Negative Bubbles*

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Abstract

We develop a macroeconomic model with credit frictions in which firms' ability to borrow depends on the value of their equity. Under irreversibility of capital investment, this framework admits negative asset price bubbles in addition to the positive ones emphasized in the literature. We show that, depending on the cost of seizing the firm in bankruptcy, negative bubbles may be contractionary or expansionary. Expansionary negative bubbles arise due to two offsetting effects. Negative bubbles reduce overall collateral which is contractionary when credit constraints bind. However, the contraction in aggregate collateral encourages the production of tangible collateral (capital) which is expansionary. When capital is highly pledgeable and, therefore, good collateral, the second effect dominates, and negative bubbles expand real economic activity, leading to dynamic inefficiency.

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1 Introduction

Asset markets go through periods of boom and bust which cannot always be easily explained with fundamentals. Several theories have been developed to account for the volatility of asset prices by appealing to ups and downs in investor sentiment that lead to deviations from fundamentals called 'bubbles'. In these models (Tirole (1985), Martin and Ventura (2012), Farhi and Tirole (2012) and Miao and Wang (2015, 2018), Hirano and Yanagawa (2016), Hirano and Toda (2024a), among many others) investors are willing to buy overvalued assets because they rationally expect to be able to sell them to others in the future. One limitation of all these models that our paper seeks to address is that they can only generate asset *over-valuations*. In contrast, there is evidence (Shiller (2000)) that asset prices can undershoot fundamentals and that sentiment can also be negative as well as positive.[1] We build a theory of asset price bubbles that admits both over-valuations (*positive bubbles*) and under-valuations (*negative bubbles*) and characterize their real and welfare effects.

Our model features credit constrained firms that use the firm's market value as collateral. As shown by Miao and Wang (2015, 2018), this can generate multiple solutions to the firm's equity price.[2] In the 'fundamental' solution, the firm's equity value is equal to the market value of tangible assets. A 'bubbly' solution also exists when credit constraints are binding because a higher firm valuation increases access to credit and boosts profitability thus validating the initial increase in the firm's equity price.[3] However, in the original model of Miao and Wang (2015, 2018), negative bubbles cannot exist because the firm's tangible assets (its capital stock) can be freely sold to other firms. The owners of a firm whose equity is trading below the market value of capital could always sell the capital and close the firm, thus ruling out the existence of a negative bubble. What our paper shows is that, if we assume that capital is firm-specific once installed, this argument does not hold and negative bubbles can exist due to the same intuition as positive ones. Pessimism about firm valuations leads to the firms' equity trading below the market value of capital. This pessimism is self-fulfilling when it leads to reduced credit access and lower profits. However, due to irreversibility, the capital cannot be sold to another firm and

[1] Shiller (2000) presents evidence for the existence of both positive and negative bubbles. Based on a questionnaire of institutional investors, he identifies 'negative bubbles' as periods in which investors expect stocks to decline sharply causing them to sell assets before the decline occurs.

[2] Gertler and Karadi (2011) also features borrowing constraints that depend on market value. In our paper we use the simpler Miao and Wang (2018) framework but we also have a version of Gertler and Karadi (2011) with negative bubbles.

[3] Azariadis et al. (2016) generate reputational collateral from a history of repayment when access to credit is profitable.

the option to close the firm is not exercised as long as the total value of the firm is positive.[4]

More specifically, our model assumes a continuum of neoclassical firms that own the capital stock but only a small subset of them have the opportunity to produce new capital goods in a given period. This means that the firms with an investment opportunity need to borrow in order to produce new capital goods. We assume that, due to a moral hazard problem, borrowing must be collateralized by the value of the firm in bankruptcy. And aggregate collateral may be insufficient to achieve the first best allocation because only a fraction of the firm's capital can be transferred to new owners (capital is only partially pledgeable). When credit constraints bind, the resulting shortage of capital elevates its price above production cost making the ability to access credit and produce more capital profitable. Like in Miao and Wang (2015, 2018), bubbles on the firm's value can exist because a more valuable firm obtains more credit and makes higher profits in the event of an investment opportunity. In a bubbly equilibrium, the firm trades above the market value of tangible assets by an amount that is equal to the net present value of the higher profits achieved through the improved credit access.[5]

Whether the bubble is positive or negative can be understood by the following thought experiment. Suppose we are in the bubbleless equilibrium and the market value of the firm increases by a small amount permanently. Does this more valuable firm with greater credit access generate a return to its owners that is more or less than their required rate of return,[6] holding all else equal? On the one hand, a higher firm valuation (holding profits constant) reduces the rate of return on equity. On the other hand, the higher profits from being able to produce more capital goods boost the rate of return on equity. When capital is scarce due to tight credit constraints, the ability to produce new capital goods is very valuable and the second effect dominates. Then the additional credit that arises from the higher firm valuation actually increases the rate of return on the firm's equity. In such an economy, the bubbly steady state features a positive bubble which increases collateral and the aggregate supply of capital goods, reducing the profit from producing new capital goods. In the end, the extra profit due to the bubble (the bubble's 'dividend') is equal to the risk free net interest rate so that the firm delivers the household's required rate of return. These are the (positive) bubbles emphasized in

[4]In the rational bubbles literature (Tirole (1985), Martin and Ventura (2012), Farhi and Tirole (2012)), bubbles arise on intrinsically worthless assets with a zero fundamental value. Hence bubbles can never be negative as long as they can be costlessly disposed of (Martin and Ventura (2018)).

[5]In this sense, the value of the firm still equals to the net present value of future dividends, even though it is higher than the market value of tangible assets. See Hirano and Toda (2024b) for discussion on 'rational' bubbles.

[6]For simplicity, we assume risk neutral households whose required rate of return is equal to the inverse of the rate of time preference. All results generalize to assuming a finite intertemporal elasticity of substitution.

the literature and they are always expansionary and welfare enhancing.

When capital is less scarce, the profits from producing new capital goods that arise from a unit bubble are lower than the net real interest rate. A bubbly equilibrium still exists but now the bubble is negative, squeezing the value of the firm and the quantity of collateral in the economy. In a bubbly equilibrium, the price of capital increases until the profit from producing new capital goods (the bubble's 'dividend') is equal to the risk free net interest rate. For intermediate values of the pledgeability of capital, these negative bubbles are contractionary: they push output further below the first best and reduce welfare.

Once capital is highly pledgeable (i.e., only a small fraction of the firm's capital is lost in bankruptcy), credit constraints do not bind and the economy is at the first best in the bubbleless equilibrium. Negative bubbles still exist but become expansionary beyond a certain level of the pledgeability of capital, causing output to rise above the first best. The intuition for this surprising result is that when the economy moves from the bubbleless equilibrium to an equilibrium with a negative bubble, this raises the scarcity of capital and its market price with two offsetting effects on the quantity of capital demanded by firms. On the one hand, the higher capital price requires a higher rental rate to ensure that the rate of return to capital for firms is preserved. This effect is contractionary. On the other hand, the higher capital price increases the profits made by firms with an investment opportunity and boosts the collateral premium attached to capital goods which can be pledged to obtain more debt funding. A higher collateral premium makes firms more willing to accumulate capital even with a lower rental rate (or 'dividend yield') because they can obtain liquidity against the capital when an investment opportunity arises. This effect is expansionary. When the pledgeability of capital is sufficiently high, the second expansionary collateral premium effect dominates and the overall level of capital (and hence output) can rise above the first best. However, even though such negative bubbles are expansionary, they are still welfare-reducing because they lead to overinvestment and dynamic inefficiency.

Finally, we examine the policy implications of our model. We show that a subsidy/tax on debt can restore the economy to the first best when there is under/overinvestment regardless of whether the economy is in a bubbly or bubbleless equilibrium. The optimal subsidy decreases following a switch to an equilibrium with positive bubbles. The subsidy increases if there is a negative contractionary bubble. Finally, a tax is imposed if there is a negative expansionary bubble which leads to overinvestment.

Negative bubbles can also exist in models with non-rational expectations. For instance, in the framework of Adam and Marcet (2011), private investors are assumed to postulate an autoregressive law of motion for the price-dividend ratio of an asset and to estimate the parameters of the law of motion from the (model-generated) data. The framework in Adam and Marcet (2011) generates momentum in asset prices leading to overvaluations when expectations are optimistic and undervaluations when they are pessimistic.

Looking at the literature more broadly, there are other models where changes in collateral values or collateral pledgeability can lead to changes in firm investment and asset demand. In Kiyotaki and Moore (2019), limited pledgeability and resaleability of physical capital creates demand for liquid assets (fiat money) which can be sold when the profitable possibility to invest in new capital goods arises. In our framework, capital cannot be resold and it has limited pledgeability but we abstract from monetary assets. For us liquidity is equivalent to pledgeability which is why capital rises in value during negative bubble episodes when overall credit supply is restricted. When capital is very pledgeable, it is also accumulated to some extent due to its liquidity properties (despite having a low conventional rate of return). In the Appendix we also develop an extension of our modelling framework with liquid short term government debt (money) and show that this addition does not remove the existence of negative bubbles as long as they are not too negative.[7]

The paper is organized as follows. Section 2 outlines the baseline model, section 3 solves for the steady state with and without bubbles and discusses the conditions under which a bubbly equilibrium exists and is expansionary or contractionary for real economic activity. Section 4 shows the transitional dynamics to steady states with positive or negative bubbles. Section 5 discusses the policy interventions that can bring the economy to the first best. Section 6 concludes.

2 The Model

Our model is a closed economy populated by a continuum of measure one of risk-neutral households who supply a unit of labour inelastically.[8] Their preferences are given by the following

[7] Miao and Wang (2014) consider (positive) sectoral asset price bubbles and show that they cause a reallocation of capital towards the sector with the bubble. In Matsuyama (2013), changes in firm net worth cause a shift in the composition of investment along the productivity/pledgeability spectrum.

[8] These assumptions are only for simplicity. More general preferences and flexible labour supply do not change our main results in any way.

value function:

$$W_t = C_t + \beta W_{t+1}, \quad 0 < \beta < 1, \tag{1}$$

where W_t is lifetime welfare, C_t is period consumption and β is the time discount factor. The household budget constraint is given below:

$$C_t + A_{t+1} = w_t + R_t A_t + S_t, \tag{2}$$

where A_t is a real bond which is traded among households and is in zero net supply. It yields a real interest rate of R_t. w_t is the real wage rate and S_t is the dividend paid to the household by firms.

There is a continuum of firms which own the capital stock. They start each period with their capital from the previous period and produce consumption goods using capital and labour. The firm's production function is given by

$$y_t = k_t^{\alpha} l_t^{1-\alpha}, \tag{3}$$

where y_t, k_t, and l_t respectively denote output, capital stock and labour in period t. Since the production function is constant returns to scale, the quasi-rent to capital is defined by

$$r_t k_t = y_t - w_t l_t.$$

In what follows, we call r_t the rental rate of capital.[9]

We assume that the firm has stochastic investment opportunity. Specifically, within the period, with a probability π, it has the possibility to produce new capital goods. We assume that one unit of new capital goods is produced from one unit of consumption goods. To finance this capital goods production, the firm can access only the quasi-rent from its old capital ($r_t k_t$) as well as intra-period loans from households (d_t).

$$i_t = r_t k_t + d_t. \tag{4}$$

These loans are risk free and carry a gross interest rate of unity but are limited to the value of

[9] Note that, even though r_t is called the rental rate, there is no unified rental market for capital because, as will be discussed shortly, capital is firm specific and irreversible.

the firm in bankruptcy:[10]

$$d_t \leqq V(\lambda k_t). \tag{5}$$

In bankruptcy, lenders can recover only a fraction λ of the firm's capital.[11]

Capital is irreversible at the firm level. This implies that capital goods are fully generic when newly produced and any firm can purchase them and install them. However, once capital is installed in one firm, it cannot be subsequently re-sold to another because it becomes fully firm-specific. We also assume that capital depreciates at the rate of δ per period. Those assumptions imply that the individual firm faces a capital accumulation equation

$$k_{t+1} = (1-\delta) k_t + m_t, \tag{6}$$

where

$$m_t \geqslant 0 \tag{7}$$

is new capital purchases.

The value function of the firm is therefore given by

$$V(k_t) = \max_{m_t, i_t} \Big[r_t k_t - q_t m_t + \pi (q_t - 1) i_t + \beta V(k_{t+1}) \Big]. \tag{8}$$

In equation (8), q_t denotes the market price of capital. Since its production cost in terms of consumption goods is unity, $q_t - 1$ represents the firm's profit from producing new capital. The firm decides how much of new capital goods i_t to produce if it gets the productive opportunity and how much new capital goods m_t to install. The maximization is subject to the flow of funds constraint (4), the borrowing constraint (5), the capital accumulation constraint (6), the irreversibility constraint (7) and the limited liability constraint

$$V(k_t) \geqslant 0.$$

Now we can proceed to solve the firm's problem for the cases when the irreversibility constraint

[10] We specify the borrowing constraint in terms of the ex ante value of the firm (i.e. before the firm knows whether it has an investment opportunity). This is done for simplicity and the interpretation is that the debt is raised ex ante and the firm can default on it only then. Then if no investment opportunity arises, the debt is immediately repaid to the household.

[11] For simplicity we abstract from liquid assets such as fiat money and government debt which can be accumulated by firms in anticipation of investment opportunities (Kiyotaki and Moore (2019)). In the Appendix we develop a version of the model with government bonds and show that this does not affect the results we will derive in this section.

binds or does not bind.

2.1 Non-binding irreversibility and borrowing constraint

When the borrowing constraint is not binding, $q_t = 1$. Then, the firm is indifferent between producing new capital goods and not. When $m_t > 0$, the irreversibility constraint does not bind and we can substitute the capital evolution equation into the value function as follows:

$$V(k_t) = \max_{k_{t+1}} \Big[(r_t + 1 - \delta)k_t - k_{t+1} + \beta V(k_{t+1}) \Big]. \tag{9}$$

Guessing that the value function is linear in capital

$$V(k_t) = \phi_t k_t, \tag{10}$$

we can easily derive the standard first order condition for capital

$$1 = \beta \left(r_{t+1} + 1 - \delta \right),$$

and the value of installed capital is given by

$$\phi_t = r_t + 1 - \delta. \tag{11}$$

This is the first best where there is ample collateral, the price of capital is equal to replacement cost and capital is valued only as a store of value.

2.2 Non-binding irreversibility constraint and binding borrowing constraint

When $q_t > 1$, the firm would like to choose an infinite level of capital production and the collateral constraint binds. Let us guess that the value function has the following form (possibly also including a bubble term b_t):

$$V(k_t) = \phi_t k_t + b_t. \tag{12}$$

Substituting this into the collateral constraint (5), we obtain

$$d_t = \lambda \phi_t k_t + b_t.$$

Substituting this into the value function, we obtain the following functional equation in ϕ_t:

$$\phi_t k_t + b_t = \max_{k_{t+1}} \Big[(r_t + q_t(1-\delta))\, k_t - q_t k_{t+1} + \pi\,(q_t - 1)\,(r_t k_t + \lambda \phi_t k_t + b_t) + \beta\,(\phi_{t+1} k_{t+1} + b_{t+1}) \Big]. \tag{13}$$

The first order condition with respect to k_{t+1} gives a condition which equates the market price of new capital with the present discounted value of installed capital next period:

$$q_t = \beta \phi_{t+1}. \tag{14}$$

The envelope condition is given by

$$\phi_t = r_t + q_t(1-\delta) + \pi\,(q_t - 1)\,(r_t + \lambda \phi_t)\,,$$

which can be written as

$$\phi_t = \frac{(1 + \pi\,(q_t - 1))\, r_t + q_t(1-\delta)}{1 - \lambda \pi\,(q_t - 1)}. \tag{15}$$

To get a bit more intuition about the meaning of the above expression, we can rewrite it as follows:

$$\phi_t = \frac{1 + \pi\,(q_t - 1)}{1 - \lambda \pi\,(q_t - 1)} r_t + \frac{\beta}{1 - \lambda \pi\,(q_t - 1)}(1-\delta)\phi_{t+1}. \tag{16}$$

When credit constraints do not bind and $q_t = 1$, equation (15) reduces to

$$\phi_t = r_t + \beta(1-\delta)\phi_{t+1}. \tag{17}$$

By comparing equation (16) and (17), we can see that when the investment flow is restricted and $q_t > 1$, this boosts the dividend term in r_t by a factor of $\frac{1+\pi(q_t-1)}{1-\lambda\pi(q_t-1)}$ through the opportunity to use the quasi-rent on already installed capital to finance further investment (the numerator) and from the possibility to use leverage when doing so (the denominator).

In addition, even though the firm cannot sell already installed capital to fund new investment, its market value can be used as collateral for loans from the household. This has the effect of reducing the effective discount rate from β to $\frac{\beta}{1-\lambda\pi(q_t-1)}$ in equation (16). The firm is willing to hold capital at a lower required rate of return than β^{-1} because capital can be pledged during an investment opportunity, giving rise to potential profits from the production of new capital goods. It therefore enjoys a collateral (or liquidity) premium driven by the 'spread' $\lambda\pi(q_t - 1)$.

Importantly, the collateral premium is large when capital is 'good collateral' (i.e. when λ is high). This feature of the model will be significant when we come to analyse the real effects of negative bubbles.

Finally, the process for the bubble (if it exists) is the following:

$$b_t = \pi (q_t - 1) b_t + \beta b_{t+1}. \tag{18}$$

Just as in Miao and Wang (2012), the bubble delivers a 'dividend' to the firm because it increases collateral and allows the firm to borrow and invest more in the event that it has an investment opportunity. This is profitable as long as $q_t > 1$. In this framework the bubble does not arise due to dynamic inefficiency and can therefore exist also when the real interest rate is equal to β^{-1} which is above the economy's growth rate (which is assumed to be unity). However, the bubble cannot deliver investors' required rate of return while remaining stable as a share of national income through capital gains alone. A 'dividend' is needed and the bubble delivers it by relaxing credit constraints and generating further profits for the firm when $q_t > 1$.

Crucially, our modified framework with irreversibility of capital at the firm level allows for negative bubbles as long as they are not so large so as to violate the limited liability constraint:

$$V(k_t) = \phi_t k_t + b_t \geqslant 0.$$

If capital were fully tradeable across firms, the firm could always sell its capital to another firm and then shut down, thus killing the negative bubble. But with irreversible capital, such a possibility no longer exists and negative bubbles can be present. We will investigate the consequences of such negative bubbles for the macroeconomy in subsequent analysis.

2.3 Binding irreversibility constraint

When

$$q_t > \beta \phi_{t+1},$$

the cost of capital is above its value, individual firms do not wish to buy new capital and the irreversibility constraint binds. Since firms are identical, when the constraint binds for one, it binds for all firms and capital purchases fall to zero. By market clearing, investment is zero too implying that $q_t \leq 1$.

The firm's value function is therefore characterized by the following functional equations while the irreversibility constraint binds:

$$\phi_t = r_t + \beta (1-\delta) \phi_{t+1},$$

and the evolution of b_t satisfies

$$b_t = \beta b_{t+1}.$$

For the parameter values we consider in the subsequent numerical solution of our model economy, the irreversibility constraint does not bind even when a large positive bubble collapses. In Figure 7 in Section B in the Appendix, we consider the collapse of a bubble when $\lambda = 0.1$. Such a bubble is positive and expansionary and its collapse reduces collateral and investment. Nevertheless, as Figure 7 shows, the fall in investment is not large enough to hit the irreversibility constraint. Hence we do not consider this constraint in the numerical exercises we perform in the rest of the paper. Irreversible capital at the individual firm level is nevertheless an important precondition for the existence of negative bubbles.

2.4 Aggregate equilibrium conditions

Now we characterise the aggregate equilibrium of the model economy. The aggregate output (Y_t) is given by

$$Y_t = K_t^{\alpha} L^{1-\alpha}, \tag{19}$$

where K_t is aggregate capital stock at time t. We assume that aggregate labour supply $L = 1$ is fixed.

In the aggregate economy, we have three more aggregate equilibrium conditions in addition to equation (15) and the aggregate version of equation (18). The rental rate of capital is given by

$$r_t = \alpha K_t^{\alpha-1}, \tag{20}$$

and the aggregate capital stock evolves as follows:

$$K_{t+1} = (1-\delta) K_t + I_t, \tag{21}$$

where I_t denotes aggregate investment. When $q_t > 1$, I_t is given by the binding aggregate collateral constraint:

$$I_t = \pi \left(\left(r_t + \lambda\beta^{-1} q_t \right) K_t + B_t \right). \tag{22}$$

When the irreversibility constraint binds, $I_t = 0$.

Finally, the aggregate dividend paid to the representative household by firms is given by:

$$S_t = r_t K_t - I_t \tag{23}$$

Rational expectations equilibrium is a sequence of endogenous variables $r_t, q_t, \phi_t, B_t, K_t, I_t, Y_t$ which satisfy (14), (15), (18), (19), (20) - (22).

3 Steady state

We start by characterizing analytically the steady state of the model. Our aim in this section is to derive the conditions under which bubbles exist, are expansionary and are negative or positive.

3.1 Bubbleless steady state

We first characterise the steady state without bubbles. In Section 3.2, we characterise the steady state with bubbles and compare the bubbly steady state with the bubbleless steady state.

3.1.1 Steady state with binding borrowing constraints

Firstly we characterize the steady state in which the borrowing constraint is binding. Combining equation (14) and (15) and imposing steady state, we get the following equation for the value of capital as a function of itself as well as the rental rate. We denote variables in the bubbleless steady state with superscript 'N'. Then we obtain

$$q^N = \beta \frac{\left(1 + \pi \left(q^N - 1\right)\right) r^N + q^N(1-\delta)}{1 - \lambda\pi \left(q^N - 1\right)}. \tag{24}$$

This is a quadratic equation in (q,r) space due to the effect of leverage (the denominator in equation (24)). Equation (14) evaluated at the steady state is given by

$$q^N = \beta\phi^N. \tag{25}$$

The capital stock evolution (21) evaluated at the steady state is given by

$$\delta = \pi \left(r^N + \lambda \phi^N \right). \tag{26}$$

From equation (24), (25) and (26), we obtain the closed-form solutions of the steady state values of ϕ, q and r as

$$\phi^N = \frac{\delta(1-\pi)}{\pi(1-\beta+\lambda)}, \tag{27}$$

$$q^N = \beta \frac{\delta(1-\pi)}{\pi(1-\beta+\lambda)}, \tag{28}$$

$$r^N = \frac{\delta(1-\beta+\lambda\pi)}{\pi(1-\beta+\lambda)}. \tag{29}$$

Equation (28) and (29) imply that q^N and r^N are both monotonically decreasing in λ. The capital stock can be computed from the expression for the rate of return on capital.

$$r^N = \alpha(K^N)^{\alpha-1}. \tag{30}$$

Condition for binding borrowing constraints. Finally we derive the condition under which the borrowing constraint is binding. The borrowing constraint binds in the steady state only if $q^N > 1$. From equation (28), the necessary and sufficient condition for $q^N > 1$ is given by

$$\lambda \leq \beta(1-\pi)\frac{\delta}{\pi} - (1-\beta) \equiv \bar{\lambda}. \tag{31}$$

Condition for $0 < \bar{\lambda} < 1$. In what follows we consider the case in which $0 < \bar{\lambda} < 1$. From equation (31), we have $\bar{\lambda} > 0$ if and only if

$$\pi < \frac{\beta\delta}{1-\beta(1-\delta)} \equiv \bar{\pi}. \tag{32}$$

Intuitively, in order for the credit constraint to be binding, the probability of investment opportunity must be small enough. Otherwise, firms would accumulate enough capital so that the collateral of firms with investment opportunities would exceed their borrowing needs. Similarly, we have $\bar{\lambda} < 1$ if and only if

$$\pi > \frac{\beta\delta}{2-\beta(1-\delta)} \equiv \underline{\pi}. \tag{33}$$

Intuitively, if the probability of investment opportunity is too small, the credit constraint will bind even when capital is fully pledgeable to outsiders and $\lambda = 1$. The above condition states that the probability of investment opportunity must be sufficiently large for an unconstrained parameter region to exist.

Combining these two conditions, we obtain $0 < \bar{\lambda} < 1$ if and only if

$$\underline{\pi} < \pi < \bar{\pi}. \tag{34}$$

In what follows, we consider the parameter space in which equation (34) holds.

3.1.2 Steady state with non-binding borrowing constraints.

Secondly we characterize the unconstrained steady state. If $\lambda > \bar{\lambda}$ then the steady state equilibrium without bubbles is unconstrained. In the unconstrained steady state equilibrium,

$$q^N = 1, \tag{35}$$

$$r^N = \beta^{-1} - (1 - \delta). \tag{36}$$

Credit constraints do not bind and capital has no liquidity premium and is valued only for its conventional return.

3.2 Bubbly steady state

Next we characterize the bubbly equilibrium. We denote variables in the bubbly steady state with superscript 'B'. The bubble arbitrage condition (18) pins down the price of capital:

$$q^B = 1 + \frac{1 - \beta}{\pi} > 1. \tag{37}$$

The price of capital delivers exactly the right amount of 'dividends' per unit of the bubble so that the bubble holder (who is also the shareholder in the representative firm) receives their required rate of return β^{-1} while the bubble remains constant as a share of national income.

Using equation (15) and substituting for q from equation (37) delivers the following expression for the capital rental rate:

$$r^B = \left(1 + \frac{1 - \beta}{\pi}\right)\left(\frac{1 - \lambda(1 - \beta) - \beta(1 - \delta)}{\beta\,(2 - \beta)}\right) \tag{38}$$

with the capital stock given by

$$r^B = \alpha(K^B)^{\alpha-1}. \tag{39}$$

Finally, the capital stock evolution equation pins down the bubble as a share of the capital stock:

$$B = \frac{\left(\delta - \pi\left(r^B + \frac{\lambda}{\beta}q^B\right)\right)}{\pi}K^B. \tag{40}$$

We can see that equation (40) allows the possibility of negative bubbles especially when λ is relatively large.

Threshold value of λ for $q^N = q^B$. When characterizing the region where bubbles are negative, the point where the capital price in the bubbly equilibrium exceeds that in the bubbleless equilibrium will turn out to be important. Therefore it will be useful in subsequent analysis to derive the value of λ that satisfies $q^N = q^B$. From equation (28) and (37), this is given by

$$\lambda^* \equiv \beta(1-\pi)\frac{\delta}{\pi + 1 - \beta} - (1-\beta). \tag{41}$$

Comparing equation (31) and (41), the following inequality holds:

$$\bar{\lambda} > \lambda^*. \tag{42}$$

This means that the parameter values for which the bubbly and bubbleless equilibria coincide occur when borrowing constraints bind. We consider a general case in which λ^* satisfies

$$0 < \lambda^* < 1. \tag{43}$$

Since $\lambda^* < \bar{\lambda}$ holds, we have $\lambda^* < 1$ under condition (33). From equation (41), the condition for $\lambda^* > 0$ is given by

$$\pi < \pi^* \equiv \frac{\delta\beta - (1-\beta)^2}{1 - \beta(1-\delta)}. \tag{44}$$

We note from equation (32) and (44) that $\pi^* < \bar{\pi}$.

We further need the condition under which $\underline{\pi} < \pi^*$ in order to ensure that the parameter space for $0 < \lambda^* < \bar{\lambda}$ is non-empty. From equation (33) and (44), we have $\underline{\pi} < \pi^*$ if and only if

$$\delta > \frac{(1-\beta)^2(2-\beta)}{\beta - (1-\beta)^2}. \tag{45}$$

This condition easily holds when β is close to unity. Therefore, in what follows, we consider the parameter space in which $\underline{\pi} < \pi < \pi^*$ so that $0 < \lambda^* < \bar{\lambda}$ holds.

3.2.1 Negative bubbles

Equation (40) implies that the bubble can be negative when:

$$\delta - \pi \left(r^B + \frac{\lambda}{\beta} q^B \right) < 0, \tag{46}$$

where q^B and r^B are respectively given by equation (37) and (38).

Proposition 1 *Consider the bubbly steady state and the no-bubble steady state. Assume that (34), (44) and (45) so that $\underline{\pi} < \pi < \pi^*$ and $0 < \lambda^* < \bar{\lambda}$ holds. Then,*

1. *If $\lambda \le \lambda^*$, then,*

$$B \geq 0, \quad q^N \geq q^B. \tag{47}$$

2. *If $\lambda > \lambda^*$, then,*

$$B < 0, \quad q^N < q^B. \tag{48}$$

Proof. Equation (28) implies that q in the bubbleless steady state is monotonically decreasing in $\lambda \in [0, \bar{\lambda}]$, and equation (35) implies that q is constant and equals to unity for $\lambda \in [\bar{\lambda}, 1]$. Equation (37) implies that q in the bubbly steady state is constant at $1 + (1 - \beta)/\pi$ regardless of the value of $\lambda \in [0, 1]$. Therefore, we establish that, under the condition $\underline{\pi} < \pi < \bar{\pi}$, $q^N > q^B$, and vice versa.

Next we turn to the sign of the bubble in the bubbly steady state. $B < 0$ if and only if inequality (46) holds. By substituting equation (37) and (38) into inequality (46), we establish that equation (46) holds if and only if

$$\lambda > \lambda^*. \tag{49}$$

■

Figure 1 explains Proposition 1 graphically. Note that condition (44) is equivalent to the condition for the positive bubble to exist. Otherwise, λ^* (defined by equation (41)) becomes negative and, as a result, the bubble is negative for all $\lambda \in [0, 1]$.

Bubbles are positive for parameter values where the price of capital is higher in the bubbleless steady state than in the bubbly one and negative when the opposite is true. Intuitively, when the

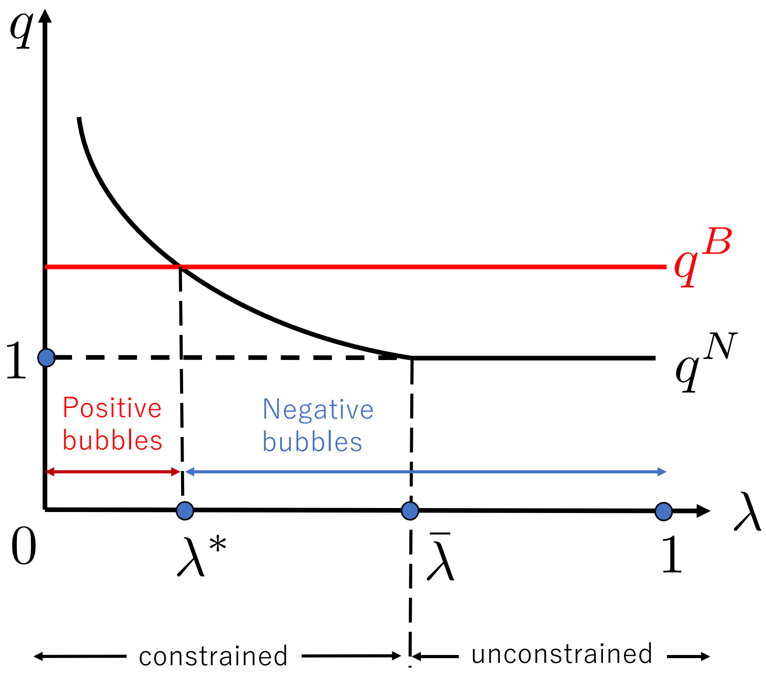

Figure 1: Positive and negative bubble regions

Notes: The figure plots the steady state comparative statics of the capital price with respect to λ. The q^B line denotes the price of capital in the bubbly equilibrium. The q^N line denotes the price of capital in the bubbleless equilibrium. $\bar{\lambda}$ denotes the value of λ beyond which credit constraints are non-binding in the bubbleless equilibrium. λ^* denotes the value of λ beyond which the bubbly equilibrium features a negative bubble.

price of capital is very high in the bubbleless steady state ($\lambda \leq \lambda^*$), this means that collateral is extremely scarce. This provides a very big 'dividend' to the bubble when it emerges. Then, for equilibrium to be restored in the collateral market, the bubble grows and adds to the aggregate stock of collateral, driving the price of capital down to $q^B = 1 + \frac{1-\beta}{\pi}$.

The opposite happens when the price of capital is relatively low in the bubbleless steady state ($\lambda > \lambda^*$). This is an economy in which collateral is relatively abundant in the bubbleless equilibrium and the price of capital is not sufficient to generate a high enough dividend so that the bubbly part of the firm earns a rate of return of β^{-1}. A negative bubble therefore appears which decreases aggregate collateral supply thus lifting the price of capital to q^B.

3.2.2 Expansionary/contractionary effects of the bubble

We turn next to the question of whether the bubble is expansionary or contractionary. Despite having a model with credit frictions, the answer to this question turns out to be subtly different from the question of whether the bubble is positive or negative. While positive bubbles are always expansionary in our framework, negative ones may be contractionary or, more surprisingly, expansionary, depending on parameter values.

In what follows, we focus on the effects of the bubble on the capital rental rate r which is equal to the marginal product of capital. Since the production function is concave and firms face identical productivity, production expands whenever the emergence of the bubble decreases the rental rate of capital.

Capital rental rate in the bubbleless equilibrium. The capital rental rate in the bubbleless equilibrium is given by equation (29) for $0 \leq \lambda \leq \bar{\lambda}$ (i.e., when the borrowing constraint (5) is binding). It is monotonically decreasing in λ. For $\bar{\lambda} < \lambda \leq 1$ (when the borrowing constraint is not binding), the rental rate is given by equation (36), which is independent of λ.

Capital rental rate in the bubbly equilibrium. The rental rate in the bubbleless equilibrium is given by equation (38) for $\lambda \in [0, 1]$. Note that the economy is constrained in the bubbly equilibrium for all $\lambda \in [0, 1]$. Equation (38) implies that the rental rate is also monotonically decreasing in λ.

The following proposition is useful for the rental rate in the bubbleless steady state and the bubbly steady state.

Proposition 2 *Suppose that* $\underline{\pi} < \pi < \pi^*$. *Then,*

$$r^N < r^B \quad \textit{for } \lambda = 0, \tag{50}$$

and

$$r^N > r^B \quad \textit{for } \lambda = 1. \tag{51}$$

Proof. Firstly, we derive the condition for $r^N < r^B$ when $\lambda = 0$. By substituting $\lambda = 0$ into equation (29) and (38), we obtain

$$r^N_{|\lambda=0} = \frac{\delta}{\pi}, \tag{52}$$

$$r^B_{|\lambda=0} = (1 + \pi - \beta)\frac{1 - (1 - \delta)\beta}{\pi\beta(2 - \beta)}, \tag{53}$$

where $r^N_{|\lambda=0}$ and $r^B_{|\lambda=0}$ respectively denote the rental rate in the bubbleless and bubbly steady state when $\lambda = 0$. From equation (52) and (53), the necessary and sufficient condition for

$r^N_{|\lambda=0} < r^B_{|\lambda=0}$ reduces to

$$\pi < \frac{\delta\beta - (1-\beta)^2}{1-\beta(1-\delta)} = \pi^*. \tag{54}$$

Secondly, we derive the condition for $r^N > r^B$ when $\lambda = 1$. When $\lambda = 1$ the bubbleless steady state is unconstrained and the rental rate is given by equation (36). We can use equation equation (38) to compute the rental rate in the bubbly steady state when $\lambda = 1$. Therefore we obtain

$$r^N_{|\lambda=1} = \beta^{-1} - (1-\delta), \tag{55}$$

$$r^B_{|\lambda=1} = \frac{1+\pi-\beta}{2-\beta}\frac{\delta}{\pi}, \tag{56}$$

where $r^N_{|\lambda=1}$ and $r^B_{|\lambda=1}$ respectively denote the rental rate in the bubbleless and bubbly steady state when $\lambda = 1$. From equation (55) and ((56)), we have $r^N_{|\lambda=1} > r^B_{|\lambda=1}$ if and only if

$$\pi > \frac{\beta\delta}{2-\beta(1-\delta)} = \underline{\pi}. \tag{57}$$

■

Note that $r^N = r^B$ for $\lambda = \lambda^*$. In addition to this, define λ^{**} such that

$$r^B_{\lambda=\lambda^{**}} = \beta^{-1} + (1-\delta). \tag{58}$$

Then, the comparison of the capital rental rate can be summarized by the following proposition.

Proposition 3 *Assume that $\underline{\pi} < \pi < \pi^*$. Then,*

1. *If $0 < \lambda < \lambda^*$, then $B > 0$ and $r^B < r^N$. Therefore the bubble is expansionary.*
2. *If $\lambda^* < \lambda < \lambda^{**}$, then $B < 0$ and $r^B > r^N$. Therefore the bubble is contractionary.*
3. *If $\lambda^{**} < \lambda < 1$, then $B < 0$ and $r^B < r^N$. Therefore the bubble is expansionary.*

Figure 2 summarizes graphically the result of Proposition 3. When the credit constraint is very tight ($\lambda < \lambda^*$), the bubble is positive and expansionary (as in Miao and Wang (2015, 2018)). The effect of the negative bubble on the economy depends on the severity of credit constraints as measured by the value of λ. When $\lambda^* \le \lambda \le \lambda^{**}$, the negative bubble is contractionary, and it is expansionary when $\lambda^{**} \le \lambda \le 1$.

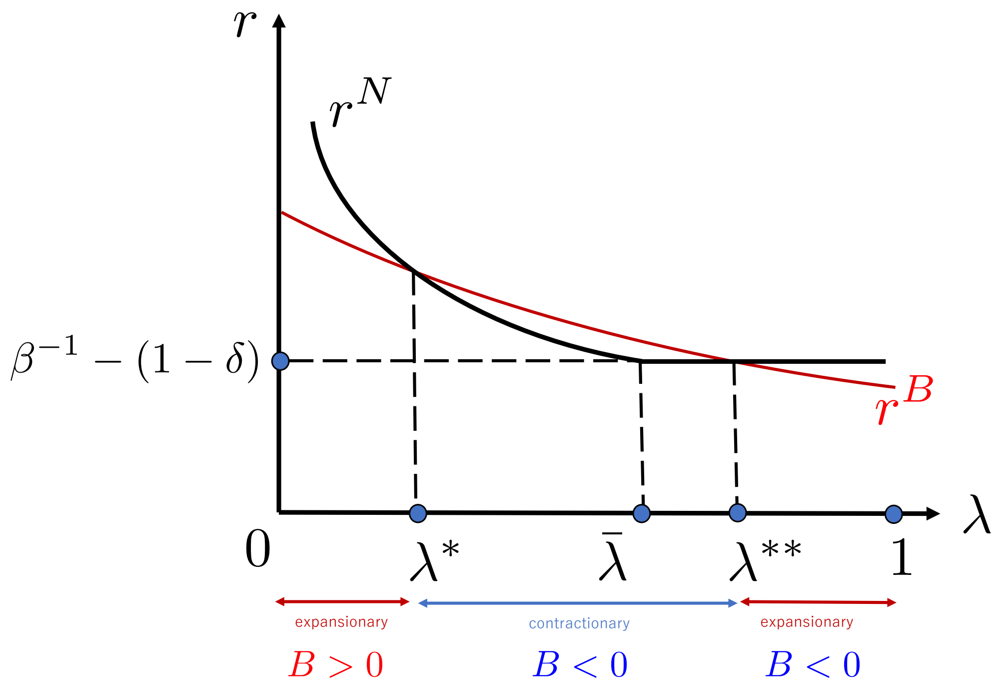

Figure 2: Real effects of the bubble

Notes: The figure plots the steady state comparative statics for the rental rate with respect to λ. The r^B line denotes the capital rental rate in the bubbly equilibrium. The r^N line denotes the rate of capital rental rate in the bubbleless equilibrium. $\bar{\lambda}$ denotes the value of λ beyond which credit constraints are non-binding in the bubbleless equilibrium. λ^* denotes the value of λ beyond which the bubbly equilibrium features a negative bubble. λ^{**} denotes the value of λ beyond which the bubbly equilibrium features a negative expansionary bubble.

This is a somewhat surprising result. Despite the bubble being negative and reducing the quantity of collateral overall, when $\lambda^{**} \leq \lambda \leq 1$, it becomes expansionary causing capital and output to rise above the first best. We know that when $\lambda > \lambda^*$ the bubble is negative and it reduces overall collateral supply leading to a higher price of capital. But what effect does a higher price have on the rental rate and hence on the level of capital and output? Above λ^{**} the negative bubble and increased capital price actually crowd capital in rather than out, while, below λ^{**}, the opposite is true.

To understand the intuition for this, we can rewrite equation (24) so that it expresses the rental rate of capital as a function of the price of capital:

$$r_t = q_t \left(\frac{\beta^{-1} \left(1 - \pi\lambda \left(q_t - 1\right)\right) - (1 - \delta)}{1 + \pi\lambda \left(q_t - 1\right)} \right). \tag{59}$$

We can see that there are two ways in which the rental rate r depends on the price q. First, there is a linear term (holding $\frac{\beta^{-1}(1-\pi\lambda(q_t-1))-(1-\delta)}{1+\pi\lambda(q_t-1)}$ fixed). This implies that higher q should lead to a higher rental rate in order to maintain the dividend yield constant. This effect is contractionary

— a higher capital price requires a lower capital quantity in order to boost the rental rate.

However, there is an important second collateral effect which works in the opposite direction. The term in the brackets in equation (59) above captures the effects of the collateral premium attached to capital and this is decreasing in the capital price. Capital generates income that can be used in an investment opportunity and its market value can also be pledged to obtain loans. This makes it more valuable. The size of the collateral premium is controlled by $\lambda(q_t - 1)$: the bigger the profits the firm can make in an investment opportunity and the more pledgeable capital is, the more valuable it is to hold for a given conventional dividend yield r_t/q_t. In equilibrium, this implies that the firm is willing to hold capital at a low dividend yield and as the profits from capital production $(q_t - 1)$ increase, the equilibrium dividend yield declines to a greater extent the more pledgeable capital is (i.e. under a higher λ).

Below λ^{**}, the first (conventional) effect dominates and the negative bubble leads to a higher rental rate and a lower capital stock. Above λ^{**}, the second (collateral premium) effect dominates and the negative bubble leads to a lower rental rate and a higher capital stock.

4 Transition to the bubbly equilibrium

Having analysed analytically the different steady states of the model, we now numerically characterize the transition from a bubbleless to a bubbly equilibrium. We do this by means of perfect foresight solutions following one-time switches from the bubbleless to the bubbly equilibrium.

We use an illustrative parameterization with parameter values popular in the wider literature. The aim is not to generate a quantitative realistic model simulation but to illustrate the workings of the model better. We set the discount factor β to 0.99 on a quarterly basis. The quarterly depreciation rate δ is 0.03 implying a 12% annual depreciation rate. The quarterly probability of a firm having an investment opportunity is 0.1. The share of capital in production α is 0.3.

We conduct the simulations under a number of values for the pledgeability of capital λ which have been motivated by the analysis in the previous section. We examine transitions in the three main bubble regions we discussed in the steady state analysis of our model - positive expansionary bubbles $(0 < \lambda < \lambda^*)$, negative contractionary bubbles $(\lambda^* \leq \lambda \leq \lambda^{**})$ and negative expansionary bubbles $(\lambda^{**} < \lambda < 1)$.

Transition to a positive expansionary bubble $(\lambda = 0.1)$ We start from a value of $\lambda = 0.1$ which, in the bubbleless steady state, leads to capital and output levels significantly below the first best steady state (shown in the red line in Figure 3). The price of capital is also significantly higher than that in the bubbly equilibrium which we already showed to be the precondition for the existence of a positive bubble. The blue line shows the transition dynamics starting from the bubbleless steady state (the first point of the blue line) to the bubbly steady state. In panels labelled 'Capital' and 'Output', we also show in red lines the levels of capital and output in the first-best steady state.

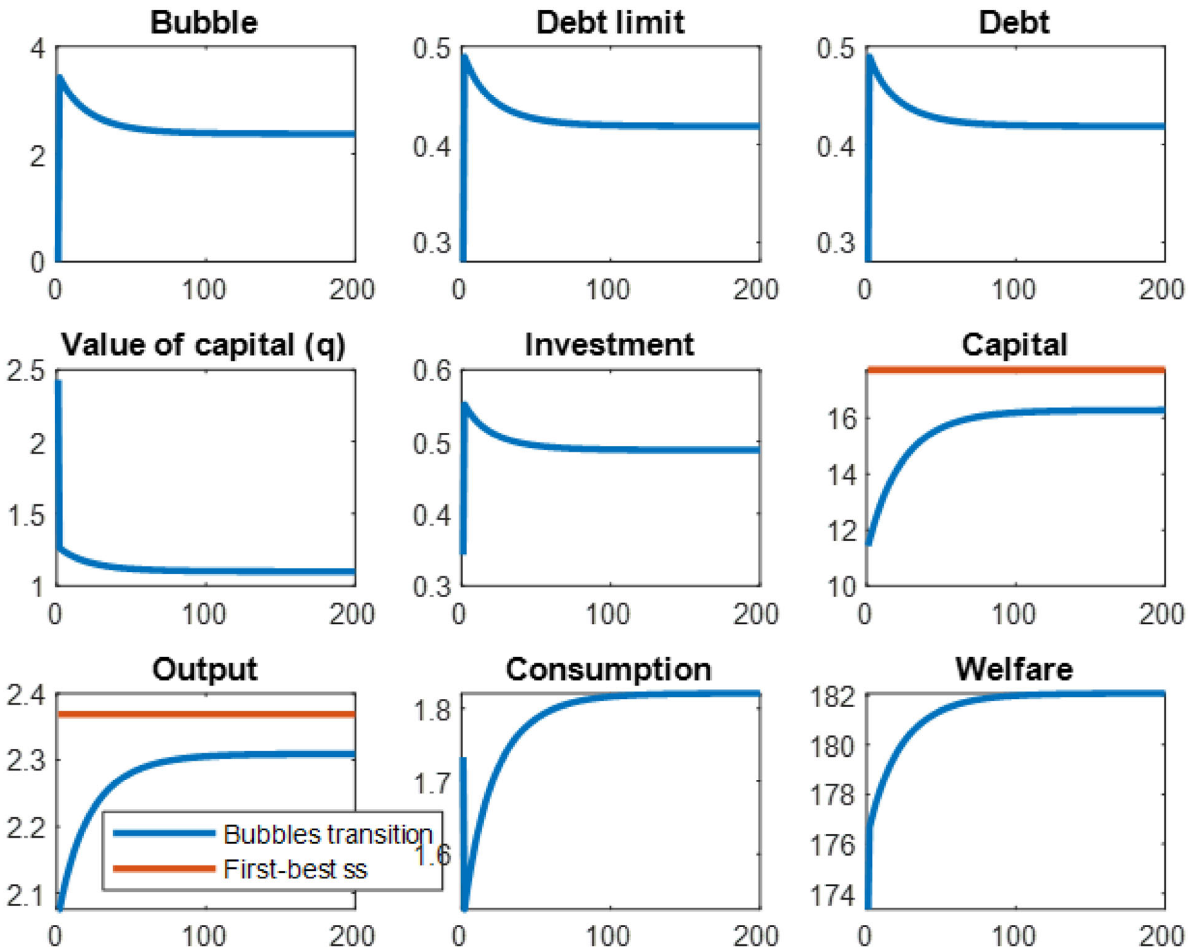

Figure 3: Transition to a positive expansionary bubble

Notes: The figure shows the perfect foresight transition dynamics of the economy to the bubbly steady state starting from the bubbleless steady state for $\lambda = 0.1$.

Once sentiment becomes optimistic, the bubble jumps and gradually settles on its long-term positive value. This increases the debt limit of firms which increase their borrowing to take advantage of profitable investment in new capital. The greater aggregate supply of collateral (due to the bubble) decreases the price of capital sharply on impact. More collateral under a binding credit constraint also increases investment, the capital stock and output. Production gets closer to the first best but stays below it in the long run as credit constraints continue to bind, limiting investment.

Consumption declines on impact to make room for investment but increases in the long run. Welfare (defined in equation (1)) jumps as the economy moves closer to the first best. Thus the long run gains in consumption outweigh any temporary declines over the transition path to the bubbly equilibrium.

Transition to a negative contractionary bubble ($\lambda = 0.25$) Figure 4 shows the macroeconomic impact of negative bubbles. We start from the bubbleless steady state where credit constraints bind but not very strongly.

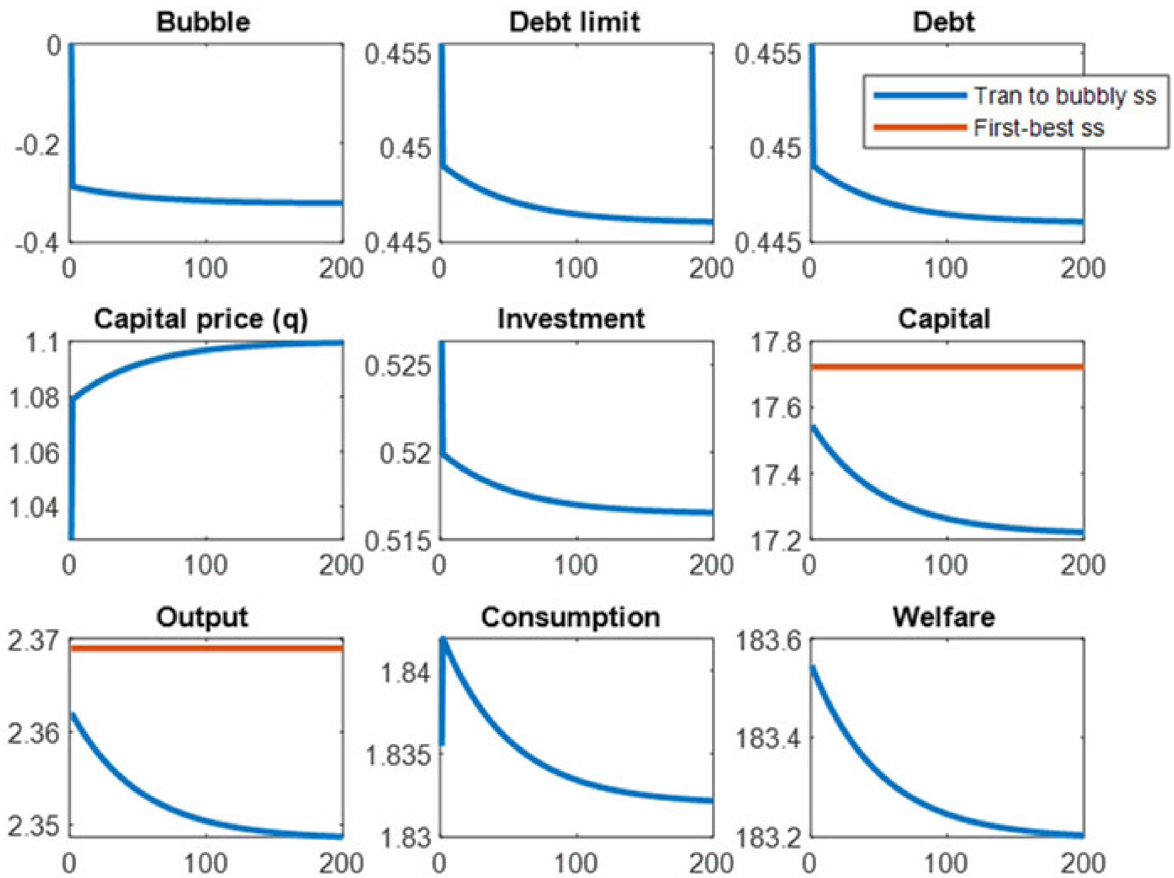

Figure 4: Transition to a negative contractionary bubble

Notes: The figure shows the transition dynamics of the economy to the bubbly steady state starting from the bubbleless steady state for $\lambda = 0.25$.

The bubble jumps to a negative value. The firm's debt limit declines and starts to bind more tightly in the bubbly equilibrium leading to a fall of the amount of debt funding the firm is able to obtain. The value of capital jumps as tangible collateral (capital stock) becomes more valuable now that the bubble has turned negative. However, due to the relatively low pledgeability of capital (low value of λ), the rise in the equilibrium value of tangible collateral requires also an increase in the rental rate. Hence, overall capital accumulation declines and output and investment fall.

On impact, the decline in investment leads to a short-lived increase in consumption but eventually it declines in line with the fall in capital and output. Welfare (defined in equation (1)) falls on impact and continues to decline as production falls further below the first best (indicated by the red line).

Transition to a negative expansionary bubble ($\lambda = 0.4$) Our last example is of the case where the borrowing constraint is very loose in the bubbleless steady state. The results are shown in Figure 5. In this example, the borrowing needs of the firm are more than covered by the collateral value of capital and output and capital are at the first best levels in the absence of bubbles.

In the bubbly equilibrium, the spread between the price of capital and its replacement cost must increase so, following on from the analysis in Section 3, the bubble is negative. However, it now becomes expansionary even though it reduces firms' overall borrowing limits (shown in the second panel of Figure 5).

The reason for this surprising result is that the collateral value of capital becomes highly sensitive to the profit from new capital goods when capital pledgeability is high (i.e. when λ is high). The holders of capital require a very low rate of return in order to hold it because it provides good access to credit and allows the holder to take advantage of highly profitable capital production opportunities. Thus, the rise in the profitability of capital in a bubbly steady state is only compatible with equilibrium once the capital rental rate has fallen below its value in the bubbleless (first best) steady state. This happens as firms accumulate additional capital goods despite the negative bubble. The negative bubble crowds in tangible collateral when the latter is sufficiently pledgeable. This happens even to the point of over-investment which reduces welfare (defined in equation (1)) on impact.[12]

5 Policy Analysis

In the baseline version of the model, the intra-period loans between firms and households carry an interest rate of unity. We now consider a tax or subsidy to debt. This makes the interest

[12] The fall in welfare on impact is small and not well visible in the figure.

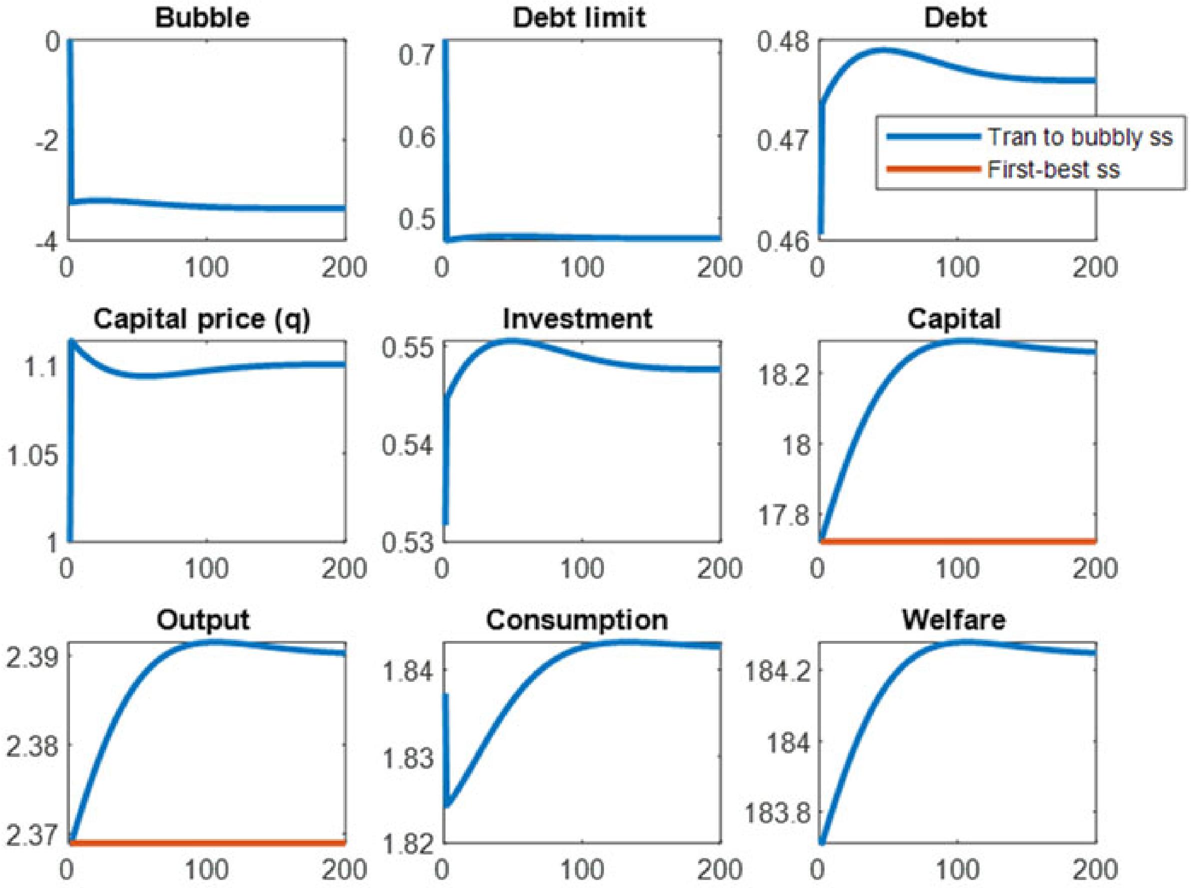

Figure 5: Transition to a negative expansionary bubble

Notes: The figure shows the transition dynamics of the economy to the bubbly steady state starting from the bubbleless steady state for $\lambda = 0.4$.

rate faced by firms equal to $1 + \tau$ where τ is the tax/subsidy rate. When $\tau > 0$ there is a tax, otherwise debt is subsidised. The policy is financed with lump-sum taxes on households in the event of a subsidy. When debt is taxed, households receive lump sum transfers.

Under the subsidy/tax, the expression for the value of installed capital becomes

$$\phi_t = r_t + q_t(1 - \delta) + \pi\left(q_t - 1 - \tau_t\right)\left(r_t + \lambda\phi_t\right).$$

We now characterize the level of the subsidy/tax required to implement the first best in the bubbleless and bubbly steady state.

5.1 Bubbleless equilibrium

In the bubbleless steady state with a subsidy, we have the following steady state solution as a function of the subsidy/tax τ:

$$\phi^N(\tau) = \frac{\delta(1-\pi(1+\tau))}{\pi(1-\beta+\lambda)}, \tag{60}$$

$$q^N(\tau) = \beta\frac{\delta(1-\pi(1+\tau))}{\pi(1-\beta+\lambda)}, \tag{61}$$

$$r^N(\tau) = \frac{\delta(1-\beta+\lambda\pi(1+\tau))}{\pi(1-\beta+\lambda)}. \tag{62}$$

To reach the first best, the social planner must therefore implement a debt subsidy which is described in the following proposition:

Proposition 4 *Assume that $\underline{\pi} < \pi < \pi^*$. If $\lambda < \bar{\lambda}$, then the collateral constraint binds and output is below the first best. Then a debt subsidy equal to $\tau = \frac{(\beta^{-1}-1+\delta)\pi(1-\beta+\lambda)-(1-\beta)\delta}{\delta\lambda\pi} - 1$ implements the first best allocation.*

Proof. Follows by setting equation (62) equal to the first best value of the rental rate $\beta^{-1}-1+\delta$ and solving for τ. ■

Proposition 4 shows that a suitably chosen subsidy can implement the first best in the bubbleless equilibrium with binding credit constraints. Since output is below the first best, a subsidy is always required to restore efficiency financed by a lump sum tax on households.

5.2 Bubbly equilibrium

Under the tax, the rental rate of capital in the bubbly equilibrium as a function of the subsidy/tax is given by the expression below.

$$r^B(\tau) = \left(1+\frac{1-\beta}{\pi}\right)\left(\frac{1-\lambda(1-\beta-\pi\tau_t)-\beta(1-\delta)}{\beta(2-\beta-\pi\tau_t)}\right). \tag{63}$$

It is clear that the subsidy (setting $\tau < 0$) would reduce the capital rental rate and would be expansionary. Then the following Proposition follows

Proposition 5 *Assume that $\underline{\pi} < \pi < \pi^*$.*

Then, a debt subsidy/tax equal to $\tau_t = \frac{(\beta^{-1}-1+\delta)\beta(2-\beta)-(1-\beta(1-\delta))\left(1+\frac{1-\beta}{\pi}\right)}{\pi\left(\lambda\left(1+\frac{1-\beta}{\pi}\right)+(\beta^{-1}-1+\delta)\beta\right)}$ implements the first best allocation.

1. *If* $0 < \lambda < \lambda^*$*, then* $B > 0$ *and* $r^B < r^N$*. The bubble is expansionary and the optimal debt subsidy is lower in the bubbly equilibrium.*

2. *If* $\lambda^* < \lambda < \lambda^{**}$*, then* $B < 0$ *and* $r^B > r^N$*. The bubble is contractionary and the optimal debt subsidy is higher in the bubbly equilibrium.*

3. *If* $\lambda^{**} < \lambda < 1$*, then* $B < 0$ *and* $r^B < r^N$*. The bubble is expansionary and a debt tax is imposed in the bubbly equilibrium.*

Proof. Follows by setting equation (63) equal to the first best value of the rental rate $\beta^{-1}-1+\delta$ and solving for τ. ■

When output is below the first best in the bubbly equilibrium ($\lambda < \lambda^{**}$), a subsidy is needed. When output is above the first best ($\lambda > \lambda^{**}$), a tax is needed. The proposition also shows that when the bubble is positive and expansionary ($0 < \lambda < \lambda^*$), the optimal subsidy is lower in the bubbly equilibrium compared to the bubbleless one. Also, when there is a negative expansionary bubble ($\lambda^{**} < \lambda < 1$), a debt tax is imposed in the bubbly equilibrium in order to remove the over-investment. In the intermediate region ($\lambda^* < \lambda < \lambda^{**}$), there is a negative contractionary bubble and the optimal subsidy increases in the bubbly equilibrium in order to correct the increased underinvestment in this part of the parameter space. Therefore the optimal subsidy/tax policy has a countercyclical character similarly to real life regulatory tools such as the Countercyclical Capital Buffer in Basel III. Proposition 5 shows that a suitably chosen subsidy or tax can implement the first best in the bubbly equilibrium

We now provide a numerical example of the way the economy adjusts to the imposition of a debt tax. We consider the case of $\lambda = 0.4$ as well as the parameter values used in the previous section. In this case, we have a negative expansionary bubble which generates over-investment and requires a tax on debt in order to bring the economy back to the first best. Figure 6 shows the transition from a bubbly equilibrium without a tax/subsidy to a bubbly equilibrium where a tax implements the first best. The figure shows that the tax reduces firms' limits and hence their investment. Capital gradually declines to the first best (shown as the solid red line in the figure). The reduction in investment initially boosts consumption although in the long run, lower output leads to lower consumption. Welfare increases on impact because the tax reduces over-investment and the short term increase in consumption dominates the long term (and hence discounted) decline in consumption.[13]

[13] Again, the welfare increase is small and not well visible in the figure.

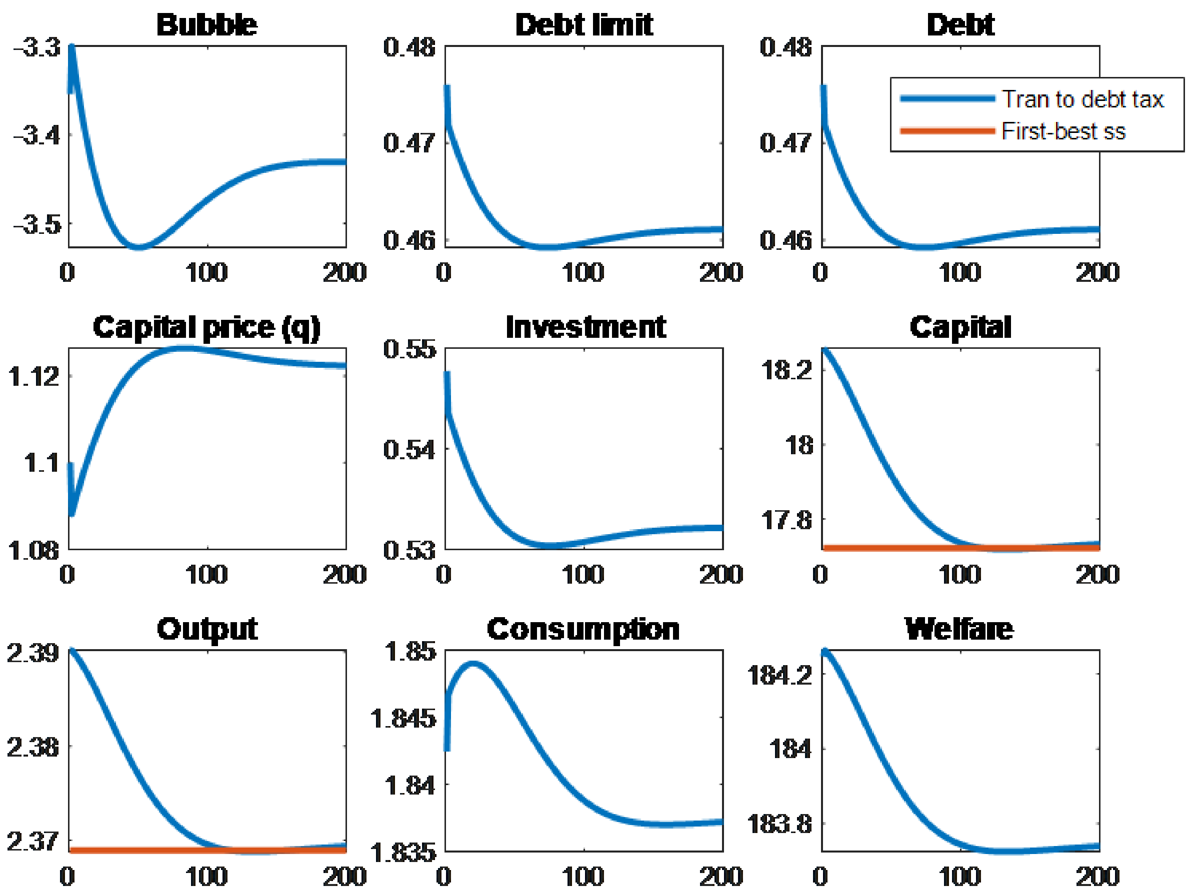

Figure 6: Transition to a negative expansionary bubble

Notes: The figure shows the transition dynamics of the economy from the bubbly steady state with a zero debt tax to a bubbly steady state with a tax rate which implements the first best real allocation. $\lambda = 0.4$.

6 Conclusions

We build a model economy in which firms' credit limits depend on stock market valuations which themselves depend on access to credit. This gives rise to bubbly equilibria where stock market values depart from strict fundamentals due to self-fulfilling optimism or pessimism about firms' credit access. We show that, when capital investment is irreversible at the firm level, these bubbles are not just positive as discussed in the wider literature but also negative. During a negative bubbly episode, equity investors become pessimistic about the firm's access to credit and hence its future profitability. This reduces the firm's share price thus confirming investors' initial pessimism. The irreversibility of capital at the firm level ensures that the value of the firm can fall below the price of new capital goods.

Positive bubbles in the model appear when credit constraints are tight (i.e. when much of the firm's capital is destroyed in the event of bankruptcy). They are also always expansionary as in the Miao and Wang (2015, 2018) framework. Negative bubbles appear when credit constraints are moderate or loose (i.e. when most the firm's capital survives in bankruptcy). They are contractionary under moderate credit constraints. The value of the firm falls, firms invest less and the economy contracts. More surprisingly, however, negative bubbles also turn out to be expansionary when credit constraints are very loose and capital is good collateral. During an expansionary negative bubble, output actually increases above the first best and the economy enters an over-investment regime. The reason for this finding is that the negative bubble leads to tighter borrowing limits and to the increase in the spread firms earn from producing capital goods. When capital is good collateral, this boosts its collateral premium so much that it actually leads to the overproduction of capital goods as the economy tries to compensate for the way the negative bubble undervalues firms.

Finally, we consider what policy interventions can restore efficiency in our model economy. A debt subsidy/tax can restore output and consumption to the first best in steady states where those are below/above the first best. When a bubble appears, the subsidy/tax needs to be adjusted in a counter-cyclical manner. For a positive expansionary bubble, the debt subsidy is optimally reduced relative to the bubbleless equilibrium. In the event of a negative contractionary bubble, the debt subsidy needs to be increased while for negative expansionary bubbles, a debt tax is needed to correct the over-investment.

References

Adam, K. and Marcet, A. (2011). Internal rationality, imperfect market knowledge and asset prices. *Journal of Economic Theory*, 146(3):1224–1252.

Azariadis, C., Kaas, L., and Wen, Y. (2016). Self-fulfilling credit cycles. *The Review of Economic Studies*, 83(4):1364–1405.

Farhi, E. and Tirole, J. (2012). Bubbly liquidity. *The Review of economic studies*, 79(2):678–706.

Gertler, M. and Karadi, P. (2011). A model of unconventional monetary policy. *Journal of monetary Economics*, 58(1):17–34.

Hirano, T. and Toda, A. A. (2024a). Bubble necessity theorem. *Journal of Political Economy*, Forthcoming.

Hirano, T. and Toda, A. A. (2024b). Rational bubbles: A clarification. *Working Paper.*

Hirano, T. and Yanagawa, N. (2016). Asset bubbles, endogenous growth, and financial frictions. *The Review of Economic Studies*, 84(1):406–443.

Kiyotaki, N. and Moore, J. (2019). Liquidity, business cycles, and monetary policy. *Journal of Political Economy*, 127(6):2926–2966.

Martin, A. and Ventura, J. (2012). Economic growth with bubbles. *American Economic Review*, 102(6):3033–58.

Martin, A. and Ventura, J. (2018). The macroeconomics of rational bubbles: A user's guide. *Annual Review of Economics*, 10(1):505–539.

Matsuyama, K. (2013). The good, the bad, and the ugly: An inquiry into the causes and nature of credit cycles. *Theoretical Economics*, 8(3):623–651.

Miao, J. and Wang, P. (2012). Bubbles and total factor productivity. *American Economic Review*, 102(3):82–87.

Miao, J. and Wang, P. (2014). Sectoral bubbles, misallocation, and endogenous growth. *Journal of Mathematical Economics*, 53:153–163.

Miao, J. and Wang, P. (2015). Banking bubbles and financial crises. *Journal of Economic Theory*, 157:763–792.

Miao, J. and Wang, P. (2018). Asset bubbles and credit constraints. *American Economic Review*, 108(9):2590–2628.

Shiller, R. J. (2000). Measuring bubble expectations and investor confidence. *The Journal of Psychology and Financial Markets*, 1(1):49–60.

Tirole, J. (1985). Asset bubbles and overlapping generations. *Econometrica: Journal of the Econometric Society*, pages 1499–1528.

A Extension: the model with government bonds

In the baseline version of the model we assumed that there is no liquid asset (money or bonds) which can be accumulated in order to take advantage of investment opportunities. In this section we investigate the consequences of adding fully pledgeable pure discount government bonds to the model. We show that adding government bonds reduces the value of the bubble throughout the parameter space of the model. This expands the region in which the bubble is negative (λ^* is lower as the quantity of government debt increases) but does not change the threshold beyond which negative bubbles become expansionary (λ^{**}). The presence of government debt does however rule out very large negative bubbles if we assume (realistically) that firms can sell their government debt holdings to other firms (i.e investment in government debt is not irreversible in the way physical capital is).

A.1 The model with government bonds

Suppose firms also have the possibility to accumulate one period government bonds s_t. The bonds pay a unit of consumption next period and cost p_t today. Then the value of the firm depends on capital and bond holdings

$$V_t(k_t, s_t, b_t) = \phi_t k_t + \psi_t s_t + b_t, \tag{64}$$

where b_t is a potential bubble. We again assume that only λ fraction of capital can be recovered in the event of bankruptcy. In contrast, we assume that government bonds are fully recoverable by creditors. The collateral constraint is therefore:

$$d_t \leqq \lambda \phi_t k_t + \psi_t s_t + b_t. \tag{65}$$

Substituting into the value function, we get the following:

$$\begin{aligned} &\phi_t k_t + \psi_t s_t + b_t \\ &= \max_{s_{t+1}, k_{t+1}} \Big\{ \left(r_t + q_t(1-\delta)\right) k_t + s_t - q_t k_{t+1} - p_t s_{t+1} \\ &+ \pi \left(q_t - 1\right) \left(r_t k_t + \lambda \phi_t k_t + \psi_t s_t + b_t\right) + \beta \left(\phi_{t+1} k_{t+1} + \psi_{t+1} s_{t+1} + b_{t+1}\right) \Big\}. \end{aligned} \tag{66}$$

The envelope condition for capital gives us the familiar capital valuation equation

$$\phi_t = (r_t + q_t(1-\delta)) + \pi\,(q_t - 1)\,(r_t + \lambda\phi_t)\,, \tag{67}$$

The envelope condition for government bonds is given by

$$\psi_t = 1 + \pi\,(q_t - 1)\,\psi_t,$$

which can be simplified as follows:

$$\psi_t = \frac{1}{1 - \pi\,(q_t - 1)}. \tag{68}$$

The first order condition for capital is given by:

$$q_t = \beta\phi_{t+1}, \tag{69}$$

while the first order condition for government bonds is:

$$\begin{aligned} p_t &= \beta\psi_{t+1} \\ &= \frac{\beta}{1 - \pi\,(q_{t+1} - 1)}. \end{aligned} \tag{70}$$

The bonds carry their own liquidity premium because they can be used as collateral to obtain credit and exploit investment opportunities. Because their price exceeds β^{-1}, households will not hold bonds and the entire bond supply will be owned by firms. Finally, we have the bubble valuation equation:

$$b_t = \pi\,(q_t - 1)\,b_t + \beta b_{t+1}. \tag{71}$$

A.2 Bubbly equilibrium

In the bubbly steady state equilibrium, the bubble valuation equation implies:

$$\pi\,(q - 1) = 1 - \beta. \tag{72}$$

This means that the price of debt is equal to unity.

$$p = \frac{\beta}{1 - \pi (q-1)} = 1. \tag{73}$$

Then the capital evolution equation is as follows:

$$\delta K = \pi \left(\left(r + \frac{\lambda}{\beta} \left(1 + \frac{1-\beta}{\pi} \right) \right) K + \frac{1}{\beta} M + B \right). \tag{74}$$

or dividing through by the capital stock, we get:

$$\delta = \pi \left(r + \frac{\lambda}{\beta} \left(1 + \frac{1-\beta}{\pi} \right) + \frac{1}{\beta} m + b \right). \tag{75}$$

Proposition 6 *The quantity of government debt M reduces the size of the bubble without affecting real allocations in a bubbly equilibrium. This expands the negative bubble region.*

We offer an informal proof below. Equation (74) pins down the value of the bubble in the absence of government debt. With government debt, equation (74) pins down $\frac{1}{\beta}M + B$. To see why this is, consider equation (38) in the main text which specifies the rental rate in the bubbly equilibrium as a function of parameters. This equation is derived using only (15) and substituting for q from equation (37). Neither the size of the bubble nor the quantity of bonds enter this expression. In a bubbly equilibrium, the size of $\frac{1}{\beta}M + B$ is then determined by (74). This means that a higher quantity of government debt will reduce the value of the bubble. This shrinks the region for which the bubble is positive and expands the region for which it is negative.

However, the λ thresholds for negative bubbles to be expansionary or contractionary remain the same regardless of the quantity of government debt as the following proposition demonstrates.

Proposition 7 *The quantity of government debt M does not affect λ^{**} - the threshold at which negative bubbles become expansionary.*

The logic for the proposition above can be seen by following similar arguments to the ones we used to show that government debt just makes negative bubbles more negative without altering real allocations. From (38) we know that the rental rate and the quantity of capital are independent of the quantity of government debt in the bubbly equilibrium. And in the region of the parameter space where negative expansionary bubbles appear, the bubbleless equilibrium is the unconstrained first best which is also independent of the quantity of government debt. Hence the critical value λ^{**} beyond which negative bubbles become expansionary is unaffected by M.

Crucially, however, the presence of government debt does put a limit on how negative bubbles can become. This result relies on the assumption that government debt could be sold to another firm, which rules out an equilibrium in which the residual value of the firm (after selling the bonds) is negative. In other words, negative bubbles which lead to a situation in which the value of the firm without bonds is negative can be ruled out.

Proposition 8 *When government debt can be freely sold to other firms, no bubbly equilibrium exists when* $\lambda > \lambda^{***}$.

With government debt, bubble existence requires that the value of the firm after selling government bonds is still positive. When this is not the case, no bubbly equilibrium exists because the firm could sell its government bonds and close itself if its residual value is negative. The condition for this is

$$V(K, 0, B) = \phi K + B < 0 \tag{76}$$

Re-arranging the capital evolution equation in steady state (75), we can get an expression for the bubble (per unit of capital).

$$b = \frac{\delta}{\pi} - r - \frac{\lambda}{\beta}\left(1 + \frac{1-\beta}{\pi}\right) - \frac{1}{\beta}m. \tag{77}$$

Substituting the bubble expression into (76) (the firm value function with debt holdings set to zero) and dividing through by K gives us the condition for the bubble not to exist.

$$\phi + \frac{\delta}{\pi} - r - \frac{\lambda}{\beta}\left(1 + \frac{1-\beta}{\pi}\right) - \frac{1}{\beta}m < 0 \tag{78}$$

After substituting out the rental rate using (38) and substituting ϕ using (37) and (14), the condition above can be expressed in terms of another threshold for λ.The negative bubble no longer exists when $\lambda > \lambda^{***}(m)$ where $\lambda^{***}(m)$ is given below.

$$\lambda^{***}(m) = \beta^{-1}(\beta(2-\beta)) - 1 + \beta(1-\delta) + \frac{\left(\frac{\delta}{\pi} - \frac{m}{\beta}\right)\beta(2-\beta)}{\left(1 + \frac{1-\beta}{\pi}\right)}$$

We can easily see that $\lambda^{***}(m)$ is decreasing in the quantity of government debt m. In other words, a higher quantity of government debt reduces the region for which negative bubbles exist.

B Transition from the bubbly to the bubbleless equilibrium

Figure 7 below computes the perfect foresight transition dynamics following the collapse of a positive expansionary bubble. The figure demonstrates that the irreversibility constraint on investment does not hold following the collapse of the bubble.

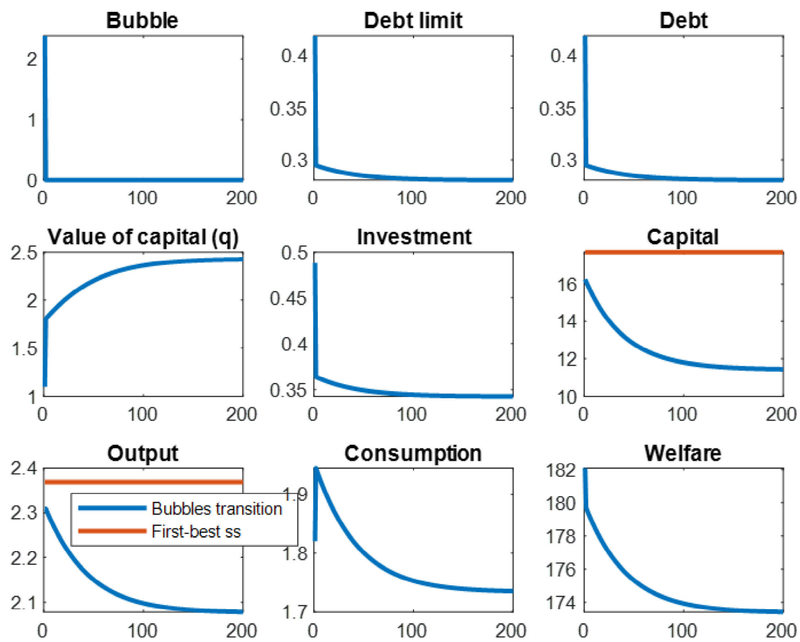

Figure 7: Transition after the collapse of a positive bubble
Notes: The figure plots the perfect foresight transition dynamics following the collapse of a positive bubble for the $\lambda = 0.1$ case.