# Negative nominal rates

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**Abstract**

We show the possibility of equilibrium negative nominal interest rates in a general equilibrium model with financial intermediation. We establish that the decentralization of the planner's steady state requires a zero nominal lending rate on bank loans to firms, as well as a negative nominal lending rate on central bank loans to banks. We also find that implementing the planner's steady state requires firms to be bound to collateral requirements that limit their leverage. The key driver of the results is the very defining characteristic of banking, namely banks' ability to create money by opening deposit accounts borrowers can withdraw from, and that are unbacked by household deposits. Our results can be used to rationalize the ultra-low rates policy implemented by major central banks in the second half of the 2010's and early 2020's.

*Keywords: negative interest rates, financial intermediation, general equilibrium*
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# 1 Introduction

Between 2014 and 2022, several major central banks have implemented ultra-low and even negative nominal rates.[1] This prolonged and unprecedented event took place in the wake of the financial crisis of 2008 —as central banks were running out of tools to support the aggregate demand— and ceased only in the face of sudden inflationary pressures allegedly due to severe disruptions of supply chains by the COVID-19 pandemic first, and later on of the energy and food markets by the war in Ukraine.

Besides the concerns about the practical implications of the policy —notably on the profitability of banks[2]— the event raises conceptual questions, since *negative nominal rates cannot be accommodated by the usual theoretical frameworks*. After all, a negative nominal interest rate amounts to an effective zero nominal price for any good that a borrower may pay for with the part of a loan that needs not be reimbursed,[3] leading to an infinite aggregate excess demand that would preclude the existence of an equilibrium in the standard general equilibrium setup.[4] In spite of this, "the theoretical literature on negative interest rates is perhaps surprisingly somewhat smaller [than the empirical one], given the high stakes in the policy debate" (Eggertsson et al., 2019a). This paper aims precisely at providing a general equilibrium model able to explain zero and negative nominal interest rates, with the hope that it is useful to shed some light on the relevant policy debate. In order to do so, we will start from scratch modeling banking in the most parsimonious possible way.

Specifically, we modify the standard stochastic infinite horizon neoclassical growth

[1] After previous short experiments with negative policy rates by the Swedish Riksbank in 2009 and 2010, and by Denmark's National Bank in 2012, the European Central Bank set on Jun 11, 2014 its deposit facility rate to -0.10% , reaching -0.50% in February 2022. The Riksbank of Sweden key rate hit again 0% on Oct 29, 2014 and turned negative to -0.10% on Feb 18, 2015, reaching -0.5% between Feb 2016 and Jan 2019. The Swiss National Bank lowered its interest rate on sight deposits to -0.75% since Jan 15, 2015. The Bank of Japan started on Jan 29, 2016 applying a -0.1% rate to new deposits by financial institutions. Denmark National Bank lowered to 0.05% on Jan 20, 2015 its already low rate, and turned it negative to -0.35% on Mar 19, 2021. In February 2022 it reached -0.45%. After June 2022 only the Bank of Japan still implemented a negative short-term nominal rate of -0.1%.

[2] For a recent review of the lessons learned from the policy in the euro area see Eisenschmidt and Smets (2019).

[3] Under such circumstances, the price is effectively zero for the buyer —and the corresponding buyer's demand therefore infinite— since the actual (positive) price is being effectively paid by the lender instead.

[4] The benchmark 10-year German bund yield had been negative for over 30 months by February 2022 —meaning that the German government will reimburse *less* than what it borrowed during that period— which effectively amounts to a zero price for the German government on anything it may have wanted to spend the share of borrowed funds not to be reimbursed, be it schools, hospitals, or any other public investment.

model to introduce *banks* that capture three key aspects of financial intermediation. First, we explicitly model the time mismatch between the need and the availability of funds for households and firms. This is done by incorporating to the model the feature that factors and output markets do not open at the same time. We capture this by splitting every period into a "morning" —in which capital and labor are traded— and an "evening" —in which output is traded. In the morning, firms need funds to hire the factors —capital and labor— that allow them to produce an output the proceeds of which will only be received in the evening market. Households, on the other hand, have no use for their morning revenues until output is traded in the evening, being thus forced to make intra-period transfers of their morning revenues to their evening expenditures. As a consequence, there is naturally room for banks to intermediate funds and extend credit.[5]

Second, our model captures too how bank lending induces an increase of available funds in the capital market. This is modeled by making explicit the implicit return on deposits created by banks for lending, without which banks would not take relatively costlier household deposits, against all evidence. The implicit return on deposits created for lending must therefore match that of household deposits, or else banks would forego the latter, which they don't. Its remuneration —which firms owning the opened deposit accounts do not receive, and pay instead interest on funds withdrawn— reaches the households owning the borrowing firms through the effective remuneration to households' own deposits, which are thus remunerated as if they brought additional funds to the capital market (see Section 2.2).

Finally, banks in our model provide maturity transformation, matching the empirical observation that most deposits are on demand while the average maturity of loans exceeds one year.[6] Maturity transformation is a crucial function of banks that allows them to invest in positive expected net present value projects requiring time before they start paying off. Indeed, direct investment from households in long-term projects is in

[5]The idea of splitting periods into sub-periods features also in some models of the new monetarist economics literature starting from Lagos and Wright (2005), but only as a means to introduce separately from centralized markets additional decentralized markets in which money is conveniently needed to deal with the friction of double coincidence of wants in the search model actually driving the model — the idea being susceptible of being extended to models with a financial market too, e.g. in Geromichalos and Herrenbrueck (2022). In our model instead, a banking sector satisfies naturally intermediation needs between firms and households in a fully fledged General Equilibrium model of competitive markets, without additional search frictions.

[6]All empirical evidence used comes from US data provided by the FRED II database of the Federal Reserve Bank of St. Louis.

most cases not possible, while banks —financed through equity and short-term debt (deposits and interbank market borrowing)— can issue longer term debt.[7]

We confront the equilibrium of the model with US data to show some of its key implications to be consistent with the empirical evidence. Thus we can use the model to draw conclusions on the role of banking in increasing output, on the efficiency of market allocations, on the decentralizabity of planner's allocations, and on the type of policies needed for that.

Specifically, we study the equilibrium and first-best allocations of the model and find that (i) while banks increase capital accumulation and output,[8] the planner's steady state cannot (generically) be reached as a market outcome if the central bank's policy focuses exclusively on nominal rates; (ii) planner allocations can, nonetheless, be decentralized by a policy of collateral requirements,[9] and finally (iii) the market implementation of the planner's steady state requires a zero nominal lending rate for bank loans to firms, as well as a negative *nominal* rate for central bank loans to banks —results that may help providing some rationale for the policy of ultra-low rates implemented by major central banks in the period 2014 – 2022.[10] The key driver of this later result turns out to be banks' ability to create money by opening deposit accounts —unbacked by households deposits— from which borrowers can withdraw, i.e. a defining characteristic of banking itself.

An important part of the analysis and its results hinges on an equilibrium wedge found between the return to capital and its marginal productivity —unlike in a friction-

[7]While maturity transformation exposes banks to the additional risks of defaults, we abstract from such risks by assuming that banks can borrow through the interbank market at a rate that, at equilibrium, has to be equivalent to the deposit rate. This way banks satisfy capital requirements and are not subject to defaults.

[8]At least, but not only, if the market steady state is sufficiently mildly inflationary or deflationary, for a sufficiently inelastic labor supply. This result is reminiscent of that of Bencivenga and Smith (1991) but without having to resort to the endogenous growth production externality that drives the result there. Berentsen and Waller (2007) also show an improvement in the allocation of resources following the introduction of financial intermediaries in a model à la Lagos and Wright (2005) in order to create through a(n unobserved) differentiation of agents a role for financial intermediation that is obtained here modeling the (observed) time mismatch between the need and the availability of funds required for production.

[9]The fact that a rate policy is not enough and that central banks should also regulate firms' leverage (and not just banks') is reminiscent of the results in Geanakoplos (2010) supporting that "central banks might consider monitoring and regulating leverage as well as interest rates". A reform introduced in Sweden in 2004 to regulate collateral values led to their reduction, which Cerqueiro et al. (2020) show to have led to a contraction in lending.

[10]We prove also that the laissez-faire equilibrium steady state delivers an output that is higher with banks than without —at least, but not only, if the market steady state is sufficiently mildly inflationary, with low lending rates, or even outright deflationary, for a sufficiently inelastic labor supply.

less model— which allows us to find naturally a role for zero and negative nominal rates in the decentralization of steady state first-best allocation. This result suits the decreasing trend in nominal rates observed since monetary policy was arguably improved through the widespread adoption central bank independence in the wake of the 1970's crises.

Few other papers have set to address rates unconstrained by the nominal zero lower bound, making modeling choices that we do not resort to. Specifically, we do not rely on putting money in the utility function —Rognlie (2016) "integrate[s] cash [...] by including [a] concave flow utility from real cash balances into household preferences"— nor on making abstraction of the maturity transformation role for banking —-in Ulate (2021) "deposits and loans have the same duration [which] sidesteps maturity transformation as an aspect of banking"— nor on assuming special costs —Brunnermeier and Koby (2018) "assume [...] that loans are priced at marginal costs that include *costs from leverage*", also Eggertsson et al. (2019a) introduces bank intermediation costs that we do not need— nor on preventing households from lending to firms —in Ulate (2021) "household[s] [...] save by depositing their money in [...] banks, or by holding cash". Only (i) the time mismatch between the need and the availability of funds faced by households and firms, and (ii) the role of banks in creating money, drive our results.

More generally, this paper relates to several strands of literature. Firstly, it relates to papers that study the effects of financial regulation using general equilibrium models —e.g. Repullo and Suarez (2012), Brunnermeier and Sannikov (2014), Malherbe (2020), Mendicino et al. (2020). While these papers focus on the effects of the capital requirements imposed on banks —which are never violated at equilibrium in our model— we study instead the equilibrium implications of financial intermediation and provide policies to implement first-best allocations.

Second, it relates to the literature that examines implications of low or negative interest rates —e.g. Heider et al. (2019), Caballero and Farhi (2017), Eggertsson et al. (2019a). We supplement this literature by investigating whether negative nominal interest rates can be a general equilibrium outcome at all. As it turns out, we not only show that negative nominal rates are consistent with equilibrium, but also that they are necessary to decentralize the planner's steady state. In particular, we find that the market implementation of a first-best steady state allocation requires negative nominal rates for households' deposits and banks' borrowing from the central bank, as well as zero nominal rates for loans from banks. Our results seem, therefore, to be in line with the secular stagnation view that a trend of low nominal rates are, leaving aside business cycle conjunctures, the ”new normal” —see e.g. Eggertsson et al. (2019b)— contributing thus to the literature that looks into the determinants of secular stagnation —e.g. Caballero et al. (2008), Philippon (2015), Marx et al. (2021).

Our paper is also related to a vast literature featuring macro-finance models — Bernanke et al. (1999) and Gertler and Karadi (2011), among many others— to which we contribute by constructing a parsimonious general equilibrium model that incorporates maturity mismatch between deposits and loans, financial regulation constraints —such as the capital requirement and the collateral constraints— and banks' role in expanding the amount of funds available in the capital markets. Incorporating these three features of banks allows us to obtain implications over observables that are consistent with the available data, yet due to our parsimonious setup we can derive all results analytically.

Last but not least, a main result of this paper relates to the point made by Geanakoplos (2010) that central banks should aim at regulating leverage in the economy as well as at conducting a rate policy, for the sake of eliminating excess volatility in the business cycle. Although in this paper we allow for productivity shocks, the fact that our results —namely on the necessity of a regulation of firms' leverage in order to decentralize the planner's steady state— do not hinge on the existence of such shocks, shows that the scope of the point made by Geanakoplos (2010) goes well beyond the business cycle considerations that are the object of that paper.

The remainder of the paper is organized as follows. Section 2 next introduces the model and its equilibria. It also shows its consistency with empirical evidence. Section 3 characterizes the planner's allocations and in particular its steady state, showing then the generic impossibility of attaining the planner's steady state as an equilibrium with banks under a rates policy only. It then shows the decentralizability of planner allocations by means of an additional policy of collateral requirements for firms' borrowing. Finally, it shows that the central bank policy rate needs to be negative (and equal to the deposits nominal rate) —as well banks' nominal rate on loans to firms needs to be zero— for the planner's steady state to be decentralized as an equilibrium with banks. A final Section 4 discusses results and concludes.

## 2 The model

The model includes three types of private agents: (i) an infinitely-lived representative household that each period works, consumes, and saves —either providing capital to firms or making bank deposits, using real balances for intra-day savings, (ii) a representative firm that each period produces consumption good —out of capital and labor using a linearly homogeneous concave production function subject to standard total factor productivity shocks— and that can take loans from (iii) banks that, with possibly different equities, offer one-period term deposit contracts to households and two-period loan contracts to firms.[11] In line with the empirical evidence, and to convey the role of banks in maturity transformation, we thus assume that loans to firms have longer maturities than households' deposits.[12]

We assume too the existence of a central bank that, on top of being able to lend funds to banks, can impose capital requirements to the latter, as well as a ceiling on *firms'* leverage by imposing collateral requirements, and hence an upper bound on the lending from banks to firms relative to some measure of the latter's ability to reimburse.

Each period has a 'morning' and an 'evening' in which different markets open. Specifically, each morning firms pay households for factors —capital and labor— while each evening firms sell the resulting output to households.[13] To that end, in the morning banks grant loans to firms and the latter repay previous loans, while in the evening banks accept deposits of unused household revenues —i.e. neither consumed nor delivered as capital to firms— and remunerate previous deposits.

Households own firms and banks, which distribute profits every morning. The objective of both firms and banks is to maximize the discounted value of profits distributed to the household, according to discount factors which may or may not coincide with the

[11]The linear homogeneity of firms' technology and objective, as well as that of banks' objective and constraints, makes their numbers irrelevant.

[12]See at https://fred.stlouisfed.org/series/EDANQ the weighted-average maturity for all commercial and industry loans to have been increasing from a year in the mid-1990's to attain about two years in the second half of the 2010's, while most of households' deposits are on demand. The assumed longer maturity of loans can straightforwardly be micro-funded by assuming that production takes two periods instead, but that would only burden the notation without providing any additional insight. Stopping at two-period maturities is without loss of generality, adding more periods to the maturity of loans would only make expressions more cumbersome without again adding insights, the important element being that the maturities of deposits and loans differ.

[13]We assume —without loss of generality and for the sake of lighter expressions— that capital is completely used up in production. Allowing for partial depreciation only adds terms lengthening the expressions without changing the nature of the results.

latter's.

In order to grasp the importance of banks for firms and households note that, in the absence of banks, firms would only be able to pay for factors in the morning —as well as for any dividends distributed— using exclusively the proceeds of the output sold in the previous evening.[14] In the presence of banks instead, firms can moreover borrow in the morning —for a return to be paid the next morning— in order to pay for additional factors, distribution of dividends, and the repayment of the previous period borrowing. Similarly, in the absence of banks, households cannot carry over to the next period any remaining balance of their morning revenues —from factors remuneration and firms dividends— after their evening consumption and setting aside, for a return, capital to be used by firms the next morning.[15] In the presence of banks instead, households can moreover deposit at the bank in the evening —for a return to be received the next evening— the value of their morning revenues that is neither used for consumption nor provided to be used as capital.

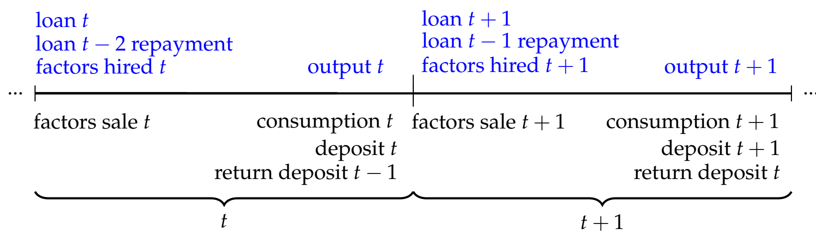
Thus, households and firms are subject to two separate morning and evening budget constraints —that take different forms depending on whether banks are present or not— while banks, on the contrary, face a single budget constraint for the whole period. Thus banks can, so to speak, be in the red (or not) at midday —but not at the end of the day— while firms and households instead can never be in the red, at no moment of the day. That banks can do so (within periods) is[16] what sets them apart from other economic agents, and summarizes the technology they embody. By resorting to banks, firms can circumvent the lack of synchronicity of their revenues and expenses, and households can carry savings forward to the next period. The timeline of two consecutive periods in the model is summarized in the figure next.

[14]Due to households inability to actually screen and monitor firms performance —which banks can, and routinely do— firms cannot issue IOU's to borrow directly from households in the morning in order to repay them in the evening. It is this missing market which allows for banks to step in between households and firms.

[15]Note that, the one-period term deposit contract offered by banks prevents firms from depositing their evening proceeds from output sales, since the latter will be withdrawn in the morning (therefore before maturing) in order to pay for factors of production and dividends distribution. Similarly, households cannot deposit morning incomes needed for output purchases in the same day evening.

[16]On top of being able to extend credit by creating deposits.

Figure 1: **Timeline.**
Factors are hired and (2-period) loans are granted in the morning, while output is traded and deposits are made in the evening.




## 2.1 Firms

Firms have —besides the proceeds from previous output sales— access to bank loans with a "long" maturity (two periods) to pay for factors, dividends, and the reimbursement of the principal and interest of previously contracted maturing loans. Firms face also a borrowing constraint determined by their collateral. Specifically, firms maximize the expected discounted value of the sequence[17] of distributed profits solving[18]

$$\max_{0\le k_t,h_t,l_t} \mathbb{E}\sum_{t=1}^{+\infty}(\delta^f)^{t-1}\pi_t^f$$

$$\pi_t^f + r_t k_t + w_t h_t + \frac{r_{t-1}^l}{\rho_t}\frac{r_{t-2}^l}{\rho_{t-1}} l_{t-2} = \frac{1}{\rho_t} m_{t-1}^f + l_t \tag{1a}$$

$$m_t^f = e^{z_t} f(k_t + e^f, h_t)$$

$$r_t^l l_t \le e^{z_t} f(k_t + e^f, h_t)\theta$$

where

$$z_t = \psi z_{t-1} + \varepsilon_t \tag{1b}$$

with and $\varepsilon_t \sim N(0,\sigma)$, and (in period $t$ real terms) $\pi_t^f$ are firm's profits, $k_t$ is period $t-1$ household savings invested to be used as capital in period $t$, $h_t$ is working hours hired, $l_t$ is firms' borrowing from banks at period $t$, $m_t^f$ is the real balances of the proceeds from output sales, and $e^f$ is the firm's equity, while $r_t$ is the real return factor (i.e. including

[17]The access to loans with maturities longer than one period makes firms' problem not to be equivalent to a sequence of "per period" problems, preventing to proceed as typically done in the real business cycles (and subsequent) literature —see, for instance, Prescott (FRB Minn. Quart. Rev., 1986).

[18]Given $r_t, w_t, r_{t-2}^l, \rho_{t-1} > 0$ for all $t$, as well as $m_0^f = l_0 = l_{-1} = 0$ .

both interest and principal) on $t-1$ savings borrowed as capital at $t$, $w_t$ is the real wage at $t$, $\rho_t$ is the price level inflation factor $p_t/p_{t-1}$ (with $p_t$ being the consumption price level at $t$) from period $t-1$ to period $t$, and $r^l_t$ is the 1-period gross *nominal* lending rate or return factor at $t$. Total factor productivity is subject each period to a shock $e^{z_t} > 0$ —where $z_t$ follows the AR(1) process in (1$b$), given some initial condition $z_0$. Finally, $\delta^f$ is the firm's discount factor,[19] $e^f$ is the firm's equity, and $\theta$ is the proportion of current output that can be pledged as collateral to borrow from banks.[20]

It is worth drawing the attention to two important modeling choices in the problem above:

1. loans to firms have a longer maturity (two periods) than households' bank deposits to be introduced in the next subsection (one period)

2. any new loan needs to be guaranteed by some collateral, expressed as some proportion of the output in *the same* period

Point 1 aims at conveying in a stylized way the empirical evidence that firms finance themselves through loans with maturities longer than those of households' deposits. It also highlights the role of banks in maturity transformation and their bearing of the risks associated to such operations, demanding the appropriate compensation as a consequence. The need for a longer maturity for loans conveys the idea that, in reality, firms often need time before investments start to pay off and generate profits, while households prefer to stay liquid and keep deposits on demand. At equilibrium, by an arbitrage argument, the return to firms' loans must be *at maturity* the compounded return of the two consecutive 1-period gross lending rates during the life of the loan.

Point 2 seems, at first sight, to be unrealistic insofar it seems to disregard future cash-flows of the firm as possible collateral. Actually, since $\theta$ is not assumed to be smaller than 1, future incomes generated by the firm are implicitly being allowed to be used as collateral. As a matter of fact, we show in section 2.4.1 below that —from the equilibrium conditions and the empirical evidence— the estimated value for $\theta$ is above 1.

[19] Note that $\delta^f$ can coincide with $\delta^h$ due to household's ownership of firms. However, the empirical evidence is that households hold only a small fraction of their asset in stocks —around 14% of total financial assets in 2016 according to Bricker et al. (2017). About half of all financial assets are held in retirement funds managed by financial intermediaries (in this framework, banks) for whom the evidence also shows a stronger discount than for households (see footnote 27). At any rate, the results in the next sections do not depend on discount factors differing or not.

[20] From the concavity of $f$, it follows that the constrained set is convex, while the presence of a fixed amount of equity $e^f$ makes the objective to be bounded in the constrained set and profits $\pi^f_t$ not to be zero, in spite of $f$ being linearly homogeneous.

The next characterization follows from firms' optimizing behavior above.[21]

**Proposition 2.1.** *Firms' profit maximizing necessarily satisfies*

*(i) the first-order conditions*

$$\frac{r_t}{f_k(k_t+e^f,h_t)} = \frac{1}{r_t^l}\Big[\delta^f \frac{r_t^l}{\rho_{t+1}}\mathbb{E}e^{z_t} + \frac{r_t^l l_t}{f(k_t+e^f,h_t)}\Big(1-\delta^f\frac{r_{t+1}^l}{\rho_{t+2}}\cdot\delta^f\frac{r_t^l}{\rho_{t+1}}\Big)\Big] = \frac{w_t}{f_h(k_t+e^f,h_t)} \tag{2}$$

*along with the budget and borrowing constraints in* (1)*,*

*(ii) the complementary slackness condition for the borrowing constraint*

$$\mathbb{E}\Big[1-\delta^f\frac{r_{t+1}^l}{\rho_{t+2}}\cdot\delta^f\frac{r_t^l}{\rho_{t+1}}\Big]\Big[r_t^l l_t - e^{z_t}f(k_t+e^f,h_t)\theta\Big] = 0 \tag{3}$$

*and*

*(ii) the non-negativity of its history-contingent multipliers, which implies*

$$\delta^f\frac{r_{t+1}^l}{\rho_{t+2}}\cdot\delta^f\frac{r_t^l}{\rho_{t+1}} \le 1 \tag{4}$$

Note that firms' optimizing requires, through condition (4), that the *discounted* value at $t$ of the reimbursed principal and interest of any loan they take is not worth more in real terms than the loan itself. The key word here is "discounted" since it underpins the firm's subjective evaluation of the cashflows across time. This still allows, of course, for real lending interest rates to be positive —i.e. for each return factor $\frac{r_t^l}{\rho_{t+1}}$ or $\frac{r_{t+1}^l}{\rho_{t+2}}$ to be above 1— but then firms' discount factor $\delta^f$ has to be small enough to make the *discounted* real return at maturity on loans taken to be negative —i.e. for the product of the two consecutive discounted real return factors $\delta^f\frac{r_t^l}{\rho_{t+1}}$ and $\delta^f\frac{r_{t+1}^l}{\rho_{t+2}}$ during the life of loan to be bounded above by 1. Equivalently, firms' optimizing imposes an upper bound equal to $(\frac{1}{\delta^f})^2$ on the real return on loans *at maturity* $\frac{r_t^l}{\rho_{t+1}}\cdot\frac{r_{t+1}^l}{\rho_{t+2}}$, that is necessarily reached whenever the firm's borrowing is not constrained by the collateral requirement. This implication of the model is shown to hold empirically in Section 2.4.1 below.

The result next follows straightforwardly as a corollary of Proposition 2.1.

[21] The derivations can be found in Appendix A.1.

**Proposition 2.2.** *The return to productivity ratio —both for capital and labor— of an optimizing firm is bounded below by the expected discounted value of the return implied by the change in the level of prices on a unit of income, i.e.*

$$\frac{r_t}{f_k(k_t + e^f, h_t)} \geq \delta^f \frac{1}{\rho_{t+1}} \mathbb{E}\, e^{z_t} \leq \frac{w_t}{f_h(k_t + e^f, h_t)} \tag{5}$$

As a consequence,

(i) with inflation of prices (or even mild deflation) factors can be remunerated below their productivity

(ii) with a strong enough deflation of prices factors have to be remunerated above their productivity.

## 2.2 Households

The representative household maximizes the expected discounted value of its sequence of utilities from consumption net of disutilities from working, subject to a morning and an afternoon budget constraint each period, by solving[22]

$$\begin{aligned} \max_{0 \leq h_t, k_{t+1}, d_t} \quad & \mathbb{E} \sum_{t=1}^{+\infty} (\delta^h)^{t-1} \left[ u(c_t) - v(h_t) \right] \\ m_t^h &= r_t k_t + w_t h_t + \pi_t^f + \sum_b \pi_t^b \\ c_t + k_{t+1} + \phi_t d_t &= m_t^h + \frac{r_{t-1}^d}{\rho_t} d_{t-1} \end{aligned} \tag{6}$$

where (in period $t$ real terms) $c_t$ is consumption at period $t$, $m_t^h$ is the real balances of intra-period income from the ownership of factors, firms, and banks carried from morning to evening by the household, $d_t$ is household's bank deposit at $t$ with short maturity (one period), $r_t^d$ is the gross *nominal* deposit rate or return factor to deposits made at $t$ (i.e. including both interest and principal), and $\pi_t^b$ are profits from bank $b$, along with variables already defined in the previous section, while $\delta^h$ is the household's discount factor.

[22] Given $r_t, w_t, r_{t-1}^d, \rho_t > 0$, $\pi_t^f, \pi_t^b \geq 0$, and $0 < \phi_t < 1$ for all $t$, as well as $k_1 \geq 0, d_0 = 0$.

The factor $\phi_t$ captures banks' money creation through lending, which often takes place by creating deposits from which borrowing firms can withdraw funds, rather than by exclusively intermediating pre-existing household deposits.[23] Indeed, the deposits created by banks for lending to firms have an implicit return (a cost for the bank) that must match that of household deposits $r_t^d$. Otherwise, banks would not (contrary to evidence) take costlier deposits from households, financing their loans exclusively by creating bank money instead. This implicit return is in reality not remunerated directly to the owners of the deposit accounts (i.e. the borrowing firms) —who instead pay interests on the amounts withdrawn from them— but indirectly to the *owners* of these firms (i.e. the households) by remunerating their own deposits $d_t$ as if they were bigger by a factor $\frac{1}{\phi_t} > 1$, so that the effective nominal rate on deposit $d_t$ is $\frac{1}{\phi_t} r_t^d$, and the real rate $\frac{1}{\phi_t} \frac{r_t^d}{\rho_{t+1}}$. Thus each $\phi_t \in (0, 1)$ holds true —which is shown to also hold empirically in Section 2.4.1 below— and all $\phi_t$'s are taken as given by households.

The next characterization follows from households' optimizing behavior above.[24]

**Proposition 2.3.** *Households' utility maximizing is necessarily characterized by the first-order conditions*

$$\frac{\mathbb{E}u'(c_t)}{\delta^h \mathbb{E}u'(c_{t+1})} = r_{t+1} = \frac{1}{\phi_t} \frac{r_t^d}{\rho_{t+1}} \tag{7}$$

$$\frac{v'(h_t)}{\mathbb{E}u'(c_t)} = w_t \tag{8}$$

*along with their budget constraints in* (6).

As usual, the fist-order conditions convey the necessary coincidence, at the optimal choice, of the household's wished trade-offs —between consecutive consumptions, and between consumption and leisure— with those made possible by the remuneration to factors and bank deposits.

## 2.3 Banks

Bank $b$ of a competitive banking sector maximizes the discounted value of the sequence of distributed profits, subject to its single budget constraint, the balance sheet constraint,

[23]See e.g. (Jakab and Kumhof, 2019). Note that these newly created deposits correspond to loans extended to firms providing financing, and are not additional resources, as made clear by the respect of the feasibility condition (14) in the equilibrium and in the planner's problem (23) below.

[24]The derivations can be found in Appendix A.2.

and a capital requirement constraint, by solving[25]

$$\max_{q^b_t,\,0\leq l^b_t,d^b_t} \quad \sum_{t=1}^{+\infty}(\delta^b)^{t-1}\pi^b_t$$

$$\pi^b_t + l^b_t - \frac{r^l_{t-1}}{\rho_t}\frac{r^l_{t-2}}{\rho_{t-1}}l^b_{t-2} = d^b_t - \frac{r^d_{t-1}}{\rho_t}d^b_{t-1} + q^b_t - \frac{r^q_{t-1}}{\rho_t}q^b_{t-1} \tag{9}$$

$$\frac{r^l_{t-1}}{\rho_t}l^b_{t-1} + l^b_t = e^b + d^b_t + q^b_t$$

$$\eta l^b_t \leq e^b$$

where (in period $t$ real terms) $d^b_t$ is the level of deposits taken by bank $b$ at period $t$, $q^b_t$ is the amount borrowed (if positive) or lent (if negative) by bank $b$ at period $t$ in the interbank market for a gross *nominal* return $r^q_t$, along with variables already defined in the previous sections, while $e^b$ is bank $b$'s equity,[26] $\delta^b$ is bank $b$'s discount factor,[27] and $\eta \in (0,1)$ is the share of lending that is covered by equity.

By means of their single budget constraint banks can circumvent the asynchronicity of revenues and payments that households and firms face. Indeed, banks face only one budget constraint, in spite of the fact that they disburse and receive funds each morning and evening, since banks :

(i) receive from firms returns to previous loans $\frac{r^l_{t-1}}{\rho_t}\frac{r^l_{t-2}}{\rho_{t-1}}l^b_{t-2}$ in the morning

(ii) grant firms new loans $l^b_t$ in the morning

(iii) distribute to households dividends $\pi^b_t$ in the morning

(iv) receive from households deposits $d^b_t$ in the evening

(v) pay households returns to previous deposits $\frac{r^d_{t-1}}{\rho_t}d^b_{t-1}$ in the evening

[25]Given $r^l_{t-2}, r^d_{t-1}, r^q_{t-1}, \rho_{t-1} > 0$ for all $t$, as well as $e^b, \eta > 0$ and $l^b_0 = l^b_{-1} = d^b_0 = 0$.

[26]The presence of a fixed amount of equity $e^b$ makes the objective to be bounded in the constrained set, and profits $\pi^b_t$ not to be zero. The heterogeneity in banks' equity makes the interbank market meaningful.

[27] Note that $\delta^b$ can coincide with $\delta^h$ due to household's ownership of banks. Other papers allow for economic agents differ in their patience, e.g. distinguishing between patient and impatient households. Angelini et al. (2014) calibrate the discount factor of patient households to 0.994 with quarterly data, while impatient households have the same discount factor as entrepreneurs in the model, that is 0.975. Iacoviello (2005) assumes a discount factor of 0.99 on quarterly data for standard households, while impatient ones discount by factor 0.95. Carroll and Samwick (1998) compute an empirical distribution of discount factors for all agents using information on the elasticity of asset demands, and report a range of values between 0.91 and 0.99. At any rate, the results in the next sections do not depend on whether discount factors differ or not.

on top of borrowing from (or lending to) the interbank market as needed. Still, banks are not bound by a morning and an evening budget constraint —as firms and households are— which would require them to satisfy[28]

$$\begin{aligned} \pi_t^b + l_t^b + m_t^b &\leq \frac{r_{t-1}^l}{\rho_t} \frac{r_{t-2}^l}{\rho_{t-1}} l_{t-2}^b \\ \frac{r_{t-1}^d}{\rho_t} d_{t-1}^b + \frac{r_{t-1}^q}{\rho_t} q_{t-1}^b &\leq d_t^b + m_t^b + q_t^b \\ 0 &\leq m_t^b \end{aligned} \tag{10}$$

instead of the first constraint in (9), with $m_t^b$ being the real balances carried over by the bank from the morning to the evening.[29] If the non-negativity constraint on $m_t^b$ is dropped —which is what the formulation of the bank's problem in (9) implicitly does by omitting $m_t^b$ altogether— then banks can effectively "transfer" funds freely within the day in any direction, i.e. not only from the morning to the same-day evening, but from the evening to the same-day morning too.

The next characterization follows from banks' optimizing behavior above.[30]

**Proposition 2.4.** *Bank b's profit maximizing necessarily satisfies*

*(i) the first order condition equating the nominal deposit and interbank market rates*[31]

$$r_t^d = r_t^q \tag{11}$$

*along with the constraints in* (9)

*(iii) the complementary slackness condition of the capital requirement constraint*

$$\left[\left(\delta^b \frac{r_t^l}{\rho_{t+1}} - \delta^b \frac{r_t^d}{\rho_{t+1}}\right) + \delta^b \frac{r_t^l}{\rho_{t+1}} \left(\delta^b \frac{r_{t+1}^l}{\rho_{t+2}} - \delta^b \frac{r_{t+1}^d}{\rho_{t+2}}\right)\right] \left(\eta l_t^b - e^b\right) = 0 \tag{12}$$

[28] For the sake of the argument, it is assumed in (10), without loss of generality, that the interbank market operates in the afternoon. The same argument can be made having it operating in the morning instead, or in both sub-periods.

[29] Note that, because of the discounting in the bank's objective, it is never optimal to carry any balance from an evening to the next morning.

[30] The detailed derivations can be found in Appendix A.3.

[31] So that, in the context of the model, the cost of funds for banks is the same whether they come as households' deposits or central bank loans. In actual economies, in which bank operating costs absent here play a role, the deposit rate can accordingly depart from the interbank rate.

*and*

*(ii) the non-negativity of its multiplier*

$$0 \leq \left(\delta^b \frac{r_t^l}{\rho_{t+1}} - \delta^b \frac{r_t^d}{\rho_{t+1}}\right) + \delta^b \frac{r_t^l}{\rho_{t+1}} \left(\delta^b \frac{r_{t+1}^l}{\rho_{t+2}} - \delta^b \frac{r_{t+1}^d}{\rho_{t+2}}\right) \tag{13}$$

Note that the last condition amounts to requiring that the value for the bank of the provision of funds is never negative. Indeed, this value is the *sum* of the discounted values at $t$ of ($i$) on the one hand, the real return (net of that of deposits) at $t+1$ to loans given at $t$, and ($ii$) on the other hand, the real return of lending at $t+1$ the discounted value at $t+1$ of the real return (net of that of deposits) at $t+2$ to loans given at $t+1$. This sum aggregates the entire value for the bank of the provision of funds at $t$ and, at the optimum, it has to be non-negative every period —although each of its components need not be.

Once the optimal behavior of all agents has been characterized in this and the previous sections, we provide next the full characterization of the equilibrium of the economy with banks.

## 2.4 Equilibrium with banks

An equilibrium with banks is characterized by non-negative real consumptions $c_t$, capital savings $k_{t+1}$, working hours $h_t$, household deposits $d_t$, and first-period incomes held as real balances $m_t^h$ for the representative household; loans obtained $l_t$, sales income held as real balances $m_t^f$, and profits $\pi_t^f$ for the firm; deposits taken $d_t^b$, loans granted $l_t^b$, borrowed or supplied funds $q_t^b$, and profits distributed $\pi_t^b$ for each bank $b$; as well as gross returns to capital $r_t$, hourly wage rates $w_t$, gross *nominal* returns on deposits $r_t^d$, and gross *nominal* returns on loans $r_t^l$, such that —given the interbank market gross *nominal* returns $r_t^q$— it holds, for all realizations of productivity shocks $z_t$, that

1. firms optimize, so that (2) , (3), and (4) hold[32]
2. households optimize, so that (7) and (8) hold[33]
3. banks optimize, so that (11) , (12), and (13) hold,[34] as well as

[32] Along with the firm's constraints.
[33] Along with the household's constraints.
[34] Along with the bank's constraints.

4. markets clear, i.e.

$$c_t + k_{t+1} = e^{z_t} f(k_t + e^f, h_t) \tag{14}$$

and

$$\begin{aligned} l_t &= \sum_b l_t^b \\ d_t &= \sum_b d_t^b \\ 0 &= \sum_b q_t^b \end{aligned} \tag{15}$$

A few remarks on the definition above are now in order. Note first that the feasibility condition (14) is equivalent, to the equilibrium in the money market

$$m_t^h + \frac{r_{t-1}^d}{\rho_t} d_{t-1} = m_t^f + \phi_t d_t \tag{16}$$

where the LHS is, at the evening of period $t$ and in real terms, the supply of money — from households' real balances and maturing deposits (respectively from the morning remuneration of the ownership of factors, firms, and banks, and from the evening gross real return on past deposits)— while the RHS is its demand (by firms for output sales, and by banks for deposit taking, respectively).[35]

Secondly, the equilibrium above is contingent to the sequence of policy rates $r_t^q$, which pins down the interbank nominal interest rate that has to match at equilibrium —and hence in the absence of arbitrage opportunities— the rate at which the central bank is ready to lend to banks. Note, however, that in order to avoid having, against all evidence, a central bank effectively injecting (or removing) real output into (or from)

[35]Equivalently, the feasibility condition (14) amounts also —from aggregating morning constraints of both households and firms, doing similarly with their evenings constraints, and then using the banks' budget constraint, the market clearing conditions, and the feasibility condition to obtain it by substitutions— to

$$e^{z_t} f(k_t + e^f, h_t) - \frac{1}{\rho_t} e^{z_{t-1}} f(k_{t-1} + e^f, h_{t-1}) = [1 - \phi_t] d_t \tag{16bis}$$

where the right-hand side of (16*bis*) is, in period $t$ real terms, the part of household deposits not coming from factor remunerations, distributed dividends, or return to previous deposits. It follows then from (16*bis*) that the additional deposits in the RHS —created by banks through lending to the firms owned by households— match the increase of income relative to that of the previous period in the LHS. Note also that it follows from (16*bis*) —i.e. from the feasibility of the allocation of resources— that the factor $\phi_t$ capturing money creation through banks' lending to firms —which households take as given— is, at equilibrium, driven by the process of productivity shocks $z_t$.

the economy,[36] no actual borrowing from the central bank takes place, since at equilibrium $\sum_b q_t^b = 0$ in (15) for all $t$. That is to say, all the additional liquidity that any bank may need, beyond its equity and households' deposits, is borrowed from other banks through the interbank market.

Remarkably, the equilibrium conditions above leave room for negative *nominal* rates to be an equilibrium outcome under the right conditions, as Proposition 2.5 next establishes. Even more notable, we show later on (see Proposition 3.6 below) that negative nominal rates are actually needed to implement the planner's steady state in the empirically relevant case.

**Proposition 2.5.** *The nominal lending rate in the interbank market needs to be negative whenever the reciprocal of the inflation factor bounds from above the expected intertemporal marginal rate of substitution.*

*Proof.* Straightforward from consumers' optimizing behavior in (7) and banks' optimization condition (11), since $\phi_t$ takes values in $(0, 1)$.

□

Finally, the equilibrium conditions imply the propositions 2.6 and 2.7 next too, which capture intuitive results.

**Proposition 2.6.** *The equilibrium nominal gross return to loans $r_t^l$ can be below that of deposits $r_t^d$ at some period $t$, but not for two consecutive periods.*

*Proof.* Straightforward from the positivity of the multiplier of the banks' capital requirement in (13). □

Note that this property fits well in the intuition that, while a bank in search for funds might be led to offer momentarily a high remuneration to deposits —even beyond that of its loans— this situation cannot last long.

The next property underlines the crucial role of banks —besides improving the efficiency of the allocation of resources— in creating and bringing funds into the capital market, boosting thus income growth.[37]

[36]This possibility could be considered should the monetary and fiscal policies be merged in the hands of a central bank/government. Such a choice would be, nonetheless, at odds with the institutional framework guaranteeing central bank independence characterizing advanced economies since the late 1980's.

[37]Other sources of growth have been left out of the model for the sake of highlighting the contribution of bank supply of loans and demand of deposits, but their incorporation to the model would not, by any means, make that contribution disappear.

**Proposition 2.7.** *At equilibrium, banks' morning net injections of funds (respectively withdrawals) in excess of their afternoon net withdrawals (resp. injections) amounts to the funds injected by banks' deposit creation when extending loans, i.e.*

$$\Big[\sum_b \pi_t^b + l_t - \frac{r_{t-2}^l}{\rho_{t-1}}\frac{r_{t-1}^l}{\rho_t} l_{t-2}\Big] + \Big[\frac{r_{t-1}^d}{\rho_t} d_{t-1} - \phi_t d_t\Big] = \big[1-\phi_t\big] d_t \tag{17}$$

*Proof.* Indeed, market clearing along with the agents' constraints implies $(16bis)$ (see footnote 34). Moreover, on the one hand households' and firms' morning constraints in (1) and (6) imply

$$m_t^h - \frac{1}{\rho_t} m_{t-1}^f = \sum_b \pi_t^b + l_t - \frac{r_{t-2}^l}{\rho_{t-1}}\frac{r_{t-1}^l}{\rho_t} l_{t-2} \tag{18}$$

i.e. banks' distributed profits and loans net of returns from previous loans (in the RHS) inject —if positive (resp. withdraw, if negative)— funds for households in excess of firms' own funds paid to households for their factors in the morning (in the LHS). On the other hand, the afternoon constraints in (1) and (6) imply —with the feasibility of the allocation—

$$m_t^f - m_t^h = \frac{r_{t-1}^d}{\rho_t} d_{t-1} - \phi_t d_t \tag{19}$$

i.e. banks' remuneration to previous deposits net of new deposits (in the RHS) withdraws —if negative (resp. injects, if positive)— funds from (resp. to) households net of firms' new funds in the afternoon (in the LHS).

Therefore, throughout the day, it holds true that

$$m_t^f - \frac{1}{\rho_t} m_{t-1}^f = \Big[\sum_b \pi_t^b + l_t - \frac{r_{t-2}^l}{\rho_{t-1}}\frac{r_{t-1}^l}{\rho_t} l_{t-2}\Big] + \Big[\frac{r_{t-1}^d}{\rho_t} d_{t-1} - \phi_t d_t\Big] \tag{20}$$

Since, from the firms' optimal behavior, it holds true that $m_t^f = e^{z_t} f(k_t + e^f, h_t)$ too, then from $(16bis)$ the RHS of (20) is equal to $\big[1-\phi_t\big] d_t$. □

Note, finally, that the conditions (2), (4), and (13) in the definition of an equilibrium impose testable implications on the remunerations to capital $r_t$ and labor $w_t$, their productivities $f_k$ and $f_h$, the gross nominal returns to loans $r_t^l$, the gross nominal returns to deposits $r_t^d$, and price level inflation factor $\rho_t$. These conditions are used next to establish, before using the model to obtain the main results, its consistency with the empirical evidence from US data in the following section.

### 2.4.1 Empirical consistency of the model

We test the empirical consistency of our model by confronting it with US data. Specifically, we measure the return to capital $r_t$ as the pre-tax return to business capital estimated in Gomme et al. (2011) —annual, updated to 2016, no capital gains. We then compute capital productivity from the data following Gomme et al. (2011) too. Specifically, marginal productivity is measured as the ratio of revenues from business capital to the value of the latter. Revenues are measured as the net operating surplus —value added net of labor expenditures and depreciation— minus the fraction of proprietor's income attributed to capital.[38] We further subtract the part of the value added and proprietor's income generated from housing. All time series are from the of the St.Louis Fed database FRED II.[39] The denominator adds capital costs: inventories —from 1960 to 1991 from NIPA, after 1992 from the U.S. Bureau of Economic Analysis (BEA)— structures from FRED II, and equipment & software from NIPA and BEA. In the resulting time series the return to capital is below its marginal productivity for the entire sample 1960–2016 (Figure 2).[40] Finally, lending and deposit rates are from the International Financial Statistics, IMF.

Equation (2) provides a relation, implied by the model, between observable time series —namely, the productivity of and return to capital (and similarly for labor), lending rates, and inflation. It involves also two parameters: the maximum proportion of output that can be pledged as collateral $\theta$, and the discount factor of firms $\delta^f$. Thus, in order to check the consistency of the model with the empirical evidence, we estimate the value of $\theta$ for various values of $\delta^f$. As a baseline, we set the value of $\delta^f$ to 0.94 —or, equivalently, a discount rate of approximately 6%. Indeed, a standard estimate for households discount factor in the literature (Gertler and Karadi, 2011) is 4% annually, and since we want to reflect that firms discount more than households,[41] we choose a

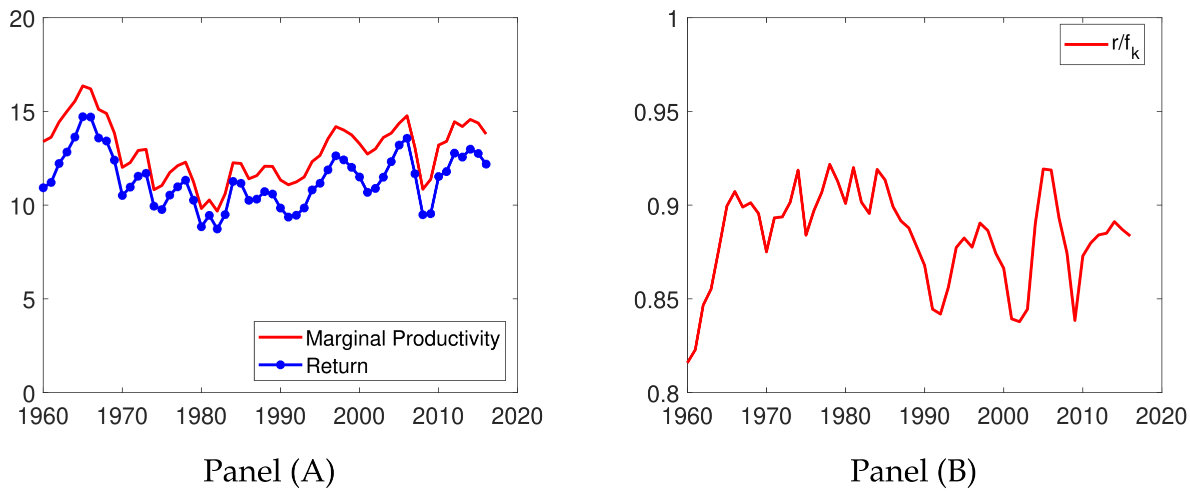
[38] We use the capital share to pin down the fraction of proprietor's income attributed to capital (roughly a third of output is generated by the input of capital).

[39] Net operating surplus is GDINOS, proprietor's income is A041RC1, and housing value is B1034C1A027NBEA. We later deflate nominal value using GDP delator (A191RI1Q225SBEA).

[40] Alternatively, we have used labor return to productivity ratio to check the empirical consistency of the model. To this end, we have measured wages as the average (per worker) hourly compensation using data from BEA, and we have computed labor productivity as output per hour worked multiplied by the labor share. Again, we observe that labor return to productivity ratio is close to one, leading to similar conclusions.

[41] Most of households investments in firms' ownership is channeled through pension plans invested in funds traded by financial intermediaries (Bricker et al., 2017), i.e. banks in this framework so that the effective discount for firms is that of banks, that can be argued to discount more than households.

Figure 2: **Remuneration of capital in the data.**
Left panel: evolution of the return on capital (blue line) and its marginal productivity (red line). Right panel: capital return over its marginal productivity. US data, 1960 - 2016.




correspondingly lower $\delta^f$. As a robustness check, we provide additional estimates with different discount rates $\delta^f = 0.93$ and 0.95. The results, summarized in Table 1 next, point to a value for $\theta$ above one.[42]

Table 1: **Model implications: pledgeable collateral**
The table shows the mean and standard deviation of the implied maximum proportion of output pledgedeable as collateral, $\theta$, as a function of the discount factor, $\delta^f$.

| Discount factor, $\delta^f$ | Mean, $\mathbb{E}(\theta)$ | Std, $\sigma(\theta)$ | Period |
|---|---|---|---|
| 0.93 | 1.00 | 3.73 | 1960-2014 |
| 0.94 | 1.69 | 3.61 | 1960-2014 |
| 0.95 | 3.28 | 34.43 | 1960-2014 |

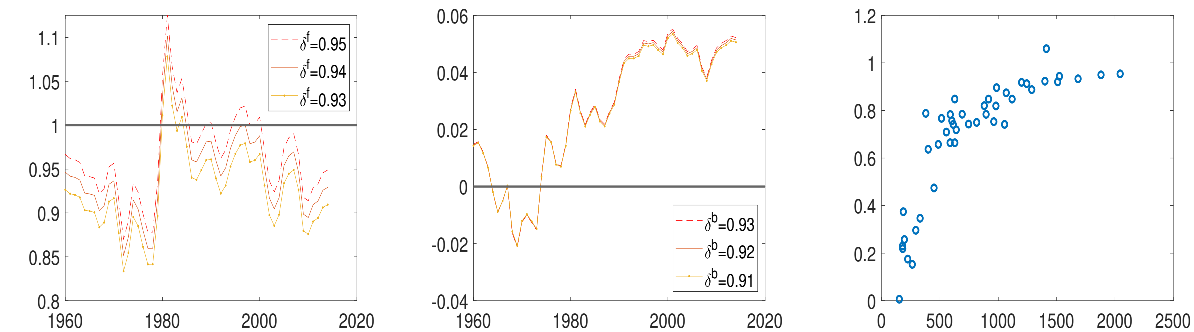
Indeed, in reality not all loans are fully collateralized, and different firms might face different collateral requirements. On average, less than 70% of loans are secured by some collateral in the US.[43] Chodorow-Reich et al. (2021) study differences in collateral requirements at the firm level and to show that the amount of collateral depends on firm characteristics. In particular, the largest US firms have access to unsecured debt, while small firms face much stricter collateral requirements. Since our model includes

[42]We exclude two values of $\theta$ in 1996 and 1997 and treat them as outliers.
[43]From Fred II, time series ESANQ.

a representative (hence average) firm, we expect it to have to provide an amount of collateral that is smaller than the size of loan. In other words, $\theta$ can be above one and thus some fraction of firms' debt can remain un-collateralized.

Regarding the positivity condition (4) on the multiplier of firms' borrowing constraint, Figure 3 presents the evolution of its LHS over time for various values of $\delta^f$. For the US the condition is almost always satisfied, except for a short period in the early 80's, right after the second oil shock of 1979 and during the ensuing spike in inflation. It also seems to be barely satisfied in the run up to the dot-com bubble at the turn of the century. Outside these episodes of abnormal behavior for the economy, condition (4) is consistent with the empirical evidence.

Figure 3: **Model implications and data: positivity of multipliers.**
Left panel: eq. (4) in firm's FOC. Centre panel: eq. (13) in banks' FOC. Right panel: scatter plot of $\phi_t$ —in ordinates— versus $l_t$ —in abscissae— implied by eq. 16. US data, 1960 – 2016.




Finally, we check the positivity condition (13) on the multiplier of banks' capital requirement. To emphasize the special role of banks we set their discount factor $\delta^b$ below that of firms or households, namely equal to 0.92, and then provide robustness checks.[44] Condition (13) is represented by the central panel in Figure 3, where it is shown to be almost always satisfied for the US throughout the sample, except at the end of the Bretton Woods regime and the first oil shock, a period of strong structural transformations for the US and world economies, which might somehow explain why.[45]

[44]There are many factors that might initiate the divergence between discount rate of households and banks: for example, banks have limited liability and are more exposed to risks.

[45]The two positivity conditions, (4) and (13), also hold with the value of $\theta = 0.96$, consistent with the standard calibration of the discount factor of household in macro models. In fact, in the context of our model banks and firms are owned by households which might imply that $\delta^f = \delta^b = \delta^h$. However, when going to the data we want to address the fact that households, firms, and banks do not need to discount at identical rates, thus we calibrate $\delta^f$, $\delta^b$, and $\delta^h$ differently.

We conclude that our model gives predictions that are most of the time consistent with the data, with only short periods —characterized by extraordinary circumstances in the 70s and 80s— in which some of the model's implications do not hold.

Table 2: **Optimal conditions of the model.**
Column *Mean* shows the mean value of the LHS of firms' and banks' optimality conditions, with the discount factors in *Discount factor*, according to US data. Column *Optimality condition* shows these conditions to hold.

| Discount Factor | Mean | Std | Optimality condition | Period |
|---|---|---|---|---|
| $\delta^f = 0.94$ | 0.95, LHS in firm's FOC (4) | 0.04 | $0.95 < 1$ | 1960-2014 |
| $\delta^f = 0.93$ | 0.93, LHS in firm's FOC (4) | 0.05 | $0.93 < 1$ | 1960-2014 |
| $\delta^b = 0.93$ | 0.03, RHS in bank's FOC (13) | 0.02 | $0.03 > 0$ | 1960-2014 |
| $\delta^b = 0.92$ | 0.03, RHS in bank's FOC (13) | 0.02 | $0.03 > 0$ | 1960-2014 |

Moreover, the feasibility condition in equation $(16bis)$ allows to plot the approximate evolution across time of $\phi_t$ —the share of deposits coming from factors revenues, dividends, and returns to previous deposits— using data on output, inflation, and aggregate deposits. A scatter plot of this graph —with monotonically increasing (during the sample period) $l_t$ in abscissae—[46] can be found in the last panel of Figure 3, showing that the empirical values for $\phi_t$ implied by the model are, as assumed, within $(0, 1)$.

Having shown the model to be consistent with the empirical evidence from the US, we use it in Section 3 to characterize the conditions guaranteeing that banking leads to a higher steady state output. To that end, we will consider the equilibria that would result from an absence of banks —in order to gauge the relevance of the latter— as well as the allocations that a utilitarian planner would choose instead. But first we characterize next the steady state equilibrium with banks.

### 2.4.2 Equilibrium steady state with banks

A steady state equilibrium with banks —hence for an economy without total factor productivity shocks—[47] is characterized by a collection of non-negative constant values for $c, k, h, d, l$, $m^h, m^f, \pi^f$, $\{d^b, q^b, l^b, \pi^b\}_{b\in B}$, $r, w, r^d, r^l$, and $\rho$, satisfying the equilibrium conditions in the previous sections and summarized in 2.4.

The next proposition follows from banks' optimizing behavior at a steady state equilibrium.

[46] Measured by the amount of loans issued by commercial banks.

[47] Specifically, such that $z_0 = 0$ and $\varepsilon_t = 0$ for all $t$.

**Proposition 2.8.** *At a steady state equilibrium with banks, it holds that*

1. *the nominal gross return to loans is at least that of deposits, i.e.*

$$r^d \leq r^l \tag{21}$$

2. *if the return to loans exceeds that of deposits, then the capital requirement constraint is binding for any bank b, i.e.*

$$\eta l^b = e^b \tag{22}$$

*Proof.* Straightforward from the non-negativity of the multiplier of the capital requirement condition (13) and its complementary slackness condition (12) respectively, at the steady state, in the previous characterization of banks' optimizing behavior in Proposition 2.4. □

In the absence of banks, firms cannot issue debt to advance evening revenue for same-day morning investments,[48] and the only possibility for firms of financing production factors is previous evening's sale proceeds. Households, in turn, also lose the possibility to save in bank deposits. Firms and households optimal choices are modified accordingly, and they therefore lead to a different equilibrium allocation.

It can be shown though that a banking steady state delivers a higher capital accumulation (and hence higher income, if not offset by labor) than without banks, if it is either deflationary or not too inflationary —in a context of low enough nominal lending rates. This requires, nonetheless, making the additional assumption of a relatively inelastic labor supply, which at the household level —i.e. at the intensive margin— is well supported by empirical evidence.[49] The result follows as a corollary from the characterization next.[50]

**Proposition 2.9.** *For a sufficiently inelastic labor supply, if the equilibrium steady state with banks is not too inflationary —i.e. either deflationary or, whenever $r^l < \frac{1}{\delta^f}$ holds, sufficiently mildly inflationary—[51] then its output is higher than without banks.*

[48]See footnote 13.

[49]See Fiorito and Zanella (2012) for an analysis of the intensive and extensive margins of the labor supply and their links.

[50]The proof is provided in Appendix A.4.

[51]That is to say, in a context of low lending rates.

While the previous results suffice to make the point of the benefits of bank intermediation —in terms of increased income, relative to the absence of banks— they actually do not rule out that the steady state banking equilibrium might still be inefficient. We therefore characterize next the allocation chosen by a utilitarian planner in order to see if the steady state that the latter would choose can be an equilibrium steady state with banks.

# 3 The social planner's solution

## 3.1 Planner allocations

We characterize the first-best allocations by solving the planner's problem next

$$\begin{aligned} \max_{0\leq k_{t+1},h_t} \mathbb{E}\sum_{t=1}^{+\infty}(\delta^h)^{t-1}\left[u(c_t)-v(h_t)\right] \\ c_t+k_{t+1}=e^{z_t}f(k_t+e^f,h_t) \end{aligned} \tag{23a}$$

where

$$z_t=\psi z_{t-1}+\varepsilon_t \tag{23b}$$

given $k_1>0$, $z_0$, and $\varepsilon_t\sim N(0,\sigma)$, from which the following necessary characterization of the planner's allocations follows.[52]

**Proposition 3.1.** *A planner's allocation is necessarily characterized by*

$$\begin{aligned} \mathbb{E}u'(c_t)&=\delta^h\mathbb{E}\left[u'(c_{t+1})\cdot e^{z_{t+1}}\right]f_k(k_{t+1}+e^f,h_{t+1}) \\ v'(h_t)&=\mathbb{E}\left[u'(c_t)\cdot e^{z_t}\right]f_h(k_t+e^f,h_t) \\ c_t+k_{t+1}&=e^{z_t}f(k_t+e^f,h_t) \end{aligned} \tag{24}$$

From the previous proposition follows straightforwardly the characterization next of a planner's steady state

$$\begin{aligned} \frac{1}{\delta^h}&=f_k(k+e^f,h) \\ \frac{v'(h)}{u'(c)}&=f_h(k+e^f,h) \\ c+k&=f(k+e^f,h) \end{aligned} \tag{25}$$

[52]The detailed derivation of the solution to the planner's problem can be found in Appendix B.1.

so that we can establish in the next section that a planner's steady state cannot be a steady state equilibrium with banks under a central banks rate policy only.

## 3.2 A planner's steady state is not an equilibrium steady state with banks

That a planner's steady state is not generically an equilibrium steady state with banks follows from the fact that, otherwise, it would have to be a solution to an overdetermined system of equations, as shown in Proposition 3.3 below. Its proof uses the Proposition 3.2 next —the proof of which is in turn provided in appendix— establishing that a market decentralization of a planner's steady state requires (interestingly enough) a zero nominal lending rate from banks to firms, as well as deflating prices.

**Proposition 3.2.** *A banking equilibrium decentralizing a planner's steady state necessarily satisfies*

*(i)* $r^l = 1$*, i.e. the nominal lending rate for bank loans to firms has to be zero*

*(ii)* $\rho = \delta^f$ *if* $r^l l \geq f(k + e^f, h)$*, i.e. prices have to deflate each period by the discount factor of firms, if the latter collateralize at least all current revenues.*[53]

In order to see why a planner's steady state allocation cannot be a laissez-faire steady state equilibrium allocation we just need to consider the implications of the previous Proposition 3.2 on the existence of a solution to the system of equations characterizing a steady state banking equilibrium, from which the result next follows.

**Proposition 3.3.** *For any generic economy,*[54] *a planner's steady state allocation is not an equilibrium steady state allocation.*

Indeed, it is straightforward to see that the system characterizing a steady state equilibrium with banks —satisfying the conditions $r^l = 1 = \delta^f \frac{1}{\rho}$ in Proposition 3.2, in order to decentralize the planner's steady state— consists of[55]

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[53]See in Appendix C a complete statement in Proposition C.1, including the case $\theta < 1$ in which firms do not collateralize the entirety of their current revenues. Given the empirical evidence about firms' borrowing levels exceeding their current output, we focus instead on the case $\theta \geq 1$ in the main body of the paper.

[54]In its fundamentals $u, v, f, \delta^h, \delta^f, \delta^b, e^f, e^b$.

[55]Along with the relevant inequalities, and after aggregation of the banking sector.

1. from households' optimizing

$$\begin{aligned} \frac{1}{\delta^h} &= r = \frac{1}{\phi} \cdot \frac{r^d}{\delta^f} \\ \frac{v'(h)}{u'(c)} &= w \end{aligned} \tag{26}$$

$$\begin{aligned} m^h &= rk + wh + \pi^f + \pi^b \\ c + k + \phi d &= m^h + \frac{r^d}{\delta^f} d \end{aligned} \tag{27}$$

2. from firms' optimizing

$$\frac{r}{f_k(k+e^f,h)} = 1 = \frac{w}{f_h(k+e^f,h)} \tag{28}$$

$$\begin{aligned} rk + wh + \pi^f + (\frac{1}{\delta^f})^2 l &= \frac{1}{\delta^f} m^f + l \\ m^f &= f(k+e^f,h) \end{aligned} \tag{29}$$

3. from banks' optimizing

$$\begin{aligned} r^d &= r^q \\ \pi^b + l - (\frac{1}{\delta^f})^2 l &= d - \frac{r^d}{\delta^f} d \\ \frac{1}{\rho} l + l &= e^b + d \end{aligned} \tag{30}$$

$$\left[1 + \delta^b \frac{1}{\delta^f}\right] \left(1 - r^d\right) \left(\eta l - e^b\right) = 0 \tag{31}$$

4. from feasibility

$$c + k = f(k+e^f,h) \tag{32}$$

in the non-negative variables $c, k, h, d, l$, $m^h, m^f, \pi^f$, $\pi^b$, $r, w$, $r^d$, and $\phi$, given a central bank policy rate $r^q$. This is a system of 14 independent equations in 13 non-negative variables. Should it have a solution for given primitives of the economy $u$, $v$, $f$, $\delta^h$, $\delta^f$, $\delta^b$, $e^f$, and $e^b$, then the latter would not be generic.

The obvious question after the previous result is whether, under some policy, a planner's steady state can be decentralized as a market equilibrium and, if that is the case, which is that policy.

### 3.3 Decentralization of planner allocations

As the next proposition establishes, any chance to attain through the market a planner's steady state will require bank intermediation. Indeed, no equilibrium without banks can deliver a planner's steady state, as the next proposition establishes.

**Proposition 3.4.** *For the economy without banks, a planner's steady state cannot be decentralized.*

*Proof.* According to the social planner's steady state characterization in (25), the marginal productivity of capital should match the reciprocal of households' discounting factor or, equivalently,

$$\frac{\frac{1}{\delta^h}}{f_k(k+e^f,h)} = 1 \tag{33}$$

while it is straightforward to show that, at an equilibrium steady state without banks, it holds —from the firms' FOCs and the feasibility of the allocation of ressources— that

$$\frac{r}{f_k(k+e^f,h)} = \delta^f \tag{34}$$

with, from the households' FOCs,

$$r = \frac{1}{\delta^h} \tag{35}$$

Decentralizing a planner's steady state would therefore require, without banks, that $\delta^f = 1$, which cannot be for the firms' problem to be well defined. □

On the contrary, with bank intermediation, the following proposition establishes the decentralizability of the planner's choice by means of an active policy on collateral requirements. Indeed, as established next, a central bank can set collateral requirements for firms' borrowing, such that households' and firms' first-order conditions imply those of the planner —while market clearing guarantees the feasibility of the allocation.

**Proposition 3.5.** *For the given economy, an equilibrium steady state with banks in which firms' borrowing is constrained by a collateral requirement implements the planner's steady state.*

*Proof.* The result follows from

(i) the decentralization of the planner's steady state [56] requires that $r^l = 1$ and

  (a) either[57] $\frac{\delta f}{\rho} = 1$

  (b) or[58] $\frac{\delta f}{\rho} = \frac{1-\theta}{\theta}$

(ii) the values for $\frac{\delta f}{\rho}$ in (a) and (b) above satisfy —at the steady state, i.e. with $r^l = 1$— the equation

$$\delta f \frac{1}{\rho_{t+1}} \mathbb{E} e^{z_t} + \theta_t \Big( 1 - \delta f \frac{r^l_t}{\rho_{t+1}} \cdot \delta f \frac{r^l_{t+1}}{\rho_{t+2}} \Big) = 1 \tag{36}$$

(iii) the condition (36) above along with the firms' and households' first-order conditions —(2) and (7)-(8) respectively— imply, at the steady state, the planner's first-order conditions in (25), while market clearing implies feasibility.

□

Now, since the market steady state depends on the gross nominal return to loans from the interbank market $r^q$ —matching the return on any loan from the central bank— one can ask what is the policy rate the central bank needs to set as nominal lending rate to banks in order to have a planner's steady state decentralized as a market steady state. The proposition next establishes that the policy rate needed is actually a *negative* nominal lending rate to banks, when firms can pledge as collateral future revenues on top of current ones, i.e. when $\theta > 1$ —the empirically relevant case according to data.

**Proposition 3.6.** *The central bank and interbank market nominal lending rate that decentralizes a planner's steady state is not positive (i.e.* $r^q \leq 1$*).*

*Moreover, it is negative (i.e.* $r^q < 1$*) whenever*

*(i) firms' pledgeable collateral can include future revenues (i.e.* $\theta \geq 1$*) and*

*(ii) households discount less (or not much more) than firms —specifically whenever* $\phi \frac{\delta^f}{\delta^h} < 1$

*Proof.* Since at a steady state equilibrium it holds that $r^d \leq r^l$ —condition (21) in Proposition 2.8— with $r^q = r^d$ —condition (11) of banks' optimizing at a steady state equilibrium— and $r^l = 1$ for the decentralization —result *(i)* in Proposition 3.3— then it needs

[56] In the absence, hence, of total factor productivity shocks, so that $z_0 = 0$ and $\varepsilon_t = 0$ for all $t$.
[57] If $\theta \geq 1$, see Proposition 3.3
[58] if $\theta < 1$, see the proof Proposition C.1 in Appendix C.

to hold that

$$r^q = r^d \leq r^l = 1 \tag{37}$$

Moreover, at a steady state, it follows from $r^d = r^q$ and consumers' first-order conditions in (26) that

$$\frac{1}{\delta^h} = \frac{1}{\phi} \cdot \frac{r^q}{\rho} \tag{38}$$

But since, from Proposition 3.3, at a banking equilibrium that decentralizes a planner's steady state it must hold that $\rho = \delta^f$ whenever $r^l l \geq f(k + e^f, h)$,[59] then it follows that

$$r^q = \phi \cdot \frac{\delta^f}{\delta^h} \tag{39}$$

so that if households discount not much more than firms —specifically, if $\phi \cdot \frac{\delta^f}{\delta^h} < 1$— then $r^q < 1$ necessarily holds. □

Note that if firms discount like households, then $r^q < 1$ holds from (39) straightforwardly —as so does the result of negative nominal rates on central bank loans to firms in order to decentralize a planner's steady state— following $\phi < 1$, i.e. as a result of banks' money creation by opening deposits accounts —unbacked by households deposits— to withdraw from for loans.

# 4 Conclusion

In this paper we have investigated whether negative *nominal* interest rates can be an equilibrium outcome. For that, we have developed a model economy where banks

[59] In case firms could only pledge a fraction of their current revenue so that $\theta < 1$ —a case that is *not* the empirically relevant one, according to the available data— then a banking equilibrium decentralizing a planner's steady state could have $\rho = \delta^f \frac{r^l l}{f(k+e^f,h) - r^l l}$ instead (see sProposition C.1 in Appendix C), from which it would follow that (39) would be instead

$$r^q = \phi \frac{\delta^f}{\delta^h} \cdot \frac{r^l l}{f(k + e^f, h) - r^l l}$$

but since $r^l = 1$ (for the decentralization, see Proposition 3.3), then if

$$\phi \frac{\delta^f}{\delta^h} \cdot \frac{l}{f(k + e^f, h) - l} < 1$$

then $r^q < 1$ follows.

play a crucial role for both production and consumption, by (i) taking deposits from households and extending loans to firms, (ii) allowing to increase funds available in the capital market by creating new funds as additional deposits, and (iii) transforming the maturity of claims on funds. We then showed that key implications of the model are consistent with empirical evidence from the US.

We characterized then the planner's solution to show that the economy with banks fails to achieve a first-best allocation if the central bank focuses on a rate policy only. Indeed, we show that the implementation of a steady state first-best allocation requires a policy of collateral requirements, a result reminiscent of that of Geanakoplos (2010). That the result does not depend on the business cycle aspects of the framework in which it is obtained shows that the scope of the point made in Geanakoplos (2010) goes beyond the leverage cycle issues addressed there.

Finally, we showed that the decentralization of a planner's steady state requires a zero nominal lending rate for loans from banks to firms, and negative nominal lending rates for central bank loans to banks —and hence for household deposits in banks. The driver of these results has been identified to be banks' capacity for money creation by opening deposits accounts to withdraw from for loans, that are unbacked by households deposits. This answers our main research question and stated goal, as we find that negative nominal interest rates are indeed consistent with a general equilibrium framework. Moreover, this result is in agreement with the recently observed behavior of interest rates: nominal lending rates have been persistently low while deposit nominal rates have been effectively negative —once considered net of fixed service fees— for most of the 2010s and the early 2020s in the wake of the financial crisis in the previous decade, before the impact on prices of extraordinary circumstances (a world pandemic and a globally destabilizing war) made their presence felt.

Our results are also in line with the theory of secular stagnation, according to which zero or low nominal interest rates are a new normal outside of periods impacted by extraordinary big shocks (pandemics, major wars,...) rather than an anomaly to be actively fought. Indeed, we show that bank intermediation —due to their ability to circumvent households' and firms' time mismatch between their needs and availability of funds, as well as to create deposits and thus leverage up funds in the capital market— allows the economy to function more efficiently in an environment of low and even negative nominal interest rates.

We additionally show that regulating the amount of collateral pledged by firms to banks plays a key role in implementing first-best allocations, which provides a rationale for similar policies implemented recently —e.g. the introduction by Sweden of a reform bounding the collateral values for firms (Cerqueiro et al., 2020).

It goes without saying that we have had to let many interesting aspects of the problem out of the picture, while making some modeling choices for the sake of simplicity. As a result, a few shortcomings need to be addressed in future work and extensions. Firstly, there is no actual borrowing from the central bank at equilibrium, all banks' funding needs, beyond deposits, being satisfied by the interbank market. The rate policy of the central bank works by the merely announced willingness of the later to lend unlimited amounts at the policy rate, which immediately pins down the interbank market nominal rate, without any bank having to actually borrow from the central bank. The reason to assume so was to avoid, in the context of a model written in real terms, having a central bank pumping into the economy (or withdrawing from it) real resources. At the same time, there is no room in the model for banks to deposit at the central bank either, while we do see banks parking significant balances at, for instance, the European Central Bank —even when the later's deposit facility rate is negative.[60] The two things put together strip the central bank in the model of any role other than setting the interbank market rate, which falls short of what we see in reality. As for the central bank policy needed, we don't see much of a role in the model for the capital requirement —probably because at a central bank negative lending rate banks cannot have a problem securing funds— in contrast with their importance for the solvency of banks in reality. Obviously, while we are not happy about these or other shortcomings of the model, we do think the results obtained justify our choices. Future research will address them all nonetheless.

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[60] A few months before the end of the ECB's negative rates policy, in Oct 29, 2021, for instance, 805.575 billion EUR at -0.05%.

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# Appendix A

## A.1 Firms' optimization

Firms solve the problem (1) given sequences $r_t, w_t, r^l_{t-2}, \rho_{t-1} > 0$ for all $t \in \mathbb{N}$, as well as $e^f, \theta > 0$, and $m^f_0 = l_0 = l_{-1} = 0$, with FOCs with respect to $k_t$, $h_t$, and $l_t$ respectively

$$\begin{aligned}(\delta^f)^t \frac{1}{\rho_{t+1}} f_k(k_t + e^f, h_t)\mathbb{E}e^{z_t} - (\delta^f)^{t-1} r_t + \theta f_k(k_t + e^f, h_t)\mathbb{E}\lambda_t e^{z_t} &= 0 \\ (\delta^f)^t \frac{1}{\rho_{t+1}} f_h(k_t + e^f, h_t)\mathbb{E}e^{z_t} - (\delta^f)^{t-1} w_t + \theta f_h(k_t + e^f, h_t)\mathbb{E}\lambda_t e^{z_t} &= 0 \\ (\delta^f)^{t-1} - (\delta^f)^{t+1} \frac{r^l_{t+1}}{\rho_{t+2}} \frac{r^l_t}{\rho_{t+1}} - r^l_t \mathbb{E}\lambda_t &= 0\end{aligned} \tag{A1}$$

along with the complementary slackness condition of the borrowing constraint

$$\mathbb{E}\lambda_t \left[ r^l_t l_t - e^{z_t} f(k_t + e^f, h_t)\theta \right] = 0 \tag{A2}$$

where $\lambda_t \geq 0$ is the multiplier of the period-$t$ borrowing constraint contingent to the history of productivity shocks up to then —the two other constraints being substituted into the objective accordingly— and from which (2) follows. Condition (4) follows from the non-negativity of the history-contingent multipliers.

## A.2 Households' optimization

Households solve the problem (6), or equivalently,

$$\max_{0 \leq k_{t+1}, h_t, d_t} \mathbb{E} \sum_{t=1}^{+\infty} (\delta^h)^{t-1} \left[ u\left( r_t k_t + w_t h_t - k_{t+1} + \pi^f_t + \sum_b \pi^b_t + \frac{r^d_{t-1}}{\rho_t} d_{t-1} - \phi_t d_t \right) - v(h_t) \right] \tag{A3}$$

given $r_t, w_t, r^d_{t-1}, \rho_t > 0$, and $\pi^f_t, \pi^b_t \geq 0$, for all $t$ and all $b$, as well as $k_1 \geq 0, d_0 = 0$, with FOCs with respect to $k_{t+1}$, $h_t$, and $d_t$ respectively

$$\begin{aligned}\delta^h r_{t+1} \mathbb{E}u'(c_{t+1}) - \mathbb{E}u'(c_t) &= 0 \\ w_t \mathbb{E}u'(c_t) - v'(h_t) &= 0 \\ \delta^h \frac{r^d_t}{\rho_{t+1}} \mathbb{E}u'(c_{t+1}) - \phi_t \mathbb{E}u'(c_t) &= 0\end{aligned} \tag{A4}$$

—where consumptions are determined by the binding budget constraints, and are therefore subject to the shocks on productivity via firms' profits— or, equivalently, (7) and (8).

## A.3 Banks' optimization

Banks solve the problem (9) given $r^l_{t-2}, r^d_{t-1}, r^q_{t-1}, \rho_{t-1} > 0$ for all $t$, as well as $e^b, \eta > 0$ and $l^b_0 = l^b_{-1} = d^b_0 = 0$, or equivalently

$$\max_{q^b_t, 0 \le l^b_t, d^b_t} \quad \sum_{t=1}^{+\infty} (\delta^b)^{t-1} \left[ d^b_t - \frac{r^d_{t-1}}{\rho_t} d^b_{t-1} + q^b_t - \frac{r^q_{t-1}}{\rho_t} q^b_{t-1} - \left( l^b_t - \frac{r^l_{t-1}}{\rho_t} \frac{r^l_{t-2}}{\rho_{t-1}} l^b_{t-2} \right) \right]$$
$$\frac{r^l_{t-1}}{\rho_t} l^b_{t-1} + l^b_t = e^b + d^b_t + q^b_t \qquad (A5)$$
$$\eta l^b_t \le e^b$$

so that FOCs are, with respect to $q^b_t$, $l^b_t$, and $d^b_t$ respectively,

$$(\delta^b)^{t-1} - (\delta^b)^t \frac{r^q_t}{\rho_{t+1}} + \lambda_t = 0$$
$$-(\delta^b)^{t-1} + (\delta^b)^{t+1} \frac{r^l_{t+1}}{\rho_{t+2}} \frac{r^l_t}{\rho_{t+1}} - \lambda_t - \lambda_{t+1} \frac{r^l_t}{\rho_{t+1}} - \mu_t = 0 \qquad (A6)$$
$$(\delta^b)^{t-1} - (\delta^b)^t \frac{r^d_t}{\rho_{t+1}} + \lambda_t = 0$$

from where (11), (12), and (11) follow.

## A.4 Proof of Proposition 2.9

*Proof.* A steady state equilibrium without banks is characterized by a collection of non-negative $c, k, h, m^h, m^f, \pi^f, \rho, r, w$ such that

1. firms optimize, i.e.

$$\frac{r}{f_k(k+e^f, h)} = \frac{\delta^f}{\rho} = \frac{w}{f_h(k+e^f, h)} \qquad (A7)$$

   hold along with the budget constraints

2. households optimize, i.e.

$$\frac{1}{\delta^h} = r$$
$$\frac{v'(h)}{u'(c)} = w \qquad (A8)$$

hold along with the budget constraints, and

3. markets clear, i.e.

$$c + k = f(k + e^f, h) \qquad (A9)$$

The feasibility condition in (A9) is equivalent[61] to $1 - \frac{1}{\rho} = 0$, i.e. to

$$\rho = 1 \qquad (A10)$$

At a steady state, in firms' FOC on the ratio of capital return to productivity —both with banks in (28) and without banks in (A7)— the return to capital in the numerator is pinned down (by housedols' FOCs) to be $\frac{1}{\delta^h}$.

Thus, comparing (2) at a steady state and (A7), whenever labor supply is inelastic —or, by continuity for an inelastic enough labor supply— the steady state capital with banks is higher than without if, and only if,

$$\frac{1}{r^l}\Big[\delta^f \frac{r^l}{\rho} + \frac{r^l l}{f(k+e^f,h)}\Big(1 - \big(\delta^f \frac{r^l}{\rho}\big)^2\Big)\Big] > \delta^f \, \frac{1}{\rho'} \qquad (A11)$$

where $\rho$ stands for the inflation factor at the steady state with banks, and $\rho'$ at the steady state without banks —which is 1, according to (A10).

Condition (A11) can easily be transformed into

$$(1-\rho)\delta^f \frac{r^l}{\rho} + \frac{r^l l}{f(k+e^f,h)}\Big(1 - \big(\delta^f \frac{r^l}{\rho}\big)^2\Big) \;\; > 0 \qquad (A12)$$

or, equivalently, into the condition that $\delta^f \frac{r^l}{\rho}$ is smaller than the biggest root of the

[61] From $(16bis)$ at the steady state —i.e. with $z_0 = 0$ and $\varepsilon_t = 0$ for all $t$— without banks, which implies a RHS equal to 0.

quadratic polynomial in $\delta^f \frac{r^l}{\rho}$ in the LHS in (A12),[62] i.e.

$$\delta^f \frac{r^l}{\rho} < \sqrt{1 + \left[\frac{1-\rho}{2\frac{r^l l}{f(k+e^f,h)}}\right]^2} + \frac{1-\rho}{2\frac{r^l l}{f(k+e^f,h)}} \tag{A13}$$

Now, since at the equilibrium steady state with banks it necessarily holds (4) —from the positivity of the multiplier of the borrowing constraint— i.e.

$$\delta^f \frac{r^l}{\rho} \leq 1 \tag{A14}$$

then the condition next suffices to imply (A13), and hence (A12),

$$1 < \sqrt{1 + \left[\frac{1-\rho}{2\frac{r^l l}{f(k+e^f,h)}}\right]^2} + \frac{1-\rho}{2\frac{r^l l}{f(k+e^f,h)}} \tag{A15}$$

which holds if $\rho < 1$. Note that, as a consequence, (A15) holds too for $\rho \geq 1$ but sufficiently close to 1 if $\delta^f r^l < 1$.

Therefore, at a deflationary steady state equilibrium with banks —and even at a sufficiently mildly inflationary steady state, for low enough nominal lending rates— the output is higher than without banks in an economy with (sufficiently) inelastic labor supply. □

# Appendix B

## B.1 The social planner's problem

The planner solves the problem (23), or equivalently

$$\max_{0 \leq k_{t+1}, h_t} \mathbb{E} \sum_{t=1}^{+\infty} (\delta^h)^{t-1} \left[u(e^{z_t} f(k_t + e^f, h_t) - k_{t+1}) - v(h_t)\right] \tag{B1}$$

[62] And bigger than the smallest root, but since the latter is negative, this is satisfied.

with FOCs

$$\mathbb{E}u'(e^{z_t}f(k_t+e^f,h_t)-k_{t+1}) = \delta^h\mathbb{E}\Big[u'(e^{z_{t+1}}f(k_{t+1}+e^f,h_{t+1})-k_{t+2})\cdot e^{z_{t+1}}\Big]f_k(k_{t+1}+e^f,h_{t+1})$$
$$v'(h_t) = \mathbb{E}\Big[u'(e^{z_t}f(k_t+e^f,h_t)-k_{t+1})\cdot e^{z_t}\Big]f_h(k_t+e^f,h_t) \tag{B2}$$

which, along with the feasibility constraint, deliver (24).

# Appendix C

## C.1 A planner's steady state is not a banking equilibrium steady state

**Proposition C.1.** *A banking equilibrium decentralizing the planner's steady state, necessarily satisfies*

1. $r^l = 1$, *i.e. the nominal lending rate for bank loans to firms has to be zero*
2. $\rho = \delta^f$ *if* $r^l l \geq f(k+e^f,h)$, *i.e. prices have to deflate each period by the discount factor of firms, if the latter collateralize at least all current revenues*

   $\rho = \delta^f$ *or* $\rho = \delta^f \frac{r^l l}{f(k+e^f,h)-r^l l}$ *if* $r^l l < f(k+e^f,h)$, *i.e. prices have to either deflate each period by the discount factor of firms, or prices evolve by a factor increasing unboundedly in the level of collateralization* $\frac{r^l l}{f(k+e^f,h)}$ *if firms can collateralize only a fraction of the current revenues.*

*Proof.* 1. At a steady state equilibrium with banks, the ratio of capital return to productivity needs to satisfy, from firms' FOC in (2),

$$\frac{r}{f_k(k+e^f,h)} = \frac{1}{r^l}\Big[\delta^f\frac{r^l}{\rho} + A\Big(1-\big(\delta^f\frac{r^l}{\rho}\big)^2\Big)\Big] \tag{C1}$$

with

$$r = \frac{1}{\delta^h} \tag{C2}$$

from households' FOC in (7) —where $A = \frac{r^l l}{f(k+e^f,h)}$, for ease of notation— and for this ratio to decentralize the planner's it needs to be equal to 1 too, since at the latter

$$\frac{\frac{1}{\delta^h}}{f_k(k+e^f,h)} = 1 \tag{C3}$$

from (25), so that at a steady state with banks decentralizing the planner's it must hold

$$\frac{1}{r^l}\Big[\delta^f\frac{r^l}{\rho}+A\Big(1-\big(\delta^f\frac{r^l}{\rho}\big)^2\Big)\Big]=1 \tag{C4}$$

Therefore, the (firm-)discounted real return on loans $\delta^f\frac{r^l}{\rho}$ at such a steady state is the biggest solution[63] —after rearranging terms— to a quadratic equation in $\delta^f\frac{r^l}{\rho}$, i.e.

$$\delta^f\frac{r^l}{\rho}=\sqrt{1+\Big[\frac{1}{2A}\cdot\frac{1-\delta^f\frac{1}{\rho}}{\delta^f\frac{1}{\rho}}\Big]^2}-\frac{1}{2A}\cdot\frac{1-\delta^f\frac{1}{\rho}}{\delta^f\frac{1}{\rho}} \tag{C5}$$

Thus, the (firm-)discounted real return on loans $\delta^f\frac{r^l}{\rho}$ at a steady state equilibrium decentralizing the planner's is above (resp. below) 1 —i.e. the (firm-)discounted real interest (lending) rate is positive (resp. negative)— if, and only if,

$$g(x)\equiv\sqrt{1+x^2}>(<)\,1+x\equiv f(x) \tag{C6}$$

where $x=\frac{1}{2A}\cdot\frac{1-\delta^f\frac{1}{\rho}}{\delta^f\frac{1}{\rho}}$. Since $g(x)=f(x)$ if, and only if, $x=0$, moreover $g'(0)=0<1=f'(0)$, $0<g'(x)<f'(x)=1$ for all $x>0$, and $g$ is strictly convex, then it follows that $g(x)>(<)\,f(x)$ if, and only if, $x<(>)\,0$, that is to say if, and only if,

$$\frac{1}{2A}\cdot\frac{1-\delta^f\frac{1}{\rho}}{\delta^f\frac{1}{\rho}}<(>)\;0 \tag{C7}$$

In other words, a $\delta^f\frac{r^l}{\rho}$ that decentralizes the planner's steady state satisfies $\delta^f\frac{r^l}{\rho}>(<)\,1$ if, and only if, $\delta^f\frac{1}{\rho}>(<)\,1$, i.e.

$$\delta^f\frac{r^l}{\rho}>(<)1\Leftrightarrow\delta^f\frac{1}{\rho}>(<)1 \tag{C8}$$

a statement that holds true only if $r^l=1$. As a consequence, a $\delta^f\frac{r^l}{\rho}$ that decentralizes the planner's steady state has to be such that $r^l=1$.

2. Since at a steady state banking equilibrium decentralizing the planner's it must

[63] The smallest is negative and hence cannot be a return factor.

hold

$$\frac{1}{r^l}\Big[\delta^f \frac{r^l}{\rho} + A\Big(1 - \big(\delta^f \frac{r^l}{\rho}\big)^2\Big)\Big] = 1 \tag{C9}$$

with $r^l = 1$, i.e.

$$A\big(\delta^f \frac{1}{\rho}\big)^2 - \delta^f \frac{1}{\rho} + 1 - A = 0 \tag{C10}$$

it follows that

$$\delta^f \frac{1}{\rho} = \begin{cases} 1 \\ \frac{1-A}{A} & \text{only if } A < 1 \end{cases} \tag{C11}$$

so that

$$\rho = \begin{cases} \delta^f \\ \delta^f \frac{r^l l}{f(k+e^f,h) - r^l l} & \text{only if } r^l l < f(k+e^f,h) \end{cases} \tag{C12}$$

□