(Strong) Implementability with Transfer[*]

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March 8, 2024

Abstract

We fully characterize (strong) implementability with transfer, without invoking quasilinearity, via a novel cyclical monotonicity condition that extends Rochet (1987). We then apply it to crack the problem of implementing monotone allocations for the general case under the assumption of the possibility of compensation: (i) the single-crossing condition, when type space is totally ordered and outcome space is partially ordered, is sufficient and necessary for strongly implementing all monotone allocation; and (ii) with the strict single-crossing condition and totally ordered type space and outcome space, an allocation is implementable if and only if it is monotone. No additional structure or regularity conditions are needed. Applications are discussed.

JEL classification: C02; D82

Keywords: Implementability; Strong Implementability; Cyclical Monotonicity; Single Crossing; Monotone Allocation; Revenue Equivalence

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[*]Previous version of this paper circulated under the title "Implementing Monotone Allocations: A Rochetian Approach." We are grateful to Larry Samuelson, Georg Nöldeke, Ludvig Sinander (and his friends, Matteo Escudé, Mathijs Janssen, and Alkis Georgiadis-Harris), and Peter Sørensen for their close reading and insightful and detailed comments, to Marek Weretka for helpful discussion, and to Alex Teytelboym for his enthusiastic comment. Special thanks to Larry and Ludvig for their excellent suggestion for the paper's exposition. Much of the work for this paper was done while Tian visited the National School of Development (NSD) and Chinese Centre for Economic Research (CCER) at Peking University. Tian thanks NSD and CCER for their hospitality.

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1 Introduction

The implementability problem is crucial in mechanism design (Myerson, 2008, p. 588). Much of the literature studies implementability (with transfer) under the assumption of quasilinearity.[1] Powerful tools are available under this assumption, such as the envelope theorem and the cyclical monotonicity condition of Rochet (1987), among others. Our understanding of implementability for the general case, however, is far less complete. The difficulty of the envelope-theory approach in addressing implementability without quasilinearity is recently discussed by Sinander (2022b). On the other hand, the cyclical monotonicity condition of Rochet (1987) is no longer applicable for the general case. The lack of a generally useful tool for studying implementability without invoking quasilinearity has greatly limited the analytic reach of the literature.

For instance, Spence (1974), Mirrlees (1976), and Rochet (1987) show an equivalence between implementability and monotonicity for the quasilinear case, when type space and outcome space are real intervals, under the Spence-Mirrlees condition. This is a key result for implementability. Applying Rochet (1987), it extends to any totally ordered type space and outcome space (Fact 1). However, the picture is quite different when we shift to the general case. Guesnerie and Laffont (1984) apply the differential approach of Laffont and Maskin (1980) to first obtain a "partial" version of this equivalence without invoking quasilinearity, by focusing on piecewisely continuously differentiable allocations. Recently, Nöldeke and Samuelson (2018) apply the optimal-transport approach to notably obtain this equivalence without invoking quasilinearity under the single-crossing condition. The same as Spence (1974) and Mirrlees (1976), they focus on the case in which type space and outcome space are real intervals. Sinander (2022a) notably advances the envelope-theorem approach to the case with a general outcome space and without quasilinearity. The set of regularity conditions needed for the envelope-theorem approach, as identified by Sinander (2022a), are somewhat heavy and not convenient to apply.[2] A natural question is: Which regularity assumptions, if any, are actually needed to implement monotone allocations when we shift to the general case?

This paper makes two main contributions to implementability. First, we offer a generally useful tool for studying (strong) implementability without invoking quasi-

[1] According to Nöldeke and Samuelson (2018), "Models based on quasilinear utility are ill-suited for mechanism design problems in which the stakes are sufficiently large to make income effects or risk aversion salient (Mirrlees (1971), Stiglitz(1977))."

[2] See Section 4.3.3 for a discussion.

linearity, which has long been missing and sought in the literature. In particular, we fully characterize (strong) implementability, without invoking quasilinearity, via a novel cyclical monotonicity condition that extends Rochet (1987) (Theorem 1). The same as Rochet (1987), Theorem 1 does not require any structure on type space or outcome space, and thus is applicable to various contexts. In Section 4, we apply Theorem 1 to crack the problem of implementing monotone allocations for the general case.

Central to our cyclical monotonicity is comparison of the surplus obtained from finite (mis)report cycles and the surplus obtained from truthfully reporting. We introduce the notion of the *generating function* associated with a given allocation, which, given each pair of types (θ, θ'), tells us the surplus obtained by the θ-type agent from (mis)reporting θ' as a function of the surplus obtained by the θ'-type agent from truthfully reporting. Importantly, it is monotone in the latter. By iteration, the generating function is applied to capture the surplus obtained from (mis)report cycles, as a function of the surplus assigned for truthfully reporting. The cyclical monotonicity condition simply requires no strict benefit from any finite (mis)report cycle. It is straightforward that cyclical monotonicity is necessary for implementability. What is less obvious is that this simple condition also generally implies implementability.

Theorem 1 has not been recognized in the literature to the best of our knowledge. This is somewhat surprising, since it has been 40 years since Rochet (1987) introduced the cyclical monotonicity condition based on Rockafellar (1970). Theorem 1 advances our understanding of cyclical monotonicity in the following two important respects.

First, the cyclical monotonicity of Rockafellar (1970) and Rochet (1987) stems from convex analysis and linear programming duality, by which it takes the specific form of the sum of (utility) differences, which is shared by all known extensions in the literature.[3] We might naturally assume that any condition that is called cyclical monotonicity and is useful should take such a form.[4] In contrast, Theorem 1 shows that the essence of such conditions is the comparison of the surplus obtained from (mis)report cycles and the surplus obtained from truthfully reporting, which is straightforward and generally applicable, irrespective of quasilinearity or non-quasilinearity. With quasilinearity, such a comparison reduces to the sign of a specific class of functions in the form of sum of differences (Section 3.2). However, this is not the case without quasilinearity. Our cyclical monotonicity condition is not

[3]See Kausamo, De Pascale, and Wyczesany (2023) for a survey of cyclical monotonicity.

[4]Conditions in a similar form are applied to address implementability without quasilinearity by some authors but with only partial success; e.g., Kos and Messner (2013b).

more difficult to verify than that of Rochet (1987), as demonstrated by our discussion of implementing monotone allocations.

Second, in the proof of Theorem 1, we intensively exploit the monotonicity of the generating function and its iteration, by which our approach to implementability is related to Tarski (1955), though we do not apply Tarski's theorem: The implementability problem is a fixed-point problem, given by (1), in which the domain is the surplus received by the agent from truthfully reporting as a function of types. The cyclical monotonicity condition for implementability arises from exploiting the order structure that is innate to the implementability problem, irrelevant to linearity. In fact, the construction in the proof of the sufficiency of Theorem 1, which modifies Rochet (1987), yields the largest incentive-compatible transfer scheme with a given initial condition (Proposition 1). This explains and also generalizes Kos and Messner (2013a)'s finding of extreme transfers from Rochet (1987)'s construction for the quasilinear case: It is not surprising to get an extreme fixed point when the condition for existence to a fixed-point problem—the cyclical monotonicity condition to the implementability problem (1) in our case—arises from exploiting an order structure.[5] The message that the implementability problem has an innate order structure that can be fruitfully exploited echoes Nöldeke and Samuelson (2018). In fact, cyclical monotonicity can be formally related to the Galois connection discussed by Nöldeke and Samuelson (2018) (Lemma 3).[6]

Nöldeke and Samuelson (2018) introduce the important notion of *strong* implementability whereby for all initial conditions that specify a transfer level for some type, there exists some incentive-compatible transfer scheme that satisfies the initial condition. As Nöldeke and Samuelson note, in general, implementability does not imply strong implementability, though they coincide for the quasilinear case. Theorem 1 characterizes both implementability and strong implementability.

The key assumption for Theorem 1 is the possibility of compensation, borrowed from Kazumura, Mishra, and Serizawa (2020). It requires that for each type and each pair of outcomes $x \neq x'$, any surplus level the agent achieves with x and some

[5]However, an important difference remains: The property of extreme transfer with a given initial condition identified in Proposition 1 is rooted in incentive compatibility, which is irrelevant to the individual rationality constraints in Kos and Messner (2013a) and Nöldeke and Samuelson (2018), among others.

[6]However, there are important differences. First, the Galois connection related to cyclical monotonicity is the version of Davey and Priestley (2002), not the version of Nöldeke and Samuelson (2018). Second, the mappings in Nöldeke and Samuelson (2018), which form a connection, are the maximum surplus of the agent and the optimal pricing scheme of the principal. In our case, the mappings that form a connection are instead the generating functions associated with each pair of different types. Third, the connection requirement in our case is symmetric and for one direction.

transfer is also achievable with x' and some (other) transfer.[7] It is weaker than the full range condition of Nöldeke and Samuelson (2018) and Sinander (2022a). As we show, it is a necessary condition for strongly implementing monotone allocations in common environments when we focus on interior types (Lemma 7).

Next, we apply Theorem 1 to crack the problem of strongly implementing monotone allocations for the general case, which is the second major contribution of this paper. Two main results are derived. First, under the assumption of the possibility of compensation, the single-crossing condition, when type space is totally ordered and outcome space is partially ordered, is sufficient and necessary for strongly implementing all monotone allocations (Theorem 2). No additional structure or regularity conditions are needed. Theorem 2 offers a structural insight on strongly implementing monotone allocations, analogous to Milgrom and Shannon (1994) on monotone comparative statics: Both show that the ordinal single-crossing condition is necessary and sufficient, and dispense with the structure and regularity conditions that are required by the traditional approach.

Theorem 2 has not yet been fully recognized in the literature. This, again, is somewhat surprising, since it has been 30 years since Milgrom and Shannon (1994) proposed the single-crossing condition. The implementability theorem of Sinander (2022a) follows from Theorem 2, since his outer Spence–Mirrlees condition implies the single-crossing condition, as Sinander notes (Claim 2).

As an immediate consequence of Theorem 2, a fully general version of the equivalence between implementability and monotonicity, without invoking quasilinearity, arises: Under the assumption of the possibility of compensation, with the strict single-crossing condition and totally ordered type space and outcome space, an allocation is implementable if and only if it is monotone (Theorem 3). Theorem 3 extends the implementation result of Nöldeke and Samuelson (2018) for real-interval-valued type space and outcome space (their Proposition 13) to any totally ordered type space and outcome space, under the weaker assumption of the possibility of compensation.

Now we turn to the question posed above: Which regularity assumptions are needed for strongly implementing monotone allocations when we shift to the general case? Based on Theorems 2 and 3, our answer is: Basically none—the condition of the possibility of compensation is commonly entailed by strongly implementing

[7] Kazumura, Mishra, and Serizawa (2020) discuss implementability without quasilinearity with a finite outcome space. They impose a richness condition on type space and show that weak monotonicity characterizes implementability, which is related to Bikhchandani, Chatterji, Lavi, Mu'alem, Nisan, and Sen (2006) for the quasilinear case.

monotone allocations (Lemma 7). Theorem 1 is crucial in order to answer this question. The cyclical monotonicity condition allows us to identify the condition needed to implement monotone allocations without any prerequisite conditions, by which a structural theorem such as Theorem 2 is possible (see Section 4.3.3 for comparison with the envelope-theorem approach).

Theorems 2 and 3 facilitate applications.[8] We can readily apply the theory of monotone comparative statics—e.g., Milgrom and Shannon (1994) and Topkis (1998)—to identify the sufficient conditions on the primitives for the single-crossing condition or the strict one to hold—the latter are the only conditions needed for implementing monotone allocations—without any technical concerns. For instance, in the quasilinear case, utility and transfer are aggregated in a linear way to obtain surplus. In a canonical case, besides increasing differences, utility is also assumed to be monotone in both types and outcomes, such as Mas-Colell, Whinston, and Green (1995). Focusing on the canonical case, we extend the implementability of monotone allocations to the broader class of convex and supermodular aggregators (Proposition 3). The proof applies Topkis (1998)'s composition result. As an application, we revisit the problem of selling information of Sinander (2022a) in Example 1.

Guesnerie and Laffont (1984) show that all piecewisely continuously differentiable monotone allocations are implementable with multidimensional Euclidean outcome space, under the Spence-Mirrlees condition (their Theorem 2). We might wonder whether the conditions needed to implement "well-behaved" monotone allocations are less restrictive. The answer is No. Applying Theorem 2, we show that commonly, partial results regarding well-behaved monotone allocations readily extend to all monotone allocations without imposing any extra conditions when we focus on interior types (Proposition 4). As an example, we extend Guesnerie and Laffont (1984) to all monotone allocations under their conditions (Corollary 1).

Finally, for a different thrust, in Appendix A we briefly discuss revenue equivalence

[8] With partially ordered outcomes, the single-crossing condition does not preclude the implementability of nonmonotone allocations. A search among the monotone ones may end up being nonoptimal. Thus, the extra usefulness of Theorem 2 over Theorem 3 is seemingly very limited. Such a comment also applies to Theorem 2 of Guesnerie and Laffont (1984) or Theorem 7.3 of Fudenberg and Tirole (1991). However, such a comment misses an important fact, whereby an optimal design problem is often solved by working on some relaxed version in which the incentive constraints are relaxed in some manner such as the Myersonian approach. A key step in such an approach is to show that the obtained solution to the relaxed problem is implementable. If it is monotone, then by Theorem 2 it is implementable and thus optimal. That is, only knowing that monotone allocations are implementable could be very helpful. Developing applications along this line goes beyond the scope of this paper. We are very grateful to Ludvig Sinander for sharing this valuable insight in his correspondence with Jianrong Tian.

for the general case. An allocation is strongly implementable and satisfies revenue equivalence if and only if for all initial conditions, there exists a unique incentive-compatible transfer scheme that satisfies the initial condition. Theorem 4 shows that, among the class of strongly implementable allocations, an intuitive so-called *inverse distance* condition fully characterizes the ones that satisfy the property of revenue equivalence. With quasilinearity, our inverse distance condition reduces to the antisymmetric distance condition of Heydenreich, Müller, Uetz, and Vohra (2009).

2 The Model

We restrict our attention to a model with a single agent. Let Θ be a set of possible types for the agent and X a set of possible outcomes. Let $(T, \succeq_T)$ be a *complete* totally ordered set of transfers.[9] Let $\phi : \Theta \times X \times T \to \mathbb{R}$ be the surplus function of the agent. The value $u = \phi(\theta, x, t)$ is the surplus obtained by the θ-type agent who gets outcome x and provides a transfer t to a principal. We impose the following assumptions on the surplus function ϕ. For each θ, let $U(\theta) = \phi(\theta, X, T) \subseteq \mathbb{R}$ be the set of all possible surplus levels of type θ.

Assumption 1. The function $\phi : \Theta \times X \times T \to \mathbb{R}$ is strictly decreasing in its third argument (with respect to $\succeq_T$), and satisfies the condition of *the possibility of compensation*: For each θ, we have $\phi(\theta, x, T) = U(\theta)$ for all $x \in X$.

Some remarks about Assumption 1 are in order. First, any real interval is complete, and it is perfectly fine to think of T as a real interval. We treat it more generally to emphasize that only the completeness property is involved. Second, the condition of the possibility of compensation, borrowed from Kazumura, Mishra, and Serizawa (2020), asserts that, for each type θ and each pair of different outcomes $x \neq x'$, any surplus level the agent can obtain with x and some transfer can also be achieved with x' and some (other) transfer. A similar condition also arises in other contexts, such as the discussion of the existence of competitive equilibrium; e.g., Mas-Colell (1977) and Baldwin, Jagadeesan, Klemperer, and Teytelboym (2023). Notice that we do not require that $U(\theta)$ be identical for all θ. Hence, this condition subsumes as a special case the full-range condition of Nöldeke and Samuelson (2018) and Sinander (2022a) which requires that $\phi(\theta, x, T) \equiv U$ for all (θ, x) and T and U be open real intervals.

[9] By completeness, we mean that $(T, \succeq_T)$ has the greatest-lower-bound property or, equivalently, has the least-upper-bound property.

Later we show that the condition of the possibility of compensation is necessary for strongly implementing monotone allocations in common environments (Lemma 7).

Remark 1. Assumption 1 can be equivalently stated in terms of preference orderings. Each θ-type agent has a rational preference ordering $\succeq_\theta$ on $X \times T$ that admits a utility function representation. The requirements of monotonicity and possibility of compensation in Assumption 1 are imposed on each $\succeq_\theta$: (i) $\forall x \in X$ and $t' \succ_T t$, $(x,t) \succ_\theta (x,t')$; and (ii) $\forall x \neq x'$ and t, there exists some t' such that $(x,t) \sim_\theta (x',t')$.

We focus on direct mechanisms. Such a mechanism consists of an allocation $\mathbf{x} : \Theta \to X$ and a transfer scheme $\mathbf{t} : \Theta \to T$. Recall that mechanism $(\mathbf{x}, \mathbf{t})$ is incentive compatible if for each $\theta \in \Theta$, we have

$$\mathbf{u}(\theta) \triangleq \phi(\theta, \mathbf{x}(\theta), \mathbf{t}(\theta)) = \max_{\theta' \in \Theta} \phi(\theta, \mathbf{x}(\theta'), \mathbf{t}(\theta')),$$

in which $\mathbf{u}(\theta)$ is the surplus obtained by the θ-type agent from truthfully reporting. An allocation $\mathbf{x}$ is *implementable* if there exists a transfer scheme $\mathbf{t}$ such that $(\mathbf{x}, \mathbf{t})$ is incentive compatible. In this case, we say that $\mathbf{t}$ *implements* $\mathbf{x}$ or that $(\mathbf{t}, \mathbf{u})$ *implements* $\mathbf{x}$ to emphasize the agent's corresponding surplus. An important stronger notion of implementability is proposed by Nöldeke and Samuelson (2018).

Definition 1 (Strong Implementability—Nöldeke and Samuelson (2018))**.** An allocation $\mathbf{x}$ is strongly implementable if for each initial condition (θ_0, u_0) with $u_0 \in U(\theta_0)$, $\mathbf{x}$ can be implemented by some $(\mathbf{t}, \mathbf{u})$ with $\mathbf{u}(\theta_0) = u_0$.

Notice that the strong implementability of $\mathbf{x}$ is equivalent to that for each initial condition (θ_0, t_0), $\mathbf{x}$ can be implemented by some transfer scheme $\mathbf{t}$ with $\mathbf{t}(\theta_0) = t_0$. By Nöldeke and Samuelson (2018), with quasilinearity, an allocation is implementable if and only if it is strongly implementable. However, this is not the case without quasilinearity. This causes some salient differences between the quasilinear and the general case. For related discussion, see Nöldeke and Samuelson (2018).

Finally, Assumption 1 ensures that for each (θ, x) and $u \in U(\theta)$, there exists a unique $t \in T$ that satisfies $u = \phi(\theta, x, t)$. We denote this t by $\psi(\theta, x, u)$. Hence, $\psi(\theta, x, \cdot) : U(\theta) \to T$ is the inverse mapping of $\phi(\theta, x, \cdot) : T \to U(\theta)$.

3 A Characterization of Implementability

In this section, under Assumption 1, we fully characterize (strongly) implementable allocations, via a novel cyclical monotonicity condition that generalizes Rochet (1987).

3.1 Generating Function and Incentive Constraints

We first introduce the important notion of the *generating function* associated with a given allocation, which is the building block of our cyclical monotonicity condition. Fix an allocation $\mathbf{x}$ and pick up any $\theta' \in \Theta$ and $u' \in U(\theta')$. If the θ'-type agent obtains a surplus u' from truthfully reporting under mechanism $\mathcal{M} = (\mathbf{x}, \mathbf{t})$, we must have $\mathbf{t}(\theta') = \psi(\theta', \mathbf{x}(\theta'), u')$. The surplus obtained by each θ-type agent from (mis)reporting θ' under $\mathcal{M}$ is thus given by

$$\Phi_{\mathbf{x}}(\theta, \theta'; u') \triangleq \phi(\theta, \mathbf{x}(\theta'), \psi(\theta', \mathbf{x}(\theta'), u')).$$

We refer to $\Phi_{\mathbf{x}}(\theta, \theta'; \cdot) : U(\theta') \to U(\theta)$ with $\Phi_{\mathbf{x}}(\theta, \theta'; u')$ defined above for each (θ, θ') and $u' \in U(\theta')$ as the generating function associated with $\mathbf{x}$. Given each pair of types (θ, θ'), it tells us the surplus obtained by the θ-type agent from (mis)reporting θ' as a function of the surplus obtained by the latter from truthfully reporting. The following lemma summarizes some obvious yet important properties of $\Phi_{\mathbf{x}}$ inherited from ϕ. The proof is omitted.

Lemma 1. *Let Assumption 1 hold. For each allocation $\mathbf{x}$ and each θ and θ', the mapping $\Phi_{\mathbf{x}}(\theta, \theta'; \cdot) : U(\theta') \to U(\theta)$ is strictly increasing on $U(\theta')$ with $\Phi_{\mathbf{x}}(\theta, \theta'; U(\theta')) = U(\theta)$. Moreover, $\Phi_{\mathbf{x}}(\theta, \theta; u) = u$ for all $u \in U(\theta)$.*

Next, we extend the arguments of $\Phi_{\mathbf{x}}$ from pairs of types to any finite chain of types by composition as follows. For each finite chain $\theta_1, \theta_2, \ldots, \theta_J$ with $J \geq 3$, let $\Phi_{\mathbf{x}}(\theta_1, \theta_2, \ldots, \theta_J; \cdot) : U(\theta_J) \to U(\theta_1)$ be the composition of $\Phi_{\mathbf{x}}(\theta_1, \theta_2; \cdot) : U(\theta_2) \to U(\theta_1)$ and $\Phi_{\mathbf{x}}(\theta_2, \ldots, \theta_J; \cdot) : U(\theta_J) \to U(\theta_2)$. For instance,

$$\Phi_{\mathbf{x}}(\theta_1, \theta_2, \theta_3; \cdot) = \Phi_{\mathbf{x}}(\theta_1, \theta_2; \Phi_{\mathbf{x}}(\theta_2, \theta_3; \cdot));$$

$$\Phi_{\mathbf{x}}(\theta_1, \theta_2, \theta_3, \theta_4; \cdot) = \Phi_{\mathbf{x}}(\theta_1, \theta_2; \Phi_{\mathbf{x}}(\theta_2, \theta_3, \theta_4; \cdot)) = \Phi_{\mathbf{x}}(\theta_1, \theta_2; \Phi_{\mathbf{x}}(\theta_2, \theta_3; \Phi_{\mathbf{x}}(\theta_3, \theta_4; \cdot))).$$

We refer to $\Phi_{\mathbf{x}}(\theta_1, \cdots, \theta_J; u)$ for $u \in U(\theta_J)$ as the surplus obtained by the (initial) type θ_1 from the (mis)report chain $\theta_1, \ldots, \theta_J$, given that the (ending) type θ_J obtains u from truthfully reporting. For instance, $\Phi_{\mathbf{x}}(\theta_1, \theta_2, \theta_3; u)$ is the surplus obtained by the θ_1-type agent from (mis)reporting θ_2, given that the θ_2-type agent obtains a surplus of $\Phi_{\mathbf{x}}(\theta_2, \theta_3, u)$ from truthfully reporting, in which $\Phi_{\mathbf{x}}(\theta_2, \theta_3, u)$ is the surplus the θ_2-type agent could obtain from (mis)reporting θ_3, given that the θ_3-type agent obtains u from truthfully reporting. By Lemma 1, $\Phi_{\mathbf{x}}(\theta_1, \cdots, \theta_J; \cdot)$ is strictly increasing on $U(\theta_J)$ with $\Phi_{\mathbf{x}}(\theta_1, \cdots, \theta_J; U(\theta_J)) = U(\theta_1)$.

We now apply the generating function to capture incentive constraints. First, it is obvious that $(\mathbf{t}, \mathbf{u})$ implements $\mathbf{x}$ if and only if

$$\mathbf{u}(\theta) = \max_{\theta'} \Phi_{\mathbf{x}}(\theta, \theta'; \mathbf{u}(\theta')), \forall \theta \in \Theta. \tag{1}$$

However, it is not obvious when such $(\mathbf{t}, \mathbf{u})$ exists. We now extend the above constraints to each finite chain $\theta_1, \ldots, \theta_J$. If the θ_J-type agent obtains $u \in U(\theta_J)$ from truthfully reporting under any incentive-compatible mechanism $\mathcal{M} = (\mathbf{x}, \mathbf{t})$, then the θ_{J-1}-type agent obtains at least $\Phi_{\mathbf{x}}(\theta_{J-1}, \theta_J; u)$ from truthfully reporting under $\mathcal{M}$, by (1). But then the θ_{J-2}-type agent obtains at least $\Phi_{\mathbf{x}}(\theta_{J-2}, \theta_{J-1}, \theta_J; u) = \Phi_{\mathbf{x}}(\theta_{J-2}, \theta_{J-1}; \Phi_{\mathbf{x}}(\theta_{J-1}, \theta_J; u))$ from truthfully reporting under $\mathcal{M}$. Because the agent obtains at least $\Phi_{\mathbf{x}}(\theta_{J-2}, \theta_{J-1}; \Phi_{\mathbf{x}}(\theta_{J-1}, \theta_J; u))$ from (mis)reporting θ_{J-1} under $\mathcal{M}$, by the monotonicity of $\Phi_{\mathbf{x}}(\theta_{J-2}, \theta_{J-1}; \cdot)$, given that the θ_{J-1}-type agent obtains at least $\Phi_{\mathbf{x}}(\theta_{J-1}, \theta_J; u)$ from truthfully reporting. Continue such an argument and we arrive at the conclusion that each θ_j-type agent should obtain at least $\Phi_{\mathbf{x}}(\theta_j, \ldots, \theta_J; u)$ under $\mathcal{M}$. The proof of Lemma 2 below is omitted.

Lemma 2. *Let Assumption 1 hold. Then* $(\mathbf{t}, \mathbf{u})$ *implements* $\mathbf{x}$ *if and only if for each finite chain* $\theta_1, \cdots, \theta_J$ *with* $J \geq 2$, *we have*

$$\Phi_{\mathbf{x}}(\theta_1, \theta_2, \ldots, \theta_J; \mathbf{u}(\theta_J)) \leq \mathbf{u}(\theta_1). \tag{2}$$

A finite chain $\theta_1, \cdots, \theta_J$ with $J \geq 2$ is a *cycle* if $\theta_1 = \theta_J$. In Section 3.2, we show that the incentive constraints (2) hold for some $(\mathbf{t}, \mathbf{u})$ if and only if (2) holds for all cycles, which is a directly verifiable condition that generalizes Rochet (1987).

3.2 Cyclical Monotonicity

Definition 2 (Cyclical Monotonicity)**.** An allocation $\mathbf{x}$ is *cyclically monotone* if there exists some θ_0 and $u_0 \in U(\theta_0)$ such that

$$\Phi_{\mathbf{x}}(\theta_0, \theta_1, \ldots, \theta_J, \theta_0; u_0) \leq u_0, \ \forall J \geq 1, \ \theta_1, \ldots, \theta_J \in \Theta. \tag{3}$$

An allocation $\mathbf{x}$ is *strongly cyclically monotone* if (3) holds $\forall \theta_0 \in \Theta$ and $u_0 \in U(\theta_0)$.

Our cyclical monotonicity requires the existence of some type θ_0 and some $u_0 \in U(\theta_0)$, such that the θ_0-type agent can never strictly benefit from any finite (mis)report cycle, given that the θ_0-type agent obtains u_0 from truthfully reporting. Our strong

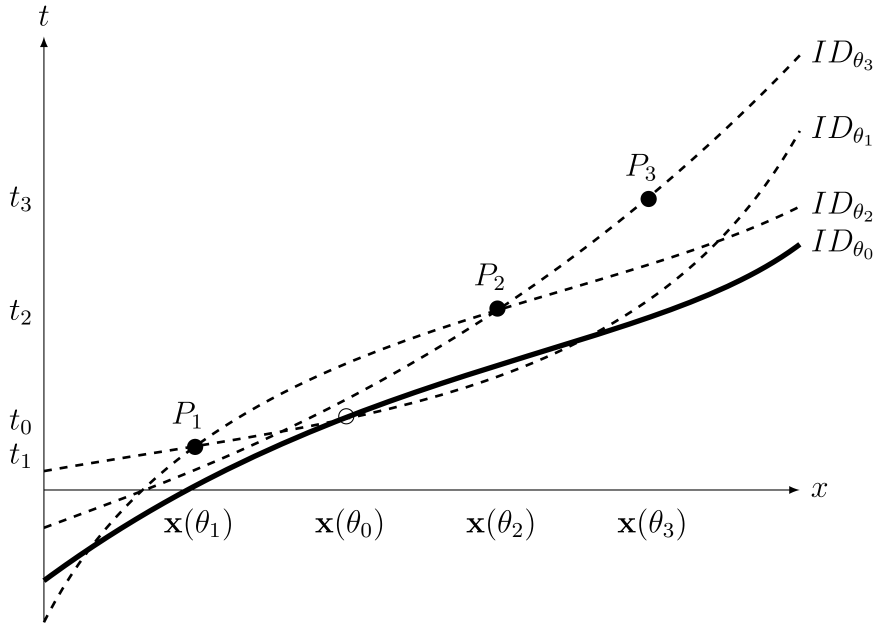

Figure 1: Illustration of Cyclical Monotonicity. The solid curve, ID_{θ_0}, represents the θ_0-type agent's indifference curve passing through $(\mathbf{x}(\theta_0), t_0)$. Consider cycle $\theta_0, \theta_3, \theta_2, \theta_1, \theta_0$. Each dashed curve, ID_{θ_j}, represents the indifference curve of the θ_j-type agent, which passes through $(\mathbf{x}(\theta_{j-1}), t_{j-1})$ and uniquely determines t_j by the indifference condition $\phi(\theta_j, \mathbf{x}(\theta_{j-1}), t_{j-1}) = \phi(\theta_j, \mathbf{x}(\theta_j), t_j)$, for $j = 1, 2, 3$. The cyclical monotonicity condition requires that the three recursively obtained points, P_1, P_2, and P_3, all lie (weakly) above ID_{θ_0}.

cyclical monotonicity requires strongly that any type cannot strictly benefit from any finite (mis)report cycle with any surplus assigned to that type for truthfully reporting.

The cyclical monotonicity proposed here can be demonstrated using indifference curves in the (x, t) plane. Notice that for each (x_0, t_0), the points in the (x, t) plane that lie *above* the indifference curve of a θ-type agent passing through (x_0, t_0) are dominated by (x_0, t_0) in the view of the θ-type agent, because $\phi(\theta, x, t)$ is strictly decreasing in t. Consider any finite cycle $\theta_0, \theta_J, \ldots, \theta_1, \theta_0$ and $u_0 \in U(\theta_0)$. Let $t_0 = \psi(\theta_0, x_0, u_0)$. Condition (3) requires that for each $j \geq 1$, $\Phi_{\mathbf{x}}(\theta_0, \theta_j, \cdots, \theta_1, \theta_0; u_0) \leq u_0$. Let $t_j = \psi(\theta_j, \mathbf{x}(\theta_j), \phi(\theta_j, \mathbf{x}(\theta_{j-1}), t_{j-1}))$ for each $j \geq 1$. Each θ_j-type agent is indifferent between $(\mathbf{x}(\theta_j), t_j)$ and $(\mathbf{x}(\theta_{j-1}), t_{j-1})$. Also, $\Phi_{\mathbf{x}}(\theta_0, \theta_j, \ldots, \theta_0; u_0) = \phi(\theta_0, \mathbf{x}(\theta_j), t_j)$. So the requirement of $\Phi_{\mathbf{x}}(\theta_0, \theta_j, \ldots, \theta_1, \theta_0; u_0) \leq u_0$ is equivalent to $\phi(\theta_0, \mathbf{x}\,(\theta_j), t_j) \leq \phi(\theta_0, \mathbf{x}(\theta_0), t_0)$. That is, the collection of recursively obtained points $\{(\mathbf{x}(\theta_j), t_j)\}_{j=1}^{J}$ in the (x, t) plane all lie weakly above the θ_0-type agent's indifference curve that passes through $(\mathbf{x}(\theta_0), t_0)$, as illustrated in Figure 1.

Rochet (1987) considers the quasilinear environment—that is, $T = \mathbb{R}$ and $\phi(\theta, x, t) =$

$v(\theta, x) - t$—and proposes the following cyclical monotonicity: For each $\theta_0, \theta_1, \ldots, \theta_J, \theta_0$, we have

$$\sum_{j=0}^{J} [v(\theta_j, \mathbf{x}(\theta_{j+1})) - v(\theta_{j+1}, \mathbf{x}(\theta_{j+1}))] \leq 0, \tag{4}$$

in which $\theta_{J+1} = \theta_0$. Our notions of cyclical monotonicity and strong cyclical monotonicity generalize Rochet (1987) to the current environment. In fact, with quasilinearity, our conditions of cyclical monotonicity and strong cyclical monotonicity coincide and reduce to (4). To see this, notice that $U(\theta) \equiv \mathbb{R}$ and $\Phi_{\mathbf{x}}(\theta, \theta'; u) = v(\theta, \mathbf{x}(\theta')) - v(\theta', \mathbf{x}(\theta')) + u$. For any finite chain $\theta_1, \cdots, \theta_J$ and $u \in U(\theta_J)$, we have

$$\Phi_{\mathbf{x}}(\theta_1, \ldots, \theta_J; u) = \sum_{j=1}^{J-1} [v(\theta_j, \mathbf{x}(\theta_{j+1})) - v(\theta_{j+1}, \mathbf{x}(\theta_{j+1}))] + u \tag{5}$$

So $\Phi_{\mathbf{x}}(\theta_0, \theta_1, \ldots, \theta_J, \theta_0; u_0) \leq u_0$ if and only if (4) holds.

Finally, the literature refers to (4) for the case $J = 1$ and each $\theta_0 \in \Theta$ as *2-cyclical monotonicity*, which is of particular interest. Both the 2-cyclical monotonicity condition and the strong cyclical monotonicity condition can be rephrased in a way reminiscent of the Galois connection discussed by Nöldeke and Samuelson (2018).[10]

Lemma 3 (Cyclical Monotonicity and Connections)**.** *Let Assumption 1 hold. The following two statements about an allocation* $\mathbf{x}$ *hold:*

(i) $\mathbf{x}$ *is 2-cyclical monotone if and only if for each* $\theta \neq \theta'$*, the two mappings* $\Phi_{\mathbf{x}}(\theta, \theta'; \cdot) : U(\theta') \to U(\theta)$ *and* $\Phi_{\mathbf{x}}(\theta', \theta; \cdot) : U(\theta) \to U(\theta')$ *satisfy: For each* $u \in U(\theta)$ *and* $v \in U(\theta')$*, we have* $u \leq (<) \Phi_{\mathbf{x}}(\theta, \theta'; v) \Longrightarrow v \geq (>) \Phi_{\mathbf{x}}(\theta', \theta; u)$*.*

(ii) $\mathbf{x}$ *is strongly cyclical monotone if and only if for each* $\theta \neq \theta'$*, each two finite chains* $\theta, \cdots, \theta'$ *and* $\theta', \cdots, \theta$*, the two mappings* $\Phi_{\mathbf{x}}(\theta, \cdots, \theta'; \cdot) : U(\theta') \to U(\theta)$ *and* $\Phi_{\mathbf{x}}(\theta', \cdots, \theta; \cdot) : U(\theta) \to U(\theta')$ *satisfy: For each* $u \in U(\theta)$ *and* $v \in U(\theta')$*, we have* $u \leq (<) \Phi_{\mathbf{x}}(\theta, \cdots, \theta'; v) \Longrightarrow v \geq (>) \Phi_{\mathbf{x}}(\theta', , \cdots, \theta; u)$*.*

3.3 An Implementation Theorem

Rochet (1987) shows that his cyclical monotonicity condition is necessary and sufficient for implementability in the quasilinear environment. Theorem 1 extends this

[10] Let A and B be two nonempty subsets of the reals. According to Davey and Priestley (2002), two functions $f : A \to B$ and $g : B \to A$ are a Galois connection between the sets A and B if for each $v \in A$ and $u \in B$ we have $u \geq f(v) \Longleftrightarrow v \leq g(u)$. The connections involved in Lemma 3, by symmetry, require $u \geq (>) f(v) \Longleftarrow v \leq (<) g(u)$ and $u \leq (<) f(v) \Longrightarrow v \geq (>) g(u)$.

result to the current environment. It states that our (strong) cyclical monotonicity condition fully characterizes (strong) implementability under Assumption 1.

Theorem 1 (An Implementation Theorem)**.** *Let Assumption 1 hold. An allocation* $\mathbf{x}$ *is (strongly) implementable if and only if it is (strongly) cyclically monotone.*

The necessity of Theorem 1 is an immediate consequence of Lemma 2, in which the monotonicity of $\Phi_{\mathbf{x}}$ is applied. The sufficiency of Theorem 1 follows from a modification of the construction of Rochet (1987), in which we continue to exploit the monotonicity of ϕ and $\Phi_{\mathbf{x}}$. For each θ and θ' in Θ and $u' \in U(\theta')$, let

$$V_{\mathbf{x}}(\theta,\theta';u') \triangleq \left\{\Phi_{\mathbf{x}}(\theta,\theta_1,\ldots,\theta_J;u') \,\middle|\, \text{chain } \theta,\theta_1,\ldots,\theta_J, \text{with } \theta_J=\theta'; J\geq 1\right\} \subseteq U(\theta)$$

and

$$T_{\mathbf{x}}(\theta,\theta';u') \triangleq \left\{\psi(\theta,\mathbf{x}(\theta),u) | u\in V_{\mathbf{x}}(\theta,\theta';u')\right\} \subseteq T.$$

A nonempty subset $V \subseteq U(\theta)$ is *bounded from above in* $U(\theta)$ if there exists some u in $U(\theta)$ such that for each $v \in V$, we have $u \geq v$. It is obvious that $T_{\mathbf{x}}(\theta,\theta';u')$ is bounded from below if and only if $V_{\mathbf{x}}(\theta,\theta';u')$ is bounded from above in $U(\theta)$.

If (3) holds, then the possible surplus levels each θ-type agent could obtain from all possible finite (mis)report chains with θ_0 as the ending type is bounded from above in $U(\theta)$, provided the θ_0-type agent obtains u_0 from truthfully reporting.

Lemma 4. *Let Assumption 1 hold. If (3) also holds, then for each* θ*,* $V_{\mathbf{x}}(\theta,\theta_0;u_0)$ *is bounded from above in* $U(\theta)$*, and* $T_{\mathbf{x}}(\theta,\theta_0;u_0)$ *is bounded from below.*

Proof. By Lemma 1, there exists some $\bar{u} \in U(\theta)$ such that $\Phi_{\mathbf{x}}(\theta_0,\theta;\bar{u}) = u_0$. For each $\theta,\theta_1,\ldots,\theta_J$ with $\theta_J = \theta_0$, we have $\Phi_{\mathbf{x}}(\theta,\theta_1,\ldots,\theta_J;u_0) \leq \bar{u}$, since $\Phi_{\mathbf{x}}(\theta_0,\theta;\Phi_{\mathbf{x}}(\theta,\theta_1,\ldots,\theta_J;u_0)) \leq u_0$ and $\Phi_{\mathbf{x}}(\theta_0,\theta;\cdot)$ is strictly increasing. So $V_{\mathbf{x}}(\theta,\theta_0;u_0)$ is bounded from above by $\bar{u}$, and thus $T_{\mathbf{x}}(\theta,\theta_0;u_0)$ is bounded from below by $\psi(\theta,\mathbf{x}(\theta),\bar{u})$. □

Next, for each θ, let

$$\mathbf{t}^*(\theta) \triangleq \inf T_{\mathbf{x}}(\theta,\theta_0;u_0), \tag{6}$$

which is welldefined when (3) holds by Lemma 4, since T is complete. The following properties of $\mathbf{t}^*$ are immediate, and the proof is omitted.

Claim 1. Let Assumption 1 hold. If (3) also holds, then for each $u \in V_{\mathbf{x}}(\theta,\theta_0;u_0)$, we have $u \leq \phi(\theta,\mathbf{x}(\theta),\mathbf{t}^*(\theta))$. Moreover, $\phi(\theta_0,\mathbf{x}(\theta_0),\mathbf{t}^*(\theta_0)) = u_0$.

The surplus obtained from truthfully reporting for each θ-type agent under mechanism $(\mathbf{x},\mathbf{t}^*)$ is (weakly) larger than the surplus he or she could obtain from *any* finite (mis)report chain with θ_0 as the ending type and with u_0 as the surplus assigned to θ_0 for truthfully reporting. Then it is obvious that $\mathbf{t}^*$ implements $\mathbf{x}$, by noticing that for each θ' and t' in $T_{\mathbf{x}}(\theta', \theta_0, u_0)$, we have $\phi(\theta, \mathbf{x}(\theta'), t') \in V_{\mathbf{x}}(\theta, \theta_0; u_0)$.

Lemma 5. *Let Assumption 1 hold. If (3) also holds, then $\mathbf{t}^*$ implements $\mathbf{x}$.*

Proof. By Lemma 4, $\mathbf{t}^*$ is welldefined, since T is complete. Consider $\theta \neq \theta'$. Let $\phi(\theta, \mathbf{x}(\theta'), \underline{t}) = \phi(\theta, \mathbf{x}(\theta), \mathbf{t}^*(\theta))$. For each $t' \in T_{\mathbf{x}}(\theta', \theta_0; u_0)$, we have $\phi(\theta, \mathbf{x}(\theta'), t') \in V_{\mathbf{x}}(\theta, \theta_0; u_0)$. Because $t' = \psi(\theta', \mathbf{x}(\theta'), \Phi_{\mathbf{x}}(\theta', \cdots, \theta_J; u_0))$ for some finite chain $\theta', \cdots, \theta_J$ with $\theta_J = \theta_0$, by the definition of $T_{\mathbf{x}}$. Thus we have $\phi(\theta, \mathbf{x}(\theta'), t') \leq \phi(\theta, \mathbf{x}(\theta), \mathbf{t}^*(\theta))$, by Claim 1. So $t' \geq \underline{t}$, since $\phi(\theta, x, \cdot)$ is strictly decreasing and $\phi(\theta, \mathbf{x}(\theta'), \underline{t}) = \phi(\theta, \mathbf{x}(\theta), \mathbf{t}^*(\theta))$. Since this holds for each $t' \in T_{\mathbf{x}}(\theta', \theta_0; u_0)$, we have $\mathbf{t}^*(\theta') \geq \underline{t}$, by which $\phi(\theta, \mathbf{x}(\theta'), \mathbf{t}^*(\theta')) \leq \phi(\theta, \mathbf{x}(\theta'), \underline{t})$. So $\phi(\theta, \mathbf{x}(\theta'), \mathbf{t}^*(\theta')) \leq \phi(\theta, \mathbf{x}(\theta), \mathbf{t}^*(\theta))$. The desired result follows. □

Proof of Theorem 1: Sufficiency follows from Lemma 5 and Claim 1. Necessity follows from Lemma 2 by taking $\theta_1 = \theta_J = \theta_0$ and $u_0 = \mathbf{u}(\theta_0)$. □

Finally, it is noteworthy that the transfer scheme $\mathbf{t}^*$ constructed in (6) is the largest among all transfer schemes that implement $\mathbf{x}$ and satisfy the initial condition (θ_0, u_0). Put $\mathbf{u}^*(\theta) \triangleq \phi(\theta, \mathbf{x}(\theta), \mathbf{t}^*(\theta))$. For each initial condition (θ_0, t_0), let

$$\mathcal{T}_{\mathbf{x}}(\theta_0, t_0) = \{(\mathbf{t}, \mathbf{u}) | \mathbf{t} \text{ implements } \mathbf{x} \text{ with } \mathbf{t}(\theta_0) = t_0; \mathbf{u}(\theta) = \phi(\theta, \mathbf{x}(\theta), \mathbf{t}(\theta)), \forall \theta\}.$$

Proposition 1 (An Extreme Transfer Property)**.** *Let Assumption 1 hold and $\mathcal{T}_{\mathbf{x}}(\theta_0, t_0)$ be nonempty. Then $(\mathbf{t}^*, \mathbf{u}^*) \in \mathcal{T}_{\mathbf{x}}(\theta_0, t_0)$, in which $\mathbf{t}^*$ is given by (6) with $u_0 = \phi(\theta_0, \mathbf{x}(\theta_0), t_0)$. Moreover, for each $(\mathbf{t}, \mathbf{u}) \in \mathcal{T}_{\mathbf{x}}(\theta_0, t_0)$, we have $\mathbf{t} \leq \mathbf{t}^*$ and $\mathbf{u} \geq \mathbf{u}^*$.*

Proof. Since $\mathcal{T}_{\mathbf{x}}(\theta_0, t_0)$ is nonempty, the cyclical monotonicity condition (3) holds, by applying Lemma 2 for the case $\theta_1 = \theta_J = \theta_0$. So $\mathbf{t}^*$ implements $\mathbf{x}$ with $\mathbf{t}^*(\theta_0) = t_0$, by Claim 1 and Lemma 5. Fix any $(\mathbf{t}, \mathbf{u})$ in $\mathcal{T}_{\mathbf{x}}(\theta_0, t_0)$. By Lemma 2 again, for each θ and each $u \in V_{\mathbf{x}}(\theta, \theta_0; u_0)$, we have $u \leq \mathbf{u}(\theta)$. Thus, for each $t \in T_{\mathbf{x}}(\theta, \theta_0; u_0)$, we have $t \geq \mathbf{t}(\theta)$ and so $\mathbf{t}^*(\theta) \geq \mathbf{t}(\theta)$. Then we have $\mathbf{u} \geq \mathbf{u}^*$. □

4 Strongly Implementing Monotone Allocations

In this section, we apply our implementation theorem to discuss the strong implementability of monotone allocations for the general case. Thus we assume, throughout this section, that the type space Θ is totally ordered by $\geq_\Theta$ and the outcome space X is partially ordered by $\geq_X$. An allocation $\mathbf{x}$ is *monotone* if $\theta' >_\Theta \theta$ implies $\mathbf{x}(\theta') \geq_X \mathbf{x}(\theta)$.

Recall that a function $v(\theta, x) : \Theta \times X \to \mathbb{R}$ has (strictly) increasing differences if for each $x' >_X x$ in X, the mapping $\theta \to v(\theta, x') - v(\theta, x)$ (strictly) increases in θ. Fact 1 below summarizes results on the implementability of monotone allocations for the quasilinear case, which extend Spence (1974), Mirrlees (1976), and Rochet (1987), by applying Rochet (1987)'s cyclical monotonicity condition.

Fact 1 (The Quasilinear Case). Let $T = \mathbb{R}$ and $\phi(\theta, x, t) = v(\theta, x) - t$. The following two statements hold:

(i) All monotone allocations are implementable if and only if v has increasing differences;

(ii) If v has strict increasing differences and X is totally ordered, then an allocation is implementable if and only if it is monotone.

Our main results in this section extend Fact 1 to the general case without invoking quasilinearity. The (strict) single-crossing condition on ϕ stated below can be equivalently rephrased as: For each pricing scheme $\mathbf{t} : X \to T$, the mapping $(\theta, x) \to \phi(\theta, x, \mathbf{t}(x))$ has (strict) single-crossing differences. It reduce to the condition of (strictly) increasing differences on v for the quasilinear case.

Definition 3 (Single Crossing)**.** The surplus function ϕ is *single crossing* if, for each $\theta' >_\Theta \theta$, each $x' >_X x$, and each t and t', $\phi(\theta, x', t') \geq (>)\phi(\theta, x, t)$ implies $\phi(\theta', x', t') \geq (>)\phi(\theta', x, t)$. It is *strictly* single crossing if for each $\theta' >_\Theta \theta$, $x' >_X x$, t and t', $\phi(\theta, x', t') \geq \phi(\theta, x, t)$ implies $\phi(\theta', x', t') > \phi(\theta', x, t)$.

4.1 A Structural Theorem and Equivalence Result

Our next theorem shows that, under Assumption 1, the single-crossing condition is both sufficient and necessary for strongly implementing all monotone allocations.

Theorem 2 (Strongly Implementing Monotone Allocations—a Structural Theorem)**.** *Let Assumption 1 hold. All monotone allocations are strongly implementable if and only if ϕ satisfies the single-crossing condition.*

By Remark 1, Theorem 2 offers an ordinal theory of implementing monotone allocations for the general case. As an immediate consequence of Theorem 2, the following equivalence between implementability and monotonicity is obtained.

Theorem 3 (Equivalence between Implementability and Monotonicity)**.** *Let X be totally ordered and ϕ satisfy the strict single-crossing condition. Under Assumption 1, an allocation is implementable if and only if it is monotone. In addition, every implementable allocation is strongly implementable.*

Proof. It follows from Milgrom and Shannon (1994) and Theorem 2. □

Theorem 3 generalizes Nöldeke and Samuelson (2018) (their Proposition 13) to any totally ordered type space and outcome space under the weaker assumption of possibility of compensation. No topological requirements are imposed; the continuity assumption of Nöldeke and Samuelson (2018) is thus dropped.

The sufficiency of Theorem 2 follows from the implementation theorem. We relegate its proof to Section 4.4, in which we show, with the single-crossing condition, that each monotone allocation is strongly cyclically monotone. Here we discuss necessity. Refer to each (θ', x') and (θ'', x'') with $\theta'' >_\Theta \theta'$ and $x'' >_X x'$ in the (θ, x) plane as an *ordered pair*. An ordered pair $(\theta', x') < (\theta'', x'')$ is *lower* (*upper*) strongly implementable if for each t' (t''), there exists some t'' (t') such that $\phi(\theta', x', t') \geq \phi(\theta', x'', t'')$ and $\phi(\theta'', x', t') \leq \phi(\theta'', x'', t'')$. An ordered pair is strongly implementable if it is both lower and upper strongly implementable. Lemma 6 relates the single-crossing condition to lower strong implementability of ordered pairs.

Lemma 6. *Let ϕ be strictly decreasing in its third argument. If all ordered pairs in the (θ, x) plane are lower strongly implementable, then ϕ is single crossing.*

Proof. Suppose the contrary—that there exists $\theta' >_\Theta \theta$, $x' >_X x$, t and t', such that $\phi(\theta, x', t') \geq \phi(\theta, x, t)$ but $\phi(\theta', x', t') < \phi(\theta', x, t)$. Since $(\theta, x) < (\theta', x')$ is lower strongly implementable, there exists some t'' such that $\phi(\theta, x, t) \geq \phi(\theta, x', t'')$ and $\phi(\theta', x', t'') \geq \phi(\theta', x, t)$. By $\phi(\theta, x, t) \geq \phi(\theta, x', t'')$ and $\phi(\theta, x', t') \geq \phi(\theta, x, t)$, we have $\phi(\theta, x', t') \geq \phi(\theta, x, t'')$ and so $t'' \succeq_T t'$. However, by $\phi(\theta', x', t'') \geq \phi(\theta', x, t)$ and $\phi(\theta', x', t') < \phi(\theta', x, t)$, we have $\phi(\theta', x', t'') > \phi(\theta', x', t')$ and thus $t' \succ_T t''$, a contradiction. The proof of the strict case is the same and omitted. □

Proof of Necessity of Theorem 2: It follows from Lemma 6: Trivially, if all monotone allocations are strongly implementable, then all ordered pairs in the (θ, x) plane are strongly implementable. □

4.2 Strongly Implementing Ordered Pairs: A Reduction

In this section, we show that commonly, the possibility of compensation is a necessary condition for strongly implementing all monotone allocations—and in particular, when we focus on interior types. As a result, commonly, strongly implementing monotone allocations reduces to strongly implementing ordered pairs in the (θ, x) plane.

Say that X is *simple order-connected* if, for each x and y in X, there exists some $z \in X$ such that x and z are ordered and y and z are ordered. Examples of simple order-connected outcome space abound: In ascending order of generality, (i) X is totally ordered; (ii) X is a lattice; and (iii) X is an upward (downward) directed set: Each pair of elements in X has an upper (lower) bound in X—e.g., experiments with Blackwell's informativeness order, which is not a lattice with multiple states.

Assumption 2. The type space Θ has neither a least nor a greatest element. The outcome space X is simple order-connected. The transfer space T is a real interval. The surplus function $\phi : \Theta \times X \times T \to \mathbb{R}$ is continuous and strictly decreasing in its third argument for all (θ, x).

When Θ is an open real interval—or when we focus on interior types—it has neither a least nor a greatest element. The requirement of the simple order-connectedness of X is commonly satisfied, as discussed above. The assumptions that T is a real interval and ϕ is continuous in transfers are made in all related discussions.

Lemma 7. *Let Assumption 2 hold. If all ordered pairs in the (θ, x) plane are strongly implementable, then ϕ satisfies the condition of the possibility of compensation.*

Proposition 2 (Reduction)**.** *Let Assumption 2 hold. All monotone allocations are strongly implementable if and only if all ordered pairs are strongly implementable.*

Proof. Necessity is trivial. Sufficiency follows from Theorem 2 and Lemmas 6 and 7. □

Finally, it is noteworthy that under Assumption 2, all monotone allocations are strongly implementable if and only if both the single-crossing condition and the condition of the possibility of compensation hold.

4.3 Discussion

In this section, we discuss several topics related to implementing monotone allocations. In Section 4.3.1, we apply Theorems 2 and 3 to extend Fact 1 to the case in which utility and transfer are aggregated by a convex and supermodular aggregator and thus allow for nonlinear aggregation. In Section 4.3.2, we apply Theorem 2 and the reduction result of Proposition 2 to discuss implementing well-behaved monotone allocations. In Section 4.3.3, we briefly comment on the envelope-theorem approach.

4.3.1 Nonlinear Aggregation

In the quasilinear case, the type-dependent utility of outcomes $v(\theta, x)$ and transfer t are aggregated in a linear way, independent of types (Fact 1). In this section, we examine the following more general aggregation:

$$\phi(\theta, x, t) \triangleq g(v(\theta, x), t), \tag{7}$$

in which $g(v, t) : \mathbb{R} \times T \to \mathbb{R}$ is an *aggregator*: It is strictly increasing in v and strictly decreasing in t with $g(\cdot, T) \equiv U$ for some $U \subseteq \mathbb{R}$. Such an aggregation first appears in Sinander (2022a)'s discussion of selling information.

By Theorem 2, all monotone allocations are strongly implementable if and only if ϕ is single crossing; Assumption 1 is satisfied for aggregation. We now identify a sufficient condition on g for the single-crossing condition on ϕ to hold, given that v has increasing differences. We focus on a *canonical* case in which, besides increasing differences, it also assumes that $v(\theta, x)$ increases in both θ and x: $v(\theta', x) \geq v(\theta, x)$ for each $\theta' >_\Theta \theta$ and x; $v(\theta, x') \geq v(\theta, x)$ for each $x' >_X x$ and θ, e.g., Mas-Colell, Whinston, and Green (1995). Proposition 3 extends Fact 1 to the broader class of convex and supermodular aggregators: The marginal contribution of v to the aggregator g increases in both v and t. It applies the composition result of Topkis (1998).

Proposition 3 (Nonlinear Aggregation)**.** *Let ϕ be given by (7), in which v is canonical and $g(v, t)$ is convex in v and supermodular in (v, t). The following statements hold:*

(i) All monotone allocations are strongly implementable;

(ii) If, additionally, v has strictly increasing differences and X is totally ordered, then an allocation is implementable if and only if it is monotone.

Proof. For the first statement, for each $\theta' >_\Theta \theta$, $x' >_X x$, and $t' \succeq_T t$, we have $\phi(\theta', x', t') - \phi(\theta', x, t) \geq \phi(\theta, x', t') - \phi(\theta, x, t)$, by Lemma 2.6.4 of Topkis (1998) (or see Claim 4 in Appendix B.4). But this implies that ϕ is single crossing, since for each θ, $x' >_X x$, and $t \succ_T t'$, we have $\phi(\theta, x', t') > \phi(\theta, x, t)$. Theorem 2 applies. If, in addition, v has strict increasing differences, then ϕ is strictly single crossing. The second statement then follows from Theorem 3. □

Example 1 (Blackwell Experiments—a Revisit of Sinander (2022a))**.** In this example, we revisit the problem of selling information discussed by Sinander (2022a).

Interpret X as a collection of Blackwell experiments on a finite set of possible states of the world Ω. Denote by Δ the set of all possible beliefs on Ω. Let $\geq_X$ be Blackwell's informativeness order—that is, for each $x' >_X x$, x' is more informative than x in the sense of Blackwell (1951).[11] Assume that all types of the agent share a common prior belief π on Ω. Let $\lambda(x, \pi)$ be the distribution of posterior beliefs induced by experiment x and prior belief π in the standard way. Each θ-type agent solves some decision problem under uncertainty of states, with A_θ as the choice set and $u_\theta : \Omega \times A_\theta \to \mathbb{R}$ as the payoff function. For each $\theta \in \Theta$ and $q \in \Delta$, let $w(\theta, q)$ be the expected value the θ-type agent achieves with posterior belief q:

$$w(\theta, q) \triangleq \sup_{a \in A_\theta} \sum_{\omega \in \Omega} u_\theta(\omega, a) q(\omega),$$

which is convex in q for each θ. Let $v(\theta, x)$ be the value of experiment x for θ-type agent:

$$v(\theta, x) \triangleq \int_\Delta w(\theta, q) d\lambda(x, \pi).$$

The agent aggregates the value of the experiment and transfer using an aggregator g:

$$\phi(\theta, x, t) \triangleq g\left(\int_\Delta w(\theta, q) d\lambda(x, \pi), t\right). \tag{8}$$

Corollary 1. *Let $w(\theta', \cdot,) - w(\theta, \cdot)$ be convex for all $\theta' >_\Theta \theta$ and $w(\cdot, q)$ be increasing for all $q \in \Delta$. If $g(v, t)$ is convex in v and supermodular in (v, t), then all monotone allocations are strongly implementable.*

Proof. $v(\theta, x)$ has increasing differences by Blackwell (1951), since $w(\theta', \cdot,) - w(\theta, \cdot)$ is convex for all $\theta' >_\Theta \theta$. Also, $v(\theta, \cdot)$ is increasing for all θ, again by Blackwell (1951). Finally, $v(\cdot, x)$ is increasing for all x, because $w(\cdot, q)$ is increasing for all q. Proposition 3 applies. □

For linear aggregation, the condition of increasing differences on v is sufficient for implementing monotone allocations by Fact 1. However, for the more general class of convex and supermodular aggregators, besides increasing differences, the monotonicity conditions on $v(\theta, x)$ are also needed. For instance, in Corollary 1, the condition of convex differences—by which the higher type gets larger marginal value from more informative experiments—ensures the implementability of monotone allocations for the quasilinear case. However, for the more general class of convex and supermodular

[11] Strictly speaking, the binary relation $\geq_X$ is a subset of Blackwell's informativeness relation.

aggregators, the extra requirement of the monotonicity of $w(\cdot, q)$ for all q—by which the higher type values each experiment in X more—is needed.

4.3.2 Well-behaved Monotone Allocations

Guesnerie and Laffont (1984) show that piecewisely continuously differentiable monotone allocations are strongly implementable with multidimensional outcomes, under the Spence-Mirrlees condition and a boundary condition (their Theorem 2).[12] In this section, we show that commonly, partial results regarding "well-behaved" monotone allocations readily extend to *all* monotone allocations without imposing any extra conditions, when we focus on interior types. For each $\theta'' >_\Theta \theta'$ in Θ, say that allocation $\mathbf{x} : [\theta, \theta'] \to X$ is *tail* strongly implementable if, for each $t \in T$, there exists $\mathbf{t} : [\theta, \theta'] \to T$ with $\mathbf{t}(\theta) = t$ and $\mathbf{t}' : [\theta, \theta'] \to T$ with $\mathbf{t}'(\theta') = t$ such that both $(\mathbf{x}, \mathbf{t})$ and $(\mathbf{x}, \mathbf{t}')$ are incentive compatible (restricted on $[\theta, \theta']$). For each $k \in \mathbb{N} \cup \{+\infty\}$, let C^k be the class of kth continuously differentiable functions.

Proposition 4 (Well-behaved Monotone Allocations)**.** *Let Θ be an open real interval, $X \subseteq \mathbb{R}^L$ be convex and simple order-connected with the usual order, and T be a real interval. Let $\phi : \Theta \times X \times T \to \mathbb{R}$ be continuous and strictly decreasing in its third argument, for all (θ, x). All monotone allocations are strongly implementable if and only if for each $\theta < \theta'$ in Θ, each monotone allocation defined on $[\theta, \theta']$ that falls into C^k (that is linear) (that is Lipschitz continuous) is tail strongly implementable.*

Proof. Consider sufficiency. For each ordered pair $(\theta', x') < (\theta'', x'')$, let $\mathbf{x} : [\theta', \theta''] \to X$ be such that $\mathbf{x}(\theta) = (\theta - \theta')/(\theta'' - \theta')x'' + (\theta'' - \theta)/(\theta'' - \theta')x'$, which is feasible since X is convex. Clearly, $\mathbf{x}$ is kth continuously differentiable (linear) (Lipschitz continuous). By the tail strong implementability of $\mathbf{x}$, the ordered pair $(\theta', x') < (\theta'', x'')$ is strongly implementable. Proposition 2 applies. □

Example 2 (A Revisit of Guesnerie and Laffont (1984))**.** In this example, we revisit Guesnerie and Laffont (1984). Let T be a real interval and $X = \prod_{l=1}^{L} X_l$ in which each X_l is a real interval. The orders on X and T are as usual. A function $\phi : \Theta \times X \times T \to \mathbb{R}$, continuously differentiable in (x, t) for each θ, satisfies the (multidimensional) Spence-Mirrlees condition if $\phi_t < 0$ and $\phi_l / |\phi_t|$ is nondecreasing in θ (with respect to $\geq_\Theta$) for each fixed (x, t) and l, in which $\phi_l = \partial\phi/\partial x_l$.[13] Following Guesnerie and Laffont, ϕ

[12] See also Theorem 7.3 of Fudenberg and Tirole (1991).

[13] We do not require real-valued types. The Spence-Mirrlees condition compares the (smooth) indifference curves in the (x, t) plane associated with each pair of different types. Conceptually, whether types are real numbers or ϕ is smooth in types should be irrelevant.

satisfies *the boundary condition* if there exists $K > 0$ and K', such that for all (θ, x, t) and l, we have $|\phi_l/\phi_t| \leq K|t| + K'$. The sufficiency of Corollary 2 extends Theorem 2 of Guesnerie and Laffont (1984) to *all* monotone allocations *under their conditions*, focusing on interior types.

Corollary 2. *Let Θ be an open real interval, $T = \mathbb{R}$, and $X = \prod_{l=1}^{L} X_l$ where each X_l is a real interval with the usual product order. Also, let $\phi : \Theta \times X \times \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable with $\phi_t < 0$ and the boundary condition hold. All monotone allocations are strongly implementable if and only if the Spence-Mirrlees condition holds.*

Proof. (Sufficiency.) With the conditions, each continuously differentiable monotone allocation defined on $[\theta, \theta']$ for each $\theta < \theta'$ in Θ is tail strongly implementable, by Guesnerie and Laffont (1984). Proposition 4 applies. (Necessity.) By Lemma 6, ϕ is single crossing. Lemma 8 below applies. $\square$

Lemma 8. *Let $X = \prod_{l=1}^{L} X_l$ in which each X_l is a real interval with the product order and T be a real interval. Also, let $\phi(\theta, x, t)$ be continuously differentiable in (x, t) for each θ with $\phi_t < 0$. If ϕ is single crossing, then the Spence-Mirrlees condition holds.*

Finally, we build an equivalence between strongly implementing monotone allocations and the Spence-Mirrlees condition by assuming that ϕ is *completely regular* in the sense of Milgrom and Shannon (1994): For each θ, each $x' >_X x$, t and t', such that $\phi(\theta, x, t) = \phi(\theta, x', t')$, the isoutility curve of the θ-type agent between (x, t) and (x', t') corresponds to a smooth path $\{(\mathbf{x}(s), t(s)) | s \in [0, 1]\}$ such that $\mathbf{x}_l'(s) \geq 0$ for each l with $(\mathbf{x}(0), \mathbf{t}(0)) = (x, t)$ and $(\mathbf{x}(1), \mathbf{t}(1)) = (x', t')$. In Proposition 5, both the boundary condition and the joint twice continuous differentiability requirement of Guesnerie and Laffont (1984) are dropped.[14]

Proposition 5 (Spence-Mirrlees Condition and Implementing Monotone Allocations)**.** *Let $X = \prod_{l=1}^{L} X_l$ in which each X_l is a real interval with the usual product order and T be a real interval. Also, let $\phi : \Theta \times X \times T \to \mathbb{R}$ be continuously differentiable in (x, t) for each θ with $\phi_t < 0$, be completely regular, and satisfy the condition of possibility of compensation. All monotone allocations are strongly implementable if and only if the Spence-Mirrlees condition holds.*

[14] Complete regularity calls for conditions on the sign of each ϕ_l—which, however, as highlighted by Guesnerie and Laffont (1984), is not assumed in their results. Nor are they assumed in Corollary 2.

4.3.3 A Comment on the Envelope-Theorem Approach

One common approach for addressing implementability with quasilinearity is based on applying the envelope theorem; e.g., Mas-Colell, Whinston, and Green (1995). It would be nice to know how far this approach could go for the general case, to which Sinander (2022a) made a notable contribution. Let $\Theta = [0, 1]$, $T = \mathbb{R}$, and $\phi(\theta, x, t) : \Theta \times X \times \mathbb{R} \to \mathbb{R}$ be strictly decreasing in t with $\phi(\theta, x, \mathbb{R}) = \mathbb{R}$ for all (θ, x).

Without quasilinearity, the first step of the envelope-theorem approach is to identify the conditions by which, for each allocation $\mathbf{x}$ among the targeted class, there exists some transfer scheme $\mathbf{t}$ such that the following envelope formula holds:

$$\phi(\theta, \mathbf{x}(\theta), \mathbf{t}(\theta)) = k + \int_0^\theta \phi_1(s, \mathbf{x}(s), \mathbf{t}(s))ds, \forall \theta \in [0, 1], \tag{9}$$

in which k is a given constant. Or equivalently,

$$\mathbf{t}(\theta) = \psi(\theta, \mathbf{x}(\theta), k + \int_0^\theta \phi_1(s, \mathbf{x}(s), \mathbf{t}(s))ds), \forall \theta \in [0, 1].$$

The conditions for the existence of such a $\mathbf{t}$, when the targeted class of allocations are the monotone ones, as identified by Sinander (2022a), are stated below.

Assumption 3 (Regularity Conditions)**.** (i) X is regular: X is order-dense-in-itself, countably chain-complete, and chain-separable. (ii) $\phi(\theta, x, t)$ is regular: ϕ is differentiable in types with bounded type derivative; $\phi_1(\theta, x, \cdot)$ is continuous in transfers for all (θ, x); for each chain $C \subseteq X$, ϕ is jointly continuous on $[0, 1] \times C \times \mathbb{R}$.

The above conditions are mild for the existence of a solution to the nontrivial fixed-point problem (9) *based on a topological approach*. However, in terms of applications, they may not be convenient to verify and easily fail.[15] For instance, of all the results in Section 4, Corollary 2 is the closest to Assumption 3. But Corollary 2 does not fit all of these requirements: Clearly, the type derivative can be unbounded, given that Θ is an open interval, when $X = [\epsilon, \infty)^L$ as in Guesnerie and Laffont (1984).

It will be instructive to compare (9) with the implementability problem (1), which is also a fixed-point problem,[16] to understand how we managed to dispense with Assumption 3 in our results. The monotonicity of the generating function $\Phi_{\mathbf{x}}$ and

[15]This is particularly the case when ϕ is endogenous.

[16]More explicitly, let $\mathcal{U} = \{\mathbf{u} : \Theta \to \mathbb{R} | \mathbf{u}(\theta) \in U(\theta), \forall \theta\}$, $\bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$, and $\Lambda_{\mathbf{x}} : \mathcal{U} \to \bar{\mathbb{R}}^\Theta$ be such that $\Lambda_{\mathbf{x}} \mathbf{u}(\theta) = \sup_{\theta'} \Phi_{\mathbf{x}}(\theta, \theta', \mathbf{u}(\theta'))$, for each $\mathbf{u}$ and θ. Then $\mathbf{x}$ is implementable if and only if $\Lambda_{\mathbf{x}}$ has a fixed point in $\mathcal{U}$.

the maximum operator in the fixed-point problem (1) offer an order structure that can be fruitfully exploited, which gives rise to the cyclical monotonicity condition for existence. The conditions in Assumption 3, entailed by a topological approach for (9), are orthogonal to our approach for (1), and thus are not needed by Theorem 1. This extends to Theorems 2 and 3 and other results based on them.

The next step of the envelope-theorem approach is to identify the conditions such that for each allocation $\mathbf{x}$ among the targeted class, the solution to (9) indeed implements $\mathbf{x}$. For this step, the converse envelope theorem of Sinander helps. For the class of monotone allocations, some version of the single-crossing condition is needed. Sinander (2022a) proposes the outer Spence–Mirrlees condition, which can be neatly applied based on his converse envelope theorem: For each monotone allocation $\mathbf{x}$ and each transfer scheme $\mathbf{t}$ and each $r < z$ in $[0,1]$, the following mapping

$$n \to \frac{\bar{d}}{\bar{d}m} \int_r^z \phi(s+n, \mathbf{x}(s+m), \mathbf{t}(s+m))ds\Big|_{m=0}$$

is single crossing, in which $\bar{d}/\bar{d}m$ denotes the upper derivative. The outer Spence-Mirrlees condition implies the single-crossing condition in Definition 3, as Sinander notes. As a result, the implementability theorem of Sinander (Section 4.3 of Sinander (2022a)) follows from Theorem 2.

Claim 2. If ϕ is regular and satisfies the outer Spence-Mirrlees condition, then all monotone allocations are strongly implementable.

Proof. By Lemma 7 of Sinander (2022a), ϕ is single crossing. Theorem 2 applies. □

The outer Spence-Mirrlees condition is not as straightforward and easy to identify as the single-crossing condition. For the latter, we can readily apply the theory of monotone comparative statics, as illustrated by Proposition 3.

4.4 Proof of the Sufficiency of Theorem 2

We show that with the single-crossing condition, each monotone allocation is strongly cyclically monotone. We first show 2-cyclical monotonicity.

Lemma 9. *Let Assumption 1 hold and ϕ be single crossing. Each monotone allocation is 2-cyclical monotone.*

Proof. We show that the condition in the first statement of Lemma 3 is satisfied. Let $\mathbf{x}$ be monotone. It is easy to see that by the single-crossing condition, for each $\theta \neq \theta'$,

t and t', we have

$$\phi(\theta, \mathbf{x}(\theta), t) \leq (<)\phi(\theta, \mathbf{x}(\theta'), t') \Longrightarrow \phi(\theta', \mathbf{x}(\theta'), t') \geq (>)\phi(\theta', \mathbf{x}(\theta), t). \tag{10}$$

Assume $u \leq (<)\Phi_{\mathbf{x}}(\theta, \theta'; v)$, or equivalently,

$$\phi(\theta, \mathbf{x}(\theta), \psi(\theta, \mathbf{x}(\theta), u)) \leq (<)\phi(\theta, \mathbf{x}(\theta'), \psi(\theta', \mathbf{x}(\theta'), v)).$$

Applying (10) for $t = \psi(\theta, \mathbf{x}(\theta), u)$ and $t' = \psi(\theta', \mathbf{x}(\theta'), v)$, we obtain

$$\phi(\theta', \mathbf{x}(\theta'), \psi(\theta', \mathbf{x}(\theta'), v)) \geq (>)\phi(\theta', \mathbf{x}(\theta), \psi(\theta, \mathbf{x}(\theta), u)),$$

or equivalently, $v \geq (>)\Phi_{\mathbf{x}}(\theta', \theta; u)$. The desired result follows from Lemma 3. $\square$

Next, we extend Rochet's induction analysis to the current environment. Given a finite chain $\theta_1, \cdots, \theta_J$ with $J \geq 3$, say that θ_j with $1 < j < J$ is a local extremum if either $\theta_j = \min\{\theta_{j-1}, \theta_j, \theta_{j+1}\}$ or $\theta_j = \max\{\theta_{j-1}, \theta_j, \theta_{j+1}\}$.

Lemma 10. *Let Assumption 1 hold and ϕ be single crossing. Also, let $\mathbf{x}$ be monotone. For each finite chain $\theta_1, \theta_2, \theta_3, \cdots, \theta_J$ with $J \geq 3$, and $u \in U(\theta_J)$, if θ_2 is a local extremum, then we have*

$$\Phi_{\mathbf{x}}(\theta_1, \theta_2, \theta_3, \cdots, \theta_J; u) \leq \Phi_{\mathbf{x}}(\theta_1, \theta_3, \cdots, \theta_J; u). \tag{11}$$

Proof. Let $\theta_2 = \max\{\theta_1, \theta_2, \theta_3\}$. Suppose to the contrary that (11) fails. That is,

$$\phi(\theta_1, \mathbf{x}(\theta_2), \psi(\theta_2, \mathbf{x}(\theta_2), \Phi_{\mathbf{x}}(\theta_2, \cdots, \theta_J; u))) > \phi(\theta_1, \mathbf{x}(\theta_3), \psi(\theta_3, \mathbf{x}(\theta_3), \Phi_{\mathbf{x}}(\theta_3, \cdots, \theta_J; u))).$$

Since $\theta_2 \geq_\Theta \theta_3$ and $\mathbf{x}$ is monotone, we have $\mathbf{x}(\theta_2) \geq_X \mathbf{x}(\theta_3)$. Then by $\theta_2 \geq \theta_1$ and the single-crossing condition, we have

$$\phi(\theta_2, \mathbf{x}(\theta_2), \psi(\theta_2, \mathbf{x}(\theta_2), \Phi_{\mathbf{x}}(\theta_2, \cdots, \theta_J; u))) > \phi(\theta_2, \mathbf{x}(\theta_3), \psi(\theta_3, \mathbf{x}(\theta_3), \Phi_{\mathbf{x}}(\theta_3, \cdots, \theta_J; u))),$$

or equivalently,

$$\Phi_{\mathbf{x}}(\theta_2, \theta_3, \cdots, \theta_J; u) > \Phi_{\mathbf{x}}(\theta_2, \theta_3, \cdots, \theta_J; u),$$

a contradiction. The case $\theta_2 = \min\{\theta_1, \theta_2, \theta_3\}$ is similar and the proof is omitted. $\square$

Proof of the Sufficiency of Theorem 2: Let $\mathbf{x}$ be monotone. We show that if condition (3) holds for some J, then it must also hold for $J + 1$. Consider any cycle

$\theta_0, \theta_1, \cdots, \theta_{J+1}, \theta_0$. If $\theta_0 >_\Theta \max\{\theta_1, \cdots, \theta_{J+1}\}$, let $j^* \in \{1, \ldots, J+1\}$ be such that $\theta_{j^*} = \min\{\theta_1, \ldots, \theta_{J+1}\}$; if $\theta_0 \leq_\Theta \max\{\theta_1, \cdots, \theta_{J+1}\}$, let $j^* \in \{1, \ldots, J+1\}$ be such that $\theta_{j^*} = \max\{\theta_1, \ldots, \theta_{J+1}\}$. Clearly, θ_{j^*} is a local extremum in the chain $\theta_{j^*-1}, \theta_{j^*}, \theta_{j^*+1}, \cdots \theta_{J+1}, \theta_0$. Thus, we have

$$\begin{aligned}
\Phi_{\mathbf{x}}(\theta_0, \theta_1, \cdots, \theta_{J+1}, \theta_0; u_0) &= \Phi_{\mathbf{x}}(\theta_0, \cdots, \theta_{j^*-1}; \Phi_{\mathbf{x}}(\theta_{j^*-1}, \theta_{j^*}, \theta_{j^*+1}, \cdots, \theta_{J+1}, \theta_0; u_0)) \\
&\leq \Phi_{\mathbf{x}}(\theta_0, \cdots, \theta_{j^*-1}; \Phi_{\mathbf{x}}(\theta_{j^*-1}, \theta_{j^*+1}, \cdots, \theta_{J+1}, \theta_0; u_0)) \\
&= \Phi_{\mathbf{x}}(\theta_0, \cdots, \theta_{j^*-1}, \theta_{j^*+1}, \cdots, \theta_{J+1}, \theta_0; u_0) \\
&\leq u_0.
\end{aligned}$$

The first inequality follows from Lemma 10 and the monotonicity of $\Phi_{\mathbf{x}}(\theta_1, \ldots, \theta_{j^*-1}; \cdot)$. The last inequality comes from the induction hypothesis. The desired result then follows from Lemma 9 and Theorem 1. □

A Revenue Equivalence

Revenue equivalence refers to the phenomenon whereby any two mechanisms that implement a given allocation give rise to the same revenue for each type to the principal, provided some type—typically, the lowest type—obtains the same surplus in these two mechanisms; e.g., Mas-Colell, Whinston, and Green (1995). In this section, we briefly discuss revenue equivalence without quasilinearity. Following Nöldeke and Samuelson (2018), we refer to $\mathbf{u} : \Theta \to \mathbb{R}$ with $\mathbf{u}(\theta) \in U(\theta)$ for each θ as a *surplus profile*; $\mathbf{u}$ is implementable under $\mathbf{x}$, if $\psi(\theta, \mathbf{x}(\theta), \mathbf{u}(\theta)) : \Theta \to T$ implements $\mathbf{x}$. Focusing on surplus profiles, we have the following definition.

Definition 4 (Revenue Equivalence)**.** An implementable allocation $\mathbf{x}$ satisfies revenue equivalence if for each two surplus profiles $\mathbf{u}$ and $\mathbf{u}'$ both implementable under $\mathbf{x}$, we have $\mathbf{u}(\theta) = \mathbf{u}'(\theta)$ for some $\theta \Longrightarrow \mathbf{u} = \mathbf{u}'$.

An allocation is strongly implementable and satisfies revenue equivalence if and only if, for each initial condition, there exists a unique transfer scheme that implements $\mathbf{x}$ and satisfies the initial condition. An equally common definition of revenue equivalence in quasilinear environments requires that two transfer schemes that implements a given allocation differ by a constant; e.g., Heydenreich, Müller, Uetz, and Vohra (2009). It is obvious that with quasilinearity, these two definitions coincide.

Heydenreich, Müller, Uetz, and Vohra (2009) provide a characterization of revenue equivalence with quasilinearity. We now extend their analysis to the current environment. Let $\mathbf{x}$ be strongly implementable, and thus strongly cyclically monotone. For each θ, θ', and $u' \in U(\theta')$ the nonempty set $V_{\mathbf{x}}(\theta, \theta'; u')$ is bounded from above in $U(\theta)$ by Lemma 4. Let $\text{dist}_{\mathbf{x}}(\theta, \theta'; u') = \sup_{U(\theta)} V_{\mathbf{x}}(\theta, \theta'; u') \in U(\theta)$ be the least upper bound of $V_{\mathbf{x}}(\theta, \theta'; u')$ in $U(\theta)$.[17] Actually, we have

$$\text{dist}_{\mathbf{x}}(\theta, \theta'; u') = \phi(\theta, \mathbf{x}(\theta), \inf T_{\mathbf{x}}(\theta, \theta'; u')).$$

By Proposition 1, for each θ' and $u' \in U(\theta')$, $\text{dist}_{\mathbf{x}}(\cdot, \theta'; u')$ is the least surplus profile implementable under $\mathbf{x}$ that satisfies the initial condition (θ', u'). We refer to $\text{dist}_{\mathbf{x}}(\theta, \theta'; \cdot) : U(\theta') \to U(\theta)$ as the *distance function* from θ' to θ associated with allocation $\mathbf{x}$. Notice that $\text{dist}_{\mathbf{x}}(\theta, \theta'; \cdot)$ is monotone for all (θ, θ') with $\text{dist}_{\mathbf{x}}(\theta, \theta; u) = u$ for each θ and $u \in U(\theta)$. Also, for each surplus profile $\mathbf{u}$ implementable under $\mathbf{x}$, we have $\text{dist}_{\mathbf{x}}(\theta, \theta'; \mathbf{u}(\theta')) \leq \mathbf{u}(\theta), \forall \theta, \theta'$. The strong implementability of $\mathbf{x}$ imposes the following requirement on the distance function associated with $\mathbf{x}$.

Claim 3. Let Assumption 1 hold and $\mathbf{x}$ be strongly implementable. Then for each θ and θ' and $u \in U(\theta)$, we have $\text{dist}_{\mathbf{x}}(\theta, \theta'; \text{dist}_{\mathbf{x}}(\theta', \theta; u)) \leq u$.

Proof. Let $\mathbf{u}$ be a surplus profile implementable under $\mathbf{x}$ with $\mathbf{u}(\theta) = u$. We have $\text{dist}_{\mathbf{x}}(\theta, \theta'; \text{dist}_{\mathbf{x}}(\theta', \theta; u)) \leq \text{dist}_{\mathbf{x}}(\theta, \theta'; \mathbf{u}(\theta')) \leq \mathbf{u}(\theta) = u$. □

Theorem 4 (Characterization of Revenue Equivalence—Inverse Distance)**.** *Let Assumption 1 hold and* $\mathbf{x}$ *be strongly implementable. Then,* $\mathbf{x}$ *satisfies revenue equivalence if and only if for all* θ *and* θ'*,* $\text{dist}_{\mathbf{x}}(\theta, \theta'; \text{dist}_{\mathbf{x}}(\theta', \theta; \cdot)) : U(\theta) \to U(\theta)$ *is an identity mapping: For each* $u \in U(\theta)$*, we have* $\text{dist}_{\mathbf{x}}(\theta, \theta'; \text{dist}_{\mathbf{x}}(\theta', \theta; u)) = u$*.*

Proof. (Necessity.) Suppose the contrary—that there exists some $u' \in U(\theta')$ such that $\text{dist}_{\mathbf{x}}(\theta', \theta''; \text{dist}_{\mathbf{x}}(\theta'', \theta'; u')) \neq u'$. Consider the surplus profiles implementable under $\mathbf{x}$: $\mathbf{u}(\cdot) \equiv \text{dist}_{\mathbf{x}}(\cdot, \theta''; \text{dist}_{\mathbf{x}}(\theta'', \theta'; u'))$ and $\mathbf{u}'(\cdot) \equiv \text{dist}_{\mathbf{x}}(\cdot, \theta'; u')$. We have $\mathbf{u}(\theta') \neq \mathbf{u}'(\theta')$ but $\mathbf{u}(\theta'') = \mathbf{u}'(\theta'')$, a contradiction. (Sufficiency.) To show that $\mathbf{x}$ satisfies the property of revenue equivalence, it suffices to show that for each surplus profile $\mathbf{u}$ implementable under $\mathbf{x}$, and each θ and θ', we have $\mathbf{u}(\theta') = \text{dist}_{\mathbf{x}}(\theta', \theta; \mathbf{u}(\theta))$. But this follows from $\mathbf{u}(\theta') \geq \text{dist}_{\mathbf{x}}(\theta', \theta; \mathbf{u}(\theta)) \geq \text{dist}_{\mathbf{x}}(\theta', \theta; \text{dist}_{\mathbf{x}}(\theta, \theta'; \mathbf{u}(\theta'))) = \mathbf{u}(\theta')$. □

[17] By completeness of T, for each θ, $U(\theta)$ with the usual order is also complete. However, notice that the least upper bound of V in $U(\theta)$ differs from its usual supremum in $\mathbb{R}$—i.e., $\sup V$—whenever $\sup V$ is not an element of $U(\theta)$. For example, let $U(\theta) = (-1, 0) \cup [1, 2)$ and $V = (-1, 0)$. V is bounded from above in $U(\theta)$ with $\sup_{U(\theta)} V = 1$. But $\sup V = 0$.

The inverse distance condition in Theorem 4 reduces to the antisymmetric distance condition of Heydenreich, Müller, Uetz, and Vohra (2009) for the quasilinear case. Let $T = \mathbb{R}$, $\phi(\theta, x, t) = v(\theta, x) - t$, and $\mathbf{x}$ be implementable. For each θ and θ',

$$\text{dist}^l_{\mathbf{x}}(\theta, \theta') \triangleq \inf_{\substack{\text{chain} \\ \theta=\theta_1,\ldots,\theta_J=\theta'}} \sum_{j=1}^{J-1} [v(\theta_{j+1}, \mathbf{x}(\theta_{j+1})) - v(\theta_{j+1}, \mathbf{x}(\theta_j))]$$

is the distance from θ to θ' (in type graph) of Heydenreich, Müller, Uetz, and Vohra (2009). By (5), for all θ and θ' and $u \in U(\theta)$, we have

$$\text{dist}_{\mathbf{x}}(\theta', \theta; u) = -\text{dist}^l_{\mathbf{x}}(\theta, \theta') + [v(\theta', \mathbf{x}(\theta')) - v(\theta, \mathbf{x}(\theta))] + u.$$

So, for all $\theta, \theta' \in \Theta$ and $u \in U(\theta)$, we have

$$\text{dist}_{\mathbf{x}}(\theta, \theta'; \text{dist}_{\mathbf{x}}(\theta', \theta; u)) = -\text{dist}^l_{\mathbf{x}}(\theta, \theta') - \text{dist}^l_{\mathbf{x}}(\theta', \theta) + u.$$

Thus, by Theorem 4, an implementable allocation $\mathbf{x}$ satisfies revenue equivalence if and only if for all θ and θ' in Θ, we have $\text{dist}^l_{\mathbf{x}}(\theta, \theta') = -\text{dist}^l_{\mathbf{x}}(\theta', \theta)$, which is the antisymmetric distance condition of Heydenreich, Müller, Uetz, and Vohra (2009).

B Omitted Proofs

B.1 Proof of Lemma 3

We only prove the second statement. The proof of the first statement is the same and omitted. (Necessity.) Let $u \leq (<)\Phi_{\mathbf{x}}(\theta, \cdots, \theta'; v)$. By strong cyclical monotonicity, we have $\Phi_{\mathbf{x}}(\theta, \cdots, \theta'; \Phi_{\mathbf{x}}(\theta', \cdots, \theta; u)) \leq u$ and so $\Phi_{\mathbf{x}}(\theta', \cdots, \theta; u) \leq (<)v$, since $\Phi_{\mathbf{x}}(\theta', \cdots, \theta; \cdot)$ is strictly increasing. (Sufficiency.) Consider any cycle $\theta_0, \theta_1, \cdots, \theta_J, \theta_0$ and $u_0 \in U(\theta_0)$. If $\theta_j \equiv \theta_0$ for all j, then (3) holds trivially, since $\Phi_{\mathbf{x}}(\theta_0, \theta_1, \cdots, \theta_J, \theta_0; u_0) = u_0$. Now consider the case $\theta_j \neq \theta_0$ for some j. Let $v = \Phi_{\mathbf{x}}(\theta_j, \cdots, \theta_J, \theta_0; u_0)$. By the proposed condition, we have $u_0 \geq \Phi_{\mathbf{x}}(\theta_0, \theta_1, \cdots, \theta_j; v)$. So $u_0 \geq \Phi_{\mathbf{x}}(\theta_0, \theta_1, \cdots, \theta_J, \theta_0; u_0)$.

B.2 Proof of Lemma 7

We show, for each θ, $x \neq x'$, and t, there exists some t' such that $\phi(\theta, x, t) = \phi(\theta, x', t')$. Let x and x' be ordered. Consider the case $x' >_X x$. Let $\theta' >_\Theta \theta$ in Θ. The existence of θ' is guaranteed by the assumption that Θ has no greatest element. Since $(\theta, x) < (\theta', x')$ is lower strongly implementable, there exists some t'' such that $\phi(\theta, x, t) \geq \phi(\theta, x', t'')$. Now let $\theta >_\Theta \theta''$ in Θ. The existence of θ'' is guaranteed by the assumption that Θ has no least element. Since $(\theta'', x) < (\theta, x')$ is lower strongly implementable, there exists some t''' such that $\phi(\theta, x', t''') \geq \phi(\theta, x, t)$. By continuity of ϕ in transfers, there exists some t' between t'' and t''' such that $\phi(\theta, x', t') = \phi(\theta, x, t)$. The case $x >_X x'$ follows from the upper strong implementability of each ordered pair. Now consider the case in which x and x' are not ordered. Since X is simple order-connected, there exists some $z \in X$ such that x and z are ordered and x' and z are ordered. By the previous argument, there exists some $\hat{t}$ and $\bar{t}$ such that $\phi(\theta, x, t) = \phi(\theta, z, \hat{t})$ and $\phi(\theta, z, \hat{t}) = \phi(\theta, x', \bar{t})$. The desired result follows.

B.3 Proof of Lemma 8

Suppose, in contradiction, that $\phi_l/|\phi_t|(\theta, \bar{x}_l, \bar{x}_{-l}, \bar{t}) > \phi_l/|\phi_t|(\theta', \bar{x}_l, \bar{x}_{-l}, \bar{t})$ for some $\theta < \theta'$ and l and $(\bar{x}, \bar{t})$. Without loss of generality, let $\bar{x}$ be in the interior of X and the strict inequality holds in a neighbourhood of $(\bar{x}_l, \bar{t})$—say, $\mathcal{N}$—when x_{-l} is fixed at $\bar{x}_{-l}$. If $\phi_l(\theta, x_l, \bar{x}_{-l}, t) \equiv 0$ on $\mathcal{N}$, then we have $\phi_l(\theta', x_l, \bar{x}_{-l}, t) < 0$ on $\mathcal{N}$. A contradiction can be drawn by noticing $\phi(\theta, x'_l, \bar{x}_{-l}, t) = \phi(\theta, x_l, \bar{x}_{-l}, t)$ but $\phi(\theta', x'_l, \bar{x}_{-l}, t) < \phi(\theta', x_l, \bar{x}_{-l}, t)$ for each (x_l, t) and (x'_l, t) with $x'_l > x_l$ in $\mathcal{N}$. Now consider the case $\phi_l(\theta, \hat{x}_l, \bar{x}_{-l}, \hat{t}) \neq 0$ for some $(\hat{x}_l, \hat{t})$ in $\mathcal{N}$. By the implicit function theorem, there exists an isoutility segment of the θ-type agent in $\mathcal{N}$, $\{x_l(s), t(s) | s \in [0, 1]\}$, such that $x'_l(s) > 0$. A contradiction can be drawn using the same argument as Milgrom and Shannon (1994).

B.4 Topkis (1998)'s Composition Result

We adapt the composition result of Topkis (1998) to our environment and give a proof for completeness—but unlike Topkis, we do not require that X be a lattice.

Claim 4 (Topkis (1998)). Let $v(\theta, x)$ in (7) have increasing differences and be increasing in both θ and x. If $g(v, t)$ is convex in v and supermodular in (v, t), then for each $\theta' >_\Theta \theta$, $x' >_X x$, and $t' \succeq_T t$, we have $\phi(\theta', x', t') - \phi(\theta', x, t) \geq \phi(\theta, x', t') - \phi(\theta, x, t)$.

Proof.

$$\begin{aligned}\phi(\theta',x',t')-\phi(\theta',x,t) &= [\phi(\theta',x',t')-\phi(\theta',x,t')]+[\phi(\theta',x,t')-\phi(\theta',x,t)]\\ &= [g(v(\theta',x'),t')-g(v(\theta',x),t')]+[g(v(\theta',x),t')-g(v(\theta',x),t)]\\ &\geq [g(v(\theta,x'),t')-g(v(\theta,x),t')]+[g(v(\theta,x),t')-g(v(\theta,x),t)]\\ &= \phi(\theta,x',t')-\phi(\theta,x,t).\end{aligned}$$

Now we explain the inequality in the above expression. By increasing differences and $v(\theta',x)\geq v(\theta,x)$, we have $g(v(\theta',x'),t')-g(v(\theta',x),t')\geq g(v(\theta,x'),t')-g(v(\theta,x),t')$, since $g(v,t)$ is convex and increasing in v. By supermodularity of g, $g(v(\theta',x),t')-g(v(\theta',x),t)\geq g(v(\theta,x),t')-g(v(\theta,x),t)$, since $t'\succeq_T t$ and $v(\theta',x)\geq v(\theta,x)$. □

B.5 Proof of Proposition 5

Necessity follows from Theorem 2 and Lemma 8. Now turn to Sufficiency. We first show that ϕ is single crossing. For each $\theta'>_\Theta \theta$, $x'>x$, t and t', such that $\phi(\theta,x',t')=\phi(\theta,x,t)$, we have $\phi(\theta',x',t')\geq\phi(\theta',x,t)$ by Milgrom and Shannon (1994), since ϕ is completely regular. Now consider the case $\phi(\theta,x',t')>\phi(\theta,x,t)$. By the condition of the possibility of compensation, there exists $t''>t'$ such that $\phi(\theta,x',t'')=\phi(\theta,x,t)$. By the previous argument, we have $\phi(\theta',x',t'')\geq\phi(\theta',x,t)$ and so $\phi(\theta',x',t')>\phi(\theta',x,t)$. This shows that ϕ is single crossing. Theorem 2 applies.

References

Baldwin, E., R. Jagadeesan, P. Klemperer, and A. Teytelboym (2023): "The equilibrium existence duality," *Journal of Political Economy*, 131(6), 000–000.

Bikhchandani, S., S. Chatterji, R. Lavi, A. Mu'alem, N. Nisan, and A. Sen (2006): "Weak monotonicity characterizes deterministic dominant-strategy implementation," *Econometrica*, 74(4), 1109–1132.

Blackwell, D. (1951): "Comparison of Experiments," *Berkeley Symposium on Mathematical Statistics and Probability*, (93-102).

Davey, B. A., and H. A. Priestley (2002): *Introduction to lattices and order.* Cambridge University Press, Cambridge.

Fudenberg, D., and J. Tirole (1991): *Game theory.* MIT Press.

Guesnerie, R., and J.-J. Laffont (1984): "A complete solution to a class of principal-agent problems with an application to the control of a self-managed firm," *Journal of Public Economics*, 25(3), 329–369.

Heydenreich, B., R. Müller, M. Uetz, and R. V. Vohra (2009): "Characterization of revenue equivalence," *Econometrica*, 77(1), 307–316.

Kausamo, A., L. De Pascale, and K. Wyczesany (2023): "60 years of cyclic monotonicity: A survey," *arXiv preprint arXiv:2308.07682*.

Kazumura, T., D. Mishra, and S. Serizawa (2020): "Mechanism design without quasilinearity," *Theoretical Economics*, 15(2), 511–544.

Kos, N., and M. Messner (2013a): "Extremal incentive compatible transfers," *Journal of Economic Theory*, 148(1), 134–164.

——— (2013b): "Incentive compatibility in non-quasilinear environments," *Economics Letters*, 121(1), 12–14.

Laffont, J.-J., and E. Maskin (1980): "A differential approach to dominant strategy mechanisms," *Econometrica*, pp. 1507–1520.

Mas-Colell, A. (1977): "Indivisible commodities and general equilibrium theory," *Journal of Economic Theory*, 16(2), 443–456.

Mas-Colell, A., M. D. Whinston, and J. R. Green (1995): *Microeconomic theory*. Oxford University Press, New York.

Milgrom, P., and C. Shannon (1994): "Monotone Comparative Statics," *Econometrica*, 62(157-180).

Mirrlees, J. A. (1976): "Optimal tax theory: A synthesis," *Journal of Public Economics*, 6(4), 327–358.

Myerson, R. B. (2008): "Perspectives on mechanism design in economic theory," *American Economic Review*, 98(3), 586–603.

Nöldeke, G., and L. Samuelson (2018): "The implementation duality," *Econometrica*, 86(4), 1283–1324.

Rochet, J.-C. (1987): "A necessary and sufficient condition for rationalizability in a quasi-linear context," *Journal of Mathematical Economics*, 16(2), 191–200.

Rockafellar, R. T. (1970): *Convex Analysis*. Princeton University Press.

Sinander, L. (2022a): "The converse envelope theorem," *Econometrica*, 90(6), 2795–2819.

——— (2022b): "Supplement to "the converse envelope theorem"," *Online Appendix*.

Spence, M. (1974): "Competitive and optimal responses to signals: An analysis of efficiency and distribution," *Journal of Economic Theory*, 7(3), 296–332.

Tarski, A. (1955): "A lattice-theoretical fixpoint theorem and its applications," *Pacific Journal of Math*, 5, 285–309.

Topkis, D. M. (1998): *Supermodularity and complementarity*. Princeton University Press.