	0.014	0.014	0.014
0.05	0.019	0.034	0.050	0.061	0.056	0.039	0.032	0.029	0.028	0.027	0.014	0.017	0.021	0.023	0.021	0.017	0.016	0.016	0.016	0.016
0.10	0.033	0.050	0.066	0.081	0.085	0.070	0.058	0.053	0.051	0.050	0.017	0.021	0.026	0.029	0.027	0.023	0.021	0.020	0.020	0.020
0.15	0.039	0.061	0.081	0.098	0.109	0.102	0.085	0.075	0.073	0.071	0.018	0.023	0.029	0.033	0.033	0.029	0.026	0.024	0.024	0.023
0.20	0.026	0.056	0.085	0.109	0.125	0.122	0.099	0.079	0.075	0.071	0.016	0.021	0.027	0.033	0.034	0.032	0.027	0.024	0.023	0.022
0.25	0.015	0.039	0.070	0.102	0.122	0.109	0.077	0.049	0.044	0.041	0.014	0.017	0.023	0.029	0.032	0.028	0.023	0.019	0.018	0.018
0.30	0.012	0.032	0.058	0.085	0.099	0.077	0.045	0.026	0.023	0.021	0.014	0.016	0.021	0.026	0.027	0.023	0.018	0.016	0.016	0.016
0.40	0.011	0.029	0.053	0.075	0.079	0.049	0.026	0.017	0.016	0.016	0.014	0.016	0.020	0.024	0.024	0.019	0.016	0.016	0.016	0.016
0.50	0.010	0.028	0.051	0.073	0.075	0.044	0.023	0.016	0.015	0.015	0.014	0.016	0.020	0.024	0.023	0.018	0.016	0.016	0.016	0.015
1.00	0.010	0.027	0.050	0.071	0.071	0.041	0.021	0.016	0.015	0.015	0.014	0.016	0.020	0.023	0.022	0.018	0.016	0.016	0.015	0.015

Table 3: $N = 100$, $T = 50$, $R = 2$

	LS										Debiased									
κ_1 \ κ_2	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.40	0.50	1.00	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.40	0.50	1.00
size																				
0.00	7.6	52.3	88.3	80.0	42.8	17.9	10.6	6.7	6.3	5.9	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.05	52.3	96.2	99.8	99.6	95.9	88.3	81.6	73.7	70.8	68.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.10	88.3	99.8	100.0	100.0	100.0	99.7	99.2	98.6	98.4	98.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.15	80.0	99.6	100.0	100.0	99.8	99.5	98.7	97.9	97.4	96.6	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.20	42.8	95.9	100.0	99.8	98.3	93.1	87.3	80.0	76.9	73.8	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.25	17.9	88.3	99.7	99.5	93.1	76.1	60.5	45.2	39.9	36.2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.30	10.6	81.6	99.2	98.7	87.3	60.5	38.9	23.3	18.9	16.2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.40	6.7	73.7	98.6	97.9	80.0	45.2	23.3	11.3	9.2	7.9	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.50	6.3	70.8	98.4	97.4	76.9	39.9	18.9	9.2	7.4	6.4	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
1.00	5.9	68.0	98.0	96.6	73.8	36.2	16.2	7.9	6.4	5.7	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
length																				
0.00	0.031	0.031	0.032	0.034	0.037	0.038	0.039	0.039	0.039	0.039	0.468	0.469	0.471	0.474	0.477	0.478	0.478	0.479	0.479	0.479
0.05	0.031	0.031	0.032	0.033	0.036	0.038	0.038	0.039	0.039	0.039	0.469	0.469	0.470	0.474	0.477	0.478	0.479	0.479	0.479	0.479
0.10	0.032	0.032	0.032	0.032	0.034	0.037	0.039	0.039	0.039	0.039	0.471	0.470	0.471	0.474	0.477	0.479	0.480	0.481	0.481	0.481
0.15	0.034	0.033	0.032	0.032	0.033	0.036	0.039	0.040	0.040	0.040	0.474	0.474	0.474	0.477	0.480	0.482	0.484	0.484	0.485	0.485
0.20	0.037	0.036	0.034	0.033	0.034	0.037	0.042	0.045	0.045	0.046	0.477	0.477	0.477	0.480	0.483	0.486	0.488	0.489	0.490	0.490
0.25	0.038	0.038	0.037	0.036	0.037	0.043	0.049	0.052	0.052	0.052	0.478	0.478	0.479	0.482	0.486	0.489	0.491	0.492	0.492	0.492
0.30	0.039	0.038	0.039	0.039	0.042	0.049	0.053	0.054	0.054	0.054	0.478	0.479	0.480	0.484	0.488	0.491	0.492	0.493	0.493	0.493
0.40	0.039	0.039	0.039	0.040	0.045	0.052	0.054	0.055	0.055	0.055	0.479	0.479	0.481	0.484	0.489	0.492	0.493	0.493	0.493	0.493
0.50	0.039	0.039	0.039	0.040	0.045	0.052	0.054	0.055	0.055	0.055	0.479	0.479	0.481	0.485	0.490	0.492	0.493	0.493	0.493	0.493
1.00	0.039	0.039	0.039	0.040	0.046	0.052	0.054	0.055	0.055	0.055	0.479	0.479	0.481	0.485	0.490	0.492	0.493	0.493	0.493	0.494
length*																				
0.00	0.345	0.346	0.353	0.373	0.406	0.420	0.424	0.427	0.427	0.428	0.114	0.114	0.114	0.115	0.115	0.115	0.115	0.115	0.115	0.115
0.05	0.346	0.346	0.349	0.360	0.391	0.416	0.424	0.427	0.428	0.429	0.114	0.114	0.114	0.115	0.115	0.115	0.115	0.115	0.115	0.115
0.10	0.353	0.349	0.348	0.354	0.375	0.409	0.424	0.430	0.432	0.432	0.114	0.114	0.114	0.115	0.115	0.115	0.115	0.115	0.115	0.116
0.15	0.373	0.360	0.354	0.354	0.365	0.399	0.428	0.441	0.443	0.445	0.115	0.115	0.115	0.115	0.115	0.116	0.116	0.116	0.116	0.116
0.20	0.406	0.391	0.375	0.365	0.371	0.410	0.459	0.491	0.497	0.503	0.115	0.115	0.115	0.115	0.116	0.116	0.116	0.116	0.116	0.116
0.25	0.420	0.416	0.409	0.399	0.410	0.475	0.536	0.568	0.573	0.576	0.115	0.115	0.115	0.116	0.116	0.116	0.116	0.116	0.116	0.116
0.30	0.424	0.424	0.424	0.428	0.459	0.536	0.579	0.595	0.598	0.599	0.115	0.115	0.115	0.116	0.116	0.116	0.116	0.116	0.116	0.117
0.40	0.427	0.427	0.430	0.441	0.491	0.568	0.595	0.604	0.606	0.607	0.115	0.115	0.115	0.116	0.116	0.116	0.116	0.117	0.117	0.117
0.50	0.427	0.428	0.432	0.443	0.497	0.573	0.598	0.606	0.608	0.609	0.115	0.115	0.115	0.116	0.116	0.116	0.116	0.117	0.117	0.117
1.00	0.428	0.429	0.432	0.445	0.503	0.576	0.599	0.607	0.609	0.610	0.115	0.115	0.116	0.116	0.116	0.116	0.117	0.117	0.117	0.117

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A Proofs

This section contains proofs of the results in the main text. Section A.1 states and proves a general result on rates of convergence using high level conditions on the covariates X and Z and the initial bound $\hat{C}$ on the nuclear norm of estimation in the initial estimate of Γ. Section A.2 proves Theorem 1. Section A.3 proves Theorem 2. Section A.4 proves Theorem 3.

A.1 General result for rates of convergence

We first prove a result giving rates of convergence for estimators $\hat{\beta} = \langle A^*_{b,c}, \tilde{Y}\rangle$ given in Definition 2.1 with weights $A^*_{b,c}$ given in (12) under high level conditions on the bound $\hat{C}$ on the initial estimation error in (4). The proofs of Theorems 2 and 3, given in Sections A.3 and A.4 below, verify the conditions of this theorem for the initial estimator and bounds used in the facor setting in Section 3.

We make the following assumption on the class of distributions of X, $Z_1, \dots, Z_k$ and U and the sequence $c = c_{N,T}$ used in the Lindeberg constraint.

Assumption 5. *There exists a sequence of $N \times T$ random matrices Ξ such that*

$$\|\Xi\|_F = \mathcal{O}_{\Theta,\mathcal{P}}(\sqrt{NT}), \quad |\langle \Xi, X\rangle_F|^{-1} = \mathcal{O}_{\Theta,\mathcal{P}}((NT)^{-1}),$$
$$s_1(\Xi) = \mathcal{O}_{\Theta,\mathcal{P}}(\max\{\sqrt{N}, \sqrt{T}\}),$$

and, with probability approaching one,

$$\mathrm{Lind}(\Xi) \leq c_{N,T} \quad \textit{and} \quad \langle \Xi, Z_k\rangle = 0 \textit{ for } k = 1, \dots, K.$$

Assumption 5 holds so long as we have $X_{it} = H_{it} + \tilde{\tilde{\Xi}}_{it}$ where $\tilde{\tilde{\Xi}}_{it}$ is mean zero given H_{it} and does not have too much dependence over i and t.

Theorem 4. *Let $\hat{\beta} = \langle A^*_{b,c}, \tilde{Y}\rangle$ for some sequences $c = c_{N,T}$ and $b = b_{N,T}$. Suppose Assumption 5 and Assumption 2(v) hold and that Assumption 1(i) holds with $C = \mathcal{O}_{\Theta,\mathcal{P}}(\overline{C}_{N,T})$ for some sequence $\overline{C}_{N,T}$. Then*

$$|\hat{\beta} - \beta| = \mathcal{O}_{\Theta,\mathcal{P}}\left(\max\left\{\overline{C}_{N,T}/b_{N,T}, 1\right\} \cdot \max\left\{(NT)^{-1/2}, b_{N,T} \cdot \max\{\sqrt{N}, \sqrt{T}\}/(NT)\right\}\right).$$

Proof. We have

$$|\hat{\beta} - \beta| \leq \left|\langle A^*_{b,c}, U\rangle_F\right| + \overline{\mathrm{bias}}_{\tilde{C}}(A^*_{b,c}) = \left|\langle A^*_{b,c}, U\rangle_F\right| + \tilde{C} s_1(A^*_{b,c})$$

where $\tilde{C} = \|\Gamma - \hat{\Gamma}\|_*$. Thus,

$$
\begin{aligned}
|\hat{\beta} - \beta|^2 &\leq 4 \left|\langle A_{b,c}^*, U\rangle_F\right|^2 + 4\tilde{C}^2 s_1(A_{b,c}^*)^2 \\
&\leq 4 \max\left\{ \frac{\left|\langle A_{b,c}^*, U\rangle_F\right|^2}{\|A_{b,c}^*\|_F^2}, \frac{\tilde{C}^2}{b^2} \right\} \cdot \left[\|A_{b,c}^*\|_F^2 + b^2 s_1(A_{b,c}^*)^2\right]. \qquad (13)
\end{aligned}
$$

Consider the oracle weights $\tilde{A} = \Xi/\langle \Xi, X\rangle$. With probability approaching one uniformly over θ, P, the weights $\tilde{A}$ are feasible for (12), so that

$$
\begin{aligned}
&\|A_{b,c}^*\|_F^2 + b^2 s_1(A_{b,c}^*)^2 \leq \|\tilde{A}\|_F^2 + b^2 s_1(\tilde{A})^2 = \frac{\|\Xi\|_F^2 + b^2 s_1(\Xi)^2}{\left|\langle \Xi, X\rangle_F\right|^2} \\
&= \mathcal{O}_{\Theta,\mathcal{P}}((NT)^{-1}) + b^2 \cdot \mathcal{O}_{\Theta,\mathcal{P}}(\max\{N,T\}/(NT)^2).
\end{aligned}
$$

Plugging this into (13) gives the result. □

A.2 Proof of Theorem 1

The probability that the upper endpoint of the CI is less than β is

$$
\begin{aligned}
&\mathbb{P}_{\theta,P}\left(\hat{\beta} + \overline{\text{bias}}_C(\hat{\beta}) + z_{1-\alpha/2}\widehat{\text{se}} < \beta\right) \\
&= \mathbb{P}_{\theta,P}\left(\langle A, X\beta + Z\cdot\delta + \Gamma - \hat{\Gamma}\rangle_F - \beta + \overline{\text{bias}}_C(\hat{\beta}) + \langle A, U\rangle_F < -z_{1-\alpha/2}\widehat{\text{se}}\right) \\
&\leq \mathbb{P}_{\theta,P}\left(\langle A, X\beta + Z\cdot\delta + \Gamma - \hat{\Gamma}\rangle_F - \beta < -\overline{\text{bias}}_C(\hat{\beta})\right) + \mathbb{P}_{\theta,P}\left(\langle A, U\rangle_F < -z_{1-\alpha/2}\widehat{\text{se}}\right).
\end{aligned}
$$

The first term is, by definition, bounded by $\mathbb{P}_{\theta,P}\left(\left\|\hat{\Gamma} - \Gamma\right\|_* > C\right)$, which converges to zero uniformly over $\theta \in \Theta, P \in \mathcal{P}$ by Assumption 1(i). The second term converges to $\alpha/2$ uniformly over $\theta \in \Theta, P \in \mathcal{P}$ by Assumption 1(ii). Applying a symmetric argument to the probability that the lower endpoint of the CI is greater than β gives the result.

A.3 Proof of Theorem 2

To prove this result, we first prove a series of lemmas. The first statement will then follow from Lemma 8 below and Theorem 4, along with Lemma 9 verifying Assumption 5. The second statement is immediate from Lemma 8 below and Theorem 1, along with Lemma 9 verifying Assumption 5.

Lemma 5. *Under Assumptions 2 (i)-(iii),*

$$
\hat{\gamma}_{\text{LS}} - \gamma_0 = \mathcal{O}_{\Theta,\mathcal{P}}\left(\frac{1}{\min\{\sqrt{N}, \sqrt{T}\}}\right).
$$

Proof. The result follows from the proof of Theorem 4.1 in Moon and Weidner (2015). Assumptions 2 (i)-(iii) are uniform analogues of Assumptions NC, SN, and EX in Moon and

Weidner (2015). The derived rate of convergence is immediately uniform over $\theta \in \Theta, P \in \mathcal{P}$ because the proof of Theorem 4.1 in Moon and Weidner (2015) explicitly bounds $\|\hat{\gamma}_{\rm LS} - \gamma_0\|$. $\square$

Lemma 6. *Under Assumption 2,* $\left\|\hat{\Gamma}_{\rm LS} - \Gamma\right\|_* = \mathcal{O}_{\Theta,\mathcal{P}}(\max\{\sqrt{N}, \sqrt{T}\})$

Proof. Since $(\widehat{\gamma}_{\rm LS}, \widehat{\Gamma}_{\rm LS})$ are defined to be the minimizers of the least-squares objective function and $(\widehat{\gamma}_{\rm LS}, \Gamma)$ are potential alternative arguments we have

$$\left\|(\widehat{\Gamma}_{\rm LS} - \Gamma) + W \cdot (\widehat{\gamma}_{\rm LS} - \gamma) - U\right\|_F^2 \leq \|W \cdot (\widehat{\gamma}_{\rm LS} - \gamma) - U\|_F^2 ,$$

and therefore

$$\begin{aligned}
\left\|\widehat{\Gamma}_{\rm LS} - \Gamma\right\|_F^2 &\leq 2\langle \widehat{\Gamma}_{\rm LS} - \Gamma, U - W \cdot (\widehat{\gamma}_{\rm LS} - \gamma)\rangle_F \\
&\leq 2\|\widehat{\Gamma}_{\rm LS} - \Gamma\|_* s_1(U - W \cdot (\widehat{\gamma}_{\rm LS} - \gamma)) \\
&\leq 2\sqrt{2R}\|\widehat{\Gamma}_{\rm LS} - \Gamma\|_F s_1(U - W \cdot (\widehat{\gamma}_{\rm LS} - \gamma))
\end{aligned}$$

where in the second step we used that $\langle A, B\rangle_F \leq \|A\|_* s_1(B)$ and in the final step we used that $\|A\|_* \leq \sqrt{\text{rank}(A)}\|A\|_F$ and $\text{rank}(\widehat{\Gamma}_{\rm LS} - \Gamma) \leq 2R$. We thus have

$$\left\|\widehat{\Gamma}_{\rm LS} - \Gamma\right\|_F \leq 2\sqrt{2R}\, s_1(U - W \cdot (\widehat{\gamma}_{\rm LS} - \gamma))$$

and therefore

$$\begin{aligned}
\left\|\widehat{\Gamma}_{\rm LS} - \Gamma\right\|_* &\leq \sqrt{2R}\left\|\widehat{\Gamma}_{\rm LS} - \Gamma\right\|_F \\
&\leq 4R s_1(U - W \cdot (\widehat{\gamma}_{\rm LS} - \gamma)) \\
&\leq 4R\left(s_1(U) + s_1(X)\left|\widehat{\beta}_{\rm LS} - \beta_0\right| + \sum_{k=1}^{K} s_1(Z_k)\left|\widehat{\delta}_{{\rm LS},k} - \delta_k\right|\right). \qquad (14)
\end{aligned}$$

Using Lemma 5 and Assumption 2 (ii) gives the result. $\square$

Lemma 7. *Suppose that Assumption 2 holds, and that Assumption 5 holds as stated and with Z_k and X interchanged for each $k = 1, \ldots, K$. Then*

$$\hat{\gamma}_{\rm pre} - \gamma = \mathcal{O}_{\Theta,\mathcal{P}}\left(1/\min\{N, T\}\right).$$

Proof. The result is immediate from Lemma 6 and Theorem 4, using the fact that b^* is bounded from above and below by a constant times $\max\{\sqrt{N}, \sqrt{T}\}$. $\square$

Lemma 8. *Suppose that Assumption 2 holds, and that Assumption 5 holds as stated and with Z_k and X interchanged for each $k = 1, \ldots, K$. Then*

$$\|\hat{\Gamma}_{\rm pre} - \Gamma\|_* \leq 4R s_1(U)(1 + o_{\Theta,\mathcal{P}}(1)) = \mathcal{O}_{\Theta,\mathcal{P}}(\max\{\sqrt{N}, \sqrt{T}\})$$

and

$$s_1(U) \le s_1(\hat{U}_{\text{pre}})(1 + o_{\Theta,\mathcal{P}}(1)) = \mathcal{O}_{\Theta,\mathcal{P}}(\max\{\sqrt{N}, \sqrt{T}\}).$$

Proof. The first statement follows by using the same arguments used to obtain (14) in the proof of Lemma 6, applied to the objective function with $\hat{\gamma}_{\text{pre}}$ plugged in, and then applying Lemma 7 and Assumption 2(ii). For the second statement, first note that, letting $\Delta_\Gamma = \hat{\Gamma}_{\text{pre}} - \Gamma$, we have

$$\begin{aligned}\left|s_1(\hat{U}_{\text{pre}}) - s_1(U - \Delta_\Gamma)\right| &\le s_1(X)\left|\widehat{\beta}_{\text{pre}} - \beta\right| + \sum_{k=1}^{K} s_1(Z_k)\left|\widehat{\delta}_{\text{pre},k} - \delta_{0k}\right| \\ &\le o_{\Theta,P}(\sqrt{NT}/\min\{\sqrt{N}, \sqrt{T}\}) = o_{\Theta,P}(\max\{\sqrt{N}, \sqrt{T}\}) \end{aligned} \tag{15}$$

where we use Lemma 7 and Assumption 2(ii) in the last inequality.

Now, using the fact that $\operatorname{rank}(\Delta_\Gamma) \le 2R$ and the general inequality $s_{i+j-1}(A+B) \le s_i(A) + s_j(B)$ with $A = U - \Delta_\Gamma$, $B = \Delta_\Gamma$, $i = 1$, $j = 2R+1$ gives $s_{2R+1}(U) \le s_1(U - \Delta_\Gamma)$. Thus,

$$s_1(U) \le s_{2R+1}(U) + o_{\Theta,P}(\max\{\sqrt{N}, \sqrt{T}\}) \le s_1(\hat{U}_{\text{pre}}) + o_{\Theta,P}(\max\{\sqrt{N}, \sqrt{T}\})$$

where we apply Assumption 2(iv) for the first inequality and (15) for the second inequality. This gives the first inequality of the second statement of the theorem. For the second inequality of the second statement of the theorem, we can again apply (15) and note that $s_1(U + \Delta_\Gamma) \le s_1(U) + s_1(\Delta_\Gamma)$ and $s_1(\Delta_\Gamma) \le \|\Delta_\Gamma\|_* = \mathcal{O}_{\Theta,\mathcal{P}}(\max\{\sqrt{N}, \sqrt{T}\})$.

□

Lemma 9. *Suppose that Assumptions 2 and 3 hold. Then Assumption 5 holds with Ξ_{it} given by the residual in the regression of V_{it} on Z_{it}, i.e., $\text{vec}(\Xi) = M_{\mathbf{Z}}\text{vec}(V)$ where $M_{\mathbf{Z}} = I_{NT} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'$.*

Proof. First, notice that $\Xi = V - Z \cdot \hat{\varphi} = V - \sum_{k=1}^{K} Z_k \hat{\varphi}_k$ where $\hat{\varphi} = (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\text{vec}(V)$. Also, it follows from Assumption 3(iii) and (iv) that $\|\hat{\varphi}\| = \mathcal{O}_{\Theta,\mathcal{P}}\left(\frac{1}{\sqrt{NT}}\right)$.

Next we verify all the conditions required by Assumption 5.

Verification of $\langle \Xi, Z_k \rangle_F = 0$ for $k = 1, \ldots, K$. By construction.

Verification of $\|\Xi\|_F = \mathcal{O}_{\Theta,\mathcal{P}}(\sqrt{NT})$. $\|\Xi\|_F = \|\text{vec}(\Xi)\| \leqslant \|\text{vec}(V)\| = \|V\|_F = \mathcal{O}_{\Theta,\mathcal{P}}(\sqrt{NT})$.

Verification of $s_1(\Xi)$. Notice that

$$s_1(\Xi) = s_1\left(V - \sum_{k=1}^{K} Z_k \hat{\varphi}_k\right) \leqslant s_1(V) + \sum_{k=1}^{K} |\hat{\varphi}_k|\, s_1(Z_k) = \mathcal{O}_{\Theta,\mathcal{P}}(\max\{\sqrt{N}, \sqrt{T}\}),$$

using the fact that $|\hat{\varphi}_k|\, s_1(Z_k) = \mathcal{O}_{\Theta,\mathcal{P}}(1)$ since $\hat{\varphi}_k = \mathcal{O}_{\Theta,\mathcal{P}}(1/\sqrt{NT})$ and $s_1(Z_k) \leq \|Z_k\|_F = \mathcal{O}_{\Theta,\mathcal{P}}(\sqrt{NT})$ under these assumptions.

Verification of $|\langle \Xi, X\rangle_F|^{-1} = \mathcal{O}_{\Theta,\mathcal{P}}((NT)^{-1})$. Using the fact that $\langle \Xi, Z_k\rangle_F = 0$ for each k, we have

$$\langle \Xi, X\rangle_F = \langle \Xi, H\rangle_F + \langle \Xi, V\rangle_F, = \langle V, H\rangle_F - \sum_{k=1}^{K} \hat{\varphi}_k \langle Z_k, H\rangle_F + \|V\|_F^2 - \sum_{k=1}^{K} \hat{\varphi}_k \langle Z_k, V\rangle_F.$$

The first term is $\mathcal{O}_{\Theta,\mathcal{P}}(\sqrt{NT}) = o_{\Theta,\mathcal{P}}(NT)$ by Assumption 3(ii). The second term is $Op(\sqrt{NT}) = o_{\Theta,\mathcal{P}}(NT)$ since $\hat{\varphi}_k = \mathcal{O}_{\Theta,\mathcal{P}}(1/\sqrt{NT})$ and $\langle Z_k, H\rangle_F \leq \|H\|_F \cdot \|Z_k\|_F = \mathcal{O}_{\Theta,\mathcal{P}}(NT)$ under these assumptions. Similarly, the fourth term is $Op(1) = o_{\Theta,\mathcal{P}}(NT)$. Thus, $\langle \Xi, X\rangle_F = \|V\|_F^2 + o_{\Theta,\mathcal{P}}(NT)$ and the result follows since $\|V\|_F^2 \asymp_{\Theta,\mathcal{P}} NT$ by Assumption 3(i).

Verification of $\mathrm{Lind}(\Xi) \leqslant c_{N,T}$ *with probability approaching one.*

$$\mathrm{Lind}(\Xi) = \frac{\max_{i,t} \Xi_{it}^2}{\|\Xi\|_F^2},$$

where

$$\|\Xi\|_F^2 = \|V\|_F^2 - 2\sum_{k=1}^{K} \hat{\varphi}_k \langle V, Z_k\rangle_F + \left\|\sum_{k=1}^{K} Z_k \hat{\varphi}_k\right\|_F^2,$$

where $\sum_{k=1}^{K} \hat{\varphi}_k \langle V, Z_k\rangle_F = \mathcal{O}_{\Theta,\mathcal{P}}(1)$ and $\left\|\sum_{k=1}^{K} Z_k \hat{\varphi}_k\right\|_F^2 = \mathcal{O}_{\Theta,\mathcal{P}}(1)$, so $\|\Xi\|_F^2 \asymp_{\Theta,\mathcal{P}} NT$. Next,

$$\begin{aligned}
\max_{i,t} \Xi_{it}^2 &= \max_{i,t} \left(V_{it} - \sum_{k=1}^{K} \hat{\varphi}_k Z_{k,it} \right)^2 \\
&\leqslant (K+1)^2 \left(\max_{i,t} V_{it}^2 + \sum_{k=1}^{K} \hat{\varphi}_k^2 \max_{i,t} Z_{k,it}^2 \right) = o_{\Theta,\mathcal{P}}(NTc_{N,T}).
\end{aligned}$$

Hence, $\mathrm{Lind}(\Xi) = o_{\Theta,\mathcal{P}}(c_{N,T})$, which completes the proof. □

A.4 Proof of Theorem 3

The result will follow from Theorem 2 once we verify Assumption 2(v) and the condition $\langle A^*_{b^*,c}, U\rangle_F / \widehat{\mathrm{se}} \xrightarrow[\Theta,\mathcal{P}]{d} N(0,1)$. Assumption 2(v) is immediate from Assumption 4 and Chebyshev's inequality. To verify $\langle A^*_{b^*,c}, U\rangle_F / \widehat{\mathrm{se}} \xrightarrow[\Theta,\mathcal{P}]{d} N(0,1)$, we show that $\langle A, U\rangle_F / \widehat{\mathrm{se}} \xrightarrow[\Theta,\mathcal{P}]{d} N(0,1)$ for $\widehat{\mathrm{se}}^2 = \sum_{i=1}^{N} \sum_{t=1}^{T} A_{it}^2 \hat{U}_{it}^2$ with any sequence of matrices A satisfying $\mathrm{Lind}(A) \leq c_{N,T}$ with $c_{N,T}$ satisfying the condition $c_{N,T} \max\{N, T\} \to 0$ given in the statement of the theorem.

To this end, we first prove a bound on $\|\hat{U} - U\|_F$ (Lemma 10), and then use this to show consistency of the standard error (Lemma 11, using a condition verified in Lemma 12). Lemma 13 completes the proof. We note that the conditions of Lemma 10 hold under the conditions of Theorem 3 by Lemma 5.

Lemma 10. *Let $\hat{U} = Y - W \cdot \hat{\gamma} - \hat{\Gamma}$, where*

$$\hat{\Gamma} = \underset{\{G \in \mathbb{R}^{N\times T} \,:\, \mathrm{rank}(G) \leq R\}}{\mathrm{argmin}} \sum_{i=1}^{N} \sum_{t=1}^{T} \left(Y_{it} - W \cdot \hat{\gamma} - G_{it}\right)^2 .$$

Suppose that

(i) $\hat{\gamma} - \gamma = \mathcal{O}_{\Theta,\mathcal{P}}\left(\frac{1}{\min\{\sqrt{N},\sqrt{T}\}}\right)$*;*

(ii) $\|X\|_F = \mathcal{O}_{\Theta,\mathcal{P}}(\sqrt{NT})$ *and* $\|Z_k\|_F = \mathcal{O}_{\Theta,\mathcal{P}}(\sqrt{NT})$ *for* $k \in \{1, \ldots, K\}$*;*

(iii) $s_1(X) = \mathcal{O}_{\Theta,\mathcal{P}}\left(\sqrt{NT}\right)$, $s_1(Z_k) = \mathcal{O}_{\Theta,\mathcal{P}}\left(\sqrt{NT}\right)$ *for* $k \in \{1, \ldots, K\}$*, and* $s_1(U) = \mathcal{O}_{\Theta,\mathcal{P}}(\max\{\sqrt{N}, \sqrt{T}\})$.

Then,

$$\left\|\hat{U} - U\right\|_F^2 = \mathcal{O}_{\Theta,\mathcal{P}}(\max\{N, T\}).$$

Proof. Using $\hat{U} = W \cdot (\gamma - \hat{\gamma}) + \Gamma - \hat{\Gamma} + U$,

$$\left\|\hat{U} - U\right\|_F^2 = \|W \cdot (\hat{\gamma} - \gamma)\|_F^2 + \left\|\hat{\Gamma} - \Gamma\right\|_F^2 + 2\langle W \cdot (\hat{\gamma} - \gamma), \hat{\Gamma} - \Gamma\rangle_F.$$

To prove the result, we show that all the terms on the right hand side of the equation above are $\mathcal{O}_{\Theta,\mathcal{P}}(\max\{N, T\})$.

First,

$$\|W \cdot (\hat{\gamma} - \gamma)\|_F \leqslant \|X\|_F \left|\hat{\beta} - \beta\right| + \sum_{k=1}^{K} \|Z_k\|_F \left|\hat{\delta}_k - \delta_k\right| = \mathcal{O}_{\Theta,\mathcal{P}}\left(\max\{\sqrt{N}, \sqrt{T}\}\right).$$

where we used conditions (i) and (ii).

Second, using the previously derived result (14) and conditions (i) and (iii),

$$\left\|\hat{\Gamma} - \Gamma\right\|_F = \mathcal{O}_{\Theta,\mathcal{P}}(\max\{\sqrt{N}, \sqrt{T}\}).$$

Third,

$$\left|\langle W \cdot (\hat{\gamma} - \gamma), \hat{\Gamma} - \Gamma\rangle_F\right| \leqslant \left|\langle X(\hat{\beta} - \beta), \hat{\Gamma} - \Gamma\rangle_F\right| + \sum_{k=1}^{K} \left|\langle Z_k(\hat{\delta}_k - \delta_k), \hat{\Gamma} - \Gamma\rangle_F\right|,$$

where

$$\left|\langle X(\hat{\beta}-\beta), \hat{\Gamma}-\Gamma\rangle_F\right| \leqslant \|X\|_F \left\|\hat{\Gamma}-\Gamma\right\|_F \left|\hat{\beta}-\beta\right| = \mathcal{O}_{\Theta,\mathcal{P}}\left(\max\{N,T\}\right).$$

Similarly,

$$\sum_{k=1}^{K} \left|\langle Z_k(\hat{\delta}_k-\delta_k), \hat{\Gamma}-\Gamma\rangle_F\right| = \mathcal{O}_{\Theta,\mathcal{P}}\left(\max\{N,T\}\right),$$

which implies

$$\left|\langle W\cdot(\hat{\gamma}-\gamma), \hat{\Gamma}-\Gamma\rangle_F\right| = \mathcal{O}_{\Theta,\mathcal{P}}(\max\{N,T\})$$

and completes the proof. □

Lemma 11. *Suppose that the hypotheses of Lemma 10 are satisfied. Suppose, in addition, that the following conditions hold:*

(i) for any collections of weights $\{\omega_{it}\}_{1\leqslant i\leqslant N, 1\leqslant t\leqslant T}$, which are non-random conditional W and Γ, such that $|\omega_{it}| \leqslant \overline{\omega}$ a.s. for all W and Γ and for all i, t, N, and T, we have

$$\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\omega_{it}U_{it}^2 - \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\omega_{it}\mathbb{E}\left[U_{it}^2|W,\Gamma\right] = \mathcal{O}_{\Theta,\mathcal{P}}\left(\frac{1}{\sqrt{NT}}\right);$$

(ii) for some $\underline{\sigma}^2>0$, $\mathbb{E}\left[U_{it}^2|W,\Gamma\right] \geqslant \underline{\sigma}^2$ a.s. for all i, t, N, and T;

(iii) $\mathrm{Lind}(A) \leqslant c_{N,T}$ and $\max\{N,T\}\, c_{N,T} \to 0$.

Then,

$$\frac{\sum_{i=1}^{N}\sum_{t=1}^{T}A_{it}^2\hat{U}_{it}^2}{\sum_{i=1}^{N}\sum_{t=1}^{T}A_{it}^2U_{it}^2} - 1 = o_{\Theta,\mathcal{P}}(1),$$

where $\hat{U}$ is defined in Lemma 10.

Proof. For simplicity of notation, we use $\sum_{i,t} \equiv \sum_{i=1}^{N}\sum_{t=1}^{T}$ and $\max_{i,t} \equiv \max_{1\leqslant i\leqslant N, 1\leqslant t\leqslant T}$ throughout the proof.

Notice that

$$\begin{aligned}\frac{\sum_{i,t}A_{it}^2\hat{U}_{it}^2}{\sum_{i,t}A_{it}^2U_{it}^2} - 1 &= \frac{\sum_{i,t}A_{it}^2\left(\hat{U}_{it}^2-U_{it}^2\right)}{\sum_{i,t}A_{it}^2U_{it}^2} \\ &= \frac{\sum_{i,t}A_{it}^2\left(\hat{U}_{it}-U_{it}\right)\left(\hat{U}_{it}-U_{it}+2U_{it}\right)}{\sum_{i,t}A_{it}^2U_{it}^2}\end{aligned}$$

$$= \frac{\sum_{i,t} A_{it}^2 \left(\hat{U}_{it} - U_{it}\right)^2}{\sum_{i,t} A_{it}^2 U_{it}^2} + \frac{2\sum_{i,t} A_{it}^2 U_{it} \left(\hat{U}_{it} - U_{it}\right)}{\sum_{i,t} A_{it}^2 U_{it}^2}. \tag{16}$$

The first term in (16) can be bounded as

$$\frac{\sum_{i,t} A_{it}^2 \left(\hat{U}_{it} - U_{it}\right)^2}{\sum_{i,t} A_{it}^2 U_{it}^2} \leqslant \frac{\max_{i,t} A_{it}^2 \left\|\hat{U} - U\right\|_F^2}{\sum_{i,t} A_{it}^2 U_{it}^2},$$

and and the second term in (16) can be bounded as

$$\begin{aligned}
\frac{\sum_{i,t} A_{it}^2 U_{it} \left(\hat{U}_{it} - U_{it}\right)}{\sum_{i,t} A_{it}^2 U_{it}^2} &\leqslant \frac{\left(\sum_{i,t} A_{it}^4 U_{it}^2\right)^{1/2} \left(\sum_{i,t} \left(\hat{U}_{it} - U_{it}\right)^2\right)^{1/2}}{\sum_{i,t} A_{it}^2 U_{it}^2} \\
&\leqslant \sqrt{\frac{\max_{i,t} A_{it}^2 \left\|\hat{U} - U\right\|_F^2}{\sum_{i,t} A_{it}^2 U_{it}^2}},
\end{aligned}$$

where the first inequality follows from the Cauchy-Schwarz inequality.

Hence, to complete the proof, it is sufficient to demonstrate

$$\frac{\max_{i,t} A_{it}^2 \left\|\hat{U} - U\right\|_F^2}{\sum_{i,t} A_{it}^2 U_{it}^2} = o_{\Theta,\mathcal{P}}(1).$$

Next, notice that

$$\begin{aligned}
\frac{1}{NT} \sum_{i,t} \frac{A_{it}^2}{\max_{i,t} A_{it}^2} U_{it}^2 &= \frac{1}{NT} \sum_{i,t} \frac{A_{it}^2}{\max_{i,t} A_{it}^2} \mathbb{E}\left[U_{it}^2 | X, \Gamma\right] + \mathcal{O}_{\Theta,\mathcal{P}}\left(\frac{1}{\sqrt{NT}}\right) \\
&\geqslant \frac{\underline{\sigma}^2}{NT\ \mathrm{Lind}(A)} + \mathcal{O}_{\Theta,\mathcal{P}}\left(\frac{1}{\sqrt{NT}}\right) \\
&\geqslant \frac{\underline{\sigma}^2}{NT\ c_{N,T}} + \mathcal{O}_{\Theta,\mathcal{P}}\left(\frac{1}{\sqrt{NT}}\right) >_{\Theta,\mathcal{P}} 0,
\end{aligned}$$

where we used condition (i), (ii), and (iii) consequently, and the last inequality (which holds holds wpa1 uniformly) is ensured by condition (iii).

Then

$$\begin{aligned}
\frac{\max_{i,t} A_{it}^2 \left\|\hat{U} - U\right\|_F^2}{\sum_{i,t} A_{it}^2 U_{it}^2} &= \frac{\frac{1}{NT} \left\|\hat{U} - U\right\|_F^2}{\frac{1}{NT} \sum_{i,t} \frac{A_{it}^2}{\max_{i,t} A_{it}^2} U_{it}^2} \\
&\leqslant \frac{c_{N,T} \left\|\hat{U} - U\right\|_F^2}{\underline{\sigma}^2 + \mathcal{O}_{\Theta,\mathcal{P}}\left(\sqrt{NT}\, c_{N,T}\right)}
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{c_{N,T}\left\|\hat{U}-U\right\|_F^2}{\underline{\sigma}^2+o_{\Theta,\mathcal{P}}(1)}\\
&= o_{\Theta,\mathcal{P}}(1),
\end{aligned}
$$

where the last inequality uses condition (iii), and the last equality follows from $\left\|\hat{U}-U\right\|_F^2 = \mathcal{O}_{\Theta,\mathcal{P}}(\max\{N,T\})$ (the result of Lemma 10) and condition (iii). This completes the proof. □

Lemma 12. *Condition (i) of Lemma 11 holds under Assumption 4.*

Proof. The quantity in condition (i) of Lemma 11 has mean zero and variance conditional on W, Γ bounded by

$$
\frac{\overline{\omega}^2}{(NT)^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbb{E}_P[U_i^4|W,\Gamma] \leq \frac{\overline{\omega}^2/\eta}{NT}.
$$

This gives the $\mathcal{O}_{\Theta,\mathcal{P}}(1/\sqrt{NT})$ rate as claimed.

□

Lemma 13. *Suppose that the hypotheses of Lemma 11 are satisfied, and that Assumption 4 holds. Then, Assumption 1 (ii) holds with* $\widehat{\mathrm{se}} = \sqrt{\sum_{i,t} A_{it}^2\hat{U}_{it}^2}$*, where* $\hat{U}$ *is defined in Lemma 10.*

Proof of Lemma 13. First, we verify

$$
\frac{\sum_{i,t} A_{it}^2 U_{it}^2}{\sum_{i,t} A_{it}^2\sigma_{it}^2} - 1 = o_{\Theta,\mathcal{P}}(1).
$$

Here $\sigma_{it}^2 \equiv \sigma_{it}^2(W,\Gamma) = \mathbb{E}[U_{it}^2|W,\Gamma]$, where we drop dependence of $\sigma_{it}^2(W,\Gamma)$ on W and Γ for brevity of notation. Notice that

$$
\frac{\sum_{i,t} A_{it}^2 U_{it}^2}{\sum_{i,t} A_{it}^2\sigma_{it}^2} - 1 = \underbrace{\frac{\sqrt{NT}\max_{i,t}A_{it}^2}{\sum_{i,t} A_{it}^2\sigma_{it}^2}}_{\overrightarrow{\Theta,\mathcal{P}}\,0}\underbrace{\frac{1}{\sqrt{NT}}\sum_{i,t}\frac{A_{it}^2}{\max_{i,t}A_{it}^2}(U_{it}^2-\sigma_{it}^2)}_{\mathcal{O}_{\Theta,\mathcal{P}}(1)} = o_{\Theta,\mathcal{P}}(1),
$$

where the first factor (uniformly) converges to zero due to conditions (ii) and (iii) of Lemma 11, and the second factor is (uniformly) bounded in probability due to condition (i) of Lemma 11. Combining this result with the result of Lemma 11, we obtain

$$
\sqrt{\frac{\sum_{it} A_{it}^2\hat{U}_{it}^2}{\sum_{it} A_{it}^2\sigma_{it}^2}} - 1 = o_{\Theta,\mathcal{P}}(1). \tag{17}
$$

Second, we demonstrate

$$\frac{\sum_{i,t} A_{it}U_{it}}{\sqrt{\sum_{i,t} A_{it}^2\sigma_{it}^2}} \underset{\Theta,\mathcal{P}}{\overset{d}{\to}} N(0,1). \tag{18}$$

Let $Q_{it} = A_{it}U_{it}/\sqrt{\sum_{i,t} A_{it}^2\sigma_{it}^2}$ and $S_{N,T} = \sum_{i,t} Q_{it}$. Following the lines of the proof of Lemma F.1 in Armstrong and Kolesár (2018) (and using Assumption 4 and conditions (ii) and (iii) of Lemma 11), we conclude that for all sequences of $W = W_{N,T}$ and $\Gamma = \Gamma_{N,T}$ we have for any fixed $\varepsilon > 0$

$$\sum_{i,t} \mathbb{E}\left[Q_{it}^2 \mathbb{1}\{|Q_{it}| > \varepsilon\}|W,\Gamma\right] \underset{\Theta,\mathcal{P}}{\to} 0. \tag{19}$$

Note that (19) is a uniform version of the Lindeberg condition (applied conditional on W and Γ). Hence, following the lines of the proof of the Lindeberg CLT (see, for example, Theorem 27.2 and its proof in Billingsley, 1995), we establish that, for any fixed $t \in \mathbb{R}$,

$$\left|\mathbb{E}\left[e^{iS_{N,T}t}|W,\Gamma\right] - e^{-t^2/2}\right| \underset{\Theta,\mathcal{P}}{\leqslant} r_{N,T} \quad \text{a.s.}$$

for some $r_{N,T} \downarrow 0$. Hence, we also have

$$\left|\mathbb{E}\left[e^{iS_{N,T}t}\right] - e^{-t^2/2}\right| \underset{\Theta,\mathcal{P}}{\to} 0,$$

which implies $S_{N,T} \underset{\Theta,\mathcal{P}}{\overset{d}{\to}} N(0,1)$ and verifies (18). (17) and (18) together deliver the result. □

B Computational details

The optimal weights A_b^* given in Definition 2.2 can be computed directly using convex programming. Alternatively, we can obtain these weights from a nuclear norm regularized "partialling out" regression of X on Z and a matrix of individual effects. This follows by applying a result from Armstrong, Kolesár and Kwon (2020) to our setting, as we now describe. We first consider the general case with covariates (Section B.1), and then obtain a further simplification by specializing to the case with no additional covariates Z.

B.1 General case

The weights A_b^* minimize $\overline{\text{bias}}_C(\hat{\beta}_A) + \sigma^2\|A\|_F$ when $C/\sigma = b$. Equivalently, we can minimize $\|A\|_F$ subject to a bound on $\overline{\text{bias}}_C(\hat{\beta}_A)$:

$$\min_A \sigma^2\|A\|_F \quad \text{s.t.} \quad \overline{\text{bias}}_C(\hat{\beta}_A) \leq B. \tag{20}$$

We can then vary the bound B to optimize any increasing function of the variance $\sigma^2\|A\|_F$ and worst-case bias $\overline{\text{bias}}_C(\hat{\beta}_A)$.

Let Π_μ^*, ψ_μ^* solve the nuclear norm regularized regression

$$\min_{\Pi,\psi} \|X - Z \cdot \psi - \Pi\|_F^2/2 + \mu\|\Pi\|_* \tag{21}$$

where μ indexes the penalty on the nuclear norm. Let

$$\Omega_\mu^* = X - Z \cdot \psi_\mu^* - \Pi_\mu^* \tag{22}$$

denote the matrix of residuals from this regression. Let

$$\hat{\beta}_{\tilde{A}_\mu^*} = \langle \tilde{A}_\mu^*, \widetilde{Y} \rangle_F = \frac{\langle \Omega_\mu^*, \widetilde{Y} \rangle_F}{\langle \Omega_\mu^*, X \rangle_F} \quad \text{where} \quad \tilde{A}_\mu^* = \frac{\Omega_\mu^*}{\langle \Omega_\mu^*, X \rangle_F} \tag{23}$$

and let

$$\overline{B}_\mu = \frac{\langle \Omega_\mu^*, \Pi_\mu^* \rangle_F}{\langle \Omega_\mu^*, X \rangle_F} \quad \text{and} \quad V_\mu = \sigma^2 \frac{\|\Omega_\mu^*\|_F^2}{\langle \Omega_\mu^*, X \rangle_F^2} \tag{24}$$

The following theorem follows immediately from applying Theorem 2.1 in Armstrong, Kolesár and Kwon (2020) to our setup (in applying the formulas from this paper, we use the fact that $\langle \Omega_\mu^*, Z \cdot \psi_\mu^* \rangle_F = 0$ by the first order conditions for ψ, since ψ is unconstrained).

Theorem 14. *Let Π_μ^*, ψ_μ^* be a solution to (21) and let Ω_μ^* be the matrix of residuals in (22), and suppose $\|\Omega_\mu^*\| > 0$. Then $\tilde{A}_\mu^*$ and the corresponding estimator $\hat{\beta}_{\tilde{A}_\mu^*}$ given in (23) solve (20) for $B = C\overline{B}_\mu$, with minimized value V_μ, where $\overline{B}_\mu$ and V_μ are given in (24).*

Thus, to compute the MSE optimizing weights A_b^*, it suffices to compute the weights $\tilde{A}_\mu^*$ for each $\mu > 0$, and then minimize $\overline{B}_\mu^2 + V_\mu$ over the one-dimensional parameter μ. We can also minimize other criteria, as in Remark 2.3 by choosing μ to minimize other functions of worst-case bias $\overline{B}_\mu$ and variance V_μ.

B.2 No additional covariates

In the case where there are no additional covariates, the nuclear norm regularized "partialling out" regression (21) reduces to

$$\min_{\Pi} \|X - \Pi\|_F^2/2 + \mu\|\Pi\|_*. \tag{25}$$

The solution Π_μ^* can then be computed using soft thresholding on the singular values of X. We describe the solution here, and refer to Moon and Weidner (2018, Lemma S.1) for a detailed derivation.

Let the singular value decomposition of X be given by $X = V_X S_X W_X'$ where V_X is an $N \times N$ orthogonal matrix (i.e. $V_X' V_X = I_N$), W_X is a $T \times T$ orthogonal matrix (i.e. $W_X' W_X = I_T$) and S_X is a $N \times T$ rectangular diagonal matrix, with jth diagonal element given by the jth singular value $s_j(X)$ of X. Let $\tilde{S}_X(\mu)$ be the $N \times T$ diagonal matrix with jth diagonal element given by $\max\{s_j(X) - \mu, 0\}$ (i.e. we perform soft thresholding on the jth singular value).

Then the solution Π_μ^* to (25) and residuals $\Omega_\mu^* = X - \Pi_\mu^*$ are given by

$$\Pi_\mu^* = V_X \tilde{S}_X(\mu) W_X', \quad \Omega_\mu^* = V_X(S_X - \tilde{S}_X(\mu)) W_X',$$

Note that $S_X - \tilde{S}_X(\mu)$ is a $N \times T$ diagonal matrix with jth diagonal element given by $\min\{s_j(X), \mu\}$. Thus, the weights $\tilde{A}_\mu^* = \Omega_\mu^* / \langle \Omega_\mu^*, X \rangle_F$ used in the estimator $\hat{\beta} = \langle \tilde{A}_\mu^*, \tilde{Y} \rangle_F$ given in (23) can be obtained by replacing the singular values $s_j(X)$ that are larger than μ with the constant μ, and then dividing by the constant $\langle \Omega_\mu^*, X \rangle_F = \langle S_X - \tilde{S}_X(\mu), S_X \rangle_F = \sum_{j=1}^{\min\{N,T\}} \min\{s_j(X), \mu\} s_j(X)$.

C Additional simulation results

Table 4: $N = 50$, $R = 1$

	LS						debiased					
κ	bias	std	rmse	size	length	length*	bias	std	rmse	size	length	length*
						$T = 20$						
0.00	-0.0006	0.0242	0.0242	7.4	0.086	0.348	-0.0007	0.0300	0.0300	0.0	0.624	0.184
0.05	0.0233	0.0249	0.0340	22.5	0.087	0.349	0.0088	0.0302	0.0314	0.0	0.625	0.184
0.10	0.0466	0.0268	0.0538	56.5	0.087	0.351	0.0177	0.0310	0.0357	0.0	0.627	0.185
0.15	0.0683	0.0309	0.0750	78.5	0.089	0.357	0.0252	0.0327	0.0413	0.0	0.630	0.186
0.20	0.0847	0.0401	0.0937	83.8	0.091	0.368	0.0293	0.0361	0.0465	0.0	0.634	0.186
0.25	0.0879	0.0555	0.1040	76.0	0.097	0.390	0.0281	0.0406	0.0494	0.0	0.639	0.187
0.50	0.0115	0.0398	0.0414	12.1	0.122	0.493	0.0025	0.0359	0.0360	0.0	0.653	0.189
1.00	0.0004	0.0330	0.0330	6.1	0.124	0.499	-0.0006	0.0346	0.0346	0.0	0.655	0.189
						$T = 50$						
0.00	0.0003	0.0147	0.0147	6.0	0.055	0.279	-0.0001	0.0213	0.0213	0.0	0.382	0.117
0.05	0.0247	0.0152	0.0290	44.0	0.055	0.280	0.0070	0.0214	0.0225	0.0	0.382	0.117
0.10	0.0487	0.0169	0.0515	87.9	0.055	0.282	0.0134	0.0220	0.0258	0.0	0.384	0.118
0.15	0.0703	0.0214	0.0734	95.9	0.056	0.287	0.0173	0.0240	0.0296	0.0	0.386	0.118
0.20	0.0784	0.0368	0.0866	86.3	0.060	0.306	0.0158	0.0269	0.0311	0.0	0.390	0.118
0.25	0.0583	0.0510	0.0774	59.2	0.068	0.344	0.0097	0.0275	0.0292	0.0	0.393	0.119
0.50	0.0035	0.0207	0.0210	6.5	0.078	0.398	0.0003	0.0238	0.0238	0.0	0.396	0.119
1.00	0.0003	0.0201	0.0201	5.0	0.078	0.399	-0.0003	0.0237	0.0237	0.0	0.396	0.119
						$T = 100$						
0.00	0.0002	0.0105	0.0105	6.0	0.039	0.229	-0.0001	0.0137	0.0137	0.0	0.295	0.080
0.05	0.0247	0.0109	0.0270	68.4	0.039	0.229	0.0064	0.0138	0.0152	0.0	0.295	0.080
0.10	0.0487	0.0124	0.0502	98.1	0.039	0.231	0.0120	0.0143	0.0187	0.0	0.296	0.080
0.15	0.0684	0.0188	0.0709	97.3	0.040	0.238	0.0134	0.0165	0.0212	0.0	0.299	0.080
0.20	0.0577	0.0390	0.0697	72.5	0.046	0.271	0.0082	0.0179	0.0197	0.0	0.301	0.081
0.25	0.0225	0.0300	0.0375	33.9	0.053	0.310	0.0031	0.0164	0.0167	0.0	0.302	0.081
0.50	0.0016	0.0145	0.0146	5.6	0.055	0.325	0.0001	0.0153	0.0153	0.0	0.303	0.081
1.00	0.0001	0.0143	0.0143	5.1	0.055	0.326	-0.0001	0.0152	0.0152	0.0	0.303	0.081
						$T = 300$						
0.00	-0.0001	0.0059	0.0059	5.8	0.023	0.170	-0.0002	0.0077	0.0077	0.0	0.225	0.049
0.05	0.0245	0.0066	0.0254	96.7	0.023	0.171	0.0060	0.0078	0.0098	0.0	0.225	0.049
0.10	0.0481	0.0088	0.0489	99.8	0.023	0.173	0.0100	0.0089	0.0134	0.0	0.227	0.049
0.15	0.0482	0.0271	0.0553	83.6	0.026	0.196	0.0059	0.0103	0.0119	0.0	0.229	0.049
0.20	0.0117	0.0143	0.0185	33.3	0.031	0.234	0.0015	0.0090	0.0091	0.0	0.229	0.049
0.25	0.0047	0.0093	0.0104	12.6	0.032	0.239	0.0006	0.0087	0.0087	0.0	0.229	0.049
0.50	0.0004	0.0084	0.0085	5.6	0.032	0.241	-0.0001	0.0086	0.0086	0.0	0.229	0.049
1.00	-0.0001	0.0084	0.0084	5.5	0.032	0.241	-0.0002	0.0086	0.0086	0.0	0.229	0.049

$\text{Lind}(A) \in \{0.0109, 0.0049, 0.0028, 0.0011\}$ for $T \in \{20, 50, 100, 300\}$.

Table 5: $N = 300,\ R = 1$

	LS						debiased					
κ	bias	std	rmse	size	length	length*	bias	std	rmse	size	length	length*
						$T = 20$						
0.00	0.0001	0.0096	0.0096	6.4	0.036	0.219	0.0001	0.0115	0.0115	0.0	0.450	0.088
0.05	0.0242	0.0106	0.0264	72.7	0.036	0.220	0.0091	0.0117	0.0148	0.0	0.451	0.088
0.10	0.0474	0.0136	0.0493	96.6	0.036	0.223	0.0169	0.0127	0.0211	0.0	0.452	0.088
0.15	0.0633	0.0235	0.0675	93.0	0.038	0.233	0.0192	0.0159	0.0249	0.0	0.455	0.088
0.20	0.0475	0.0382	0.0610	65.4	0.044	0.267	0.0120	0.0182	0.0218	0.0	0.459	0.088
0.25	0.0192	0.0276	0.0336	31.4	0.049	0.297	0.0047	0.0155	0.0162	0.0	0.460	0.088
0.50	0.0015	0.0134	0.0135	6.3	0.051	0.310	0.0004	0.0134	0.0134	0.0	0.461	0.089
1.00	0.0002	0.0132	0.0132	5.6	0.051	0.310	0.0001	0.0133	0.0133	0.0	0.462	0.089
						$T = 50$						
0.00	-0.0001	0.0060	0.0060	5.8	0.023	0.173	-0.0002	0.0078	0.0078	0.0	0.225	0.048
0.05	0.0246	0.0067	0.0254	96.8	0.023	0.173	0.0060	0.0079	0.0099	0.0	0.225	0.048
0.10	0.0482	0.0090	0.0490	99.8	0.023	0.175	0.0100	0.0090	0.0134	0.0	0.226	0.049
0.15	0.0478	0.0272	0.0550	83.6	0.026	0.199	0.0058	0.0103	0.0118	0.0	0.228	0.049
0.20	0.0117	0.0144	0.0186	32.5	0.031	0.237	0.0014	0.0090	0.0091	0.0	0.229	0.049
0.25	0.0047	0.0093	0.0105	12.8	0.032	0.243	0.0005	0.0088	0.0088	0.0	0.229	0.049
0.50	0.0004	0.0085	0.0085	5.7	0.032	0.245	-0.0001	0.0087	0.0087	0.0	0.229	0.049
1.00	-0.0001	0.0084	0.0084	5.6	0.032	0.245	-0.0002	0.0086	0.0086	0.0	0.229	0.049
						$T = 100$						
0.00	0.0001	0.0041	0.0041	5.0	0.016	0.123	0.0001	0.0055	0.0055	0.0	0.138	0.033
0.05	0.0248	0.0046	0.0253	100.0	0.016	0.123	0.0047	0.0056	0.0073	0.0	0.138	0.033
0.10	0.0482	0.0071	0.0488	99.9	0.016	0.125	0.0056	0.0067	0.0088	0.0	0.139	0.033
0.15	0.0178	0.0172	0.0248	61.1	0.021	0.163	0.0015	0.0063	0.0065	0.0	0.140	0.033
0.20	0.0048	0.0064	0.0080	16.3	0.022	0.172	0.0006	0.0060	0.0061	0.0	0.140	0.033
0.25	0.0024	0.0059	0.0064	7.6	0.023	0.173	0.0003	0.0060	0.0060	0.0	0.140	0.033
0.50	0.0004	0.0057	0.0057	4.8	0.023	0.174	0.0001	0.0060	0.0060	0.0	0.140	0.033
1.00	0.0001	0.0057	0.0057	4.9	0.023	0.174	0.0001	0.0060	0.0060	0.0	0.140	0.033
						$T = 300$						
0.00	-0.0000	0.0024	0.0024	5.5	0.009	0.105	-0.0001	0.0037	0.0037	0.0	0.072	0.018
0.05	0.0249	0.0028	0.0250	100.0	0.009	0.106	0.0030	0.0038	0.0048	0.0	0.072	0.018
0.10	0.0310	0.0169	0.0352	95.9	0.011	0.123	0.0011	0.0041	0.0042	0.0	0.073	0.018
0.15	0.0036	0.0037	0.0051	21.8	0.013	0.148	0.0002	0.0039	0.0039	0.0	0.073	0.018
0.20	0.0014	0.0034	0.0037	7.5	0.013	0.149	0.0000	0.0039	0.0039	0.0	0.073	0.018
0.25	0.0007	0.0033	0.0034	5.2	0.013	0.149	-0.0000	0.0039	0.0039	0.0	0.073	0.018
0.50	0.0000	0.0033	0.0033	4.5	0.013	0.149	-0.0001	0.0039	0.0039	0.0	0.073	0.018
1.00	-0.0000	0.0033	0.0033	4.5	0.013	0.149	-0.0001	0.0039	0.0039	0.0	0.073	0.018

$\text{Lind}(A) \in \{0.0025, 0.0011, 0.0006, 0.0002\}$ for $T \in \{20, 50, 100, 300\}$.

Table 6: $N = 100$, $T = 20$, $R = 2$

$\kappa_1 \backslash \kappa_2$	LS										Debiased									
	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.40	0.50	1.00	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.40	0.50	1.00
bias																				
0.00	0.000	0.016	0.031	0.040	0.037	0.024	0.014	0.005	0.003	0.000	0.000	0.008	0.015	0.019	0.016	0.010	0.006	0.002	0.001	0.000
0.05	0.016	0.033	0.048	0.061	0.064	0.054	0.042	0.030	0.027	0.025	0.008	0.017	0.024	0.029	0.029	0.023	0.018	0.013	0.012	0.011
0.10	0.031	0.048	0.065	0.080	0.088	0.084	0.071	0.055	0.051	0.048	0.015	0.024	0.032	0.038	0.040	0.036	0.030	0.024	0.022	0.021
0.15	0.040	0.061	0.080	0.097	0.109	0.111	0.100	0.080	0.073	0.069	0.019	0.029	0.038	0.046	0.049	0.047	0.041	0.033	0.030	0.029
0.20	0.037	0.064	0.088	0.109	0.125	0.131	0.124	0.097	0.087	0.081	0.016	0.029	0.040	0.049	0.055	0.055	0.050	0.038	0.034	0.031
0.25	0.024	0.054	0.084	0.111	0.131	0.140	0.132	0.095	0.080	0.071	0.010	0.023	0.036	0.047	0.055	0.057	0.052	0.036	0.030	0.026
0.30	0.014	0.042	0.071	0.100	0.124	0.132	0.118	0.074	0.056	0.045	0.006	0.018	0.030	0.041	0.050	0.052	0.044	0.027	0.020	0.016
0.40	0.005	0.030	0.055	0.080	0.097	0.095	0.074	0.032	0.020	0.013	0.002	0.013	0.024	0.033	0.038	0.036	0.027	0.011	0.007	0.005
0.50	0.003	0.027	0.051	0.073	0.087	0.080	0.056	0.020	0.012	0.006	0.001	0.012	0.022	0.030	0.034	0.030	0.020	0.007	0.004	0.002
1.00	0.000	0.025	0.048	0.069	0.081	0.071	0.045	0.013	0.006	0.001	0.000	0.011	0.021	0.029	0.031	0.026	0.016	0.005	0.002	0.001
std																				
0.00	0.015	0.016	0.018	0.024	0.031	0.030	0.024	0.019	0.018	0.018	0.019	0.019	0.020	0.022	0.025	0.024	0.023	0.021	0.021	0.021
0.05	0.016	0.015	0.016	0.020	0.027	0.030	0.026	0.020	0.019	0.018	0.019	0.019	0.019	0.021	0.024	0.025	0.023	0.021	0.021	0.021
0.10	0.018	0.016	0.016	0.018	0.024	0.030	0.030	0.022	0.021	0.020	0.020	0.019	0.020	0.021	0.023	0.025	0.024	0.022	0.022	0.022
0.15	0.024	0.020	0.018	0.019	0.023	0.030	0.034	0.027	0.026	0.025	0.022	0.021	0.021	0.022	0.023	0.026	0.026	0.024	0.024	0.023
0.20	0.031	0.027	0.024	0.023	0.026	0.034	0.043	0.040	0.038	0.038	0.025	0.024	0.023	0.023	0.025	0.028	0.030	0.029	0.028	0.028
0.25	0.030	0.030	0.030	0.030	0.034	0.047	0.060	0.060	0.055	0.053	0.024	0.025	0.025	0.026	0.028	0.032	0.036	0.035	0.033	0.033
0.30	0.024	0.026	0.030	0.034	0.043	0.060	0.076	0.070	0.059	0.054	0.023	0.023	0.024	0.026	0.030	0.036	0.042	0.038	0.034	0.033
0.40	0.019	0.020	0.022	0.027	0.040	0.060	0.070	0.049	0.036	0.031	0.021	0.021	0.022	0.024	0.029	0.035	0.038	0.031	0.027	0.026
0.50	0.018	0.019	0.021	0.026	0.038	0.055	0.059	0.036	0.027	0.025	0.021	0.021	0.022	0.024	0.028	0.033	0.034	0.027	0.025	0.025
1.00	0.018	0.018	0.020	0.025	0.038	0.053	0.054	0.031	0.025	0.024	0.021	0.021	0.022	0.023	0.028	0.033	0.033	0.026	0.025	0.024
rmse																				
0.00	0.015	0.022	0.036	0.047	0.049	0.039	0.028	0.020	0.018	0.018	0.019	0.021	0.025	0.029	0.030	0.027	0.023	0.021	0.021	0.021
0.05	0.022	0.036	0.051	0.064	0.070	0.062	0.049	0.036	0.033	0.031	0.021	0.025	0.031	0.036	0.037	0.034	0.029	0.025	0.024	0.024
0.10	0.036	0.051	0.067	0.082	0.092	0.090	0.077	0.060	0.055	0.052	0.025	0.031	0.038	0.044	0.046	0.044	0.038	0.032	0.031	0.030
0.15	0.047	0.064	0.082	0.098	0.111	0.115	0.106	0.084	0.078	0.074	0.029	0.036	0.044	0.051	0.055	0.054	0.049	0.041	0.038	0.037
0.20	0.049	0.070	0.092	0.111	0.127	0.136	0.131	0.105	0.095	0.089	0.030	0.037	0.046	0.055	0.061	0.062	0.058	0.048	0.044	0.042
0.25	0.039	0.062	0.090	0.115	0.136	0.147	0.145	0.112	0.097	0.089	0.027	0.034	0.044	0.054	0.062	0.065	0.063	0.050	0.045	0.042
0.30	0.028	0.049	0.077	0.106	0.131	0.145	0.140	0.102	0.081	0.071	0.023	0.029	0.038	0.049	0.058	0.063	0.061	0.047	0.040	0.036
0.40	0.020	0.036	0.060	0.084	0.105	0.112	0.102	0.059	0.041	0.034	0.021	0.025	0.032	0.041	0.048	0.050	0.047	0.033	0.028	0.027
0.50	0.018	0.033	0.055	0.078	0.095	0.097	0.081	0.041	0.030	0.026	0.021	0.024	0.031	0.038	0.044	0.045	0.040	0.028	0.026	0.025
1.00	0.018	0.031	0.052	0.074	0.089	0.089	0.071	0.034	0.026	0.024	0.021	0.024	0.030	0.037	0.042	0.042	0.036	0.027	0.025	0.024

Table 7: $N = 100$, $T = 20$, $R = 2$

	LS										Debiased									
κ_1 \ κ_2	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.40	0.50	1.00	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.40	0.50	1.00
size																				
0.00	9.0	31.0	63.0	72.6	59.7	38.2	22.8	11.7	9.3	7.9	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.05	31.0	70.9	91.0	94.8	90.5	79.3	66.4	50.2	44.3	38.8	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.10	63.0	91.0	98.4	99.3	98.4	96.0	92.3	86.0	82.3	79.2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.15	72.6	94.8	99.3	99.7	99.6	98.6	97.5	94.4	92.7	91.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.20	59.7	90.5	98.4	99.6	99.0	97.8	95.6	91.6	89.0	86.4	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.25	38.2	79.3	96.0	98.6	97.8	94.4	88.7	78.0	72.4	67.9	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.30	22.8	66.4	92.3	97.5	95.6	88.7	77.2	58.7	50.2	43.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.40	11.7	50.2	86.0	94.4	91.6	78.0	58.7	31.2	21.4	15.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.50	9.3	44.3	82.3	92.7	89.0	72.4	50.2	21.4	12.9	8.7	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
1.00	7.9	38.8	79.2	91.0	86.4	67.9	43.1	15.0	8.7	6.4	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
length																				
0.00	0.049	0.050	0.050	0.052	0.055	0.058	0.060	0.061	0.061	0.062	0.831	0.832	0.834	0.839	0.844	0.849	0.852	0.854	0.854	0.855
0.05	0.050	0.049	0.050	0.051	0.053	0.057	0.059	0.061	0.061	0.062	0.832	0.832	0.834	0.838	0.843	0.848	0.852	0.854	0.855	0.856
0.10	0.050	0.050	0.050	0.050	0.052	0.055	0.059	0.061	0.062	0.062	0.834	0.834	0.835	0.838	0.843	0.848	0.853	0.856	0.857	0.858
0.15	0.052	0.051	0.050	0.051	0.052	0.054	0.058	0.062	0.063	0.063	0.839	0.838	0.838	0.840	0.845	0.850	0.855	0.860	0.861	0.862
0.20	0.055	0.053	0.052	0.052	0.052	0.054	0.058	0.064	0.066	0.067	0.844	0.843	0.843	0.845	0.848	0.854	0.859	0.865	0.867	0.869
0.25	0.058	0.057	0.055	0.054	0.054	0.056	0.060	0.069	0.072	0.073	0.849	0.848	0.848	0.850	0.854	0.859	0.865	0.872	0.875	0.877
0.30	0.060	0.059	0.059	0.058	0.058	0.060	0.066	0.076	0.079	0.080	0.852	0.852	0.853	0.855	0.859	0.865	0.871	0.878	0.881	0.883
0.40	0.061	0.061	0.061	0.062	0.064	0.069	0.076	0.084	0.086	0.086	0.854	0.854	0.856	0.860	0.865	0.872	0.878	0.885	0.886	0.888
0.50	0.061	0.061	0.062	0.063	0.066	0.072	0.079	0.086	0.087	0.087	0.854	0.855	0.857	0.861	0.867	0.875	0.881	0.886	0.888	0.889
1.00	0.062	0.062	0.062	0.063	0.067	0.073	0.080	0.086	0.087	0.088	0.855	0.856	0.858	0.862	0.869	0.877	0.883	0.888	0.889	0.890
length*																				
0.00	0.438	0.440	0.448	0.463	0.491	0.518	0.534	0.543	0.546	0.547	0.211	0.211	0.212	0.213	0.213	0.214	0.214	0.215	0.215	0.215
0.05	0.440	0.440	0.443	0.453	0.475	0.506	0.528	0.543	0.546	0.548	0.211	0.211	0.212	0.212	0.213	0.214	0.214	0.215	0.215	0.215
0.10	0.448	0.443	0.443	0.449	0.464	0.493	0.522	0.545	0.550	0.553	0.212	0.212	0.212	0.213	0.213	0.214	0.215	0.215	0.215	0.216
0.15	0.463	0.453	0.449	0.450	0.459	0.482	0.516	0.551	0.559	0.563	0.213	0.212	0.213	0.213	0.214	0.215	0.215	0.216	0.216	0.216
0.20	0.491	0.475	0.464	0.459	0.463	0.481	0.516	0.569	0.584	0.592	0.213	0.213	0.213	0.214	0.215	0.216	0.216	0.217	0.217	0.217
0.25	0.518	0.506	0.493	0.482	0.481	0.499	0.537	0.614	0.637	0.650	0.214	0.214	0.214	0.215	0.216	0.216	0.217	0.218	0.218	0.218
0.30	0.534	0.528	0.522	0.516	0.516	0.537	0.585	0.673	0.700	0.714	0.214	0.214	0.215	0.215	0.216	0.217	0.218	0.218	0.218	0.219
0.40	0.543	0.543	0.545	0.551	0.569	0.614	0.673	0.746	0.762	0.769	0.215	0.215	0.215	0.216	0.217	0.218	0.218	0.219	0.219	0.219
0.50	0.546	0.546	0.550	0.559	0.584	0.637	0.700	0.762	0.773	0.778	0.215	0.215	0.215	0.216	0.217	0.218	0.218	0.219	0.219	0.219
1.00	0.547	0.548	0.553	0.563	0.592	0.650	0.714	0.769	0.778	0.783	0.215	0.215	0.216	0.216	0.217	0.218	0.219	0.219	0.219	0.219

Table 8: $N = 100$, $T = 100$, $R = 2$

κ_1 \ κ_2	LS										Debiased									
	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.40	0.50	1.00	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.40	0.50	1.00
bias																				
0.00	0.000	0.017	0.030	0.023	0.008	0.004	0.002	0.001	0.000	0.000	-0.000	0.005	0.007	0.004	0.002	0.001	0.000	0.000	0.000	-0.000
0.05	0.017	0.033	0.049	0.056	0.038	0.030	0.027	0.026	0.025	0.025	0.005	0.010	0.013	0.011	0.008	0.006	0.006	0.006	0.006	0.005
0.10	0.030	0.049	0.066	0.080	0.072	0.056	0.052	0.050	0.049	0.049	0.007	0.013	0.017	0.017	0.013	0.011	0.011	0.010	0.010	0.010
0.15	0.023	0.056	0.080	0.098	0.103	0.080	0.070	0.064	0.063	0.062	0.004	0.011	0.017	0.019	0.016	0.012	0.011	0.010	0.009	0.009
0.20	0.008	0.038	0.072	0.103	0.095	0.054	0.036	0.029	0.026	0.025	0.002	0.008	0.013	0.016	0.012	0.007	0.005	0.004	0.003	0.003
0.25	0.004	0.030	0.056	0.080	0.054	0.020	0.013	0.010	0.009	0.008	0.001	0.006	0.011	0.012	0.007	0.003	0.002	0.001	0.001	0.001
0.30	0.002	0.027	0.052	0.070	0.036	0.013	0.009	0.006	0.005	0.004	0.000	0.006	0.011	0.011	0.005	0.002	0.001	0.001	0.001	0.001
0.40	0.001	0.026	0.050	0.064	0.029	0.010	0.006	0.003	0.003	0.002	0.000	0.006	0.010	0.010	0.004	0.001	0.001	0.000	0.000	0.000
0.50	0.000	0.025	0.049	0.063	0.026	0.009	0.005	0.003	0.002	0.001	0.000	0.006	0.010	0.009	0.003	0.001	0.001	0.000	0.000	0.000
1.00	0.000	0.025	0.049	0.062	0.025	0.008	0.004	0.002	0.001	0.000	-0.000	0.005	0.010	0.009	0.003	0.001	0.001	0.000	0.000	-0.000
std																				
0.00	0.006	0.007	0.009	0.016	0.009	0.008	0.007	0.007	0.007	0.007	0.010	0.010	0.011	0.012	0.011	0.011	0.011	0.011	0.011	0.011
0.05	0.007	0.006	0.007	0.013	0.013	0.009	0.008	0.008	0.008	0.008	0.010	0.010	0.011	0.012	0.011	0.011	0.011	0.011	0.011	0.011
0.10	0.009	0.007	0.007	0.009	0.016	0.011	0.010	0.009	0.009	0.009	0.011	0.011	0.011	0.012	0.012	0.012	0.012	0.012	0.012	0.012
0.15	0.016	0.013	0.009	0.009	0.019	0.022	0.021	0.022	0.022	0.022	0.012	0.012	0.012	0.013	0.014	0.014	0.014	0.014	0.014	0.014
0.20	0.009	0.013	0.016	0.019	0.047	0.046	0.034	0.029	0.028	0.027	0.011	0.011	0.012	0.014	0.015	0.015	0.014	0.013	0.013	0.013
0.25	0.008	0.009	0.011	0.022	0.046	0.023	0.015	0.013	0.013	0.012	0.011	0.011	0.012	0.014	0.015	0.013	0.012	0.012	0.012	0.012
0.30	0.007	0.008	0.010	0.021	0.034	0.015	0.012	0.011	0.011	0.011	0.011	0.011	0.012	0.014	0.014	0.012	0.012	0.012	0.012	0.012
0.40	0.007	0.008	0.009	0.022	0.029	0.013	0.011	0.011	0.011	0.011	0.011	0.011	0.012	0.014	0.013	0.012	0.012	0.012	0.012	0.012
0.50	0.007	0.008	0.009	0.022	0.028	0.013	0.011	0.011	0.011	0.010	0.011	0.011	0.012	0.014	0.013	0.012	0.012	0.012	0.012	0.012
1.00	0.007	0.008	0.009	0.022	0.027	0.012	0.011	0.011	0.010	0.010	0.011	0.011	0.012	0.014	0.013	0.012	0.012	0.012	0.012	0.012
rmse																				
0.00	0.006	0.018	0.032	0.028	0.012	0.009	0.008	0.007	0.007	0.007	0.010	0.011	0.013	0.013	0.011	0.011	0.011	0.011	0.011	0.011
0.05	0.018	0.034	0.050	0.057	0.040	0.031	0.028	0.027	0.026	0.026	0.011	0.014	0.017	0.016	0.014	0.013	0.013	0.012	0.012	0.012
0.10	0.032	0.050	0.066	0.080	0.074	0.057	0.053	0.051	0.050	0.050	0.013	0.017	0.021	0.021	0.018	0.016	0.016	0.016	0.015	0.015
0.15	0.028	0.057	0.080	0.098	0.104	0.083	0.073	0.068	0.067	0.066	0.013	0.016	0.021	0.023	0.021	0.018	0.017	0.017	0.016	0.016
0.20	0.012	0.040	0.074	0.104	0.106	0.071	0.050	0.041	0.039	0.037	0.011	0.014	0.018	0.021	0.019	0.016	0.014	0.014	0.014	0.013
0.25	0.009	0.031	0.057	0.083	0.071	0.031	0.020	0.016	0.015	0.015	0.011	0.013	0.016	0.018	0.016	0.013	0.013	0.012	0.012	0.012
0.30	0.008	0.028	0.053	0.073	0.050	0.020	0.015	0.013	0.012	0.012	0.011	0.013	0.016	0.017	0.014	0.013	0.012	0.012	0.012	0.012
0.40	0.007	0.027	0.051	0.068	0.041	0.016	0.013	0.011	0.011	0.011	0.011	0.012	0.016	0.017	0.014	0.012	0.012	0.012	0.012	0.012
0.50	0.007	0.026	0.050	0.067	0.039	0.015	0.012	0.011	0.011	0.010	0.011	0.012	0.015	0.016	0.014	0.012	0.012	0.012	0.012	0.012
1.00	0.007	0.026	0.050	0.066	0.037	0.015	0.012	0.011	0.010	0.010	0.011	0.012	0.015	0.016	0.013	0.012	0.012	0.012	0.012	0.012

Table 9: $N = 100$, $T = 100$, $R = 2$

$\kappa_1 \backslash \kappa_2$	LS										Debiased									
	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.40	0.50	1.00	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.40	0.50	1.00
size																				
0.00	5.9	79.3	97.3	69.9	24.7	11.3	7.7	6.1	5.8	5.7	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.05	79.3	100.0	100.0	99.9	98.8	96.8	95.0	92.6	91.8	90.9	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.10	97.3	100.0	100.0	100.0	100.0	100.0	99.9	99.9	99.9	99.9	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.15	69.9	99.9	100.0	100.0	99.3	97.8	96.0	94.4	93.9	93.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.20	24.7	98.8	100.0	99.3	89.1	73.7	62.8	54.0	51.0	48.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.25	11.3	96.8	100.0	97.8	73.7	45.0	32.3	23.4	21.0	18.8	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.30	7.7	95.0	99.9	96.0	62.8	32.3	19.9	13.2	11.4	10.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.40	6.1	92.6	99.9	94.4	54.0	23.4	13.2	8.1	7.3	6.8	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.50	5.8	91.8	99.9	93.9	51.0	21.0	11.4	7.3	6.6	6.2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
1.00	5.7	90.9	99.9	93.1	48.5	18.8	10.1	6.8	6.2	5.7	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
length																				
0.00	0.022	0.022	0.023	0.025	0.027	0.027	0.027	0.028	0.028	0.028	0.327	0.327	0.329	0.331	0.332	0.332	0.332	0.332	0.332	0.332
0.05	0.022	0.022	0.023	0.024	0.026	0.027	0.027	0.028	0.028	0.028	0.327	0.327	0.328	0.331	0.332	0.332	0.332	0.333	0.333	0.333
0.10	0.023	0.023	0.023	0.023	0.025	0.027	0.028	0.028	0.028	0.028	0.329	0.328	0.329	0.332	0.333	0.334	0.334	0.334	0.334	0.334
0.15	0.025	0.024	0.023	0.023	0.024	0.028	0.029	0.029	0.030	0.030	0.331	0.331	0.332	0.334	0.336	0.337	0.338	0.338	0.338	0.338
0.20	0.027	0.026	0.025	0.024	0.027	0.033	0.035	0.036	0.036	0.036	0.332	0.332	0.333	0.336	0.338	0.339	0.340	0.340	0.340	0.340
0.25	0.027	0.027	0.027	0.028	0.033	0.037	0.038	0.038	0.038	0.039	0.332	0.332	0.334	0.337	0.339	0.340	0.340	0.340	0.340	0.340
0.30	0.027	0.027	0.028	0.029	0.035	0.038	0.039	0.039	0.039	0.039	0.332	0.332	0.334	0.338	0.340	0.340	0.340	0.340	0.340	0.340
0.40	0.028	0.028	0.028	0.029	0.036	0.038	0.039	0.039	0.039	0.039	0.332	0.333	0.334	0.338	0.340	0.340	0.340	0.340	0.340	0.340
0.50	0.028	0.028	0.028	0.030	0.036	0.038	0.039	0.039	0.039	0.039	0.332	0.333	0.334	0.338	0.340	0.340	0.340	0.340	0.340	0.341
1.00	0.028	0.028	0.028	0.030	0.036	0.039	0.039	0.039	0.039	0.039	0.332	0.333	0.334	0.338	0.340	0.340	0.340	0.340	0.341	0.341
length*																				
0.00	0.285	0.287	0.293	0.323	0.346	0.350	0.352	0.352	0.353	0.353	0.079	0.079	0.079	0.079	0.079	0.079	0.079	0.080	0.080	0.080
0.05	0.287	0.286	0.289	0.304	0.339	0.349	0.352	0.353	0.354	0.354	0.079	0.079	0.079	0.079	0.079	0.080	0.080	0.080	0.080	0.080
0.10	0.293	0.289	0.288	0.294	0.326	0.349	0.354	0.356	0.357	0.357	0.079	0.079	0.079	0.080	0.080	0.080	0.080	0.080	0.080	0.080
0.15	0.323	0.304	0.294	0.293	0.312	0.355	0.370	0.377	0.379	0.380	0.079	0.079	0.080	0.080	0.080	0.080	0.080	0.080	0.080	0.080
0.20	0.346	0.339	0.326	0.312	0.351	0.426	0.452	0.463	0.465	0.467	0.079	0.079	0.080	0.080	0.080	0.080	0.080	0.080	0.080	0.080
0.25	0.350	0.349	0.349	0.355	0.426	0.480	0.488	0.492	0.493	0.493	0.079	0.080	0.080	0.080	0.080	0.080	0.080	0.080	0.080	0.080
0.30	0.352	0.352	0.354	0.370	0.452	0.488	0.494	0.497	0.498	0.498	0.079	0.080	0.080	0.080	0.080	0.080	0.080	0.080	0.080	0.080
0.40	0.352	0.353	0.356	0.377	0.463	0.492	0.497	0.500	0.500	0.501	0.080	0.080	0.080	0.080	0.080	0.080	0.080	0.080	0.080	0.080
0.50	0.353	0.354	0.357	0.379	0.465	0.493	0.498	0.500	0.501	0.501	0.080	0.080	0.080	0.080	0.080	0.080	0.080	0.080	0.080	0.080
1.00	0.353	0.354	0.357	0.380	0.467	0.493	0.498	0.501	0.501	0.502	0.080	0.080	0.080	0.080	0.080	0.080	0.080	0.080	0.080	0.080

Table 10: $N = 100$, $T = 300$, $R = 2$

$\kappa_1 \backslash \kappa_2$	LS										Debiased									
	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.40	0.50	1.00	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.40	0.50	1.00
bias																				
0.00	-0.000	0.016	0.025	0.006	0.002	0.001	0.001	0.000	0.000	-0.000	0.000	0.004	0.004	0.001	0.000	0.000	0.000	0.000	0.000	0.000
0.05	0.016	0.033	0.048	0.037	0.028	0.026	0.026	0.025	0.025	0.025	0.004	0.009	0.010	0.007	0.005	0.005	0.005	0.005	0.005	0.005
0.10	0.025	0.048	0.066	0.074	0.054	0.051	0.049	0.048	0.048	0.048	0.004	0.010	0.013	0.011	0.008	0.007	0.007	0.007	0.007	0.007
0.15	0.006	0.037	0.074	0.082	0.037	0.025	0.022	0.020	0.019	0.019	0.001	0.007	0.011	0.008	0.004	0.003	0.002	0.002	0.002	0.002
0.20	0.002	0.028	0.054	0.037	0.011	0.008	0.006	0.005	0.005	0.005	0.000	0.005	0.008	0.004	0.001	0.001	0.001	0.001	0.001	0.001
0.25	0.001	0.026	0.051	0.025	0.008	0.005	0.004	0.003	0.003	0.002	0.000	0.005	0.007	0.003	0.001	0.001	0.001	0.000	0.000	0.000
0.30	0.001	0.026	0.049	0.022	0.006	0.004	0.003	0.002	0.002	0.001	0.000	0.005	0.007	0.002	0.001	0.001	0.000	0.000	0.000	0.000
0.40	0.000	0.025	0.048	0.020	0.005	0.003	0.002	0.001	0.001	0.001	0.000	0.005	0.007	0.002	0.001	0.000	0.000	0.000	0.000	0.000
0.50	0.000	0.025	0.048	0.019	0.005	0.003	0.002	0.001	0.001	0.000	0.000	0.005	0.007	0.002	0.001	0.000	0.000	0.000	0.000	0.000
1.00	-0.000	0.025	0.048	0.019	0.005	0.002	0.001	0.001	0.000	0.000	0.000	0.005	0.007	0.002	0.001	0.000	0.000	0.000	0.000	0.000
std																				
0.00	0.003	0.004	0.009	0.005	0.004	0.004	0.004	0.004	0.004	0.004	0.005	0.005	0.006	0.006	0.006	0.006	0.006	0.006	0.006	0.006
0.05	0.004	0.004	0.005	0.009	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.006	0.006	0.006	0.006	0.006	0.006	0.006	0.006
0.10	0.009	0.005	0.004	0.010	0.007	0.007	0.007	0.007	0.007	0.007	0.006	0.006	0.007	0.007	0.007	0.007	0.007	0.007	0.007	0.007
0.15	0.005	0.009	0.010	0.030	0.030	0.022	0.020	0.019	0.019	0.018	0.006	0.006	0.007	0.008	0.007	0.007	0.007	0.007	0.007	0.007
0.20	0.004	0.005	0.007	0.030	0.008	0.007	0.007	0.007	0.007	0.007	0.006	0.006	0.007	0.007	0.006	0.006	0.006	0.006	0.006	0.006
0.25	0.004	0.005	0.007	0.022	0.007	0.006	0.006	0.006	0.006	0.006	0.006	0.006	0.007	0.007	0.006	0.006	0.006	0.006	0.006	0.006
0.30	0.004	0.005	0.007	0.020	0.007	0.006	0.006	0.006	0.006	0.006	0.006	0.006	0.007	0.007	0.006	0.006	0.006	0.006	0.006	0.006
0.40	0.004	0.005	0.007	0.019	0.007	0.006	0.006	0.006	0.006	0.006	0.006	0.006	0.007	0.007	0.006	0.006	0.006	0.006	0.006	0.006
0.50	0.004	0.005	0.007	0.019	0.007	0.006	0.006	0.006	0.006	0.006	0.006	0.006	0.007	0.007	0.006	0.006	0.006	0.006	0.006	0.006
1.00	0.004	0.005	0.007	0.018	0.007	0.006	0.006	0.006	0.006	0.006	0.006	0.006	0.007	0.007	0.006	0.006	0.006	0.006	0.006	0.006
rmse																				
0.00	0.003	0.017	0.026	0.008	0.005	0.004	0.004	0.004	0.004	0.004	0.005	0.007	0.008	0.006	0.006	0.006	0.006	0.006	0.006	0.006
0.05	0.017	0.033	0.049	0.038	0.029	0.027	0.026	0.025	0.025	0.025	0.007	0.010	0.011	0.009	0.008	0.008	0.008	0.008	0.008	0.007
0.10	0.026	0.049	0.066	0.075	0.055	0.051	0.050	0.049	0.049	0.049	0.008	0.011	0.014	0.012	0.010	0.010	0.010	0.010	0.010	0.010
0.15	0.008	0.038	0.075	0.087	0.048	0.033	0.030	0.027	0.027	0.026	0.006	0.009	0.012	0.011	0.008	0.007	0.007	0.007	0.007	0.007
0.20	0.005	0.029	0.055	0.048	0.013	0.010	0.009	0.009	0.008	0.008	0.006	0.008	0.010	0.008	0.006	0.006	0.006	0.006	0.006	0.006
0.25	0.004	0.027	0.051	0.033	0.010	0.008	0.007	0.007	0.007	0.007	0.006	0.008	0.010	0.007	0.006	0.006	0.006	0.006	0.006	0.006
0.30	0.004	0.026	0.050	0.030	0.009	0.007	0.007	0.006	0.006	0.006	0.006	0.008	0.010	0.007	0.006	0.006	0.006	0.006	0.006	0.006
0.40	0.004	0.025	0.049	0.027	0.009	0.007	0.006	0.006	0.006	0.006	0.006	0.008	0.010	0.007	0.006	0.006	0.006	0.006	0.006	0.006
0.50	0.004	0.025	0.049	0.027	0.008	0.007	0.006	0.006	0.006	0.006	0.006	0.008	0.010	0.007	0.006	0.006	0.006	0.006	0.006	0.006
1.00	0.004	0.025	0.049	0.026	0.008	0.007	0.006	0.006	0.006	0.006	0.006	0.007	0.010	0.007	0.006	0.006	0.006	0.006	0.006	0.006

Table 11: $N = 100$, $T = 300$, $R = 2$

κ_1 \ κ_2	LS										Debiased									
	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.40	0.50	1.00	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.40	0.50	1.00
size																				
0.00	5.8	99.4	96.5	33.0	9.9	6.2	5.3	4.9	4.8	4.7	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.05	99.4	100.0	100.0	100.0	100.0	99.9	99.9	99.9	99.9	99.9	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.10	96.5	100.0	100.0	100.0	100.0	100.0	100.0	100.0	99.9	99.9	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.15	33.0	100.0	100.0	97.7	85.1	75.2	70.2	65.8	64.6	63.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.20	9.9	100.0	100.0	85.1	47.0	30.5	23.8	19.8	18.4	17.4	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.25	6.2	99.9	100.0	75.2	30.5	16.1	11.8	9.5	8.7	7.9	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.30	5.3	99.9	100.0	70.2	23.8	11.8	8.7	6.9	6.3	6.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.40	4.9	99.9	100.0	65.8	19.8	9.5	6.9	5.8	5.6	5.4	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.50	4.8	99.9	99.9	64.6	18.4	8.7	6.3	5.6	5.4	5.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
1.00	4.7	99.9	99.9	63.5	17.4	7.9	6.0	5.4	5.1	5.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
length																				
0.00	0.013	0.013	0.014	0.016	0.016	0.016	0.016	0.016	0.016	0.016	0.231	0.231	0.233	0.234	0.234	0.234	0.234	0.234	0.234	0.234
0.05	0.013	0.013	0.013	0.015	0.016	0.016	0.016	0.016	0.016	0.016	0.231	0.231	0.233	0.234	0.234	0.234	0.234	0.234	0.234	0.234
0.10	0.014	0.013	0.013	0.014	0.016	0.016	0.016	0.016	0.016	0.016	0.233	0.233	0.234	0.235	0.236	0.236	0.236	0.236	0.236	0.236
0.15	0.016	0.015	0.014	0.015	0.020	0.021	0.021	0.021	0.021	0.021	0.234	0.234	0.235	0.237	0.237	0.237	0.237	0.238	0.238	0.238
0.20	0.016	0.016	0.016	0.020	0.022	0.022	0.022	0.022	0.022	0.022	0.234	0.234	0.236	0.237	0.238	0.238	0.238	0.238	0.238	0.238
0.25	0.016	0.016	0.016	0.021	0.022	0.022	0.022	0.023	0.023	0.023	0.234	0.234	0.236	0.237	0.238	0.238	0.238	0.238	0.238	0.238
0.30	0.016	0.016	0.016	0.021	0.022	0.022	0.023	0.023	0.023	0.023	0.234	0.234	0.236	0.237	0.238	0.238	0.238	0.238	0.238	0.238
0.40	0.016	0.016	0.016	0.021	0.022	0.023	0.023	0.023	0.023	0.023	0.234	0.234	0.236	0.238	0.238	0.238	0.238	0.238	0.238	0.238
0.50	0.016	0.016	0.016	0.021	0.022	0.023	0.023	0.023	0.023	0.023	0.234	0.234	0.236	0.238	0.238	0.238	0.238	0.238	0.238	0.238
1.00	0.016	0.016	0.016	0.021	0.022	0.023	0.023	0.023	0.023	0.023	0.234	0.234	0.236	0.238	0.238	0.238	0.238	0.238	0.238	0.238
length*																				
0.00	0.213	0.215	0.227	0.258	0.261	0.262	0.263	0.263	0.263	0.263	0.047	0.047	0.047	0.047	0.047	0.047	0.047	0.047	0.047	0.047
0.05	0.215	0.214	0.217	0.249	0.261	0.263	0.263	0.263	0.264	0.264	0.047	0.047	0.047	0.047	0.047	0.047	0.047	0.047	0.047	0.047
0.10	0.227	0.217	0.216	0.229	0.261	0.266	0.267	0.268	0.268	0.268	0.047	0.047	0.047	0.047	0.047	0.047	0.047	0.047	0.047	0.047
0.15	0.258	0.249	0.229	0.245	0.323	0.340	0.344	0.346	0.347	0.347	0.047	0.047	0.047	0.047	0.047	0.047	0.047	0.047	0.047	0.047
0.20	0.261	0.261	0.261	0.323	0.363	0.366	0.367	0.368	0.368	0.368	0.047	0.047	0.047	0.047	0.047	0.047	0.047	0.047	0.047	0.047
0.25	0.262	0.263	0.266	0.340	0.366	0.369	0.370	0.371	0.371	0.371	0.047	0.047	0.047	0.047	0.047	0.047	0.047	0.047	0.047	0.047
0.30	0.263	0.263	0.267	0.344	0.367	0.370	0.371	0.372	0.372	0.372	0.047	0.047	0.047	0.047	0.047	0.047	0.047	0.047	0.047	0.047
0.40	0.263	0.263	0.268	0.346	0.368	0.371	0.372	0.372	0.373	0.373	0.047	0.047	0.047	0.047	0.047	0.047	0.047	0.047	0.047	0.047
0.50	0.263	0.264	0.268	0.347	0.368	0.371	0.372	0.373	0.373	0.373	0.047	0.047	0.047	0.047	0.047	0.047	0.047	0.047	0.047	0.047
1.00	0.263	0.264	0.268	0.347	0.368	0.371	0.372	0.373	0.373	0.373	0.047	0.047	0.047	0.047	0.047	0.047	0.047	0.047	0.047	0.047