Updating Under Imprecise Information

Yi-Hsuan Lin and Fernando Payró*

January, 2024

Abstract

This paper models an agent that ranks actions with uncertain payoffs after observing a signal that could have been generated by multiple objective information structures. Under the assumption that the agent's preferences conform to the multiple priors model (Gilboa and Schmeidler (1989)), we show that a simple behavioral axiom characterizes a generalization of Bayesian updating. Our axiom requires that whenever all possible sources of information agree that it is more "likely" for an action with uncertain payoffs to be better than one with certain payoffs, the agent prefers the former. We also provide axiomatizations for various special cases. Additionally, we explore a scenario where the informational content of a signal is purely subjective. We analyze the existence of a subjective set of information structures under full Bayesian updating for two extreme cases: (i) No ex-ante state ambiguity, and (ii) No signal imprecision.

*Lin: Institute of Economics, Academia Sinica, 128 Academia Road, Section 2, Nankang, Taipei 11529, Taiwan. Email: yihsuanl@econ.sinica.edu.tw. Payró: Department of Economics and Economic History, Universitat Autonoma de Barcelona, Barcelona School of Economics and the Center for the Study of Organizations and Decisions in Economics, Campus de la UAB, Edifici B, Bellaterra, 08193. Email: fernando.payro@uab.cat. The authors would like to thank seminar audiences at Boston University and Academia Sinica, and participants of Hitotsubashi's annual Decision Theory Conference 2023, ASSET 2023, SAe 2023, and TUS 2023 for helpful feedback. Payró gratefully acknowledges financial support from the Ministerio de Economía y Competitividad and Feder (PGC2018-094348-B-I00 and PID2020-116771GB-I0) and the Spanish Agencia Estatal de Investigación (AEI), through the Severo Ochoa Programme for Centres of Excellence in R&D (CEX2019-000915-S). Lin gratefully acknowledges financial support from Academia Sinica and Taiwan's National Science and Technology Council. All errors are our own.

1 Introduction

Individuals often acquire information before making a choice. However, in many situations, they are not able to completely pin down the source of the acquired information. Since the same information can have different meanings depending on its source, they may have a hard time processing it if they are not sure which one is the true source. The goal of this paper is to provide a theory of how agents process such information.

To illustrate, we provide an example inspired by the early research conducted during the COVID-19 pandemic.[1] Consider a policy-maker that must choose to adopt a costly novel security measure. The policy-maker is aware that a study has shown the security measure reduces the risk of an accident. However, she is unsure about the quality of the data employed in the study. If the study was done properly, then the results suggest she should in fact adopt the measure. However, if the quality is poor, then the results are meaningless. Indeed, if the sample employed in a study is too small, then it is equally likely to get results that suggest the measure is effective than results that say it has no effect. How should the policy-maker process such information?

We take as primitives and outcome space X, an objective state space Ω, an objective signal space S, and a set of information structures $\mathcal{L} \subseteq \{\ell : \Omega \to \Delta(S)\}$, where $\Delta(S)$ is the set of probability measures over S. We refer to $\mathcal{L}$ as an imprecise information structure. The idea behind our definition is that $\mathcal{L}$ is the set of identifiable sources of information. The agent is assumed to have ex-ante preference $\succeq_0$ and signal-conditional preferences $(\succeq_s)_{s\in S}$ over actions that have uncertain payoffs. We interpret $\succeq_s$ to be the agent's preferences conditional on observing signal s generated by some $\ell \in \mathcal{L}$. We assume that ex-ante $\succeq_0$ and ex-post $\succeq_s$ conforms to Gilboa and Schmeidler (1989) maxmin model (MEU).

Our theory is built on the premise that if all identifiable information structures recommend choosing an action with uncertain payoffs over one with deterministic payoffs, then the agent should select the former. Imposing this condition as an axiom yields a representation (Theorem 2.1) and consequently, an updating rule. For any action $f : \Omega \to X$, the utility of f after observing s is given by:

$$U_s(f) = \min_{q\in\rho(\mathcal{M}_0,\mathcal{L},s)} \int u(f)dq$$

Here, $\mathcal{M}_0$ represents the set of priors, and $\rho(\mathcal{M}_0, \mathcal{L}, s)$ is a subset of the convex hull of all posteriors generated by "point-wise" Bayesian updating. We term this

[1] During the early days of the pandemic, over 66 clinical prediction models were proposed in peer-reviewed settings to study the effect of a mask mandate. However, it was later established that all suffered a high risk of bias due to concerns surrounding the data quality, statistical analysis, and reporting. See Collins et al. (2020) for a discussion.

updating rule the Generalized Bayesian Updating (GBU) rule.

In our context, signals are not considered payoff-relevant, meaning that the payoff of actions does not depend on them. This departure from the typical axiomatic literature on updating, where signals are often modeled as subsets of Ω, is a distinctive feature of our framework. Unlike the conventional approach, our setting does not assume signals to be subsets of Ω. This departure is crucial for our purposes as it allows us to disentangle the agent's attitude toward imprecise information from her attitude toward uncertainty. If signals were modeled as subsets of Ω, this separation would not be feasible. Additionally, our approach facilitates the testing of our axioms in a laboratory setting, as our framework serves as a theoretical representation of experiments conducted in Epstein and Halevy (2022) and Shishkin et al. (2021).

Our updating rule generalizes both *Full Bayesian updating* (FBU) and *Maximum Likelihood updating* (MLU), both of which are widely popular. However, the conceptual rationale behind why an agent might adopt either of these rules is not entirely clear. FBU allows the agent to have a subjective view of the uncertainty about the state space but not about the source of information. On the other hand, MLU utilizes the realized signal to *jointly* discriminate among priors and information structures, evaluating each information structure using the prior that maximizes the likelihood of the observed signal.

A more conservative or intermediate approach would involve evaluating each information structure using *all* priors. In other words, an agent may prefer considering an information structure that has a reasonable likelihood according to all priors rather than one that has maximal likelihood according to a single prior and minimal likelihood according to another. We establish axiomatic foundations for such a rule in Theorem 4.1.

The psychology literature has consistently demonstrated that individuals, when updating information, can exhibit susceptibility to certain biases. A prevalent finding is the tendency to selectively consider information structures that align with their pre-existing beliefs, a phenomenon commonly referred to as confirmation bias.[2] In our framework, given a signal, the probability that the signal originated in each state induces a likelihood ranking over states. If the agent is able to articulate a likelihood ranking over states and is prone to confirmation bias, she may limit her consideration to information structures that align with her pre-existing likelihood ranking. GBU allows for such updating, its axiomatic foundations are described in Theorem 4.2.

Our analysis hinges on the assumption that imprecise information is objective. However, in numerous situations, information may be private or lack a reliable description. In such cases, assuming observability of $\mathcal{L}$ is not appropriate, and

[2]See Rabin and Schrag (1999) for a review of evidence in psychology.

its nature should be inferred from behavior. We demonstrate that under FBU, $\mathcal{L}$ can be recovered in scenarios where there is a single prior belief or when $\mathcal{L}$ is a singleton.

The paper proceeds as follows: The introduction concludes with a literature review. Axioms and the implied updating rule are described in Sections 2.2 and 2.3 respectively. Section 3 contains a detailed discussion of the properties of the model. Special cases of the GBU rule are investigated in Section 4. Section 5 delves into the consideration of subjective information structures. Concluding remarks are presented in Section 6. All proofs are provided in Appendix A.

1.1 Related Literature

In recent years, there has been a growing interest in decision-making under uncertainty, particularly in the context of incorporating objective but imprecise information. Our work aligns closely with several recent papers, including Gul and Pesendorfer (2021), Cheng (2022), Tang (2022), and Kovach (2023).

Gul and Pesendorfer (2021) presents a theory of updating for the Choquet Expected Utility model (Schmeidler (1989)). On the other hand, Cheng (2022), Tang (2022), and Kovach (2023) offer distinct axiomatic updating rules for the Maxmin Expected Utility model. Notably, the primary distinctions between our work and theirs lie in the framework and the nature of the information considered.

In the frameworks of Cheng (2022), Tang (2022), and Kovach (2023), information is considered payoff-relevant, and the information itself is precise in the sense that it is an event. In contrast, our work focuses on imprecise information that is not necessarily payoff-relevant. The motivation for their work stems from the descriptive shortcomings of the Maximum Likelihood Updating and Full Bayesian updating rules, axiomatized by Gilboa and Schmeidler (1993) and Pacheco Pires (2002), respectively

Dominiak et al. (2021) model imprecise information as sets of probability measures that contain the "true" distribution. They provide an updating rule for the Subjective Expected Utility model that selects the posterior that minimizes the distance between her prior and the set she is provided with. Their updating rule has a somewhat similar flavor to ours due to the fact that they allow the distance to be subjective.

Jaffray (1989), Ahn (2008), Dumav and Stinchcombe (2013), Olszewski (2007), Gajdos et al. (2008), and Riedel et al. (2018) consider settings in which the objects of choice are "merged" with imprecise information. More specifically, Jaffray considers a preference over belief functions. Gajdos et al. (2008) and Riedel et al. (2018) assume the agent can choose the set of data-generating processes that contains the true law. The rest of the authors study preferences over sets of lotteries. All of these assumptions imply that imprecise information is payoff-relevant.

Epstein and Schneider (2007,8,10) provide a non-axiomatic updating rule for imprecise information (therein referred to as ambiguous signals). They introduce a thought experiment that highlights its importance. In a dynamic setting, they provide conditions under which their updating rule delivers convergence in beliefs after repeated sampling. Reshidi et al. (2022) further investigate when beliefs converge under a more general data-generating process. Like Epstein and Schneider (2007), Lanzani (2023) also studies a learning problem. However, unlike Epstein and Schneider, he assumes robust control preferences. He shows how a misspecification concern can lead to different preferences under uncertainty arising in the limit.

Building on Epstein and Schneider's thought experiment, Epstein and Halevy (2022) provides a definition of aversion towards signal ambiguity and tests it in an experimental setting. The key difference between our work and theirs is that they study the attitude towards the information as opposed to *how* to process it, which is our focus.

Finally, outside the updating literature but within the imprecise information literature, Wang et al. (2023) axiomatize a selection criterion for imprecise information. They provide conditions under which a choice function over "theories" always selects the ones that pass a likelihood ratio test.

2 General Model

2.1 Preliminary Definitions

We consider a set Ω of states of the world, a set S of signals, and a set X of consequences. We assume Ω and S are finite and X is a compact and convex subset of a linear space. This is the case of Anscombe and Aumann (1963) where X is the set of all lotteries over some finite set of prizes. Let n denote the cardinality of Ω.

An act is a function $f : \Omega \to X$. The set of all acts is denoted by $\mathcal{F}$. We write x for the constant act f such that $f(\omega) = x$ for all $\omega \in \Omega$. Given $f, g \in \mathcal{F}$ and $\alpha \in [0, 1]$, we denote by $\alpha f + (1-\alpha)g$ the act in $\mathcal{F}$ that takes value $\alpha f(\omega)+(1-\alpha)g(\omega)$ in state ω.

An *information structure* is a function $\ell : \Omega \to \Delta(S)$ where $\Delta(S)$ is the set of all probability measures over S. We only consider information structures that satisfy the following property:

$$\forall\ s \in S, \text{ there exists } \omega \in \Omega \text{ such that } \ell(s|\omega) > 0.$$

As it will become clear later, this assumption is made purely for exposition purposes. An *imprecise information structure* is a finite set of such information structures. If the set only contains a single information structure, then we refer to it as a *precise information structure*. Generic imprecise information structures are denoted by $\mathcal{L}$.

$\Delta(\Omega)$ denotes the set of all probability measures on Ω. Given a probability measure $q \in \Delta(\Omega)$, an information structure ℓ, and a signal s, we denote by $BU(q, \ell, s)$ the probability measure given by Bayesian updating:

$$BU(q, \ell, s)(\omega) = \frac{\ell(s|\omega)q(\omega)}{\sum_{\omega} \ell(s|\omega)q(\omega)}$$

whenever it exists. Furthermore, for any set of measures $\mathcal{M} \subseteq \Delta(\Omega)$, imprecise information structure $\mathcal{L}$, and signal $s \in S$, $BU(\mathcal{M}, \mathcal{L}, s)$ denotes the set of all posteriors generated by *point-wise Bayesian updating*:

$$BU(\mathcal{M}, \mathcal{L}, s) = \{BU(q, \ell, s) \mid q \in \mathcal{M}, \ell \in \mathcal{L}\}.$$

Finally, for any set $C \subseteq \Delta(\Omega)$, $ch(C)$ denotes its convex hull.

2.2 Axioms and Representation

Our primitive is a family of preferences over acts $(\succeq_0, (\succeq_s)_{s\in S})$ and an imprecise information structure $\mathcal{L}$. We impose two axioms on $(\succeq_0, (\succeq_s)_{s\in S})$ of which the first is that the preferences admit a MEU representation.

A utility function $U : \mathcal{F} \to \mathbb{R}$ is *MEU* if there exists an affine function $u : X \to \mathbb{R}$ and a closed and convex set of probability measures $\mathcal{M}$ over Ω such that

$$U(f) = \min_{q\in\mathcal{M}} \int_{\Omega} u(f)dq. \tag{1}$$

Say that $(\mathcal{M}, u)$ represents $\succeq$ if the utility function given by (1) represents $\succeq$.

MEU Utility $\succeq_s$ admits representation by $(\mathcal{M}_s, u)$ for all $s \in S\cup\{0\}$. Moreover, each $q_0 \in \mathcal{M}_0$ has full support.

This axiom is not stated in terms of behavior which is presumably the only observable. However, its behavioral foundations are widely known. It also imposes that all priors have full support; our model has nothing to say about updating zero-probability events.

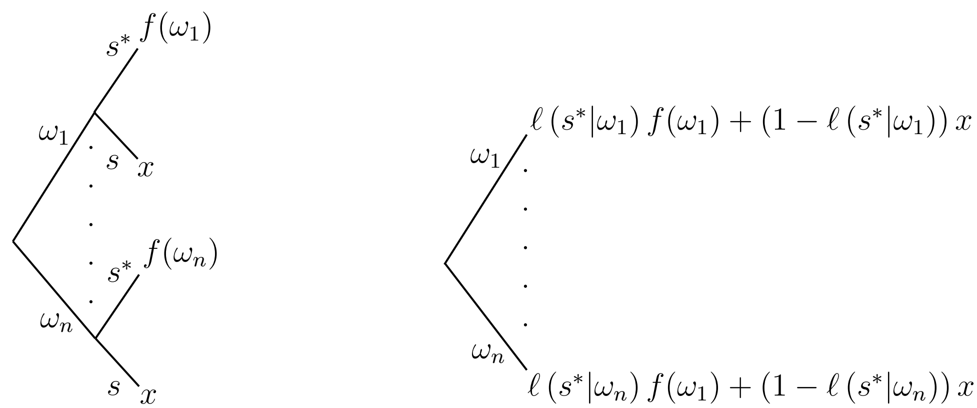
Our main axiom is based on the idea that if for a given signal all the identifiable information structures agree that it is more likely that an act is better than a constant act, then the agent should choose the act. We begin by demonstrating that our primitives are sufficiently rich to identify this condition if the agent satisfies *Reduction*: she is indifferent between a two-stage lottery and its equivalent one-stage counterpart.

Consider an act f and a constant act x, along with an information structure $\ell \in \mathcal{L}$. Extend the ex-ante preference to acts in which the signals are payoff-relevant: $\{F : \Omega \times S \to X\}$. Suppose the agent is told that ℓ is the true information structure and is asked to choose between a constant act x and a *signal act*:

$$F(\omega, s) = \begin{cases} f(\omega) & \text{if } s = s^* \\ x & \text{if } s \neq s^* \end{cases}.$$

Figure 1(a) visually represents this signal act. Notably, given a state of the world, there is no ambiguity about the probability of observing s—it is precisely $\ell(\cdot|\omega) \in \Delta(S)$. As both acts yield identical payoffs when ℓ generates $s \neq s^*$, the agent only needs to compare them under the assumption that s^* occurred. Consequently, she would prefer F to x if she believes that, conditional on observing s^*, she would prefer f to x.

For a given state ω, the signal act F yields the same expected payoffs as a lottery that pays $f(\omega)$ with probability $\ell(s^*|\omega)$ and x with probability $1 - \ell(s^*|\omega)$. Therefore, under Reduction, she is indifferent between F and an act that pays $\ell(s^*|\omega)f(\omega) + (1 - \ell(s^*|\omega))x$ for each state ω. Figure 1(b) illustrates such act.



(a) Signal Act (b) AA Act

Figure 1

Thus,

$$F \succeq_0 x \iff \begin{bmatrix} \ell(s^*|\omega_1)f(\omega_1) + (1 - \ell(s^*|\omega_1))x \\ \vdots \\ \ell(s^*|\omega_n)f(\omega_n) + (1 - \ell(s^*|\omega_n))x \end{bmatrix} \succeq_0 x$$

which means we can identify when the agent expects to prefer f over x after observing s if she knew ℓ is the true source of information.

To introduce our main axiom, we establish some notation. For each $f, x \in \mathcal{F}$, $\ell \in \mathcal{L}$, and $s \in S$, let $f_x^{\ell,s}$ denote the act that takes the value $\ell(s|\omega)f(\omega) + (1 - \ell(s|\omega))x$ in state ω.

Total Information Agreement For any $f, x \in \mathcal{F}$ and $s \in S$,

$$f_x^{\ell,s} \succeq_0 x \text{ for all } \ell \in \mathcal{L} \implies f \succeq_s x.$$

Given our discussion, Total Information Agreement asserts that if, under all information structures, the act f is deemed better than the constant act x after observing signal s, then the agent should prefer f over x after observing s.

2.3 Representation: Generalized Bayesian Updating

Our first result is an axiomatization of the Generalized Bayesian Updating described in the introduction.

Theorem 2.1. *Let $(\succeq_0, (\succeq_s)_{s \in S})$ be a family of preferences over $\mathcal{F}$ that satisfies MEU Utility and $\mathcal{L}$ an imprecise information structure. Then, $(\succeq_0, (\succeq_s)_{s \in S})$ satisfies Total Information Agreement if and only if $\mathcal{M}_s \subseteq ch(BU(\mathcal{M}_0, \mathcal{L}, s))$ for all $s \in S$.*

Theorem 2.1 only provides conditions for the set of posteriors to be a subset of $ch(BU(\mathcal{M}_0, \mathcal{L}, s))$. We now provide the necessary axiom required to replace "$\subseteq$" with "$=$".

Default to Certainty For all $f, x \in \mathcal{F}$ and $s \in S$,

$$x \succeq_0 f_x^{\ell,s} \text{ for some } \ell \in \mathcal{L} \implies x \succeq_s f.$$

To interpret Default to Certainty, it is crucial to understand that Total Information Agreement imposes constraints on the agent's behavior in a highly specific scenario. It becomes relevant only when every conceivable source of information unanimously indicates a preference for an uncertain payoff act over a certain one. Nevertheless, this constraint will not have an impact when different sources of information lead to divergent preferences based on the generated signal. For instance, an agent might favor an act f over a constant act x if she believes ℓ is the source of information, but her preference could reverse if she believes ℓ' is the source. Default to Certainty asserts that in such situations, where preferences depend on the source of information, the agent consistently opts for the constant act.[3]

Proposition 2.1. *Suppose $(\succeq_0, (\succeq_s)_{s \in S})$ and $\mathcal{L}$ satisfy the axioms of Theorem 2.1. Then, $(\succeq_0, (\succeq_s)_{s \in S})$ satisfies Default to Certainty if and only if $\mathcal{M}_s = ch(BU(\mathcal{M}_0, \mathcal{L}, s))$ for all $s \in S$.*

[3] Faro and Lefort (2019) uses a similar axiom to characterize FBU in a precise information context.

3 Discussion

The relationship between Theorem 2.1 and the standard Bayesian model deserves special attention. The latter arises when we enhance MEU Utility by requiring that belief sets are singletons and assume $\mathcal{L}$ is also a singleton. This adjustment has a significant implication for interpreting Total Information Agreement. Specifically, under the Subjective Expected Utility (SEU) axioms, Bayesian updating is characterized by Consequentialism and Dynamic Consistency (see Ghirardato (2002)). Given that signals in our context are not payoff-relevant, Consequentialism is implicitly assumed. Therefore, Total Information Agreement reduces to Dynamic Consistency when ex-ante and ex-post preferences adhere to SEU. Consequently, it can be seen as a relaxation of Dynamic Consistency, whenever ex-ante and ex-post preferences are MEU.

Theorem 2.1 delivers a representation, and thus an updating rule, for imprecise information. Formally, a *Generalized Bayesian Updating rule* (GBU) is a function $\rho : (\mathcal{M}, \mathcal{L}, s) \mapsto \mathcal{M}'$ such that $\mathcal{M}' \subseteq ch(BU(\mathcal{M}, \mathcal{L}, s))$.

Because the GBU rule imposes little structure on the set of posteriors for the conditional preferences, the model can accommodate a wide range of behavior. On the other hand, some may view the model as "too general" in that it permits the posteriors not to be generated by Bayesian updating of any of the feasible information sources. Indeed, it may be the case that

$$\rho(\mathcal{M}_0, \mathcal{L}, s) \subseteq ch(BU(\mathcal{M}_0, \mathcal{L}, s)) \backslash BU(\mathcal{M}_0, \mathcal{L}, s). \tag{2}$$

One reason to allow (2) is to nest the case in which the agent has, possibly non-singleton, beliefs over $\mathcal{L}$. To illustrate, consider a probability measure λ over $\mathcal{L}$. We interpret λ as the agent's beliefs about the true information source. When updating her set of priors $\mathcal{M}_0$, such an agent may use her beliefs to determine her set of posteriors:

$$\rho(\mathcal{M}_0, \mathcal{L}, s) = \{p \in \Delta(\Omega) | p = \sum_{\ell \in \mathcal{L}} \lambda(\ell) BU(q, \ell, s), q \in \mathcal{M}_0\}.$$

Another potential concern is that the GBU allows the agent to use different ℓ's depending on the signal. If one follows a "maximum likelihood" type of rule, then the ℓ's that have the maximal likelihood of generating s may be different than the ones generating s'. Similarly, an agent who only considers information structures that "confirm" her beliefs may consider different information structures depending on the realization of the signal. We further investigate these types of rules in Section 4. In the next section, we discuss the necessary conditions needed to rule out this feature of the model.

Next, we discuss the uniqueness properties of the model. Because we assumed the preferences conform to the MEU model, $(\mathcal{M}_0, (\mathcal{M}_s)_{s\in S})$ are unique.[4] This does not mean that the information structures that were used to construct the posterior set are unique. In general, there is no hope for an identification result. As the following example shows, Bayesian updating of a set of priors $\mathcal{M}_0$ and two different subsets of a given $\mathcal{L}$ may lead to the same set of posteriors.

Example 3.1. *Consider a binary state space $\Omega = \{\omega_1, \omega_2\}$ and a binary signal space $S = \{s_1, s_2\}$. The prior belief q_0 is uniform, $q_0 = (1/2, 1/2)$. Let $\mathcal{M}_i$ be the set of posteriors given signal s_i. Since there are only two states, we can identify a belief by its assessment on ω_1. Suppose that $\mathcal{M}_1 = [5/8, 7/8]$ and $\mathcal{M}_2 = [1/8, 3/8]$.*

Consider the following two distinct sets of information structures:

$$\mathcal{L}_1 = \{\ell : \ell(s_1|\omega_1) \in [5/8, 7/8];\ \ell(s_1|\omega_2) = \ell(s_2|\omega_1)\},\ \textit{and}$$
$$\mathcal{L}_2 = \{\ell : k \in [5/8, 7/8];\ \ell(s_1|\omega_1) = 4k(1-k);\ \ell(s_1|\omega_2) = 4(1-k)^2\}.$$

Then under FBU, these two sets of information structures both induce $\mathcal{M}_1$ and $\mathcal{M}_2$. That is, given the unconditional and conditional preferences, from which we can identify q_0, $\mathcal{M}_1$ and $\mathcal{M}_2$, we cannot distinguish if the set of information structures adopted by the agent is $\mathcal{L}_1$ or $\mathcal{L}_2$. Hence, if $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$ there is no hope for identifying the set of information structures used.

We conclude the discussion with a few words on the impossibility of providing a measure of aversion to updating imprecise information in the current framework. The main reason is that the current framework does not allow the agent to choose between facing imprecise information and precise information. Indeed, any measure of aversion towards a phenomenon requires a comparison between the phenomenon and an object that does not suffer from the phenomenon.[5] Therefore, in order to provide such a measure, we would need to consider a different setting. One possibility would be pairs of menus of acts and imprecise information structures. Although it is possible to adapt our analysis to such a setting, because most of the experimental literature does not consider menus, we feel it would obscure the message of the paper.

3.1 Consistent Updating

As discussed in the previous section, our model allows the agent to use different information structures for different signals. The following axiom makes sure that it can never be the case that ℓ plays a role after observing s' if it is ignored after observing s.

[4] See the uniqueness result in Gilboa and Schmeidler (1989).

[5] For example, in order to check if an agent is averse to ambiguity one needs to observe a choice between acts and lotteries.

Consistency Across Signals For any $f, g, x, x' \in \mathcal{F}$ and $s, s' \in S$,

$$\text{if } x \succeq_0 f_x^{\ell,s} \text{ and } f \succeq_s x \text{, then}$$
$$g_{x'}^{\ell,s'} \succeq_0 x' \text{ and } g \succeq_{s'} x' \implies g_{x'}^{\ell',s'} \succeq_0 x' \text{ for some } \ell' \in \mathcal{L},$$

where $g_{x'}^{\ell',s'} \succeq_0 x'$ holds with indifference (strictness) if $g \succeq_{s'} x'$ holds with indifference (strictness).

To interpret, recall that from the discussion of Total Information Agreement, $x \succeq_0 f_x^{\ell,s}$ and $f \succeq_s x$ reveal the agent is ignoring the possibility that ℓ generated s. Therefore, any behavior that considering ℓ would lead to after observing another signal s', such as $g_{x'}^{\ell,s'} \succeq_0 x$ and $g \succeq_s x'$, has to be rationalizable by another information structure ℓ'.

To state the result we need some notation. For any closed and convex set of priors $\mathcal{M}$, $\mathcal{E}(\mathcal{M})$ denotes all of its extreme points.

Proposition 3.1. *Suppose $(\succeq_0, (\succeq_s)_{s\in S})$ and $\mathcal{L}$ satisfy the axioms of Theorem 2.1. Then, $(\succeq_0, (\succeq_s)_{s\in S})$ satisfies Consistency Across Signals if and only if for all $s, s' \in S$ and $\ell \in \mathcal{L}$*

$$BU(q_0, \ell, s) \notin \mathcal{M}_s \ \forall\ q_0 \in \mathcal{M}_0 \implies BU(q_0', \ell, s') \notin \mathcal{E}(\mathcal{M}_{s'}) \ \forall\ q_0' \in \mathcal{M}_0.$$

Proposition 3.1 states that if ℓ is ignored after observing s, then it cannot be an extreme point of $\mathcal{M}_{s'}$. It is well known that only the extreme points of $\mathcal{M}_{s'}$ are important in the MEU model. Specifically,

$$\min_{q\in\mathcal{M}_s} \int_\Omega u(f)dq = \min_{q\in\mathcal{E}(\mathcal{M}_s)} \int_\Omega u(f)dq$$

for all f. Therefore, our axiom ensures that if the agent does not consider ℓ after s, she never considers it whenever it could affect her preferences.

4 Specializations

In this section, we discuss several special cases of the GBU. We separate the discussion into two classes, the ones motivated by statistics and the ones motivated by behavioral biases.

4.1 Statistical GBU

Taking a statistics point of view, it is natural to only consider pairs (q, ℓ) that pass some statistical test. We model a test as a function $\phi : (\mathcal{M}, \mathcal{L}, s) \mapsto (\mathcal{M}', \mathcal{L}') \subseteq$

$\mathcal{M} \times \mathcal{L}$. We refer to such rules as Statistical Generalized Bayesian Updating rules (SGBU):

$$\rho(\mathcal{M}_0, \mathcal{L}, s) = ch(BU(\phi(\mathcal{M}_0, \mathcal{L}, s), s)).$$

It is easy to see that FBU (3) and MLU (4) are special cases of SGBU:

$$\phi(\mathcal{M}_0, \mathcal{L}, s) = (\mathcal{M}_0, \mathcal{L}) \tag{3}$$

$$\phi(\mathcal{M}_0, \mathcal{L}, s) = \{(q_0, \ell) | (q_0, \ell) \in \arg \max_{(q_0, \ell) \in \mathcal{M}_0 \times \mathcal{L}} \int_\Omega \ell(s|\omega) dq_0\}. \tag{4}$$

In the introduction, we discussed how MLU uses the signals to *jointly* discriminate among priors and information structures. Each information structure is evaluated according to the prior that maximizes the likelihood of the observed signal. We propose the following conservative approach: evaluate each information structure using *all* priors. The following special case of SGBU, referred to as Maximum Robust Likelihood Updating (MRLU), formalizes this idea:

$$\phi_{MRLU}(\mathcal{M}_0, \mathcal{L}, s) = \{(q_0, \ell) | q_0 \in \mathcal{M}_0, \ell \in \arg \max_{\ell \in \mathcal{L}} \min_{q \in \mathcal{M}} \int_\Omega \ell(s|\omega) dq\}). \tag{5}$$

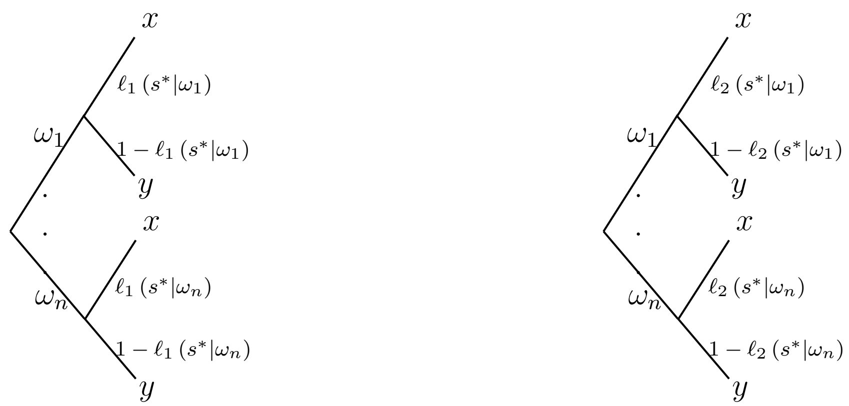
The axioms that characterize (5) are a strengthenings of Total Information Agreement and Default to Certainty. They basically require the agent to only consider information structures she believes are the most likely to generate the observed signal. We now show how this can be identified in our framework if the agent satisfies Reduction.

Fix a signal $s^* \in S$, two constant acts x, y such that $x \succ y$. Consider an extension of $\succeq_0$ to pairs of signal acts and information structures (F, ℓ). The idea is that conditional on each state ω, $\ell(\cdot|\omega)$ describes the probability law on S.

Suppose we ask the agent to choose between (F, ℓ_1) and (F, ℓ_2) where

$$F(\omega, s) = \begin{cases} x & s = s^* \\ y & s \neq s^*. \end{cases}$$

Figure 2 (a) and Figure 2 (b) illustrate (F, ℓ_1) and (F, ℓ_2) respectively. The agent will choose (F, ℓ_2) over (F, ℓ_1) if and only if she thinks ℓ_2 is more likely to generate s^* than ℓ_1. Further, by an identical argument to the one in the motivation of Total Information Agreement, $(F, \ell_i) \sim_0 x_y^{\ell_i, s^*}$ for $i = 1, 2$. Hence, for a given x, y, we can identify which information structures the agent believes are more likely to generate a given signal.



(a) Signal Act 1 (b) Signal Act 2

Figure 2

To state the next axiom we need some notation. Let $\geq^s_{x,y}$ be the binary relation over $\mathcal{L}$ such that $\ell \geq^s_{x,y} \ell'$ if $x^{\ell,s}_y \succeq_0 x^{\ell',s}_y$ and let

$$\arg\max(\mathcal{L}, \geq^s_{x,y}) = \{\ell \in \mathcal{L} | \ell \geq^s_{x,y} \ell' \text{ for all } \ell' \in \mathcal{L}\}.$$

Likelihood Information Agreement For any $x, y, z, f \in \mathcal{F}$ such that $x \succ_0 y$ and $s \in S$,

$$f^{\ell,s}_z \succeq_0 z \text{ for all } \ell \in \arg\max(\mathcal{L}, \geq^s_{x,y}) \implies f \succeq_s z.$$

Likelihood Information Agreement (LIA) guarantees that the set of posteriors is a subset of the set of the convex hull of the set of posteriors generated by pointwise Bayesian updating of (5). The following axiom strengthens set contention to equality.

Likelihood Default to Certainty For any $x, y, z, f \in \mathcal{F}$ such that $x \succ_0 y$ and $s \in S$,

$$z \succeq_0 f^{\ell,s}_z \text{ for some } \ell \in \arg\max(\mathcal{L}, \geq^s_{x,y}) \implies z \succeq_s f.$$

The interpretation of both axioms is analogous to the one of Total Information Agreement and Default to Certainty.

Theorem 4.1. *Let $(\succeq_0, (\succeq_s)_{s\in S})$ be a family of preferences over $\mathcal{F}$ that satisfies MEU Utility and $\mathcal{L}$ an imprecise information structure. Then, $(\succeq_0, (\succeq_s)_{s\in S})$ satisfies Likelihood Information Agreement if and only if*

$$\mathcal{M}_s \subseteq ch(BU(\phi_{MRLU}(\mathcal{M}_0, \mathcal{L}, s), s)) \tag{6}$$

for all $s \in S$. Moreover, the set contention in (6) is replaced with equality if and only if $(\succeq_0, (\succeq_s)_{s\in S})$ also satisfies Likelihood Default to Certainty.

We conclude this section by noting that for the case in which the agent holds a single prior q_0, Theorem 4.1 delivers a characterization of a version of the maximum likelihood updating rule. Specifically, the agent will only consider the information structures $\ell \in \mathcal{L}$ such that

$$\ell \in \arg\max_{\ell' \in \mathcal{L}} \int_{\Omega} \ell(s|\omega) dq_0.$$

Hence, our result can be viewed as an imprecise information counterpart of Gilboa and Schmeidler (1993) result for precise information.

4.2 Behavioral GBU

One of the more robust findings in the empirical literature on updating is that people tend to only update information that is consistent with their prior beliefs. This phenomenon is called confirmatory bias (Rabin and Schrag (1999)). In this section, we argue our model is well-suited to accommodate it. Because confirmatory bias and state space ambiguity do not share a conceptual link, we focus on the case in which $\mathcal{M}_0 = \{q_0\}$.

When the agent's ex-ante prior beliefs can be described by a single probability distribution we can recover the agent's probabilistic ranking among the states of the world. Indeed, for any $x, y \in \mathcal{F}$ such that $x \succ_0 y$, one can recover the ranking by observing the agent's preferences over the following acts:

$$f_\omega(\omega') = \begin{cases} x & \omega' = \omega \\ y & \omega' \neq \omega \end{cases}.$$

Let $\succeq^*$ be the binary relation over Ω induced by the ranking.

Intuitively, we can use $\succeq^*$ to define whether an information structure ℓ provides information consistent with the agents prior: For a given signal s, $\ell(s|\omega)$ is the likelihood it was generated by state ω. Hence, the information will confirm the agent's ex-ante beliefs if the order among the $\ell(s|\omega)$'s is the same as her ranking among states. Formally, we say that $\ell(s|\cdot)$ *s-confirms* $\succeq^*$ if

$$\omega \succeq^* \omega' \iff \ell(s|\omega) \geq \ell(s|\omega').$$

An agent who suffers from confirmatory bias would only consider information that confirms her beliefs. The following axiom captures this.

Confirmatory Information Agreement For any $f, x \in \mathcal{F}$,

$$f_x^{\ell,s} \succeq_0 x \text{ for all } \ell \in \mathcal{L} \text{ that } s\text{-confirms } \succeq^* \implies f \succeq_s x.$$

Confirmatory Information Agreement guarantees that the set of posteriors is a subset of the convex hull of the posteriors generated by point-wise Bayesian updating of the information structures that confirm the agent's beliefs (whenever it is non-empty). The following axiom strengthens set contention to equality.

Confirmatory Default to Certainty For any $f, x \in \mathcal{F}$ such that $x \succ_0 y$ and $s \in S$,

$$x \succeq_0 f_x^{\ell,s} \ell \in \mathcal{L} \text{ that } s\text{-confirms } \succeq^* \implies x \succeq_s f.$$

Theorem 4.2. *Let $(\succeq_0, (\succeq_s)_{s\in S})$ be a family of preferences over $\mathcal{F}$ that satisfies MEU Utility and $\mathcal{L}$ an imprecise information structure. Assume $\mathcal{M}_0$ is a singleton and for each $s \in S$ there exists some $\ell \in \mathcal{L}$ that s-confirms $\succeq^*$. Then, $(\succeq_0, (\succeq_s)_{s\in S})$ satisfies Confirmatory Information Agreement if and only if*

$$\mathcal{M}_s \subseteq ch(BU(\mathcal{M}_0, \{\ell \in \mathcal{L} | \ell \ s\text{-confirms } \succeq^*\}, s)) \tag{7}$$

for all $s \in S$. Moreover, the set contention in (7) is replaced with equality if and only if $(\succeq_0, (\succeq_s)_{s\in S})$ also satisfies Confirmatory Default to Certainty.

5 Subjective Information Structures

So far, we have assumed $\mathcal{L}$ is both observable and objective. This is not always an appropriate assumption. For example, an agent might receive a signal without a clear description of its content and may construct $\mathcal{L}$ herself. In such cases, $\mathcal{L}$ becomes purely subjective and can only be inferred from observable behavior. In this section, we study when the ex-ante and ex-post preferences are consistent with the existence of a *subjective* $\mathcal{L}$.

As in the previous analysis, we consider a family of preferences over acts $(\succeq_0, (\succeq_s)_{s\in S})$ that admit a MEU representation. We establish necessary and sufficient conditions on behavior for these preferences to be linked by FBU of a subjective $\mathcal{L}$ under two specific scenarios: (i) No ex-ante state ambiguity, and (ii) No signal imprecision. The general case poses a significant challenge and is left for future research.

To introduce our axioms we need some preliminaries. For any MEU preference $\succeq$ represented by $(\mathcal{M}, u)$, let $\bar{\succeq}$ denote the preference over acts represented by

$$\max_{q\in\mathcal{M}} \int_\Omega u(f)dq.$$

Observe that for any acts f, g, and constant act x satisfying $x = \alpha f + (1-\alpha)g$ for some $\alpha \in (0,1)$, we have

$$f \bar{\succeq} x \text{ if and only if } x \succeq g.$$

Thus $\bar{\succeq}$ can be understood as a conjugate of $\succeq$ and can be recovered from $\succeq$.

5.1 No ex-ante state ambiguity

First, we consider the situation where the state space is ex-ante unambiguous, meaning that the agent possesses a single prior belief, and thus the ex-ante preference $\succeq_0$ is SEU with belief q_0.

We begin by considering the case in which the ex-post preference $\succeq_s$ are all SEU with belief p_s. It is well known that a necessary and sufficient condition for the existence of a likelihood function ℓ such $q_s = BU(q_0, \ell, s)$ for all $s \in S$ is that q_0 must lie in the convex hull of $\{p_s | s \in S\}$. This property is equivalent to an adaptation of Dynamic Consistency to our setting: For any act f and constant act x, if $f \succeq_s x$ for all $s \in S$, then $f \succeq_0 x$. Intuitively, if f is preferred to x under any signal realization, then f is also preferred to x ex-ante.

In scenarios where ex-post preferences can be MEU, the above dominance property is insufficient to determine the existence of $\mathcal{L}$ such that $\mathcal{M}_s = BU(q_0, \mathcal{L}, s)$. Our main finding in this section demonstrates that the following stronger dominance property is a necessary and sufficient condition for this relationship.

P1 For any $f, x \in \mathcal{F}$,

$$f \bar{\succeq}_s x \text{ for some } s \in S \text{ and } f \succeq_{s'} x \text{ for all } s' \neq s \implies f \succeq_0 x.$$

If, in addition, $f \bar{\succ}_s x$ or $f \succ_{s'} x$ for some $s' \neq s$, then $f \succ_0 x$.

To see why P1 is stronger, observe that $f \succeq x$ implies $f \bar{\succeq} x$, but the converse fails. However under SEU, $\succeq = \bar{\succeq}$ making P1 equivalent to Dynamic Consistency.

Proposition 5.1. *Let $(\succeq_0, (\succeq_s)_{s\in S})$ be a family of preferences over $\mathcal{F}$. Assume (u, q_0) represents $\succeq_0$ and $(u, \mathcal{M}_s)$ represents $\succeq_s$ for all $s \in S$. Then, $(\succeq_0, (\succeq_s)_{s\in S})$ satisfies P1 if and only if there exists $\mathcal{L}$ such that $\mathcal{M}_s = BU(q_0, \mathcal{L}, s)$ for all s.*

To aid intuition for the result, we describe the necessity of P1. Given $\succeq$, $f \bar{\succeq}_s x$ requires the existence of a belief p_s in $\mathcal{M}_s$ under which the utility of f is no less than the utility of x. Under full Bayesian updating, $p_s = BU(q_0, \ell, s)$ for some $\ell \in \mathcal{L}$. Moreover, for any other signal s', the Bayesian posterior of q_0 and ℓ, say $p_{s'}$, also lies in $\mathcal{M}_{s'}$. Thus, $f \succeq_{s'} x$ implies that the utility of f under $p_{s'}$ is no less than the utility of x. Since the prior q_0 must lie in the convex hull of the posteriors $\{p_s | s \in S\}$, we obtain $f \succeq_0 x$.

5.2 No signal imprecision

We now consider the case in which ex-ante and ex-post preferences are MEU and focus on the existence of a precise information structure.

To state our axioms, we need to introduce some notation. For any $x, y, z \in X$ and states ω, ω', let $[x, \omega | y, \omega' | z]$ denote the act that yields x in state ω, y in state ω', and z in any other state.

P2 For any $f \in \mathcal{F}$, state ω and outcomes$(y^0_{\omega'})_{\omega' \in \Omega \setminus \{\omega\}}, (y^s_{\omega'})_{\omega' \in \Omega \setminus \{\omega\}}$ such that $f(\omega) = \sum_{\omega' \neq \omega} \frac{y^s_{\omega'}}{|\Omega|-1} = \sum_{\omega' \neq \omega} \frac{y^0_{\omega'}}{|\Omega|-1}$ for all $s \in S$,

$$\begin{cases} x \succeq_s [f(\omega'), \omega' | y^s_{\omega'}, \omega | x]; \\ x \bar{\succeq}_s [f(\omega'), \omega' | y^s_{\omega'}, \omega | x] \text{ if } f(\omega') \succeq_s x \end{cases} \quad \text{for all } s \in S \text{ and } \omega' \neq \omega$$

$$\Longrightarrow \begin{cases} x \succeq_0 [f(\omega''), \omega'' | y^s_{\omega''}, \omega | x]; \\ x \bar{\succeq}_0 [f(\omega''), \omega' | y^s_{\omega''}, \omega | x] \text{ if } f(\omega'') \succeq_0 x \end{cases} \quad \text{for some } \omega''.$$

To understand P2, consider utility acts (i.e. $X \subset \mathbb{R}$ and $u(x) = x$) for simplicity. Suppose that $|\Omega| = 3$. Thus an act is an element of $\mathbb{R}^3$. Suppose that, for instance, we have for all $s \in S$,

$$x \succeq_s (x - a_1, x, x + b^s_1) \text{ and } x \succeq_s (x, x - a_2, x + b^s_2),$$

where $a_1, a_2 > 0$ and $b^s_1 + b^s_2 = k$ for all s. We can interpret these rankings as follows. Given a constant act x, we lower its payoff in ω_1 by a_1, and we increase its payoff in ω_3 by b^s_1 as compensation. However, the compensation is not large enough to fully compensate for the loss in ω_1. Thus the first ranking follows. Similarly, if we lower the payoff in ω_2 by a_2, then we increase the payoff in ω_3 by b^s_2 as compensation. The second-ranking suggests that this compensation is not large enough. As $b^s_1 + b^s_2 = k$ for all s, we can say that k is too small as a total stake for compensation for any signal realization. Then P2 says that k is also too small from the ex-ante perspective. We can never split k into b^0_1 and b^0_2 such that

$$(x - a_1, x, x + b^0_1) \succeq_0 x \text{ and } (x, x - a_2, x + b^0_2) \succeq_0 x.$$

In terms of the MEU model, P2 captures the following "convex hull" implication of a single subjective information structure. Fix a state ω. As we show in the proof, if $\mathcal{M}_s = BU(\mathcal{M}_0, \ell, s)$ for all $s \in S$, then for all $\omega' \neq \omega$,

$$\sum_s \ell(s|\omega) \max_{p \in \mathcal{M}_s} \frac{p(\omega')}{p(\omega)} = \sum_s \ell(s|\omega) \times \frac{\ell(s|\omega')}{\ell(s|\omega)} \times \max_{q \in \mathcal{M}_0} \frac{q(\omega')}{q(\omega)} = \max_{q \in \mathcal{M}_0} \frac{q(\omega')}{q(\omega)}.$$

This means that the vector $(\max_{q \in \mathcal{M}_0} \frac{q(\omega')}{q(\omega)})_{\omega' \neq \omega}$ lies in the convex hull of the set $\{(\max_{p \in \mathcal{M}_{s_1}} \frac{p(\omega')}{p(\omega)})_{\omega' \neq \omega}, ..., \max_{p \in \mathcal{M}_{s_{|S|}}} \frac{p(\omega')}{p(\omega)})_{\omega' \neq \omega}\}$. An implication is that for all reals k and $(r_{\omega'})_{\omega' \neq \omega}$,

$$\left[\sum_{\omega' \neq \omega} r_{\omega'} \max_{p \in \mathcal{M}_s} \frac{p(\omega')}{p(\omega)} \geq k \;\forall\; s \in S \right] \Rightarrow \sum_{\omega' \neq \omega} r_{\omega'} \times \max_{q \in \mathcal{M}_0} \frac{q(\omega')}{q(\omega)} \geq k.$$

P2 follows from this dominance property. Specifically, the necessity of P2 follows from the fact that for any MEU preference $\succeq$ represented by $(\mathcal{M}, u)$,

$$\begin{cases} x \succeq [f(\omega'), \omega' | y_{\omega'}, \omega | x]; \\ x \bar{\succeq} [f(\omega'), \omega' | y_j, \omega | x] \text{ if } f(\omega') \succeq x \end{cases}$$

if and only if

$$[u(x) - u(f(\omega'))] \max_{p \in \mathcal{M}} \frac{p(\omega')}{p(\omega)} \geq u(y_{\omega'}) - u(x).$$

Hence, P2 allows us to find an information structure ℓ such that

$$\max_{p \in \mathcal{M}_s} \frac{p(\omega')}{p(\omega)} = \frac{\ell(s|\omega')}{\ell(s|\omega)} \times \max_{q \in \mathcal{M}_0} \frac{q(\omega')}{q(\omega)}$$

for all ω, ω' and all $s \in S$. Yet, this is not enough to establish $\mathcal{M}_s = BU(\mathcal{M}_0, \ell, s)$. By Proposition 2.1, we need that $f_x^{\ell,s} \succeq_0 x \Leftrightarrow f \succeq_s x$. This implication is captured by the following axiom.

P3 For any state ω, acts f, g, x such that $f(\omega) = g(\omega)$, and outcomes $(y_{\omega'})_{\omega' \in \Omega \setminus \{\omega\}}$ the following two statements are true:

(i) If for all $\omega' \neq \omega$,

$$\begin{cases} [y_{\omega'}, \omega' | g(\omega'), \omega | x] \bar{\succeq}_s x; \\ [y_{\omega'}, \omega' | g(\omega'), \omega | x] \succeq_s x \text{ if } x \succeq_s y_{\omega'} \end{cases} \quad \text{and} \quad \begin{cases} x \succeq_0 [y_{\omega'}, \omega' | f(\omega'), \omega | x]; \\ x \bar{\succeq}_0 [y_{\omega'}, \omega' | f(\omega'), \omega | x] \text{ if } y_{\omega'} \succeq_0 x \end{cases}$$

then $x \succeq_0 g$ implies $x \succeq_s f$.

(ii) If for all $\omega' \neq \omega$,

$$\begin{cases} x \succeq_s [y_{\omega'}, \omega' | g(\omega'), \omega | x]; \\ x \bar{\succeq}_s [y_{\omega'}, \omega' | g(\omega'), \omega | x] \text{ if } y_{\omega'} \succeq_s x \end{cases} \quad \text{and} \quad \begin{cases} [y_{\omega'}, \omega' | f(\omega'), \omega | x] \bar{\succeq}_0 x; \\ [y_{\omega'}, \omega' | f(\omega'), \omega | x] \succeq_0 x \text{ if } x \succeq_0 y_{\omega'} \end{cases}$$

then $g \succeq_0 x$ implies $f \succeq_s x$.

To illustrate P3, consider again utility acts and assume $|\Omega| = 3$. Fix $f = (f_1, f_2, f_3)$ and $g = (g_1, g_2, g_3)$ where $f_3 = g_3$. Suppose that

$$(a_1, x, g_1) \succeq_s x \text{ and } (x, a_2, g_2) \succeq_s x$$

where $x > a_1$ and $x > a_2$. The former suggests that g_1, under signal s, is good enough in the following sense: if we lower the payoff of a constant act x in ω_2 to a_2 and replace its payoff in ω_3 by g_1 as a compensation, we improve it. The latter ranking also suggests that g_2 is good enough under signal realization s. Suppose also that

$$x \succeq_0 (a_1, x, f_1) \text{ and } x \succeq_0 (x, a_2, f_2).$$

Again, they suggest that f_1 and f_2 are not good enough from ex-ante perspective. Now P3 requires that we cannot have g worse than x ex-ante but f better than x ex-post.

The existence of a precise information structure under MEU preferences and full-Bayesian updating is characterized by P2 and P3, as shown in the following proposition.

Proposition 5.2. *Let $(\succeq_0, (\succeq_s)_{s\in S})$ be a family of preferences over $\mathcal{F}$ that satisfies MEU Utility. Then $(\succeq_0, (\succeq_s)_{s\in \mathcal{S}})$ satisfies P2 and P3 if and only if there exists ℓ such that $\mathcal{M}_s = BU(\mathcal{M}_0, \ell, s)$ for all $s \in \mathcal{S}$.*

The proposition illustrates the challenge in identifying ℓ from $(\succeq_0, (\succeq_s)s \in S)$. Indeed, P2 and P3 are not straightforward axioms. Each axiom governs, in terms of the ex-ante preferences, the extent to which an agent is willing to transfer utility across states for different signal realizations. The more general case in which $\mathcal{L}$ is allowed to not be a singleton also requires an understanding of how discipline the transfers. Imposing these regularities is particularly challenging due to the non-payoff-relevance of information. Indeed, taking as primitive an ex-ante preference $\succeq_0$ over signal acts $F : \Omega \times S \to X$ would simplify the analysis. In this framework, $\succeq_s$ could be regarded as the preference induced by $\succeq_0$ over acts that yield the same payoff whenever s is not realized. However, when attempting to understand the behavioral implications of imprecise information in non-payoff-relevant contexts, such complexities cannot be avoided. Moreover, our analysis suggests that there is no simple way to capture the trade-offs.

6 Concluding Remarks

In this paper, we have provided a theory of updating under imprecise information that generalizes FBU and MLU. Although both of these rules are widely popular, there is no conceptual reason to adopt either.[6]

The ability to accommodate different attitudes towards updating is particularly significant in applications For instance Beauchêne et al. (2019) shows that under FBU a sender can extract full surplus from a receiver in a Bayesian Persuasion style game.[7] However, once we allow for a more general updating rule, such a result may not hold. This opens the door for a richer theory of persuasion under imprecise information.

[6]One can draw a parallel with the 3-color Ellsberg paradox. As formalized by Gajdos et al. (2008), there is no reason for the agent to consider *all* the objectively possible probabilistic beliefs (priors). The same intuition applies to our setting, there is no reason for the agent to consider all objectively possible information structures.

[7]An equally striking result holds in a cheap talk game with FBU. See Kellner and Le Quement (2018).

The framework we adopted was inspired by the experimental literature, where observations are typically limited to ex-ante and conditional on signal preferences. In these experiments, signals are often considered to be payoff irrelevant, and information structures are commonly employed. We hope that the constructive nature of our axioms offers some guidance on how to test behavior when information is imprecise.

Lastly, our analysis has been normative, with Total Information Agreement presented as an attractive property that delivers a generalization of Bayesian updating. Consequently, it serves as a test for any updating theory that challenges Bayesianism within the MEU framework. In essence, any updating rule that contravenes Total Information Agreement must be supported by an example demonstrating its unreasonableness. We view this aspect of the paper as a separate contribution as it can offer guidance for future research.

Appendix A Proofs

The necessity of the axioms is obvious in each of our representation theorems. Therefore, we only prove sufficiency.

Throughout, it is assumed that $\succeq_s$ is represented by $(u, \mathcal{M}_s)$ for all $s \in S \cup \{0\}$. Finally, each proof of sufficiency makes use of the following lemma.

Lemma A.1. *Assume $\succeq$ and $\succeq'$ admit MEU representations $(u, \mathcal{M})$, $(u, \mathcal{M}')$ respectively such that $\mathcal{M} \not\subseteq \mathcal{M}'$. Then there exists an act f and a constant act x such that $f \sim x$ and $f \succ' x$.*

The proof follows from an identical argument from the uniqueness result in Gilboa and Schmeidler (1989).

A.1 Proof of Theorem 2.1

Let $(\mathcal{M}_s)_{s \in S \cup \{0\}}$ be the sets of probability measures described by MEU Utility. Suppose $(\succeq_s)_{s \in S \cup \{0\}}$ satisfies Total Information Agreement (TIA) and assume by way of contradiction that $\mathcal{M}_s \not\subseteq ch(BU(\mathcal{M}_0, \mathcal{L}, s))$ for some $s \in S$.

By Lemma A.1, there exists an act f and a constant act x such that

$$u(x) < \int_\Omega u(f) dBU(q', \ell, s) \text{ for all } (q', \ell) \in \mathcal{M}_0 \times \mathcal{L}.$$

Fix $(q, \ell) \in \mathcal{M}_0 \times \mathcal{L}$, then

$$\begin{aligned} u(x) &< \int_\Omega u(f) dBU(q, \ell, s) \\ \left(\int_\Omega \ell(s|\omega) dq\right) u(x) &< \int_\Omega u(f) \ell(s|\omega) dq \\ \left(\int_\Omega \ell(s|\omega) dq\right) u(x) + \left(1 - \int_\Omega \ell(s|\omega) dq\right) u(x) &< \int_\Omega u(f) \ell(s|\omega) dq + \left(1 - \int_\Omega \ell(s|\omega) dq\right) u(x) \\ u(x) &< \int_\Omega u(\ell(s|\omega) f(\omega) + (1 - \ell(s|\omega))x) dq. \end{aligned}$$

Thus, $u(x) < \int_\Omega u(\ell(s|\omega) f(\omega) + (1 - \ell(s|\omega)x)) dq$ for all $q \in \mathcal{M}_0$ and $\ell \in \mathcal{L}$. Therefore, $u(x) < \min_{q \in \mathcal{M}_0} \int_\Omega u(\ell(s|\omega) f(\omega) + (1 - \ell(s|\omega)x)) dq$ for all $\ell \in \mathcal{L}$. Hence, $f_x^{\ell,s} \succ_0 x$ for all $\ell \in \mathcal{L}$ and $x \succeq_s f$, a contradiction of TIA.

A.2 Proof of Proposition 2.1

Given Theorem 2.1, we only need to show that if $(\succeq, (\succeq_s)_{s \in S})$ also satisfies Default to Certainty (DTC), then $ch(BU(\mathcal{M}_0, \mathcal{L}, s)) \subseteq \mathcal{M}_s$.

Fix $s \in S$ and assume by way of contradiction there exists $p^* \in ch(BU(\mathcal{M}_0, \mathcal{L}, s))$ such that $p^* \notin \mathcal{M}_s$. By Lemma A.1, there exists f and x such that $\int_\Omega u(f)p^* < \min_{q' \in \mathcal{M}_s} \int_\Omega u(f)dq' = u(x)$.

Since $p^* \in ch(BU(\mathcal{M}_0, \mathcal{L}, s))$, then there exist $(q_1, \ell_1), ..., (q_n, \ell_n) \in \mathcal{M}_0 \times \mathcal{L}$ such that $p^* = \sum_i \alpha_i BU(q_i, \ell_i, s)$ and

$$\int_\Omega u(f)p^* = \sum_i \alpha_i \int_\Omega u(f)dBU(q_i, \ell_i, s).$$

Hence, there exists i such that $\int_\Omega u(f)dBU(q_i, \ell_i, s) \leq \int_\Omega u(f)dp^*$. Let $q = q_i$ and $\ell = \ell_i$. Then,

$$\int_\Omega u(f)dBU(q, \ell, s) < u(x)$$

$$\int_\Omega u(f)\ell(s|\omega)dq + \left(1 - \int_\Omega \ell(s|\omega)dq\right)u(x) < u(x)\left(\int_\Omega \ell(s|\omega)dq\right) + \left(1 - \int_\Omega \ell(s|\omega)dq\right)u(x)$$

$$\int_\Omega u(f(\omega)\ell(s|\omega) + (1 - \ell(s|\omega))x)dq < u(x).$$

Hence, $\min_{q \in \mathcal{M}_0} \int_\Omega u(f(\omega)\ell(s|\omega) + (1 - \ell(s|\omega))x)dq < u(x)$. This implies that there exists $\ell \in \mathcal{L}$ such that $x \succ_0 f_x^{\ell,s}$. Thus, by DTC, $x \succ_s f$, a contradiction.

A.3 Proof of Proposition 3.1

By assumption, $BU(\mathcal{M}_0, \ell, s) \cap \mathcal{M}_s = \emptyset$. Thus, by a hyperplane separating argument, there exists f such that

$$\int_\Omega u(f)dq_s < c < \int_\Omega u(f)dBU(q_0, \ell, s)$$

for all $q_0 \in \mathcal{M}_0$ and $q_s \in \mathcal{M}_s$. Let x be such that $x \sim_s f$. Then $f_x^{l,s} \succ_0 x$ and $x \succeq_s f$.

Next, assume that there exists $\ell \in \mathcal{L}$, $s' \in S$ and $q_0' \in \mathcal{M}_0$ such that $BU(q_0', \ell, s') \in \mathcal{E}(\mathcal{M}_{s'})$. Observe that $BU(q_0', \ell, s') \in \mathcal{E}(\mathcal{M}_{s'})$ implies there exists an act g such that

$$\min_{q \in \mathcal{M}_{s'}} \int_\Omega u(g)dq = \int_\Omega u(g)dBU(q_0', \ell, s') < \int_\Omega u(g)dq \text{ for all } q \in \mathcal{M}_{s'} \backslash \{BU(q_0', \ell, s')\}.$$

Let y be the constant act such that $u(y) = \int_\Omega u(g)dBU(q_0', \ell, s')$. Then, $g_x^{\ell,s} \sim_0 x$ and $g \sim_s x$. Moreover, $g_x^{\ell',s} \succ_0$ for all $\ell' \in \mathcal{L}$. Hence, for Consistency Across Signals to hold, either $BU(\mathcal{M}_0, \ell, s) \cap \mathcal{M}_s \neq \emptyset$ or $BU(q_0', \ell, s') \notin \mathcal{E}(\mathcal{M}_{s'})$ for all q_0'.

A.4 Proof of Theorem 4.1

Suppose $(\succeq_0, (\succeq_s)_{s\in S})$ satisfies Likelihood Information Agreement (LIA) and assume by way of contradiction that

$$\mathcal{M}_s \not\subseteq ch(\{BU(q,\ell,s)|q \in \mathcal{M}_0, \ell \in \arg\max_{\ell\in\mathcal{L}} \max_{q\in\mathcal{M}_0} \int_\Omega \ell(s|\omega)dq\})$$

for some $s \in S$.

Let $\mathcal{L}_s^{**} = \arg\max_{\ell\in\mathcal{L}} \min_{q\in\mathcal{M}_0} \int_\Omega \ell(s|\omega)dq$. By Lemma A.1, there exists f and x such that

$$u(x) = \min_{q\in ch(BU(\mathcal{M}_0,\mathcal{L}_s^{**},s))} \int_\Omega u(f)dq \leq \min_{(q,\ell)\in\mathcal{M}_0\times\mathcal{L}_s^{**}} \int_\Omega u(f)dBU(q,\ell,s)$$

Hence, for all $q \in \mathcal{M}_0$ and $\ell \in \mathcal{L}_s^{**}$

$$\begin{aligned} u(x) &< \int_\Omega u(f)dBU(q,\ell,s) \\ u(x) &< \int_\Omega u(f)\ell(s|\omega)dq + u(p)(1 - \int_\Omega \ell(s|\omega)dq). \end{aligned}$$

Thus,

$$u(x) < \min_{q\in\mathcal{M}_0} \int_\Omega u(f)\ell(s|\omega)dq + u(x)(1 - \int_\Omega \ell(s|\omega)dq).$$

If we can show that $\ell \in \arg\max(\mathcal{L}, \geq_{x,y}^s)$ implies $\ell \in \mathcal{L}_s^{**}$ for any x, y such that $u(x) > u(y)$, we will have a contradiction of LIA. To see that $\ell \in \max(\mathcal{L}, \geq_{x,y}^s)$ implies $\ell \in \mathcal{L}_s^{**}$ note that $\ell \geq_{x,y}^s \ell'$ if and only if

$$\min_{q\in\mathcal{M}_0} [u(x) \int_\Omega \ell(s|\omega)dq + u(y)(1 - \int_\Omega \ell(s|\omega)dq)] \geq \min_{q\in\mathcal{M}_0} [u(x) \int_\Omega \ell'(s|\omega)dq + u(y)(1 - \int_\Omega \ell'(s|\omega)dq)]$$

which holds if and only if

$$\min_{q\in\mathcal{M}_0} \int_\Omega \ell(s|\omega)dq \geq \min_{q\in\mathcal{M}_0} \int_\Omega \ell'(s|\omega)dq.$$

Hence, $\ell \in \max(\mathcal{L}, \geq_{x,y}^s)$ implies $\ell \in \arg\max_{\ell\in\mathcal{L}} \min_{q\in\mathcal{M}_0} \int_\Omega \ell(s|\omega)dq = \mathcal{L}_s^{**}$.

Next, we prove that Likelihood Default to Certainty (LDC) implies equality of the sets described in the Theorem.

Let $ch(\{BU(q,\ell,s)|\ell \in \arg\max_{\ell\in\mathcal{L}} \min_{q\in\mathcal{M}_0} \int_\Omega \ell(s|\omega)dq \text{ and } q \in \mathcal{M}_0\}) \equiv \mathcal{M}_s^{MM}$. We only need to show that $\mathcal{M}_s^{MM} \subseteq \mathcal{M}_s$.

Assume by way of contradiction that $\mathcal{M}_s^{MM} \not\subseteq \mathcal{M}_s$. Then, by Lemma A.1, there exists f and x such that

$$\min_{q \in \mathcal{M}_s^{MM}} \int_\Omega u(f)dq < \min_{q \in \mathcal{M}_s} \int_\Omega u(f)dq = u(x).$$

Fix $q' \in \arg \min_{q \in \mathcal{M}_s^{MM}} \int_\Omega u(f)dq$. Then, by an analogous argument as in the proof of Proposition 2.1, there exists $(q, \ell) \in \mathcal{M} \times \mathcal{L}^{**}$ such that $\int_\Omega u(f)dBU(q, \ell, s) \leq \int_\Omega u(f)dq'$. Hence,

$$\begin{aligned} \int_\Omega u(f)dBU(q, \ell, s) &< u(x) \\ \int_\Omega \ell(s|\omega)u(f)dq + (1 - \int_\Omega \ell(s|\omega)dq)u(x) &< u(x) \\ \min_{q \in \mathcal{M}} \int_\Omega \ell(s|\omega)u(f)dq + (1 - \int_\Omega \ell(s|\omega)dq)u(x) &< u(x). \end{aligned}$$

Thus, $x \succ_0 f_x^{\ell_i, s}$. Moreover, $\ell \in \arg\max_{\ell \in \mathcal{L}} \min_{q \in \mathcal{M}_0} \int_\Omega \ell(s|\omega)dq$, thus, $\ell \geq_{x,y}^s l'$ for all $\ell' \in \mathcal{L}$ and x, y sych that $u(x) > u(y)$. By LDC, $x \succ_s f$, a contradiction.

A.5 Proof of Theorem 4.2

Suppose $(\succeq_s)_{s \in S \cup \{0\}}$ the satisfies Confirmatory Information agreement (CIA) and assume by way of contradiction $\mathcal{M}_s \not\subseteq ch(BU(\mathcal{M}_0, \{\ell \in \mathcal{L} | \ell \text{ confirms } \succeq^*\}, s)))$. Then, by Lemma A.1, there exists f and x such that

$$u(x) = \min_{q \in \mathcal{M}_s} \int_\Omega u(f)dq < \int_\Omega u(f)dBU(q, \ell, s) \text{ for all } (q, \ell) \in \mathcal{M}_0 \times \{\ell \in \mathcal{L} | \ell \text{ confirms } \succeq^*\}.$$

An identical argument as in Theorem 2.1 shows that $f_x^{\ell, s} \succ_0 x$ for all $\ell \in \{\ell \in \mathcal{L} | \ell \text{ confirms } \succeq^*\}$. This contradicts CIA as $x \sim_s f$.

Next, we show that $\mathcal{M}_s^{CM} \equiv ch(BU(\mathcal{M}_0, \{\ell \in \mathcal{L} | \ell \text{ confirms } \succeq^*\}, s))) \subseteq \mathcal{M}_s$ under Confirmatory Default to Certainty (CDC). Assume by way of contradiction that this is not the case. Then, by Lemma A.1, there exists f, x such that

$$\min_{p \in \mathcal{M}_s^{CM}} \int_\Omega u(f)dp < \min_{q \in \mathcal{M}_s} \int_\Omega u(f)dp = u(x).$$

Fix $p \in \arg\min_{p \in \mathcal{M}_s^{CM}}$. Then an analogous argument as in the proof of Proposition 2.1, establishes that there exists $(q, \ell) \in \mathcal{M}_0 \times \{\ell \in \mathcal{L} | \ell \text{ confirms } \succeq^*\}$ such that $\int_\Omega u(f) dBU(q,\ell,s) \le \int_\Omega u(f)dp$. Hence,

$$\begin{aligned}
\int_\Omega u(f)dBU(q,\ell,s) &< u(x)\\
\int_\Omega \ell(s|\omega)u(f)dq + (1-\int_\Omega \ell(s|\omega)dq)u(x) &< u(x)\\
\min_{q\in\mathcal{M}_0}\int_\Omega \ell(s|\omega)u(f)dq + (1-\int_\Omega \ell(s|\omega)dq)u(x) &< u(x).
\end{aligned}$$

Thus, $x \succ_0 f_x^{\ell,s}$. Moreover, $\ell \in \{\ell \in \mathcal{L} | \ell \text{ confirms } \succeq^*\}$. Therefore, CDC, $x \succ_s f$, a contradiction.

A.6 Proof of Proposition 5.1

We first prove the necessity of statement 1. Take any $s \in S$. Suppose that $f \bar{\succsim}_s x$ and $f \succeq_{s'} x$ for all $s' \neq s$. Since $f \bar{\succsim}_s x$, there exists $p_s \in \mathcal{M}_s$ such that $p_s \cdot (u \circ f) \ge u(x)$. Since $\mathcal{M}_s = BU(q_0, \mathcal{L}, s)$, there exists $\ell \in \mathcal{L}$ such that $p_s = BU(q_0, \ell, s)$. Let $p_{s'} = BU(q_0, \ell, s')$ for all $s' \neq s$. Since $\mathcal{M}_{s'} = BU(q_0, \mathcal{L}, s')$, $p_{s'} \in \mathcal{M}_{s'}$. Now for all $s' \neq s$, because $f \succeq_{s'} x$, we have $p_{s'} \cdot (u \circ f) \ge u(x)$. Since q_0 lies in the convex hull of $p_s, p_{s'}, \ldots$, we have $q_0 \cdot (u \circ f) \ge u(x)$. In addition, if $f \bar{\succ}_s x$ or $f \succ_{s'} x$ for some $s' \neq s$, we obtain $q_0 \cdot (u \circ f) > u(x)$. This proves the necessity of statement 1.

Next, we prove the sufficiency of statement 1. The following claim will be useful.

Claim 1. *Suppose that $A_1, \cdots, A_I$ are closed and convex sets in $\mathbb{R}^N$. Then*

$$ri(ch(\cup_i A_i)) \subset \left\{ v \in \mathbb{R}^N : \exists w_i \in A_i,\ \lambda_i > 0\ \forall\ i\ s.t.\ \sum_i \lambda_i w_i = v,\ \sum_i \lambda_i = 1 \right\}.$$

Proof. Suppose $v \in ri(ch(\cup_i A_i))$. Take any $v' = \sum_i \lambda_i w_i$ with $w_i \in A_i$, $\lambda_i > 0$, and $\sum_i \lambda_i = 1$. Since $v \in ri(ch(\cup_i A_i))$, there exists $v'' \in ch(\cup_i A_i)$ and $k \in (0,1)$ such that $v = kv' + (1-k)v''$. Since A_i is convex for all i, v'' can be expressed as $v'' = \sum_i \lambda'_i w'_i$ with $w'_i \in A_i$, $\lambda'_i \ge 0$, and $\sum_i \lambda'_i = 1$. Thus,

$$v = kv' + (1-k)v'' = \sum_i (k\lambda_i + (1-k)\lambda'_i) \frac{k\lambda_i w_i + (1-k)\lambda'_i w'_i}{k\lambda_i + (1-k)\lambda'_i}.$$

Since $\lambda_i > 0$, $k\lambda_i + (1-k)\lambda'_i > 0$. Since A_i is convex, $\frac{k\lambda_i w_i + (1-k)\lambda'_i w'_i}{k\lambda_i + (1-k)\lambda'_i} \in A_i$. Therefore v belongs to the set on the right-hand side. $\square$

We will also use the following separating hyperplane theorem: Two non-empty convex sets A and B can be separated properly if and only if their relative interiors do not intersect. Here, proper separation means that there is a hyperplane H such that A and B lie in opposite closed half-spaces with respect to H, and at least one of the sets A, B is not contained in H.

Suppose that statement 2 fails. There exists a signal realization s and $q_s \in \mathcal{M}_s$ such that there exist no $\in \mathcal{M}_{s'}$ for each $s' \neq s$ such that q_0 equals a convex combination of $q_s, q_{s'}, \cdots$. We want to establish a violation of statement 1. Consider two cases: $q_s = q_0$ and $q_s \neq q_0$.

Assume $q_s = q_0$. By the claim, $q_0 \notin ri(ch(\cup_{s' \neq s} \mathcal{M}_{s'}))$. By the aforementioned separating hyperplane theorem, there exist a vector $(v_\omega)_{\omega \in \Omega} \equiv v$ and a real r such that $q_0 \cdot v = r \leq q \cdot v$ for all $q \in ch(\cup_{s' \neq s} \mathcal{M}_{s'})$ where the inequality holds strictly for some q. Note that we are free to take a positive linear transformation on v and r, which means that we can choose v and r such that r and v_ω all lie in the range of u. Therefore, there exist an act f and a constant act x such that $q_0 \cdot (u \circ f) = u(x) \leq q \cdot (u \circ f)$ for all $q \in ch(\cup_{s' \neq s} \mathcal{M}_{s'})$ where the inequality holds strictly for some q. Now we have $f \succeq_{s'} x$ for all $s' \neq s$ and $f \succ_{s'} x$ for some $s' \neq s$. Since $q_0 \cdot (u \circ f) = u(x)$, $f \sim_0 x$. Since $q_s = q_0 \in \mathcal{M}_s$ by assumption, $f \bar{\succeq} x$. Hence statement 1 fails.

Assume instead $q_s \neq q_0$. Consider the convex set

$$\{q \in \Delta(S) : \exists \lambda > 1 \ s.t. \ q - q_s = \lambda(q_0 - q_s)\} \equiv C.$$

By the claim, C is disjoint with $ri(ch(\cup_{s' \neq s} \mathcal{M}_{s'}))$. By a separating hyperplane theorem, there exist an act f and an constant act x such that $q_0 \cdot (u \circ f) = u(x) > q \cdot (u \circ f)$ for all $q \in C$, and $q_0 \cdot (u \circ f) = u(x) \leq q \cdot (u \circ f)$ for all $q \in ch(\cup_{s' \neq s} \mathcal{M}_{s'})$. It follows that $f \sim_0 x$ and $f \succeq_{s'} x$ for all $s' \neq s$. Notice that C and $\{q_s\}$ are also separated by f and x. So $q_s \cdot (u \circ f) > u(x)$, implying $f \bar{\succ}_s x$. Hence statement 1 fails.

A.7 Proof of Proposition 5.2

A.7.1 Preliminaries

For ease of exposition, we enumerate the states $\Omega = \{\omega_1, ..., \omega_n\}$ and write P2 and P3 using this notation.

P2 Take any f and fix ω_i. Let $y_j^0, y_j^s \in X$ be such that $f(\omega_i) = \sum_{j \neq i} \frac{y_j^s}{n-1} =$

$\sum_{j \neq i} \frac{y_j^0}{n-1}$ for all $s \in S$. Suppose that for all $s \in S$ and all $j \neq i$

$$\begin{cases} x \succeq_s [f(\omega_j), \omega_j | y_j^s, \omega_i | x]; \\ x \bar{\succeq}_s [f(\omega_j), \omega_j | y_{\omega_j}^s, \omega_i | x] \text{ if } f(\omega_j) \succeq_s x \end{cases} \quad \text{for all } s \in S \text{ and } j \neq i$$

$$\implies \begin{cases} x \succeq_0 [f(\omega_j), \omega_j | y_{\omega_j}^s, \omega_i | x]; \\ x \bar{\succeq}_0 [f(\omega_j), \omega_j | y_{\omega_j}^s, \omega_i | x] \text{ if } f(\omega_j) \succeq_0 x \end{cases} \quad \text{for some } j \neq i$$

P3 Take any f, g and fix ω_i. Suppose that $f(\omega_i) = g(\omega_i)$. Take any $y_j \in X$ for each $j \in \{1, ..., n\} \backslash \{i\}$. Then the following two statements are true:

(i) For any $x \in X$ if for all $j \neq i$,

$$\begin{cases} [y_{\omega_j}, \omega_j | g(\omega_j), \omega_i | x] \bar{\succeq}_s x; \\ [y_{\omega_j}, \omega_j | g(\omega_j), \omega_i | x] \succeq_s x \text{ if } x \succeq_s y_{\omega_j} \end{cases} \quad \text{and} \quad \begin{cases} x \succeq_0 [y_{\omega_j}, \omega_j | f(\omega_j), \omega_i | x]; \\ x \bar{\succeq}_0 [y_{\omega_j}, \omega_j | f(\omega_j), \omega_i | x] \text{ if } y_{\omega_j} \succeq_0 x \end{cases}$$

then $x \succeq_0 g$ implies $x \succeq_s f$.

(ii) For any $x \in X$ if for all $j \neq i$,

$$\begin{cases} x \succeq_s [y_{\omega_j}, \omega_j | g(\omega_j), \omega_i | x]; \\ x \bar{\succeq}_s [y_{\omega_j}, \omega_j | g(\omega_j), \omega_i | x] \text{ if } y_{\omega_j} \succeq_s x \end{cases} \quad \text{and} \quad \begin{cases} [y_{\omega_j}, \omega_j | f(\omega_j), \omega_i | x] \bar{\succeq}_0 x; \\ [y_{\omega_j}, \omega_j | f(\omega_j), \omega_i | x] \succeq_0 x \text{ if } x \succeq_0 y_{\omega_j} \end{cases}$$

then $g \succeq_0 x$ implies $f \succeq_s x$.

Next, we write the implications of the preferences expressed in P2 and P3 in terms of the MEU representation: Take any MEU preference $\succeq$ represented by $(\mathcal{M}, u)$. Take any $x, y, z \in X$ and any distinct states $\omega_i, \omega_j \in \Omega$. We have

$$\begin{aligned} & x \succeq [y, \omega_j | z, \omega_i | x] \\ & \Leftrightarrow \exists p \in \mathcal{M}, \; u(x) \geq p(\omega_j) u(y) + p(\omega_i) u(z) + [1 - p(\omega_i) - p(\omega_j)] u(x) \\ & \Leftrightarrow \exists p \in \mathcal{M}, \; [u(x) - u(y)] \frac{p(\omega_j)}{p(\omega_i)} \geq [u(z) - u(x)] \\ & \Leftarrow [u(x) - u(y)] \max_{p \in \mathcal{M}} \frac{p(\omega_j)}{p(\omega_i)} \geq [u(z) - u(x)]. \end{aligned}$$

When $u(x) - u(y) \geq 0$,

$$x \succeq [y, \omega_j | z, \omega_i | x] \Leftrightarrow [u(x) - u(y)] \max_{p \in \mathcal{M}} \frac{p(\omega_j)}{p(\omega_i)} \geq [u(z) - u(x)].$$

Similarly, we have

$$
\begin{aligned}
&x\bar{\succeq}[y,\omega_j|z,\omega_i|x] \\
&\Leftrightarrow \forall p\in\mathcal{M},\ u(x)\geq p(\omega_j)u(y)+p(\omega_i)u(z)+[1-p(\omega_i)-p(\omega_j)]u(x) \\
&\Leftrightarrow \forall p\in\mathcal{M},\ [u(x)-u(y)]\frac{p(\omega_j)}{p(\omega_i)}\geq[u(z)-u(x)] \\
&\Rightarrow [u(x)-u(y)]\max_{p\in\mathcal{M}}\frac{p(\omega_j)}{p(\omega_i)}\geq[u(z)-u(x)].
\end{aligned}
$$

When $u(x)-u(y)\leq 0$,

$$
x\bar{\succeq}[y,\omega_j|z,\omega_i|x]\Leftrightarrow[u(x)-u(y)]\max_{p\in\mathcal{M}}\frac{p(\omega_j)}{p(\omega_i)}\geq[u(z)-u(x)].
$$

Consequently,

$$
\begin{cases}
x\succeq[y,\omega_j|z,\omega_i|x]; \\
x\bar{\succeq}[y,\omega_j|z,\omega_i|x] \text{ if } y\succeq x
\end{cases}
\Leftrightarrow [u(x)-u(y)]\max_{p\in\mathcal{M}}\frac{p(\omega_j)}{p(\omega_i)}\geq u(z)-u(x),
$$

and

$$
\begin{cases}
[y,\omega_j|z,\omega_i|x]\bar{\succeq}x; \\
[y,\omega_j|z,\omega_i|x]\succeq x \text{ if } x\succeq y
\end{cases}
\Leftrightarrow [u(x)-u(y)]\max_{p\in\mathcal{M}}\frac{p(\omega_j)}{p(\omega_i)}\leq u(z)-u(x).
$$

A.7.2 Necessity

Now we prove the necessity of P2 and P3. Consider P2 first. For any belief $q\in\Delta(\Omega)$, if $p\in\Delta(\Omega)$ is the Bayesian updating of q given signal s and likelihood function ℓ, then for any states ω_i,ω_j,

$$
\frac{p(\omega_j)}{p(\omega_i)}=\frac{\ell(s|\omega_j)}{\ell(s|\omega_i)}\times\frac{q(\omega_j)}{q(\omega_i)}.
$$

Hence $\mathcal{M}_s=BU(\mathcal{M}_0,\ell,s)$ implies

$$
\max_{p\in\mathcal{M}_s}\frac{p(\omega_j)}{p(\omega_i)}=\frac{\ell(s|\omega_j)}{\ell(s|\omega_i)}\times\max_{q\in\mathcal{M}_0}\frac{q(\omega_j)}{q(\omega_i)}.
$$

Fixing any i, we have

$$
\sum_{s\in S}\ell(s|\omega_i)\max_{p\in\mathcal{M}_s}\frac{p(\omega_j)}{p(\omega_i)}=\sum_{s\in S}\ell(s|\omega_i)\times\frac{\ell(s|\omega_j)}{\ell(s|\omega_i)}\times\max_{q\in\mathcal{M}_0}\frac{q(\omega_j)}{q(\omega_i)}=\max_{q\in\mathcal{M}_0}\frac{q(\omega_j)}{q(\omega_i)}
$$

for any $j \neq i$. Hence the vector $\left(\max_{q\in\mathcal{M}_0} \frac{q(\omega_j)}{q(\omega_i)}\right)_{j\neq i} \in \mathbb{R}^{|\Omega|-1}$ is in the convex hull of $\left\{\left(\max_{p\in\mathcal{M}_s} \frac{p(\omega_j)}{p(\omega_i)}\right)_{j\neq i} : s \in \mathcal{S}\right\}$.

For all $s \in \mathcal{S}$ and all $j \neq i$, because

$$\begin{cases} x \succeq_s [f(\omega_j), \omega_j | y_j^s, \omega_i | x]; \\ x \bar{\succeq}_s [f(\omega_j), \omega_j | y_j^s, \omega_i | x] \text{ if } f(\omega_j) \succeq_s x \end{cases},$$

we have

$$[u(x) - u(f(\omega_j))] \max_{p\in\mathcal{M}_s} \frac{p(\omega_j)}{p(\omega_i)} \geq u(y_j^s) - u(x).$$

Thus, for all $s \in \mathcal{S}$,

$$\sum_{j\neq i} [u(x) - u(f(\omega_j))] \max_{p\in\mathcal{M}_s} \frac{p(\omega_j)}{p(\omega_i)} \geq \sum_{j\neq i} [u(y_j^s) - u(x)].$$

Because $\sum_{j\neq i} \frac{y_j^s}{|\Omega|-1} = \sum_{j\neq i} \frac{y_j^0}{|\Omega|-1}$, we have $\sum_{j\neq i} u(y_j^s) = \sum_{j\neq i} u(y_j^0)$. Consequently,

$$\sum_{j\neq i} [u(x) - u(f(\omega_j))] \max_{q\in\mathcal{M}_0} \frac{q(\omega_j)}{q(\omega_i)} \geq \sum_{j\neq i} [u(y_j^0) - u(x)].$$

This implies that

$$\exists j \neq i,\ [u(x) - u(f(\omega_j))] \max_{q\in\mathcal{M}_0} \frac{q(\omega_j)}{q(\omega_i)} \geq u(y_j^0) - u(x).$$

Therefore,

$$\exists j \neq i,\ \begin{cases} x \succeq_0 [f(\omega_j), \omega_j | y_j^s, \omega_i | x]; \\ x \bar{\succeq}_0 [f(\omega_j), \omega_j | y_j^s, \omega_i | x] \text{ if } f(\omega_j) \succeq_0 x. \end{cases}$$

We have established P2.

Now we check P3. By Proposition 2.1, $\mathcal{M}_s = BU(\mathcal{M}_0, \ell, s)$ implies that $f_x^{\ell,s} \succeq_0 x \Leftrightarrow f \succeq_s x$. Fix any i. Suppose that

$$\begin{cases} [y_j, \omega_j | g(\omega_j), \omega_i | x] \bar{\succeq}_s x; \\ [y_j, \omega_j | g(\omega_j), \omega_i | x] \succeq_s x \text{ if } x \succeq_s y_j \end{cases} \quad \text{and} \quad \begin{cases} x \succeq_0 [y_j, \omega_j | f(\omega_j), \omega_i | x]; \\ x \bar{\succeq}_0 [y_j, \omega_j | f(\omega_j), \omega_i | x] \text{ if } y_j \succeq_0 x \end{cases}$$

for all $j \neq i$. Then

$$u(g(\omega_j)) - u(x) \geq [u(x) - u(y_j)] \max_{p\in\mathcal{M}_s} \frac{p(\omega_j)}{p(\omega_i)} = [u(x) - u(y_j)] \frac{\ell(s|\omega_j)}{\ell(s|\omega_i)} \max_{q\in\mathcal{M}_0} \frac{q(\omega_j)}{q(\omega_i)}$$
$$\geq \frac{\ell(s|\omega_j)}{\ell(s|\omega_i)} [u(f(\omega_j)) - u(x)].$$

Hence

$$u(g(\omega_j)) \geq \frac{\ell(s|\omega_j)}{\ell(s|\omega_i)} u(f(\omega_j)) + \left[1 - \frac{\ell(s|\omega_j)}{\ell(s|\omega_i)}\right] u(x) \ \forall\ j \neq i.$$

Moreover, $f(\omega_i) = g(\omega_i)$ by assumption. Thus, g state-by-state dominates the act $\left(\frac{\ell(s|\omega_j)}{\ell(s|\omega_i)} f(\omega_j) + \left[1 - \frac{\ell(s|\omega_j)}{\ell(s|\omega_i)}\right] x\right)_{j=1}^{|\Omega|}$. Note that

$$\ell(s|\omega_i) \left(\frac{\ell(s|\omega_j)}{\ell(s|\omega_i)} f(\omega_j) + \left[1 - \frac{\ell(s|\omega_j)}{\ell(s|\omega_i)}\right] x\right)_{j=1}^{|\Omega|} + (1 - \ell(s|\omega_i))x = f_x^{\ell,s}.$$

Hence

$$x \succeq_0 g \Rightarrow x \succeq_0 \left(\frac{\ell(s|\omega_j)}{\ell(s|\omega_i)} f(\omega_j) + \left[1 - \frac{\ell(s|\omega_j)}{\ell(s|\omega_i)}\right] x\right)_{j=1}^{|\Omega|} \Rightarrow x \succeq_0 f_x^{\ell,s} \Rightarrow x \succeq_s f.$$

Suppose instead

$$\begin{cases} x \succeq_s [y_j, \omega_j | g(\omega_j), \omega_i | x]; \\ x \bar{\succeq}_s [y_j, \omega_j | g(\omega_j), \omega_i | x] \text{ if } y_j \succeq_s x \end{cases} \quad \text{and} \quad \begin{cases} [y_j, \omega_j | f(\omega_j), \omega_i | x] \bar{\succeq}_0 x; \\ [y_j, \omega_j | f(\omega_j), \omega_i | x] \succeq_0 x \text{ if } x \succeq_0 y_j \end{cases}$$

for all $j \neq i$. Then

$$\begin{aligned} u(g(\omega_j)) - u(x) &\leq [u(x) - u(y_j)] \max_{p \in \mathcal{M}_s} \frac{p(\omega_j)}{p(\omega_i)} = [u(x) - u(y_j)] \frac{\ell(s|\omega_j)}{\ell(s|\omega_i)} \max_{q \in \mathcal{M}_0} \frac{q(\omega_j)}{q(\omega_i)} \\ &\leq \frac{\ell(s|\omega_j)}{\ell(s|\omega_i)} [u(f(\omega_j)) - u(x)]. \end{aligned}$$

Hence

$$u(g(\omega_j)) \leq \frac{\ell(s|\omega_j)}{\ell(s|\omega_i)} u(f(\omega_j)) + \left[1 - \frac{\ell(s|\omega_j)}{\ell(s|\omega_i)}\right] u(x) \ \forall\ j \neq i.$$

Thus, the act $\left(\frac{\ell(s|\omega_j)}{\ell(s|\omega_i)} f(\omega_j) + \left[1 - \frac{\ell(s|\omega_j)}{\ell(s|\omega_i)}\right] x\right)_{j=1}^{|\Omega|}$ state-by-state dominates g. We have

$$g \succeq_0 x \Rightarrow \left(\frac{\ell(s|\omega_j)}{\ell(s|\omega_i)} f(\omega_j) + \left[1 - \frac{\ell(s|\omega_j)}{\ell(s|\omega_i)}\right] x\right)_{j=1}^{|\Omega|} \succeq_0 x \Rightarrow f \succeq_s x.$$

We have established P3.

A.7.3 Sufficiency

Let $|\Omega| = N$. For each $j = 1, \cdots, N-1$, let

$$\phi_j^0 = \max_{q \in \mathcal{M}_0} \frac{q(\omega_j)}{q(\omega_N)} \text{ and } \phi_j^s = \max_{p \in \mathcal{M}_s} \frac{p(\omega_j)}{p(\omega_N)} \ \forall\ s \in \mathcal{S}.$$

Let $\phi^0 = (\phi^0_j)_{j=1}^{N-1} \in \mathbb{R}^{N-1}$ and $\phi^s = (\phi^s_j)_{j=1}^{N-1} \in \mathbb{R}^{N-1}$.

Claim that P2 implies that ϕ^0 lies in the relative interior of the convex hull of $\{\phi^s : s \in \mathcal{S}\}$. If not, then by a separating hyperplane theorem, there exists $r \in \mathbb{R}^{N-1}$ and $k \in \mathbb{R}$ such that (i) $\phi^s \cdot r \geq k$ for all $s \in \mathcal{S}$, (ii) $\phi^s \cdot r > k$ for some $s \in \mathcal{S}$, and (iii) $\phi^0 \cdot r \leq k$.

For any $s \in \mathcal{S}$, take $(k^s_j)_{j=1}^{N-1} \in \mathbb{R}^{N-1}$ such that (i) $\phi^s_j r_j \geq k^s_j$ for all $j = 1, \cdots, N-1$, and (ii) $\sum_{j=1}^{N-1} k^s_j = k$.

Take $(k^0_j)_{j=1}^{N-1} \in \mathbb{R}^{N-1}$ such that (i) $\phi^0_j r_j \leq k^0_j$ for all $j = 1, \cdots, N-1$, and (ii) $\sum_{j=1}^{N-1} k^0_j = k$.

Fix $x \in X$. Consider any $s \in \mathcal{S}$. Let f be an act such that $u(x) - u(f(\omega_j)) = r_j$ for all $j = 1, \cdots, N-1$. Let $y^s_j \in X$ satisfying $u(y^s_j) - u(x) = k^s_j$ for all $j = 1, \cdots, N-1$. We have

$$
\begin{aligned}
&\forall\, j \in \{1, \cdots, N-1\},\ \phi^s_j r_j \geq k^s_j \\
\Leftrightarrow &\forall\, j \in \{1, \cdots, N-1\},\ [u(x) - u(f(\omega_j))] \max_{p \in \mathcal{M}_s} \frac{p(\omega_j)}{p(\omega_N)} \geq u(y^s_j) - u(x) \\
\Rightarrow &\forall\, j \in \{1, \cdots, N-1\}, \begin{cases} x \succeq_s [f(\omega_j), \omega_j | y^s_j, \omega_N | x]; \\ x \bar{\succeq}_s [f(\omega_j), \omega_j | y^s_j, \omega_N | x] \text{ if } f(\omega_j) \succeq_s x. \end{cases}
\end{aligned}
$$

Let $y^0_j \in X$ satisfying $u(y^0_j) - u(x) = k^0_j$ for all $j = 1, \cdots, N-1$. We have

$$
\begin{aligned}
&\forall\, j \in \{1, \cdots, N-1\},\ \phi^0_j r_j \leq k^0_j \\
\Leftrightarrow &\forall\, j \in \{1, \cdots, N-1\},\ [u(x) - u(f(\omega_j))] \max_{q \in \mathcal{M}_0} \frac{q(\omega_j)}{q(\omega_N)} \leq u(y^0_j) - u(x) \\
\Rightarrow &\forall\, j \in \{1, \cdots, N-1\}, \begin{cases} [f(\omega_j), \omega_j | y^0_j, \omega_N | x] \bar{\succeq}_0 x; \\ [f(\omega_j), \omega_j | y^0_j, \omega_N | x] \succeq_0 x \text{ if } x \succeq_0 f(\omega_j). \end{cases}
\end{aligned}
$$

We have found a violation of P2. Therefore, we must have ϕ^0 in the relative interior of the convex hull of $\{\phi^s : s \in \mathcal{S}\}$.

Let $(\lambda_s)_{s \in \mathcal{S}}$ be such that (i) $\lambda_s \in (0, 1)$ for all $s \in \mathcal{S}$, (ii) $\sum_{s \in \mathcal{S}} \lambda_s = 1$, and (iii) $\phi^0 = \sum_{s \in \mathcal{S}} \lambda_s \phi^s$. Let $\ell(s|\omega_N) = \lambda_s$ and $\ell(s|\omega_j) = \lambda_s \phi^s_j / \phi^0_j$ for each $s \in \mathcal{S}$ and each $j \in \{1, \cdots, N-1\}$. Observe that $\ell(s|\omega) > 0$ and $\sum_{s \in \mathcal{S}} \ell(s|\omega) = 1$ for all $s \in \mathcal{S}$ and $\omega \in \Omega$.

Now we verify that $\mathcal{M}_s = BU(\mathcal{M}_0, \ell, s)$. By Proposition 2.1, it is equivalent to show $f^{\ell,s}_x \succeq_0 x \Leftrightarrow f \succeq_s x$. Take any act f and constant act x. Let g be such that

$$
g(\omega) := \frac{\ell(s|\omega)}{\ell(s|\omega_N)} f(\omega) + \left[1 - \frac{\ell(s|\omega)}{\ell(s|\omega_N)}\right] x \ \forall\, \omega \in \Omega.
$$

For every $j < N$, pick y_j such that

$$u(x) = \frac{u(y_j) + \frac{1}{\phi_j^0} u(f(\omega_j))}{1 + \frac{1}{\phi_j^0}}.$$

Then we have

$$\frac{u(f(\omega_j)) - u(x)}{u(x) - u(y_j)} = \phi_0^j$$

and

$$\frac{\frac{\ell(s|\omega_j)}{\ell(s|\omega_N)} u(f(\omega_j)) + \left[1 - \frac{\ell(s|\omega_j)}{\ell(s|\omega_N)}\right] u(x) - u(x)}{u(x) - u(y_j)} = \frac{\ell(s|\omega_j)}{\ell(s|\omega_N)} \times \phi_j^0 = \phi_j^s.$$

Thus,

$$u(f(\omega_j)) - u(x) = [u(x) - u(y_j)] \max_{q \in \mathcal{M}_0} \frac{q(\omega_j)}{q(\omega_N)}$$

and

$$u(g(\omega_j)) - u(x) = [u(x) - u(y_j)] \max_{p \in \mathcal{M}_s} \frac{p(\omega_j)}{p(\omega_N)}.$$

Therefore, we have

$$\begin{cases} [y_j, \omega_j | g(\omega_j), \omega_i | x] \bar{\succeq}_s x; \\ [y_j, \omega_j | g(\omega_j), \omega_i | x] \succeq_s x \text{ if } x \succeq_s y_j \end{cases} \quad \text{and} \quad \begin{cases} x \succeq_0 [y_j, \omega_j | f(\omega_j), \omega_i | x]; \\ x \bar{\succeq}_0 [y_j, \omega_j | f(\omega_j), \omega_i | x] \text{ if } y_j \succeq_0 x \end{cases}.$$

We also have

$$\begin{cases} x \succeq_s [y_j, \omega_j | g(\omega_j), \omega_i | x]; \\ x \bar{\succeq}_s [y_j, \omega_j | g(\omega_j), \omega_i | x] \text{ if } y_j \succeq_s x \end{cases} \quad \text{and} \quad \begin{cases} [y_j, \omega_j | f(\omega_j), \omega_i | x] \bar{\succeq}_0 x; \\ [y_j, \omega_j | f(\omega_j), \omega_i | x] \succeq_0 x \text{ if } x \succeq_0 y_j \end{cases}.$$

By P3, we have $g \succeq_0 x$ if and only if $f \succeq_s x$. Since $\ell(s|\omega_N) g + (1 - \ell(s|\omega_N)) x = f_x^{\ell,s}$, we obtain $f_x^{\ell,s} \succeq_0 x \Leftrightarrow f \succeq_s x$. This completes the proof.

References

D. S. Ahn. Ambiguity without a state space. *The Review of Economic Studies*, 75(1):3–28, 2008.

F. Anscombe and R. Aumann. A definition of subjective probability. *The Annals of Mathematical Statistics*, 34:199–205, 1963.

D. Beauchêne, J. Li, and M. Li. Ambiguous persuasion. *Journal of Economic Theory*, 179:312–365, 2019.

X. Cheng. Relative maximum likelihood updating of ambiguous beliefs. *Journal of Mathematical Economics*, 99:102587, 2022.

G. S. Collins, M. van Smeden, and R. D. Riley. Covid-19 prediction models should adhere to methodological and reporting standards. *European Respiratory Journal*, 56(3), 2020.

A. Dominiak, M. Kovach, and G. Tserenjigmid. Minimum distance belief updating with general information. Technical report, Working paper, 2021.

M. Dumav and M. Stinchcombe. The von neumann/morgenstern approach to ambiguity. *Institute of Mathematical Economics Working Paper*, (480), 2013.

L. G. Epstein and Y. Halevy. Hard-to-interpret signals. *mimeo*, 2022.

L. G. Epstein and M. Schneider. Learning under ambiguity. *The Review of Economic Studies*, 74(4):1275–1303, 2007.

L. G. Epstein and M. Schneider. Ambiguity, information quality, and asset pricing. *The Journal of Finance*, 63(1):197–228, 2008.

L. G. Epstein, M. Schneider, et al. Ambiguity and asset markets. *Annual Review of Financial Economics*, 2(1):315–346, 2010.

J. H. Faro and J.-P. Lefort. Dynamic objective and subjective rationality. *Theoretical Economics*, 14(1):1–14, 2019.

T. Gajdos, T. Hayashi, J.-M. Tallon, and J.-C. Vergnaud. Attitude toward imprecise information. *Journal of Economic Theory*, 140(1):27–65, 2008.

P. Ghirardato. Revisiting savage in a conditional world. *Economic theory*, 20: 83–92, 2002.

I. Gilboa and D. Schmeidler. Maxmin expected utility with non-unique prior. *Journal of Mathematical Economics*, 18(2):141–153, 1989.

I. Gilboa and D. Schmeidler. Updating ambiguous beliefs. *Journal of economic theory*, 59(1):33–49, 1993.

F. Gul and W. Pesendorfer. Evaluating ambiguous random variables from choquet to maxmin expected utility. *Journal of Economic Theory*, 192:105129, 2021.

J.-Y. Jaffray. Linear utility theory for belief functions. *Operations Research Letters*, 8(2):107–112, 1989.

C. Kellner and M. T. Le Quement. Endogenous ambiguity in cheap talk. *Journal of Economic Theory*, 173:1–17, 2018.

M. Kovach. Ambiguity and partial bayesian updating. *Economic Theory*, pages 1–26, 2023.

G. Lanzani. Dynamic concern for misspecification. *Available at SSRN 4454504*, 2023.

W. Olszewski. Preferences over sets of lotteries. *The Review of Economic Studies*, 74(2):567–595, 2007.

C. Pacheco Pires. A rule for updating ambiguous beliefs. *Theory and Decision*, 53(2):137–152, 2002.

M. Rabin and J. L. Schrag. First impressions matter: A model of confirmatory bias. *The quarterly journal of economics*, 114(1):37–82, 1999.

P. Reshidi, J. Thereze, and M. Zhang. Asymptotic learning with ambiguous information. *mimeo*, 2022.

F. Riedel, J.-M. Tallon, and V. Vergopoulos. Dynamically consistent preferences under imprecise probabilistic information. *Journal of Mathematical Economics*, 79:117–124, 2018.

D. Schmeidler. Subjective probability and expected utility without additivity. *Econometrica: Journal of the Econometric Society*, pages 571–587, 1989.

D. Shishkin, P. Ortoleva, et al. Ambiguous information and dilation: An experiment. Technical report, 2021.

R. Tang. A theory of contraction updating. *Available at SSRN*, 2022.

F. Wang, I. Gilboa, and S. Minardi. Likelihood regions: An axiomatic approach. *mimeo*, 2023.