Breaking the Curse of Dimensionality in Heterogeneous-Agent Models: A Deep Learning-Based Probabilistic Approach

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Abstract

Dynamic heterogeneous-agent models share two features: 1) high-dimensional aggregate states that are beyond the control of individual agents, and 2) low-dimensional aggregate shocks. This paper exploits these two features using a deep learning-based probabilistic approach and demonstrates that it is possible to solve for the global solution of these models without compromising dimensionality reduction. The computational advantage of the probabilistic approach lies in converting a conditional expectation equation into multiple equations of shock realizations, significantly enhancing evaluation efficiency. As illustration, I solve two models: the continuous-time version of Krusell and Smith (1997) with a two-asset portfolio choice and nonlinear debt market clearing condition, and an extension of a search-and-matching model (Duffie, Gârleanu and Pedersen, 2007) with a continuum of heterogeneous investors and anticipated aggregate risks.

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1 Introduction

The curse of dimensionality has long impeded economists from pursuing large-scale, and potentially more realistic, models. This constraint poses a significant challenge for heterogeneous-agent models with aggregate shocks, as the distribution of individual characteristics, such as wealth levels, serves as a state variable. This paper will demonstrate that the infinite-dimensional state variable is less formidable than commonly believed within the profession.

I will exploit two common features of heterogeneous-agent models to solve for global solutions. Firstly, the infinite-dimensional state variable is uncontrollable from an individual's perspective. Therefore, the primary computational challenge is purely about calculating conditional expectations while solving for dynamic optimization. Secondly, the dynamics of the infinite-dimensional state variable are driven by low-dimensional aggregate shocks. This paper will illustrate that the probabilistic formulation significantly enhances the computation efficiency of conditional expectations when the dimensionality of aggregate shocks is not too high.

The probabilistic formulation, widely used in the study of stochastic controls, posits that a forward-looking random variable can be expressed as the sum of its conditional expectation and the linear impact of exogenous shocks within a short time interval. This transforms the equation for a random variable's conditional expectation into infinitely many equations, each corresponding to a realization of exogenous shocks. Consequently, the probabilistic formulation yields a significantly greater number of equations to identify conditional expectations compared to the conventional analytic approach. The trade-off involves estimating the "coefficients" of exogenous shocks, which is generally less problematic when aggregate shocks are low-dimensional. Moreover, since these "coefficients" explicitly influence agents' portfolio choices, directly solving them proves to be a more efficient alternative to conventional methods. In the remainder of the introduction, I will provide a one-dimensional state variable example to exemplify the computational advantage of the probabilistic formulation.

Suppose X_t is an uncontrolled Itô process described by

$$X_{t+\Delta} = X_t + \mu\left(X_t\right)\Delta + \sigma\left(X_t\right)\left(W_{t+\Delta} - W_t\right), \tag{1}$$

where Δ represents the length of time period, and $W_{t+\Delta} - W_t$ follows a normal distribution with a mean of zero and variance of Δ. In economics, we are interested in a forward-looking process $V(X_t)$ defined as the fixed point $V(\cdot)$ of the following functional equation

$$V\left(X_t\right) = u\left(X_t\right)\Delta + E\left[V\left(X_{t+\Delta}\right)\middle| X_t\right], \tag{2}$$

where $u(\cdot)$ can be interpreted as the utility flow, and $V(\cdot)$ represents the value function. In the discrete-time setting, it becomes increasingly more challenging to compute conditional expectation if X_t has higher dimensions.

In the continuous-time framework where $\Delta \to 0$, the evaluation of conditional expectation can be simplified since the integration is pre-computed.[1] The coefficient of the Δ term yields the differential equation

$$0 = u\left(x\right) + V'\left(x\right)\mu\left(x\right) + 0.5V''\left(x\right)\sigma^2\left(x\right). \tag{3}$$

Although only $V'(x)$ and $V''(x)$ need to be evaluated, the computational workload still increases significantly if X_t's dimensionality rises. This issue becomes particularly challenging in heterogeneous-agent models where the state variable, such as wealth distribution, has infinite dimensions.

In contrast to the analytic approach, the probabilistic approach focuses on the integrand of the recursive formulation. First, equation (2) can be rewritten as:

$$E\left[V\left(X_{t+\Delta}\right)\middle| X_t\right] = V\left(X_t\right) - u\left(X_t\right)\Delta.$$

According to the Martingale Representation Theorem in stochastic calculus, when Δ is sufficiently small, there exists a function $z(\cdot)$ such that

$$V(X_{t+\Delta}) = V\left(X_t\right) - u\left(X_t\right)\Delta + z(X_t)(W_{t+\Delta} - W_t). \tag{4}$$

The coefficient of the shock, $z(\cdot)$, is an unknown function of the current state X_t that needs to be solved for, along with $V(\cdot)$. In mathematics, equation (4) is referred to as a Backward Stochastic Differential Equations (BSDE), which effectively provides infinitely

[1] By substituting (1) with $X_t = x$ into equation (2), we obtain

$$\begin{aligned} 0 &= u\left(x\right)\Delta + E\left[V\left(x + \mu\left(x\right)\Delta + \sigma\left(x\right)\left(W_{t+\Delta} - W_t\right)\right) - V\left(x\right)\middle| X_t = x\right] \\ &= u\left(x\right)\Delta + \int_{-\infty}^{\infty}\left(V\left(x + \mu\left(x\right)\Delta + \sigma\left(x\right)w\right) - V\left(x\right)\right)\frac{1}{\sqrt{2\pi\Delta}}e^{-\frac{w^2}{2\Delta}}\mathrm{d}w \\ &= u(x)\Delta + \int_{-\infty}^{\infty}\left(V'(x)\mu(x)\Delta + V'(x)\sigma(x)w + 0.5V''(x)\sigma^2(x)w^2 + o(\Delta)\right)\frac{1}{\sqrt{2\pi\Delta}}e^{-\frac{w^2}{2\Delta}}\mathrm{d}w, \end{aligned}$$

where the last equation applies Taylor expansion, and $o(\Delta)$ denotes terms of order higher than Δ. Dividing both sides of the above equation by Δ and taking Δ to the limit zero yields equation (3).

many equations because it holds for any realization of $W_{t+\Delta} - W_t$. To demonstrate the numerical advantages of this formulation, let's consider using value function iteration to find $V(\cdot)$. With $V^n(\cdot)$, I only need to sample two realizations of $W_{t+\Delta} - W_t$, w^1 and w^2, which yield x^1 and x^2 according to equation (1). The two-by-two linear system

$$V^n(x^j) = V^{n+1}(x) - u(x)\Delta + z^{n+1}(x)w^j, j = 1, 2.$$

is sufficient to determine $V^{n+1}(x)$ and $z^{n+1}(x)$. In fact, regardless of the dimensionality of X_t, only two evaluations of $V(\cdot)$ to are required to determine $V^{n+1}(x)$ and $z^{n+1}(x)$. The probabilistic formulation offers more equations or restrictions than the analytic formulation (2), which facilitates the identification of conditional expectations.

To implement the probabilistic approach, I seek parametric approximations of $V(\cdot)$ and $z(\cdot)$, represented as $\tilde{V}(\cdot;\Theta)$ and $\tilde{z}(\cdot;\Theta)$, respectively. Equation (4) suggests that the parameters Θ should solve the following optimization problem

$$\begin{aligned} \min_{\Theta}: \quad & \frac{1}{NM}\sum_{i=1}^{N}\sum_{j=1}^{M}\left(\tilde{V}(\hat{x}^{i,j};\Theta) + u(x^i)\Delta - \tilde{V}(x^i;\Theta) - \tilde{z}(\cdot;\Theta)w^{i,j}\right)^2 \\ \text{s.t.} \quad & \hat{x}^{i,j} = x^i + \mu(x^i)\Delta + \sigma(x^i)w^{i,j} \\ & w^{i,j} \text{ is sampled independently from } N(0,\Delta) \\ & x^i \text{ is from a given set.} \end{aligned} \tag{5}$$

This formulation makes the difficulty of solving the fixed-point problem (2) much less sensitive to the dimensionality of the state variable X_t. One only needs to increase the size of the sample, i.e., N and M, to ensure the accuracy of the probabilistic approach when the dimensionality of X_t increases. Moreover, this formulation can make use of parallel computing as the evaluation of each sample path is independent of others.

In the discrete-time setting, if one uses Monte Carlo simulation to evaluate the conditional expectation, the objective function becomes

$$\frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{M}\sum_{j=1}^{M}\tilde{V}(\hat{x}^{i,j};\Theta) + u(x^i)\Delta - \tilde{V}(x^i;\Theta)\right)^2. \tag{6}$$

In this case, only N equations are utilized to determine the value function. However, the probabilistic formulation takes advantage of $N \times M$ equations with the same number of $V(\cdot)$ evaluations. In the continuous-time setting, the analytic formulation leads to the objective function

$$\frac{1}{N}\sum_{i=1}^{N}\left(u\left(x^{i}\right)+\tilde{V}'\left(x^{i};\Theta\right)\mu\left(x^{i}\right)+0.5\tilde{V}''\left(x^{i};\Theta\right)\sigma^{2}\left(x^{i}\right)\right)^{2}. \tag{7}$$

The number of $V(\cdot)$ evaluations increases with the dimension of X_t because of numerical derivative calculations. However, for the probabilistic approach, the minimum number of evaluations remains unchanged regardless of the dimensionality of X_t.

The volatility term, $z(\cdot)$, is crucial in transforming the conditional expectation equation (2) into the probabilistic equation (4). It also plays a vital role in solving portfolio choice problems in heterogeneous-agent models. To show this point clearly, I rearrange equation (4) and apply Taylor expansion at $X_t = x$

$$\begin{aligned}\left(u\left(x\right)+V'\left(x\right)\mu\left(x\right)\right)\Delta+\frac{1}{2}V''\left(x\right)\sigma^{2}\left(x\right)\left(W_{t+\Delta}-W_{t}\right)^{2}+V'\left(x\right)\sigma\left(x\right)\left(W_{t+\Delta}-W_{t}\right)+o(\Delta)\\ =z(x)\left(W_{t+\Delta}-W_{t}\right).\end{aligned}$$

Taking the expectation or integration on both sides yields the differential equation (3). However, the expectation hides a crucial relationship: $V'(x)\sigma(x) = z(x)$. When solving for optimal strategies, such as portfolio choices, it is necessary to capture the impact of exogenous shocks on agents' lifetime expected utility, i.e., $V'(x)\sigma(x)$. The conventional approach is to evaluate $V'(x)$ and $\sigma(x)$ separately, and calculate their inner product, which becomes increasingly cumbersome if the state variable is high-dimensional. The advantage of the probabilistic approach is that one can bypass the evaluation of $V'(x)$ and directly solve for its inner product with $\sigma(x)$, i.e., $z(x)$.

To demonstrate the application of the probabilistic approach, I solve two models involving distributions as state variables: an incomplete-market heterogeneous-agent model with two assets and a nonlinear debt market clearing condition, based on Krusell and Smith (1997); and a search-and-matching model of over-the-counter (OTC) markets with a continuum of heterogeneous agents and anticipated aggregate risks, based on Duffie, Gârleanu and Pedersen (2007). The model in Krusell and Smith (1997) is more challenging to solve than the one in the well-celebrated Krusell and Smith (1998) using their approximate equilibrium approach. In Krusell and Smith (1998), the bond price is an explicit function of the first moment of households' wealth distribution. However, in Krusell and Smith (1997), the nonlinear bond market clearing condition implies that the bond price is an implicit function of households' wealth distribution, i.e., its moments. To solve the approximate equilibrium, households' perceived mapping from the moments of wealth distribution to the bond price ought to coincide with the implicit function implied by the bond market clearing. However, this challenge does not apply to the probabilistic approach, as households have the true mapping from the wealth distribution to the bond price in their expectation.

In models with search and matching frictions, it is crucial for agents to understand the distribution of potential counterparties' characteristics, because the matching outcome depends on the entire distribution rather than its moments. In the model I consider, the cross-sectional distribution of investors' characteristics follows a law of motion driven by an anticipated aggregate risk, as the common shock influences the dynamics of individual agents' characteristics. Therefore, when investors solve their dynamic optimization problem, it is necessary to consider the cross-sectional distribution of individual characteristics as a state variable.

Literature. Given economists' decades-long battle with the curse of dimensionality, it is impractical to review all related papers. The spirit of my numerical scheme can be traced back to the Parameterized Expectation Approach in Marcet (1988), which utilizes simulated sample paths in equilibrium to train the parameterized conditional expectation. Notably, a recent advancement in this area can be found in Judd, Maliar and Maliar (2011). Given the recent progress in Machine Learning, numerous authors have employed deep neural networks to approximate the conditional expectation and solve optimization problems using techniques like stochastic gradient descent, e.g., Maliar, Maliar and Winant (2021), Azinovic, Gaegauf and Scheidegger (2022), and Han, Yang and E (2021). It is worth noting that most, if not all, numerical methods in the economics literature rely on the analytic formulation (2). This formulation gives rise to an objective function similar to (6) that needs to be minimized.

The probabilistic approach to the study of stochastic processes was suggested by Paul Lévy and carried out by Kiyosi Itô in 1940s via the machinery of stochastic calculus and Stochastic Differential Equations (SDEs). In the study of optimal stochastic control, Bismut (1973, 1978) first introduced linear Backward SDEs, which characterize forward-looking stochastic processes. The mathematical foundation for the probabilistic approach employed in my paper's numerical scheme, namely nonlinear BSDEs, was established in the seminal work of Pardoux and Peng (1990). While theoretically appealing, numerically solving BSDEs poses challenges. Recently, E, Han and Jentzen (2017) and Han, Jentzen and E (2018) showcased the power of deep learning in solving high-dimensional BSDEs. The idea of formulating the solvability of Forward-Backward SDEs as an optimal control problem traces back to Ma and Yong (1995). My contribution, building upon the work of Han et al. (2018), lies in adapting their algorithm to the Markov equilibrium of an infinite-horizon time-homogeneous system.[2]

BSDEs were born for characterizing dynamic economic models. Independently of Pardoux and Peng (1990), Duffie and Epstein (1992) introduced a nonlinear BSDE for analyzing recursive utility. Since then, economists have sporadically utilized BSDEs to derive analytic results in studies such as Schroder and Skiadas (1999), Chen and Epstein (2002), Williams (2011). A series of my works is the first in the economics literature to emphasize the supremacy of the probabilistic formulation for numerical schemes, as well as its flexibility in accommodating various modeling ingredients. Huang (2023) considers high-dimensional macro-finance models with asset pricing and endogenous risks. Huang and Yu (2024) applies the approach to combinatorial problems often encountered in fields like international trade, industrial organization, and production networks.

The continuous-time setting is convenient as it allows for local approximation at any state, making the pre-computation of integrals advocated by Judd, Maliar, Maliar and Tsener (2017) in the discrete-time setting a natural fit for numerical schemes of continuous-time models. Nevertheless, the curse of dimensionality still applies to the analytic formulation in the continuous-time setting. Similar to the discrete-time setting, several authors have utilized deep learning to solve high-dimensional partial differential equations arising from the analytic formulation. Examples include Duarte (2018), Gopalakrishna (2021), Sauzet (2021), and Gu, Laurière, Merkel and Payne (2023), where the objective function to minimize resembles (7).

As its title indicates, my paper contributes to the dynamic heterogeneous-agent literature starting with Bewley-Huggett-Aiyagari models. Recently, Achdou, Han, Lasry, Lions and Moll (2022) cast these models in the continuous-time setting using the analytic formulation. Similar to my paper, both Fernández-Villaverde, Hurtado and Nũno (2022) and Bilal (2023) consider aggregate shocks. The former adopts the dimensionality reduction approach proposed by Krusell and Smith (1998), while the later focuses on the local approximation around the steady state. The mathematics community studies the dynamic heterogeneous-agent models under the framework of Mean Field Games, initiated by Lasry and Lions (2007). Both analytic and probabilistic approaches are widely employed to study mean field

[2]See Duffie, Geanakoplos, Mas-Colell and McLennan (1994) for the general treatment of time-homogeneous Markov equilibrium within a discrete-time setting. To the best of my knowledge, corresponding results in the continuous-time setting are still absent. I boldly speculate that it is plausible to establish the existence and uniqueness of Markov equilibria for a specific class of heterogeneous-agent continuous-time models with aggregate shocks, by framing these models within Forward-Backward Stochastic Differential Equation (FBSDE) systems. For most economic models, the correspondent FBSDE systems are fully coupled, and existence and uniqueness results for fully coupled FBSDEs are known to be extremely challenging to obtain in a time-inhomogeneous setting. However, most economic systems are time-homogeneous, a setting does not draw enough attention from scholars in BSDE and mean-field game communities. I reckon that one could extend the proofs for the well-posedness of fully coupled FBSDEs in small-time duration, as found in Antonelli (1993), to time-homogeneous Markov equilibria.

games (Carmona and Delarue, 2018).

Last but not least, this paper also contributes to the literature on search and matching frictions. Taking the study of OTC markets as an example, due to the curse of dimensionality, the literature — beginning with the pioneering work of Duffie, Gârleanu and Pedersen (2005) — has primarily focused on the steady states of carefully constructed models in the absence of aggregate risks. I hope that the probabilistic approach could lend scholars in the fields of labor search, money search, and OTC markets more freedom to explore a wider range of models for their research.

The organization of the paper is as follows. In Section 2, I provide a comprehensive explanation of solving dynamic optimization using the probabilistic approach when dealing with a high-dimensional uncontrolled state variable. Section 3 focuses on numerical methods for addressing market clearing conditions and establishing the law of motion for distributions driven by aggregate risks. Section 4 presents a detailed illustration of solving a modified version of the Krusell and Smith (1997) model, incorporating a two-asset portfolio choice problem and a nonlinear market clearing condition. In Section 5, I apply the approach to an asset valuation model of OTC markets with a continuum of heterogeneous investors and aggregate risks. Lastly, Section 6 contains some remarks on wider applications of the probabilistic approach.

2 Dynamic Optimization of an Individual

This section focuses on the application of a deep learning-based probabilistic approach in solving the dynamic optimization problem of an individual agent while facing a high-dimensional uncontrolled state variable.

Consider an agent maximizing the objective function

$$V\left(X_0, G_0\right) = \max_{\alpha_t} E_0\left[\int_0^\infty e^{-\rho s} f\left(X_s, G_s, \alpha_s\right) \mathrm{d}s\right],$$

where ρ is the discount factor, the unidimensional control variable α_t takes values in a convex set, and X_t and G_t represent individual and aggregate state variables, respectively. To simplify, assume that X_t is one-dimensional and G_t has N dimensions

$$G_t \equiv \left[G_t^1, G_t^2, \cdots, G_t^N\right]^T,$$

where ^{T} denotes transpose. The controlled process governing the individual state variable

(X_t) is given by

$$\mathrm{d}X_t = \mu\left(X_t, G_t, \alpha_t\right)\mathrm{d}t + \sigma\left(X_t, G_t, \alpha_t\right)\mathrm{d}W_t + \sigma^0\left(X_t, G_t, \alpha_t\right)\mathrm{d}W_t^0, \tag{8}$$

where W_t and W_t^0 are independent standard one-dimensional Brownian motions. W_t and W_t^0 represent idiosyncratic and systematic shocks, respectively. The uncontrolled aggregate state variable (G_t) follows the stochastic process

$$\mathrm{d}G_t = b\left(G_t\right)\mathrm{d}t + \Sigma\left(G_t\right)\mathrm{d}W_t^0, \tag{9}$$

where $b(\cdot)$ and $\Sigma(\cdot)$ are vector-valued functions with ranges in N-dimensional space.

I consider an N-dimensional aggregate state instead of an infinite-dimensional one for two reasons. First, the differentiation in an infinite-dimensional space is technically challenging for general audience in the economics community.[3] Second, when solving models numerically, I will discretize the state space and convert an infinite-dimensional state variable such as wealth distribution into a finite-dimensional state variable.

2.1 BSDEs

To convey the key insight of BSDEs, I eliminate the controlled individual state and discretize time with a step size Δ. The recursive formulation of the value function $V(G_t)$ is

$$V\left(G_t\right) = f\left(G_t\right)\Delta + E_t\left[e^{-\rho\Delta}V\left(G_{t+\Delta}\right)\right],$$

where G_t follows

$$G_{t+\Delta} = G_t + b\left(G_t\right)\Delta + \Sigma\left(G_t\right)\left(W_{t+\Delta}^0 - W_t^0\right).$$

To prepare for taking $\Delta \to 0$, I rearrange the above recursive formulation

$$0 = f\left(G_t\right) + E_t\left[\frac{e^{-\rho\Delta} - 1}{\Delta}V\left(G_{t+\Delta}\right) + \frac{V\left(G_{t+\Delta}\right) - V\left(G_t\right)}{\Delta}\right],$$

which leads to a partial differential equation (PDE) in the limit

$$\rho V = f(g) + \nabla_g V \cdot b(g) + \frac{1}{2}\mathrm{tr}\left(\Sigma(g)\,\Sigma(g)^T\,\nabla_{gg}V\right), \tag{10}$$

[3] Readers who are interested in the differentiation of functions of probability distribution may find Chapter 5 (Vol 1) of Carmona and Delarue (2018) helpful.

where $\cdot$ denotes inner product and $\operatorname{tr}(\cdot)$ represents the trace operator. The challenge in solving PDE (10) lies in evaluating $\nabla_g V$ and $\nabla_{gg} V$ when the dimension of the state variable G_t is high.

The intuition behind BSDEs is to project the random variable $V(G_{t+\Delta}) - V(G_t)$ onto the deterministic trend (Δ) and the exogenous shock to the system $\left(W^0_{t+\Delta} - W^0_t\right)$. This projection becomes arbitrarily accurate as Δ approaches zero.

$$V(G_{t+\Delta}) = V(G_t) - (f(G_t) - \rho V(G_t))\,\Delta + z(G_t)\left(W^0_{t+\Delta} - W^0_t\right) \tag{11}$$

The coefficient of the shock, $z(\cdot)$, is endogenous and depends fully on the current state G_t. The solution of the BSDE consists of a pair $(V(\cdot), z(\cdot))$, where $V(\cdot)$ refers to the fixed point as defined in the recursive formulation, and $z(\cdot)$ represents the volatility term. The solution pair has the following intuitive interpretation: for any current state G_t and innovation $W^0_{t+\Delta} - W^0_t$, the updated $V_{t+\Delta}$ of $V_t = V(G_t)$, according to

$$V_{t+\Delta} = V_t - (f(G_t) - \rho V_t)\,\Delta + z(G_t)\left(W^0_{t+\Delta} - W^0_t\right)$$

always satisfies the mapping $V_{t+\Delta} = V(G_{t+\Delta})$. The volatility term $z(\cdot)$ essentially controls the impact of the shock $W^0_{t+\Delta} - W^0_t$ on V_t, ensuring that $V_{t+\Delta}$ hits the target $V(G_{t+\Delta})$ for any current state G_t. This insight is initially highlighted by Ma and Yong (1995), which leads to the design of the deep learning problem shown later.

The Taylor expansion of $V(G_{t+\Delta})$ around G_t (or Itô's formula) reveals the connection between $z(\cdot)$ and $V(\cdot)$, as well as the relationship between the analytic and probabilistic formulations

$$\begin{aligned} V(G_{t+\Delta}) = {} & V + \nabla_g V \cdot b(G_t)\,\Delta + \nabla_g V \cdot \Sigma(G_t)\left(W^0_{t+\Delta} - W^0_t\right) \\ & + \frac{1}{2}\operatorname{tr}\left(\Sigma(G_t)\,\Sigma(G_t)^T \nabla_{gg} V\right)\left(W^0_{t+\Delta} - W^0_t\right)^2 + o(\Delta), \end{aligned}$$

where $o(\Delta)$ represents higher order terms. By combining this with BSDE (11), two observations can be made. First, the BSDE implies the PDE (10) when taking the expectation and letting $\Delta \to 0$. Second, the coefficients of two $W^0_{t+\Delta} - W^0_t$ terms, which vanish in expectation, imply that

$$z(G_t) = \nabla_g V \cdot \Sigma(G_t).$$

This observation indicates that when the shock is unidimensional, computing a single term $z(\cdot)$ is sufficient to reveal the stochastic properties of a high-dimensional dynamic system. While the exogenous shock $W^0_{t+\Delta} - W_t$ affects the high-dimensional state variable G_t through

$\Sigma(G_t)$, what truly matters in a dynamic economic model is the shock's impact on the forward-looking stochastic process $\{V_t\}$. This impact is fully captured by the inner product of $\Sigma(G_t)$ and $\nabla_g V$ or simply $z(G_t)$ in the language of BSDEs. The second observation reveals the key idea of dimensionality reduction in the probabilistic numerical approach.

2.2 Dynamic Optimization

Next, I reintroduce the controlled state X_t and apply dynamic programming to establish the Hamilton-Jacobi-Bellman (HJB) equation

$$\rho V = \max_{\alpha}\Big\{ f(x,g,\alpha) + \mu(x,g,\alpha)\nabla_x V + \frac{1}{2}\sigma^2(x,g,\alpha)\nabla_{xx}V + \frac{1}{2}\left(\sigma^0(x,g,\alpha)\right)^2\nabla_{xx}V \\ + \sigma^0(x,g,\alpha)\nabla_{xg}V\cdot\Sigma(G_t)\Big\} + \nabla_g V\cdot b(g) + \frac{1}{2}\mathrm{tr}\left(\Sigma(g)\Sigma(g)^T\nabla_{gg}V\right)$$

The challenge in numerically solving the HJB equation arises from evaluating derivatives in the space of aggregate state variables, which can have very high dimensions in heterogeneous-agent models. However, among the three relevant terms ($\nabla_{xg}V$, $\nabla_g V$, and $\nabla_{gg}V$), only $\nabla_{xg}V$ plays a role in determining the optimal dynamic choice. Moreover, $\nabla_{xg}V$ does not independently affect the optimal choice; it is the inner product with $\Sigma(G_t)$ that impacts the optimality condition. Therefore, it is ideal to compute $\nabla_{xg}V\cdot\Sigma(G_t)$ directly as a single term. Fortunately, the stochastic maximum principle can serve this purpose.

To apply the stochastic maximum principle, I define the generalized current-value Hamiltonian

$$H\left(x,g,\alpha,y,z,z^0\right) = f(x,g,\alpha) + \mu(x,g,\alpha)\,y + \sigma(x,g,\alpha)\,z + \sigma^0(x,g,\alpha)\,z^0.$$

The optimal control $\hat{\alpha}_t$ satisfies

$$\hat{\alpha}_t = \arg\max_{\alpha} : H\left(X_t, G_t, \alpha, Y_t, Z_t, Z_t^0\right),$$

where (Y_t, Z_t, Z_t^0) follows the BSDE

$$-\mathrm{d}Y_t = \left(\nabla_x H\left(X_t, G_t, \hat{\alpha}_t, Y_t, Z_t, Z_t^0\right) - \rho Y_t\right)\mathrm{d}t - Z_t\mathrm{d}W_t - Z_t^0\mathrm{d}W_t^0.$$

Combined with SDE (8) under the optimal control ($\alpha_t = \hat{\alpha}_t$) and SDE (9), the above BSDE gives rise to the function $Y(X_t, G_t)$. Itô's formula implies

$$
\begin{aligned}
Z_t &= \nabla_x Y(X_t, G_t)\, \sigma(X_t, G_t, \hat{\alpha}_t),\\
Z_t^0 &= \nabla_x Y(X_t, G_t)\, \sigma^0(X_t, G_t, \hat{\alpha}_t) + \nabla_g Y(X_t, G_t) \cdot \Sigma(G_t).
\end{aligned}
$$

Given the optimal control process $(\hat{\alpha}_t)_{t\geq 0}$, the principle of optimality indicates that the value function can take the following recursive form

$$
V(X_t, G_t) = E_t\left[\int_t^T e^{-\rho(s-t)} f(X_s, G_t, \hat{\alpha}_s)\, \mathrm{d}s + e^{-\rho(T-t)} V(X_T, G_T)\right]
$$

for an arbitrary T. The stochastic process $V(X_t, G_t)$ satisfies a BSDE

$$
-\mathrm{d}V_t = (f(X_t, G_t, \hat{\alpha}_t) - \rho V_t)\, \mathrm{d}t - Z_t^V \mathrm{d}W_t - Z_t^{0,V} \mathrm{d}W_t^0.
$$

Assuming the smoothness of $V(X_t, G_t)$, there exists an important relationship between $Y(\cdot)$ and $V(\cdot)$[4]

$$
Y(X_t, G_t) = \nabla_x V(X_t, G_t),
$$

which implies

$$
\begin{aligned}
Z_t &= \sigma(X_t, G_t, \hat{\alpha}_t)\, \nabla_{xx} V(X_t, G_t),\\
Z_t^0 &= \sigma^0(X_t, G_t, \hat{\alpha}_t)\, \nabla_{xx} V(X_t, G_t) + \nabla_{xg} V(X_t, G_t) \cdot \Sigma(G_t).
\end{aligned}
$$

It is straightforward to observe that the HJB equation and the Hamiltonian yield the same optimality condition w.r.t. α_t

$$
\begin{aligned}
0 = {} & \nabla_\alpha f(X_t, G_t, \hat{\alpha}_t) + \nabla_\alpha \mu(X_t, G_t, \hat{\alpha}_t)\, \nabla_x V + \sigma(X_t, G_t, \hat{\alpha}_t)\, \nabla_\alpha \sigma(X_t, G_t, \hat{\alpha}_t)\, \nabla_{xx} V\\
& + \sigma^0(X_t, G_t, \hat{\alpha}_t)\, \nabla_\alpha \sigma^0(X_t, G_t, \hat{\alpha}_t)\, \nabla_{xx} V + \nabla_\alpha \sigma^0(X_t, G_t, \hat{\alpha}_t)\, \nabla_{xg} V \cdot \Sigma(G_t) \qquad (12)
\end{aligned}
$$

To numerically compute $\hat{\alpha}_t$, I need to evaluate $\nabla_x V$ and $\nabla_{xx} V$ (or Y and $\nabla_x Y$) as well as $\nabla_{xg} V \cdot \Sigma(G_t)$. A key insight from BSDEs is that the volatility terms like Z_t^0 are part of the solution. Therefore, I explicitly calculate $\sigma^0(X_t, G_t, \hat{\alpha}_t)\, \nabla_x Y(X_t, G_t)$ and treat $\nabla_{xg} V \cdot \Sigma(G_t)$ as part of the solution, as shown by

$$
\nabla_{xg} V \cdot \Sigma(G_t) = Z_t^0 - \sigma^0(X_t, G_t, \hat{\alpha}_t)\, \nabla_x Y(X_t, G_t).
$$

[4]For nonsmooth cases, a similar relationship still exists, and readers are referred to Chapter 5 of Yong and Zhou (1999).

In this way, I can avoid the computation of the high-dimensional object $\nabla_{xg}V$. Hereafter, the inner product term $\nabla_{xg}V \cdot \Sigma(G_t)$ is denoted as $\tilde{Z}^0(X_t, G_t)$. The previous subsection has indicated that evaluating both $\nabla_g V$ and $\nabla_{gg}V$ are unnecessary if I solve for $V(X_t, G_t)$ via its BSDE.

2.3 Numerical Scheme

The probabilistic numerical scheme takes advantage of the property that the dynamic paths of forward-looking variables (e.g., Y_t and V_t) generated by BSDEs and the dynamic paths of backward-looking variables (e.g., X_t and G_t) generated by SDEs always satisfy the fixed-point mapping defined by the Forward-Backward SDE system. For the above dynamic optimization problem, the FBSDE system is

$$\begin{aligned}
-\mathrm{d}V_t &= \left(f\left(X_t, G_t, \hat{\alpha}_t\right) - \rho V_t\right)\mathrm{d}t - Z_t^V \mathrm{d}W_t - Z_t^{0,V}\mathrm{d}W_t^0, \\
-\mathrm{d}Y_t &= \left(\nabla_x H\left(X_t, G_t, \hat{\alpha}_t, Y_t, Z_t, Z_t^0\right) - \rho Y_t\right)\mathrm{d}t - Z_t \mathrm{d}W_t - Z_t^0 \mathrm{d}W_t^0, \\
\mathrm{d}X_t &= \mu\left(X_t, G_t, \hat{\alpha}_t\right)\mathrm{d}t + \sigma\left(X_t, G_t, \hat{\alpha}_t\right)\mathrm{d}W_t + \sigma^0\left(X_t, G_t, \hat{\alpha}_t\right)\mathrm{d}W_t^0, \\
\mathrm{d}G_t &= b\left(G_t\right)\mathrm{d}t + \Sigma\left(G_t\right)\mathrm{d}W_t^0, \\
Y_t &= Y\left(X_t, G_t\right), V_t = V\left(X_t, G_t\right),
\end{aligned}$$

where $\hat{\alpha}_t$ is given by the optimality condition (12), $Y(x, g)$ and $V(x, g)$ are the fixed-point mappings defined by the FBSDE system. The solutions of BSDEs include their volatility functions $\left(Z(x, g), \tilde{Z}^0(x, g)\right)$ and $\left(Z^V(x, g), Z^{0,V}(x, g)\right)$.

To handle the high dimensionality of G_t, I will use deep neural networks to approximate these functions. The approximations are denoted as $y(\cdot, \cdot; \Theta), z(\cdot, \cdot; \Theta), \tilde{z}^0(\cdot, \cdot; \Theta)$, $v(\cdot, \cdot; \Theta), z^v(\cdot, \cdot; \Theta)$, and $z^{0,v}(\cdot, \cdot; \Theta)$, where Θ represents the set of parameters that identify the neural network.

For an arbitrary T, I discretize time interval $[0, T]$ evenly into $\mathbf{I}$ subintervals: $0 = t_0 < t_1 < t_2 < \cdots < t_{\mathbf{I}} = T$,. Then, I simulate M sample paths of Brownian motions $\left\{W_{i,m}, W_{i,m}^0\right\}_{i=0,m=1}^{\mathbf{I}-1,M}$ and initial states $\{x_{0,m}, g_{0,m}\}_{m=1}^M$. Here, I drop the subscript m to simplify notation. Let $\Delta \equiv t_{i+1} - t_i$, $w_i \equiv W_{i+1} - W_i$, and $w_i^0 = W_{i+1}^0 - W_i^0$. Initialize $v_0 = v(x_0, g_0; \Theta)$ and $y_0 = y(x_0, g_0; \Theta)$. Next, I will repeat the following procedure for i ranging from 0 to $\mathbf{I} - 1$.

1. Compute $z_i = z(x_i, g_i; \Theta), \tilde{z}_i^0 = \tilde{z}^0(x_i, g_i; \Theta), z_i^v = z^v(x_i, g_i; \Theta)$, and $z_i^{0,v} = z^{0,v}(x_i, g_i; \Theta)$;
2. Evaluate $y_{x,i} \equiv \nabla_x y(x_i, g_i; \Theta) = \frac{1}{2k}\left(y(x_i + k, g_i; \Theta) - y(x_i - k, g_i; \Theta)\right)$ and $v_{x,i} \equiv \nabla_x v(x_i, g_i; \Theta) = \frac{1}{2k}\left(v(x_i + k, g_i; \Theta) - v(x_i - k, g_i; \Theta)\right)$

3. Compute $\hat{\alpha}_i$ according to the optimality condition (12) and compute $z_i^0 = \sigma^0(x_i, g_i, \hat{\alpha}_i)\, y_{x,i} + \tilde{z}_i^0$;

4. Calculate $v_{i+1}, y_{i+1}, x_{i+1}$, and g_{i+1} according to

$$\begin{aligned}
v_{i+1} &= v_i - \left(f(x_i, g_i, \hat{\alpha}_i) - \rho v_i\right)\Delta + z_i^v w_i + z_i^{0,v} w_i^0, \\
y_{i+1} &= y_i - \left(\nabla_x H\left(x_i, g_i, \hat{\alpha}_i, y_i, z_i, z_i^0\right) - \rho y_i\right)\Delta + z_i w_i + z_i^0 w_i^0, \\
x_{i+1} &= x_i + \mu(x_i, g_i, \hat{\alpha}_i)\Delta + \sigma(x_i, g_i, \hat{\alpha}_i) w_i + \sigma^0(x_i, g_i, \hat{\alpha}_i) w_i^0, \\
g_{i+1} &= g_i + b(g_i)\Delta + \Sigma(g_i) w_i^0;
\end{aligned}$$

5. Compute $\tilde{v}_{i+1} = v(x_{i+1}, g_{i+1}; \Theta)$ and $\tilde{y}_{i+1} = y(x_{i+1}, g_{i+1}; \Theta)$.

Given the simulated paths, the loss function is constructed as follow:

$$\text{Loss}\left(\Theta; \left\{x_{0,m}, g_{0,m}, w_{i,m}, w_{i,m}^0\right\}_{i=1,m=1}^{\mathbf{I}-1,M}\right) = \frac{1}{M\mathbf{I}} \sum_{m=1}^{M} \sum_{i=1}^{\mathbf{I}-1} \left\|v_i - \tilde{v}_i\right\|^2 + \omega_1 \left\|y_i - \tilde{y}_i\right\|^2 + \omega_2 \left\|y_i - v_{x,i}\right\|^2,$$

where $\|\cdot\|$ denotes the square norm and ω_1 and ω_2 are weight parameters.

The core of the numerical scheme is to construct the loss function, and the minimization of the loss over the parameter Θ is outsourced to deep learning packages in Python, such as TensorFlow or PyTorch. Goodfellow, Bengio and Courville (2016) is the standard reference for deep neural networks, optimization algorithms and related topics on deep learning.

3 Aggregation and Dynamics of Distributions

In this section, I will first discuss the numerical treatment of market clearing conditions. Second, I will present how to use the finite volume method to approximate the law of motion for an infinite-dimensional state variable, such as the wealth distribution. The finite volume method was first introduced to computational fluid dynamics in the 1970s (McDonald, 1971; MacCormack and Paullay, 1972).

3.1 Market Clearing

In many heterogeneous-agent models, the distribution of individual states, which I denote as $G_t(\cdot)$, influences an individual's dynamics through market-clearing prices. A common equilibrium price is the risk-free r_t, which directly affects the law of motion for a household's wealth. Consequently, SDE (8) can be expressed as

$$\mathrm{d}X_t = \mu\left(X_t, r_t, \alpha_t\right)\mathrm{d}t + \sigma\left(X_t, r_t, \alpha_t\right)\mathrm{d}W_t + \sigma^0\left(X_t, r_t, \alpha_t\right)\mathrm{d}W_t^0$$

instead. The optimality condition (12) can then be written as follows

$$\begin{aligned} 0 = &\nabla_\alpha f\left(X_t, r_t, \hat{\alpha}_t\right) + \nabla_\alpha \mu\left(X_t, r_t, \hat{\alpha}_t\right)\nabla_x V + \sigma\left(X_t, r_t, \hat{\alpha}_t\right)\nabla_\alpha \sigma\left(X_t, r_t, \hat{\alpha}_t\right)\nabla_{xx} V \\ &+ \sigma^0\left(X_t, r_t, \hat{\alpha}_t\right)\nabla_\alpha \sigma^0\left(X_t, r_t, \hat{\alpha}_t\right)\nabla_{xx} V + \nabla_\alpha \sigma^0\left(X_t, r_t, \hat{\alpha}_t\right)\nabla_{xg} V \cdot \Sigma\left(G_t\right), \end{aligned}$$

which defines the policy function $\hat{\alpha}\left(x, r_t, G_t\right)$. Aggregating individual policy functions requires satisfying a market clearing condition

$$0 = S\left(\int \hat{\alpha}\left(x, r_t, G_t\right)\mathrm{d}G_t\left(x\right), G_t\right), \tag{13}$$

where $S(\cdot,\cdot)$ is a function dependent on the integration of $\hat{\alpha}(x, r_t, G_t)$ over $G_t(\cdot)$ and the distribution of individual states. Thus, given the policy function, the market-clearing condition establishes a mapping from the distribution $G_t(\cdot)$ to the equilibrium price r_t.

In some models such as Krusell and Smith (1998), the equilibrium price can be written as an explicit function of the distribution or its moment like

$$r_t = S\left(\int x \mathrm{d}G_t(x)\right).$$

This characteristic simplifies the numerical computation process. However, not all models offer such simplifications. For example, in Krusell and Smith (1997), households allocate their wealth between risky and safe assets while facing a borrowing constrain and a short-sale constraint on the risky asset. As a result, the policy function $\hat{\alpha}\left(x, r_t, G_t\right)$ becomes nonlinear with respect to the risk-free rate r_t, and no explicit function of r_t is available. In such cases, it is possible to leverage the analytic properties of the policy function $\hat{\alpha}\left(x, r_t, G_t\right)$ and $S\left(\cdot,\cdot\right)$ and develop specific model algorithms to solve for the equilibrium r_t given $G_t\left(\cdot\right)$. For instance, in Krusell and Smith (1997), households' demand for the risky asset decreases with r_t. Hence, a simple bisection method is sufficient to determine the market-clearing r_t. Unfortunately, there is no universally effective numerical treatment, except using neural networks to approximate the price function of aggregate state variables and incorporating the error of the market-clearing condition into the loss function. A crucial consideration for implementation is determining the weight assigned to the loss of the market-clearing condition.

In **asset markets**, returns are typically not locally deterministic. As a result, investors' demand for an asset is influenced by both its instantaneous return and volatility. Simultaneously, the asset price and its return are endogenously determined within the general equilibrium framework. Consequently, one must solve for investors' policy functions, asset pricing equations, and market clearing conditions collectively. This implies the need to incorporate BSDEs, which describe the forward-looking processes of asset prices, into the Forward-Backward SDE framework. Those interested in resolving large-scale asset pricing or macro-finance models can refer to my paper Huang (2023) for further details on establishing BSDEs for asset prices. In Appendix B, I consider a variant of Krusell and Smith (1997) with durable assets and asset pricing.[5]

In models with **search and matching** frictions, the absence of centralized markets prohibits any simplification of individual states' laws of motion (8) because their dynamics depend on the outcomes of search and matching, which in turn rely on the entire distribution instead of on a clearing price of a centralized market. In Section 5, I consider an asset valuation model of OTC markets based on Duffie et al. (2007).[6]

3.2 Kolmogorov Forward Equation and Finite Volume Method

Given that equilibrium prices are expressed as functions of aggregate states and individual policy functions, the law of motion for an individual household's state X_t depends on X_t itself and the distribution of X_t represented as $G_t(\cdot)$ in the economy. To simplify the notation, I rewrite the SDE governing X_t in Section 2 as follows

$$\mathrm{d}X_t = \mu\left(X_t, G_t(\cdot)\right)\mathrm{d}t + \sigma\left(X_t, G_t(\cdot)\right)\mathrm{d}W_t + \sigma^0\left(X_t, G_t(\cdot)\right)\mathrm{d}W_t^0.$$

The law of motion for $g_t(\cdot)$, the density function of X_t, is governed by the stochastic Kolmogorov forward equation (KFE), also known as the Fokker-Planck equation[7]

$$\begin{aligned}\mathrm{d}g_t(x) = &-\nabla_x\left(\mu\left(x, G_t(\cdot)\right)g_t(x)\right)\mathrm{d}t - \nabla_x\left(\left(\sigma^0\left(x, G_t(\cdot)\right)\mathrm{d}W_t^0\right)g_t(x)\right)\\ &+\frac{1}{2}\nabla_{xx}\left\{\left[\left(\sigma\left(x, G_t(\cdot)\right)\right)^2 + \left(\sigma^0\left(x, G_t(\cdot)\right)\right)^2\right]g_t(x)\right\}\mathrm{d}t.\end{aligned}$$

[5] A recent paper by Gopalakrishna, Gu and Payne (2024) solves the heterogeneous-agent asset pricing in an incomplete market via the analytic (PDE) approach and deep learning.

[6] A recent paper by Payne, Rebei and Yang (2024) solves a labor search model with aggregate risks via the analytic approach and deep learning.

[7] This KFE is stochastic due to the influence of an aggregate risk $\{W_t^0\}$, which represents common noise in the study of mean-field games. A heuristic derivation of the stochastic KFE can be found on Page 111 in Volume II of Carmona and Delarue (2018).

As it is impractical to input an infinite-dimensional object into a computer, I need to discretize the individual state space and the density function. I employ the finite volume method for this purpose, which involves dividing the space domain of $g_t(\cdot)$ into a finite number of intervals, such as $(x_0, x_1), \cdots, (x_{N-1}, x_N)$, and approximating the density $g_t(\cdot)$ using a finite number of probabilities over these intervals, denoted as G_t^n

$$G_t^n = \int_{x_{n-1}}^{x_n} g_t(x)\,\mathrm{d}t.$$

The cumulative distribution function $G_t(\cdot)$ can be approximated as

$$G_t(x) = \sum_{n=1}^{N} G_t^n \mathbf{1}\{x_{n-1} \le x\}.$$

By performing integration on both sides of the KFE over an interval (x_{n-1}, x_n), we obtain a forward SDE with respect to G_t^n

$$\begin{aligned}
\mathrm{d}G_t^n = &-\left(\mu(x_n, G_t(\cdot))\, g_t(x_n) - \mu(x_{n-1}, G_t(\cdot))\, g_t(x_{n-1})\right)\mathrm{d}t \\
&-\left(\sigma^0(x_n, G_t(\cdot))\, g_t(x_n) - \sigma^0(x_{n-1}, G_t(\cdot))\, g_t(x_{n-1})\right)\mathrm{d}W_t^0 \\
&+\frac{1}{2}\nabla_x\left\{\left[(\sigma(x_n, G_t(\cdot)))^2 + \left(\sigma^0(x_n, G_t(\cdot))\right)^2\right] g_t(x_n)\right\}\mathrm{d}t \\
&-\frac{1}{2}\nabla_x\left\{\left[(\sigma(x_{n-1}, G_t(\cdot)))^2 + \left(\sigma^0(x_{n-1}, G_t(\cdot))\right)^2\right] g_t(x_{n-1})\right\}\mathrm{d}t.
\end{aligned}$$

Thus, the flow of the distribution $g_t(\cdot)$ can be approximated using a finite number of forward SDEs with respect to G_t^n, and the accuracy of the approximation increases with finer intervals.

Figure 1 and the law of motion for G_t^n highlight that the dynamics of households near the boundaries at x_{n-1} and x_n influence the evolution of G_t^n over time. For example, if $\mu(x_n, G_t(\cdot))$ or $\sigma^0(x_n, G_t(\cdot))\,\mathrm{d}W_t^0$ is positive, households in the vicinity of x_n will move upward and exit the interval (x_{n-1}, x_n). Consequently, the probability G_t^n will decrease. This intuition also explains the application of the upwind scheme to approximate $g_t(x_n)$. Specifically, if the sign of $\mu(x_n, G_t(\cdot))$ or $\sigma^0(x_n, G_t(\cdot))\,\mathrm{d}W_t^0$ is positive, then $g_t(x_n)$ is approximated by ${G_t^n}/{(x_n - x_{n-1})}$; otherwise, if $\mu(x_n, G_t(\cdot))$ or $\sigma^0(x_n, G_t(\cdot))\,\mathrm{d}W_t^0$ is negative, then $g_t(x_n)$ is approximated by ${G_t^{n+1}}/{(x_{n+1} - x_n)}$.

Next, I outline the numerical scheme for updating G_t^n over time, using the same time discretization as in the previous section. For brevity, I only display the case $1 < n < N$ with

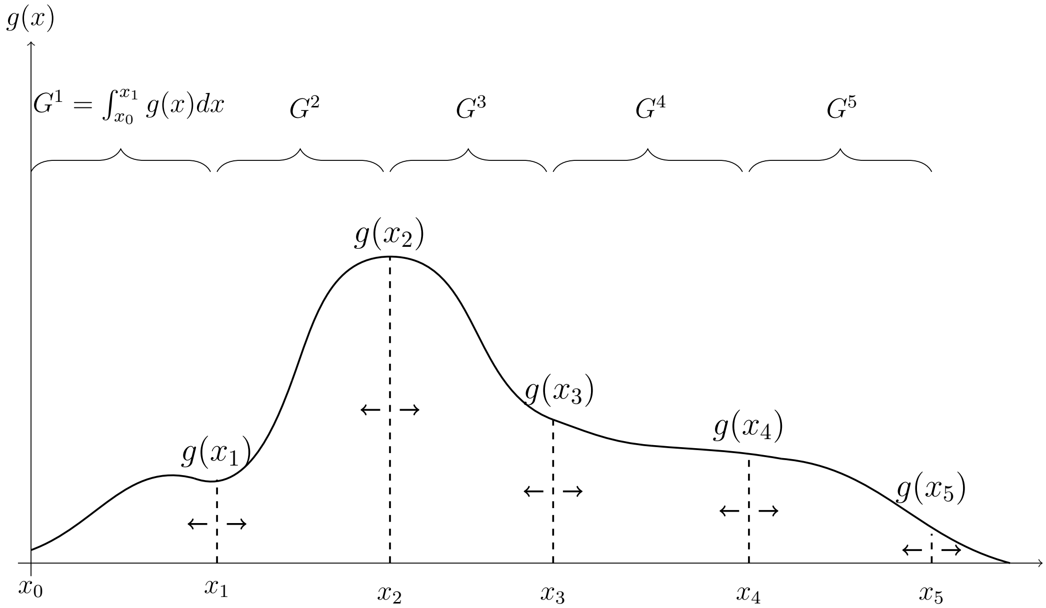

Figure 1: Finite Volume Method

boundary cases left in Appendix A

$$
\begin{aligned}
& G_{t+\Delta}^{n}-G_{t}^{n} \\
& =-\mu\left(x_{n}, G_{t}(\cdot)\right) \Delta\left(\frac{G_{t}^{n+1}}{x_{n+1}-x_{n}} \mathbf{1}\left\{\mu\left(x_{n}, G_{t}(\cdot)\right)<0\right\}+\frac{G_{t}^{n}}{x_{n}-x_{n-1}} \mathbf{1}\left\{\mu\left(x_{n}, G_{t}(\cdot)\right) \geq 0\right\}\right) \\
& \quad+\mu\left(x_{n-1}, G_{t}(\cdot)\right) \Delta\left(\frac{G_{t}^{n}}{x_{n}-x_{n-1}} \mathbf{1}\left\{\mu\left(x_{n-1}, G_{t}(\cdot)\right)<0\right\}+\frac{G_{t}^{n-1}}{x_{n-1}-x_{n-2}} \mathbf{1}\left\{\mu\left(x_{n-1}, G_{t}(\cdot)\right) \geq 0\right\}\right) \\
& \quad-\sigma^{0}\left(x_{n}, G_{t}(\cdot)\right) \mathrm{d} W_{t}^{0}\left(\frac{G_{t}^{n+1}}{x_{n+1}-x_{n}} \mathbf{1}\left\{\sigma^{0}\left(x_{n}, G_{t}(\cdot)\right) \mathrm{d} W_{t}^{0}<0\right\}\right. \\
& \qquad\qquad\qquad\left.+\frac{G_{t}^{n}}{x_{n}-x_{n-1}} \mathbf{1}\left\{\sigma^{0}\left(x_{n}, G_{t}(\cdot)\right) \mathrm{d} W_{t}^{0} \geq 0\right\}\right) \\
& \quad+\sigma^{0}\left(x_{n-1}, G_{t}(\cdot)\right) \mathrm{d} W_{t}^{0}\left(\frac{G_{t}^{n}}{x_{n}-x_{n-1}} \mathbf{1}\left\{\sigma^{0}\left(x_{n-1}, G_{t}(\cdot)\right) \mathrm{d} W_{t}^{0}<0\right\}\right. \\
& \qquad\qquad\qquad\left.+\frac{G_{t}^{n-1}}{x_{n-1}-x_{n-2}} \mathbf{1}\left\{\sigma^{0}\left(x_{n-1}, G_{t}(\cdot)\right) \mathrm{d} W_{t}^{0} \geq 0\right\}\right) \\
& \quad+\frac{\left[\left(\sigma\left(\hat{x}_{n+1}, G_{t}(\cdot)\right)\right)^{2}+\left(\sigma^{0}\left(\hat{x}_{n+1}, G_{t}(\cdot)\right)\right)^{2}\right] G_{t}^{n+1}}{\left(x_{n+1}-x_{n-1}\right)\left(x_{n+1}-x_{n}\right)}-\frac{\left[\left(\sigma\left(\hat{x}_{n}, G_{t}(\cdot)\right)\right)^{2}+\left(\sigma^{0}\left(\hat{x}_{n}, G_{t}(\cdot)\right)\right)^{2}\right] G_{t}^{n}}{\left(x_{n+1}-x_{n-1}\right)\left(x_{n}-x_{n-1}\right)} \\
& \quad-\frac{\left[\left(\sigma\left(\hat{x}_{n}, G_{t}(\cdot)\right)\right)^{2}+\left(\sigma^{0}\left(\hat{x}_{n}, G_{t}(\cdot)\right)\right)^{2}\right] G_{t}^{n}}{\left(x_{n}-x_{n-2}\right)\left(x_{n}-x_{n-1}\right)}+\frac{\left[\left(\sigma\left(\hat{x}_{n-1}, G_{t}(\cdot)\right)\right)^{2}+\left(\sigma^{0}\left(\hat{x}_{n-1}, G_{t}(\cdot)\right)\right)^{2}\right] G_{t}^{n-1}}{\left(x_{n}-x_{n-2}\right)\left(x_{n-1}-x_{n-2}\right)},
\end{aligned}
$$

where $\hat{x}_n = 0.5\,(x_n + x_{n-1})$. [8]

3.3 Numerical Implementation

The numerical scheme for solving a heterogeneous-agent dynamic general equilibrium model is based on that of solving an individual agent's dynamic optimization, as outlined in Section 2.3, with three main differences. First, the law of motion for aggregate state variables is endogenous in dynamic general equilibrium (as shown in step *5* of Section 2.3), and their numerical updates are demonstrated in Section 3.2.

Second, when simulating samples, I will track M economies and simultaneously simulate the dynamics of H heterogeneous households for each economy. In other words, the H households share the same aggregate states. Therefore, individual state variables or choice variables have three subscripts: i captures time, h represents the household index, and m indicates the economy.

The third difference, which also presents the most computational challenge, involves solving for equilibrium prices. With the state discretization of the finite volume method, I integrate individual policy functions using the simple midpoint rule. Step *3* in Section 2.3 requires solving for equilibrium prices and agents' policy functions simultaneously while maintaining market clearing conditions, such as equation (13). As discussed in Section 3.1, one must exploit the property of the market clearing condition to simplify the procedure of jointly solving for policy functions and equilibrium prices.

4 A Variant of Krusell and Smith (1997)

In this section, I will present how to apply the probabilistic approach to solving a modified version of Krusell and Smith (1997). Compared to Krusell and Smith (1998), the model

---

[8]

$$
\begin{aligned}
&\nabla_x \left\{ \left[\left(\sigma\left(x_n, G_t\right)\right)^2 + \left(\sigma^0\left(x_n, G_t\right)\right)^2 \right] g_t\left(x_n\right) \right\} \\
=& \frac{\left[\left(\sigma\left(0.5\left(x_{n+1}+x_n\right), G_t\right)\right)^2 + \left(\sigma^0\left(0.5\left(x_{n+1}+x_n\right), G_t\right)\right)^2 \right] g_t\left(0.5\left(x_{n+1}+x_n\right)\right)}{0.5\left(x_{n+1}-x_{n-1}\right)} \\
&\quad - \frac{\left[\left(\sigma\left(0.5\left(x_n+x_{n-1}\right), G_t\right)\right)^2 + \left(\sigma^0\left(0.5\left(x_n+x_{n-1}\right), G_t\right)\right)^2 \right] g_t\left(0.5\left(x_n+x_{n-1}\right)\right)}{0.5\left(x_{n+1}-x_{n-1}\right)} \\
=& \frac{\left[\left(\sigma\left(\hat{x}_{n+1}, G_t\right)\right)^2 + \left(\sigma^0\left(\hat{x}_{n+1}, G_t\right)\right)^2 \right] G_t^{n+1}}{0.5\left(x_{n+1}-x_{n-1}\right)\left(x_{n+1}-x_n\right)} - \frac{\left[\left(\sigma\left(\hat{x}_n, G_t\right)\right)^2 + \left(\sigma^0\left(\hat{x}_n, G_t\right)\right)^2 \right] G_t^n}{0.5\left(x_{n+1}-x_{n-1}\right)\left(x_n-x_{n-1}\right)}
\end{aligned}
$$

I consider incorporates a portfolio choice between a risky asset and risk-free debt, as well as a nonlinear debt market clearing condition. These two features often present significant challenges for other numerical algorithms intended for heterogeneous-agent models.

4.1 Model

The economy consists of a continuum of households who supply labor inelastically. The aggregate production function takes the aggregate physical capital K_t (in efficiency units) and aggregate labor supply L_t as inputs

$$Y_t = zK_t^{\alpha}L_t^{1-\alpha},$$

where the TFP parameter z remains constant over time. The labor wage is determined by

$$w\left(K_t, L_t\right) = \left(1-\alpha\right) z \left(\frac{K_t}{L_t}\right)^{\alpha},$$

and the return to physical capital is given by

$$r\left(K_t, L_t\right) = \alpha z \left(\frac{K_t}{L_t}\right)^{\alpha-1}.$$

Households can transform final goods into physical capital or reversely at the one-for-one ratio. Hence, the price of physical capital is a constant one over time. In Appendix B, I relax this assumption and introduce durable assets and asset pricing to Krusell and Smith (1997). The net supply of risk-free debt is zero in the economy.

A household's life-time expected discounted utility is

$$E_0\left[\int_0^{\infty} e^{-\rho t}\ln\left(c_t\right)\mathrm{d}t\right].$$

The employment status of a household follows a two-state Markov chain: $\epsilon_t \in \{0, 1\}$. When $\epsilon_t = 1$, the household supplies $\underline{l}$ efficiency units of labor inelastically; when $\epsilon_t = 0$, the household is unemployed and receives an exogenous amount g of goods. The transition intensity from $\epsilon_t = 0$ to $\epsilon_t = 1$ is λ^0; while the intensity from $\epsilon_t = 1$ to $\epsilon_t = 0$ is λ^1. Assuming λ^0 and λ^1 are constants, I posit that the aggregate labor supply reaches a stationary level of ${}^{\lambda_1 \underline{l}}\!/\!\left(\lambda_0+\lambda_1\right)$ without loss of generality.

Households can only allocate their wealth between risky physical capital and risk-free debt. Therefore, the idiosyncratic risk is partially insurable. The instantaneous rate of return for holding physical capital is $r(K_t, L_t) - \delta$, where δ represents the depreciation rate, and the risk of holding physical capital is driven by an aggregate Brownian shock W_t with a constant percentage volatility σ. Given the holding of physical capital k_t, consumption rate c_t, and risk-free rate r_t, a household's wealth x_t evolves as

$$\mathrm{d}x_t = \underbrace{\left(r_t x_t + \left(r\left(K_t, L_t\right) - \delta - r_t\right) k_t + \epsilon_t \underline{l} w\left(K_t, L_t\right) + \left(1 - \epsilon_t\right) g - c_t\right)}_{\equiv \mu_t(x,j)} \mathrm{d}t + \sigma k_t \mathrm{d}W_t.$$

Additionally, households are not allowed to short physical capital, i.e., $k_t \geq 0$, and there is an upper bound for borrowing such that $k_t - x_t \leq \underline{b}$. Considering these two restrictions, there is a constraint on a household's individual state variable: $x_t \geq -\underline{b}$.

Remark on the Aggregate Shock. In Krusell and Smith (1997), the aggregate shock is driven by the Markov chain of the TFP z_t. An implicit assumption in discrete-time models is that agents cannot adjust their holdings of physical capital for a "long" period, such as a quarter or a year. Under this assumption, the capital holding decision made at the beginning of a period (or, equivalently, at the end of the previous period) is risky because the realization of TFP in the current period is stochastic, and agents cannot adjust their holdings throughout the period. In the continuous-time setting, unless explicitly stated, agents can adjust their capital holding choice instantaneously, making it locally risk-free.[9] As a result, I incorporate the aggregate capital quality shock into the model to introduce local risk into the households' capital holding decision.

4.2 Markov Equilibrium

I will solve for the Markov equilibrium of the model. With the presence of both idiosyncratic and aggregate shocks, market incompleteness implies that the distribution of households' wealth levels x_t will serve as an aggregate state variable denoted as $G_t(\cdot, j)$, where $j = 0, 1$. The Markov equilibrium will be characterized by value functions and policy functions of individual wealth level x_t, employment status ϵ_t, and the aggregate state variable G_t, as well as price functions.

I follow the procedure outlined in Section 2.2 and apply stochastic maximum principle

[9]To understand this point clearly, let's consider the expected output of a firm with a given capital stock k_t, labor l_t, and the pre-production TFP z_h. The realized TFP can be either z_l with a probability of $1 - e^{-\lambda\Delta}$ or z_h with a probability of $e^{-\lambda\Delta}$. The expected output is

$$e^{-\lambda\Delta} z_h k_t^{\alpha} l_t^{1-\alpha} \Delta + \left(1 - e^{-\lambda\Delta}\right) z_l k_t^{\alpha} l_t^{1-\alpha} \Delta.$$

The second term above, which captures the risk of capital holding k_t, approaches zero at a rate of $\left(\Delta^2\right)$.

to characterize the optimal decisions of a household. The Hamiltonian function is

$$H(t,x,c,k,y,z) = \ln(c) + \Big(r_t x + (r(K_t,L_t) - \delta - r_t)k + \epsilon_t \underline{l} w(K_t,L_t) + (1-\epsilon_t) g - c\Big) y \\ + \sigma k z,$$

when the state constraint $x_t \geq -\underline{b}$ is not binding. The co-state variable of type-j household, denoted by Y_t^j, follows a BSDE

$$\mathrm{d}Y_t^j = -(r_t - \rho) Y_t^j \mathrm{d}t + U_t^{Y,j} \mathrm{d}\Lambda_t^j + Z_t^{Y,j} \mathrm{d}W_t, \tag{14}$$

where Λ_t^j represents a Poisson process with an arrival rate of λ^j. The solution to the BSDE is given by $\left(Y_t^j, U_t^{Y,j}, Z_t^{Y,j}\right)$. The first-order conditions are

$$c_t^j = \frac{1}{Y_t^j}$$
$$r(K_t,L_t) - \delta - r_t = -\frac{\sigma Z_t^{Y,j}}{Y_t^j} \quad \text{if } \underline{k} < k < x_t + \underline{b}$$

Next, I analyze the portfolio choice k_t. In the Markov equilibrium, there exists a mapping from (x_t, G_t) to Y_t^j denoted as $Y^j(x_t, G_t)$. By applying Ito's formula, I decompose the volatility term $Z_t^{Y,j}$ into two components: $\frac{\partial Y^j}{\partial n}(x_t, G_t)\sigma k_t$, which represents the contribution of x_t, and $\tilde{Z}_t^{Y,j}$, which represents the contribution of G_t

$$Z_t^{Y,j} = \nabla_x Y^j(x_t, G_t)\sigma k_t + \tilde{Z}_t^{Y,j}.$$

If the excess return $r(K_t,L_t) - \delta - r_t$ is high enough,

$$r(K_t,L_t) - \delta - r_t > -\frac{\sigma}{Y_t^j}\left(\nabla_x Y^j(x_t,G_t)\sigma(x_t + \underline{b}) + \tilde{Z}_t^{Y,j}\right)$$

then the optimal choice of k_t is $k_t = x_t + \underline{b}$. If the excess return is too low,

$$r(K_t,L_t) - \delta - r_t < -\frac{\sigma}{Y_t^j}\left(\nabla_x Y^j(x_t,G_t)\sigma\underline{k} + \tilde{Z}_t^{Y,j}\right),$$

then the optimal $k_t = \underline{k}$. In cases where it falls between these boundaries, the optimal k_t is determined by

$$k_t = -\frac{1}{\sigma^2 \nabla_x Y^j(x_t,G_t)}\left(Y_t^j(r(K_t,L_t) - \delta - r_t) + \sigma\tilde{Z}_t^{Y,j}\right)$$

Overall, the policy function for a household's portfolio choice $k(x_t, G_t)$ is given by

$$\begin{aligned} k^j(x_t, G_t, r_t) &= \min\left\{\max\left\{\tilde{k}^j(x_t, G_t), \underline{k}\right\}, x_t + \underline{b}\right\}, \text{ where} \\ \tilde{k}^j(x_t, G_t, r_t) &\equiv -\frac{1}{\sigma^2 \nabla_x Y^j(x_t, G_t)}\left(Y_t^j\left(r(K_t, L_t) - \delta - r_t\right) + \sigma \tilde{Z}_t^{Y,j}\right). \end{aligned} \tag{15}$$

Given the consumption and portfolio choice policy functions, a household's life-time expected utility can be characterized by the solution to a BSDE, i.e., stochastic differential utility

$$\mathrm{d}V_t^j = -\left(\ln(c_t) - \rho V_t^j\right)\mathrm{d}t + U_t^{V,j}\mathrm{d}\Lambda_{i,t}^j + Z_t^{V,j}\mathrm{d}W_t. \tag{16}$$

The recursive utility defined above yields a Markov solution $V^j(x_t, G_t)$, which has a tight connection with the co-state variable Y_t^j

$$Y_t^j = \nabla_x V^j(x_t, G_t).$$

Given a household's demand for risk-free debt, represented as $x_t - k^j(x_t, G_t)$, the zero net debt supply condition is expressed as

$$\sum_{j=0,1}\int\left(x - k^j(x, G_t, r_t)\right)\mathrm{d}G_t(x, j) = 0. \tag{17}$$

The risk-free rate r_t will adjust the aggregate demand for physical capital and ensure equilibrium in the debt market. It's important to note that both the aggregate physical capital and the aggregate labor supply are entirely determined by the state variable G_t

$$K_t = \sum_{j=0,1}\int x\mathrm{d}G_t(x, j), \qquad L_t = \underline{l}\int \mathrm{d}G_t(x, 1). \tag{18}$$

To fully capture the dynamics of the economy, it is necessary to characterize the law of motion of $G_t(\cdot, j)$ or its density function $g_t(x, j)$, with $j = 0, 1$. This is determined by the stochastic KFE

$$\begin{aligned} \mathrm{d}g_t(n, j) = &-\nabla_x\left(\mu_t(x, j)\, g_t(x, j)\right)\mathrm{d}t - \nabla_x\left(\left[\sigma k_t(x, j)\,\mathrm{d}W_t\right] g_t(x, j)\right) \\ &+ \frac{1}{2}\nabla_{xx}\left\{\sigma^2 k_t^2(x, j)\, g_t(x, j)\right\}\mathrm{d}t - \lambda^j g_t(x, j)\,\mathrm{d}t + \lambda^{1-j} g_t(x, 1-j)\,\mathrm{d}t. \end{aligned} \tag{19}$$

Here, $\mu_t(x, j)$ and $\sigma k_t(x, j)$ represent the drift and volatility terms for an individual household of type-j, with wealth x. It is crucial to emphasize that the dynamics of a household's wealth depend on the distribution G_t through the return to capital $r(K_t, L_t)$, the wage $w(K_t, L_t)$, and the forward-looking variables Y_t^j, along with their volatility terms $\nabla_x Y^j$ and $\tilde{Z}^{Y,j}$. Consequently, I need to solve a fully-coupled forward-backward dynamic system.

4.3 Finite Volume Method

To implement the finite volume method, the individual state space is divided into N intervals or cells using $N+1$ points: $-\underline{b} = x_0 < x_1 < x_2 < \cdots < x_N$. It is assumed that $G_t(x_N, j) = G_t(x, j)$ for any $x > x_N$, where $j = 0, 1$. Previous research, such as Achdou et al. (2022), suggests that the distribution $G_t(\cdot, 0)$ for type-0 households has a mass point at x_0. Interval n refers to $(x_{n-1}, x_n]$. $G_t(\cdot, 0)$ is approximated using $N+1$ scalars, while $G_t(\cdot, 1)$ is approximated using N scalars.

$$G_t^{n,j} = \int_{x_{n-1}}^{x_n} \mathrm{d}G_t(x, j) = \int_{x_{n-1}}^{x_n} g_t(x, j)\, \mathrm{d}x$$

Given the stochastic KFE (19), the law of motion for $G_t^{n,j}$ with $1 < n < N$

$$\begin{aligned}\int_{x_{n-1}}^{x_n} \mathrm{d}g_t(x, j)\, \mathrm{d}x = &-\int_{x_{n-1}}^{x_n} \nabla_x \left(\mu_t(x, j)\, g_t(x, j)\, \mathrm{d}t\right) \mathrm{d}x - \int_{x_{h-1}}^{x_h} \nabla_x \left(\left[\sigma k_t(x, j)\, \mathrm{d}W_t\right] g_t(x, j)\right) \mathrm{d}x \\ &+ \int_{x_{n-1}}^{x_n} \frac{1}{2} \nabla_{xx} \left\{\sigma^2 k_t^2(x, j)\, g_t(x, j)\right\} \mathrm{d}t\mathrm{d}x - \lambda^j \mathrm{d}t \int_{x_{n-1}}^{x_n} g_t(x, j)\, \mathrm{d}x \\ &+ \lambda^{1-j} \mathrm{d}t \int_{x_{n-1}}^{x_n} g_t(x, 1-j)\, \mathrm{d}x\end{aligned}$$

$$\begin{aligned}\mathrm{d}G_t^{n,j} = &-\left(\mu_t(x_n, j)\, g_t(x_n, j) - \mu_t(x_{n-1}, j)\, g_t(x_{n-1}, j)\right) \mathrm{d}t \\ &- \sigma \left(k_t(x_n, j)\, g_t(x_n, j) - k_t(x_{n-1}, j)\, g_t(x_{n-1}, j)\right) \mathrm{d}W_t \\ &+ \frac{\sigma^2}{2} \nabla_x \left\{k_t^2(x_n, j)\, g_t(x_n, j)\right\} \mathrm{d}t - \frac{\sigma^2}{2} \nabla_x \left\{k_t^2(x_{n-1}, j)\, g_t(x_{n-1}, j)\right\} \mathrm{d}t \\ &- \lambda^j G_t^{n,j} \mathrm{d}t + \lambda^{1-j} G_t^{n,1-j} \mathrm{d}t\end{aligned}$$

Assuming that the probability density within an interval is constant, the upwind scheme leads to the numerical scheme

$$
\begin{aligned}
&G_{t+\Delta}^{n,j} - G_t^{n,j} \\
&= -\mu_t(x_n, j)\,\Delta\left(\frac{G_t^{n+1,j}}{x_{n+1}-x_n}\mathbf{1}\{\mu_t(x_n,j)<0\} + \frac{G_t^{n,j}}{x_n-x_{n-1}}\mathbf{1}\{\mu_t(x_n,j)\ge 0\}\right) \\
&+ \mu_t(x_{n-1}, j)\,\Delta\left(\frac{G_t^{n,j}}{x_n-x_{n-1}}\mathbf{1}\{\mu_t(x_{n-1},j)<0\} + \frac{G_t^{n-1,j}}{x_{n-1}-x_{n-2}}\mathbf{1}\{\mu_t(x_{n-1},j)\ge 0\}\right) \\
&- \sigma k_t(x_n, j)\,\Delta W_t\left(\frac{G_t^{n+1,j}}{x_{n+1}-x_n}\mathbf{1}\{k_t(x_n,j)\,\Delta W_t<0\} + \frac{G_t^{n,j}}{x_n-x_{n-1}}\mathbf{1}\{k_t(x_n,j)\,\Delta W_t\ge 0\}\right) \\
&+ \sigma k_t(x_{n-1}, j)\,\Delta W_t\left(\frac{G_t^{n,j}}{x_n-x_{n-1}}\mathbf{1}\{k_t(x_{n-1},j)\,\Delta W_t<0\} + \frac{G_t^{n-1,j}}{x_{n-1}-x_{n-2}}\mathbf{1}\{k_t(x_{n-1},j)\,\Delta W_t\ge 0\}\right) \\
&+ \sigma^2\left[\frac{k_t^2(\hat{x}_{n+1},j)\,G_t^{n+1,j}}{(x_{n+1}-x_n)(x_{n+1}-x_{n-1})} - \frac{k_t^2(\hat{x}_n,j)\,G_t^{n,j}}{(x_n-x_{n-1})(x_{n+1}-x_{n-1})}\right]\Delta \\
&- \sigma^2\left[\frac{k_t^2(\hat{x}_n,j)\,G_t^{n,j}}{(x_n-x_{n-1})(x_n-x_{n-2})} - \frac{k_t^2(\hat{x}_{n-1},j)\,G_t^{n-1,j}}{(x_{n-1}-x_{n-2})(x_n-x_{n-2})}\right]\Delta - \lambda^j G_t^{n,j}\Delta + \lambda^{1-j} G_t^{n,1-j}\Delta
\end{aligned}
$$

for $1 < n < N$, where $\hat{x}_n = 0.5\,(x_n + x_{n-1})$.[10] Section A in Appendix contains the detailed law of motion for boundary cases such as $n = 0, 1$ or N, and the matrix operations used for updating vectors $\mathbf{G}_t^0$ and $\mathbf{G}_t^1$ in the algorithm, where

$$
\mathbf{G}_t^0 = \left[G_t^{0,0}, G_t^{1,0}, \cdots, G_t^{N,0}\right]^T, \mathbf{G}_t^1 = \left[G_t^{1,1}, \cdots, G_t^{N,1}\right]^T.
$$

4.4 Numerical Implementation

This section sketches the numerical implementation of the probabilistic approach for solving the model. The state variables of the numerical solution are individual wealth level x_t, individual employment status ϵ_t, and aggregate state $\mathbf{G}_t = [\mathbf{G}_t^0; \mathbf{G}_t^1]$. A Markov equilibrium is characterized by functions $V(\cdot), Y(\cdot)$ and the volatility terms $U^V(\cdot), Z^V(\cdot), U^Y(\cdot), \tilde{Z}^Y(\cdot)$ of corresponding BSDEs (16) and (14). These functions are approximated using a feedforward neural network. Specifically, the state variables x_t and $\mathbf{G}_t$ are inputs through two shared

[10] In particular, to approximate

$$
\begin{aligned}
&\nabla_x\left\{k_t^2(x_n,j)\,g_t(x_n,j)\right\} \\
=&\frac{k_t^2(0.5(x_{n+1}+x_n),j)\,g_t(0.5(x_{n+1}+x_n),j) - k_t^2(0.5(x_n+x_{n-1}),j)\,g_t(0.5(x_n+x_{n-1}),j)}{0.5(x_{n+1}-x_{n-1})} \\
=&\frac{2k_t^2(\hat{x}_{n+1},j)\,G_t^{n+1,j}}{(x_{n+1}-x_n)(x_{n+1}-x_{n-1})} - \frac{2k_t^2(\hat{x}_n,j)\,G_t^{n,j}}{(x_n-x_{n-1})(x_{n+1}-x_{n-1})}
\end{aligned}
$$

hidden layers, which then produce outputs for three independent sets of two hidden layers. The first set generates V_t^0 and V_t^1; the second produces Y_t^0 and Y_t^1; the third yields $U_t^{v,j}, U_t^{y,j}, Z_t^{v,j}$, and $\tilde{Z}_t^{y,j}, j = 0, 1$. Each hidden layer comprises 256 nodes with sigmoid activation functions.[11] The most critical component of the numerical scheme is to construct the loss function, which takes the initial individual and aggregate states, as well as the paths of aggregate and idiosyncratic shocks, as data inputs. Minimization of the loss is implemented with TensorFlow in Python.

Section C in Appendix contains the detailed procedure for calculating the loss. Here, I outline major steps. Since the state space of $\mathbf{G}_t$ is large, I generate a large number of samples, and each sample refers to an economy of a given initial $\mathbf{G}_t$. For a sample point or within an economy, I also consider a large number of households to cover the individual state x_t and ϵ_t. Along the time path of an economy, I first use $\mathbf{G}_t$ and conditions (18) to calculate the aggregate stock of physical capital and labor while entering a period. Aggregate K_t and L_t give rise to the return on physical capital $r(K_t, L_t)$ and the wage $w(K_t, L_t)$. To determine individual demand for physical capital, I evaluate $\nabla x Y^j$, $\tilde{Z}^{Y,j}$, and Y_t^j (from the equation 15). Since demand for physical capital decreases with the risk-free interest rate r_t, I use the bisection method to find the r_t that clears the debt market (from equation 17). Given r_t, it is straightforward to calculate the drift and volatility terms of a household's wealth x_t, as well as the law of motion for $\mathbf{G}_t$. While deriving state variables for the next period, I also update forward-looking variables according to their BSDEs (14) and (16). The values of state variables and forward-looking variables are supposed to satisfy the mappings $V(\cdot)$ and $Y(\cdot)$, which give rise to the primary components of the loss.

There are several **boundary conditions** used to fix the solution. Due to the state constraint $x_t \geq -\underline{b}$, the drift term of x_t at $-\underline{b}$ cannot be negative. Moreover, Achdou et al. (2022) shows that the drift term of x_t at the state constraint is zero for unemployed households. When households' wealth levels are high enough, the effects of their borrowing constraint become negligible, and the logarithmic utility function implies that their consumption is linear in wealth with coefficient ρ. The last boundary condition of my numerical scheme is to impose an upper bound on the consumption policy function for large x_t.

4.5 Numerical Solution

In this section, I illustrate some details of the numerical exercises and present the numerical results. Parameter values that are annualized follow Krusell and Smith (1997): $z = 4, \alpha =$

[11] Refer to Goodfellow et al. (2016), particularly Chapter 6, which discusses feedforward neural networks and deep learning terminology.

$0.36, \rho = 0.04, \lambda^0 = 2.667, \lambda^1 = 0.111, \underline{l} = 0.3271, g = 0.8, \underline{b} = 2.4$, and $\sigma = 0.023$. To cover the aggregate state space, I simulate 8,000 economies, and each economy contains 150 households of each employment status. While the support of households' wealth distribution is $[-2.4, 25]$, I also consider those with wealth level from 25 to 100 whose policy and value functions must satisfy boundary conditions. The length of each period Δ is 0.01, the arbitrary $T = 0.4$, and the number of bi-section iterations is 25 while solving for market-clearing risk-free rate. The training process or the numerical implementation is undertaken on two Nvidia A800 GPUs, and each GPU is allocated with 64 samples in each batch. All variables take values of float64. The average loss is at the magnitude of 10^{-6}.

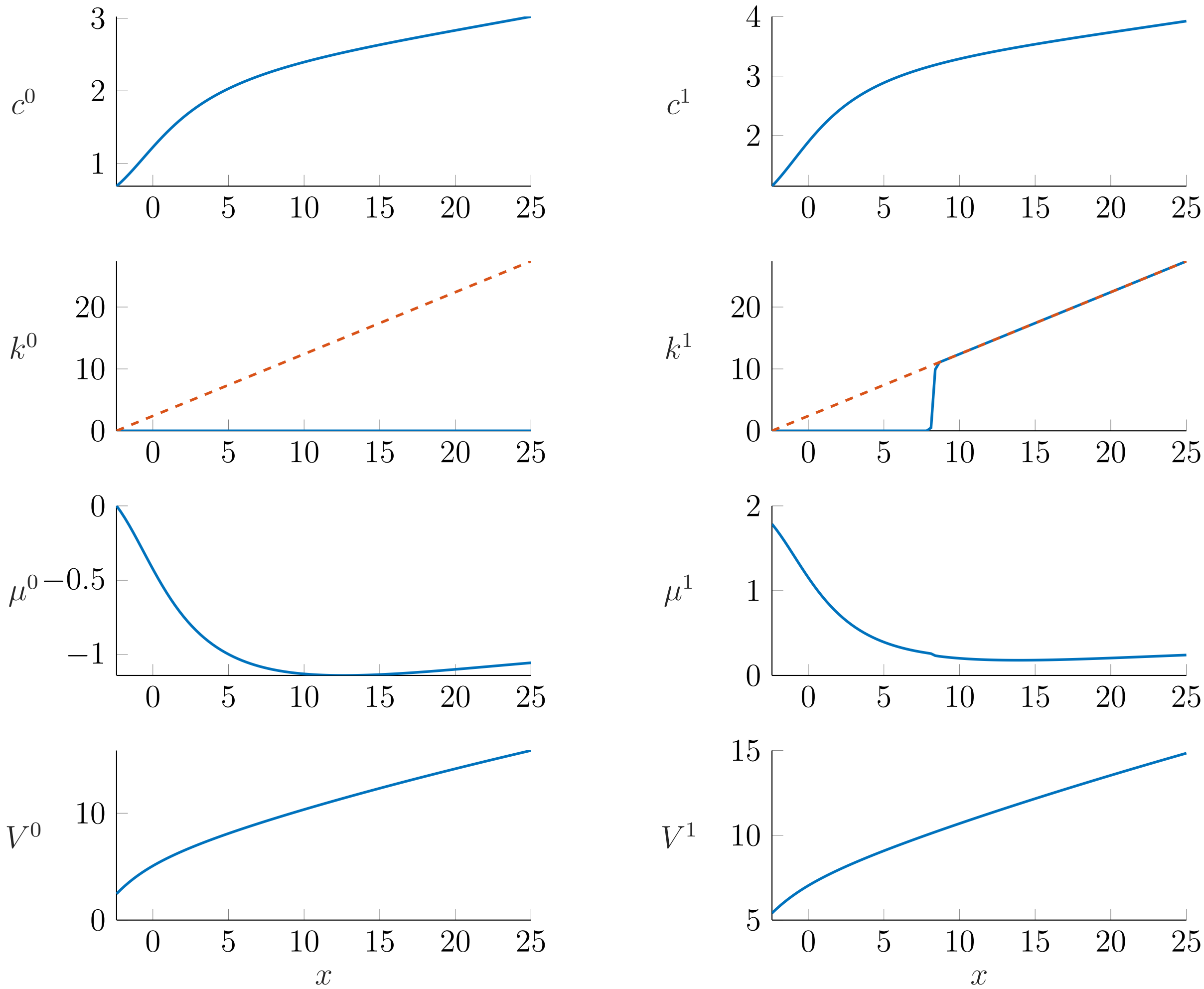

Figure 2: Policy and Value Functions

Figure 2 displays households' value and policy functions under a specific aggregate state $\mathbf{G}_t$, which leads to $r(K_t, L_t) = 0.145, w(K_t, L_t) = 9.316$, and the risk-free rate $r_t = 0.047$. Top panels show that households' propensity to consume is higher when their wealth or income levels are relatively low. For employed households, they choose to hold risky physical assets when they are sufficiently wealthy, and unemployed households only hold risk-free bonds instead (see the second row where the dashed line is $k_t = x_t$).

5 Scaling up Duffie, Gârleanu and Pedersen (2007)

This section focuses on an asset pricing model of over-the-counter (OTC) markets with search-and-bargaining frictions. Specifically, I extend Duffie et al. (2007) to allow for a continuum of heterogeneous investors and expected aggregate shocks. In this model, the cross-sectional distribution over investors' types and asset ownership is a crucial state variable.

5.1 Model

There exits a long-lived indivisible asset with a total supply denoted by θ. These assets generate nondurable consumption goods in each period. The asset's cumulative dividend process, represented by D_t, evolves according to

$$\mathrm{d}D_t = \mu_D \mathrm{d}t + \sigma_D \mathrm{d}B_t,$$

where B_t is a standard Brownian motion, and μ_D and σ_D are positive constants, characterizing the drift and volatility of the dividend process, respectively.

There exits a unit measure of infinitely-lived agents, who have constant-absolute-risk-averse (CARA) additive utility characterized by the absolute risk aversion coefficient γ and the time discount rate β. Agent h is equipped with a stochastic flow of consumption goods, governed by the cumulative endowment process η_t^h, which evolves according to

$$\mathrm{d}\eta_t^h = \mu_\eta \mathrm{d}t + \sigma_\eta \rho_t^h \mathrm{d}B_t + \sigma_\eta \sqrt{1 - \left(\rho_t^h\right)^2} \mathrm{d}Z_t^h,$$

where μ_η and σ_η are the same positive constants across all agents. Here, Z_t^h denotes the idiosyncratic shock derived from a standard Brownian motion, while ρ_t^h is a stochastic process driving the asset's hedging motives. Notably, Z_t^h is independent of both B_t and Z_t^j for $j \neq h$. The dynamics of ρ_t^h follow

$$\mathrm{d}\rho_t^h = -\phi \left(\rho_t^h - \mu_t\right) \mathrm{d}t + \sigma \mathrm{d}B_t^{1,h},$$

where $B_t^{1,h}$ is a standard Brownian motion and μ_t alternates between two states, μ_0 or μ_1, governed by a two-state Markov chain.[12] The shift in μ_t constitutes an **aggregate** shock, affecting all agents uniformly. The transition intensity of μ_t is δ in both directions. $B_t^{1,h}$, an agent-specific idiosyncratic shock, remains independent of $B_t, Z_t^h, Z_t^j, B_t^{1,j}$ for $j \neq h$.

Due to the idiosyncratic shock, there exists a cross-sectional distribution of type ρ_t^h at any time t, which is denoted as $G_t(\cdot)$. Given the asset's indivisibility, an agent's holdings of the asset can either be 0 or 1. Thus, $G_t(\cdot, 0)$ specifies the cumulative distribution of types among non-owners, while $G_t(\cdot, 1)$ outlines that of the owners. The asset's market-clearing condition is

$$G_t(1,1) = \theta.$$

In essence, the aggregate state of the economy is determined by μ_t and $G_t(\cdot)$.

Search and Matching. Agents trade the indivisible asset in an OTC market. During the time interval $(t, t+\mathrm{d}t]$, an agent meets a counterparty with a probability of $\lambda \mathrm{d}t$. A transaction is initiated if it enhances the welfare of both participating parties. The final transaction price is determined through the process of Nash Bargaining.

Suppose each agent has access to a liquid money-market account that offers a constant risk-free rate r. Then, the wealth of agent h, denoted by W_t^h, follows the law of motion

$$\mathrm{d}W_t^h = \left(rW_t^h - c_t^h\right)\mathrm{d}t + \theta_t^h \mathrm{d}D_t + \mathrm{d}\eta_t^h - P_t^h \mathrm{d}\theta_t^h,$$

where c_t^h represents the consumption flow, θ_t^h denotes the agent's asset holdings, and P_t^h is the transaction price when the agent adjusts their asset holdings. Note that the transaction price depends on the aggregate state and the types of both the agent and the counterparty.

Following Duffie et al. (2007), I utilize the CARA utility and conjecture that an agent's continuation value function can be expressed as

$$J^j(w, \rho, G, \mu) = -e^{-r\gamma\left(w + a^j(\rho, G, \mu)\right)}. \tag{20}$$

Hereafter, I will omit the agent index, and the superscript j indicates asset ownership, which can take two values: 0 or 1. $J^0(w_t, \rho_t, G_t, \mu_t)$ represents the continuation value of an asset non-owner with wealth w_t and type ρ_t under aggregate states G_t and μ_t; and $J^1(w_t, \rho_t, G_t, \mu_t)$ represents that of an asset owner.

[12] ρ_t^h has the reflecting upper bound 1 and the reflecting lower bound -1.

5.2 Bargaining

When an asset owner of type ρ_t^1 encounters a non-owner of type ρ_t^0, a Pareto optimal transaction is feasible if there exists a price p such that

$$J^0\left(w_t + p, \rho_t^1, G_t, \mu_t\right) > J^1\left(w_t, \rho_t^1, G_t, \mu_t\right) \quad \text{and } J^1\left(\tilde{w}_t - p, \rho_t^0, G_t, \mu_t\right) \quad > J^0\left(\tilde{w}_t, \rho_t^0, G_t, \mu_t\right),$$

where w_t represents the asset owner's wealth, and $\tilde{w}_t$ denotes the non-owner's wealth. Given the functional form of $J^j(\cdot)$, the condition above is equivalent to the following inequality

$$a^1\left(\rho_t^0, G_t, \mu_t\right) - a^0\left(\rho_t^0, G_t, \mu_t\right) > a^1\left(\rho_t^1, G_t, \mu_t\right) - a^0\left(\rho_t^1, G_t, \mu_t\right), \tag{21}$$

which implies that the non-owner's willingness to pay exceeds the owner's reservation value.

The transaction price results from Nash bargaining, wherein the owner is assumed to have bargaining power denoted by q. Let $P\left(\rho_t^0, \rho_t^1, G_t, \mu_t\right)$ denote the transaction price, which is well-defined if condition (21) is satisfied. Then, $P\left(\rho_t^0, \rho_t^1, G_t, \mu_t\right)$ solves

$$\max_p \left(J^0\left(w_t + p, \rho_t^1, G_t, \mu_t\right) - J^1\left(w_t, \rho_t^1, G_t, \mu_t\right)\right)^q \left(J^1\left(\tilde{w}_t - p, \rho_t^0, G_t, \mu_t\right) - J^0\left(\tilde{w}_t, \rho_t^0, G_t, \mu_t\right)\right)^{1-q}.$$

Given the functional form of $J^h(\cdot)$, the first-order condition of the above optimization implies that $P\left(\rho_t^0, \rho_t^1, G_t, \mu_t\right)$ satisfies

$$\frac{q}{1-q} = \frac{1 - \exp\left(-r\gamma\left(-P\left(\rho_t^0, \rho_t^1, G_t, \mu_t\right) + a^1\left(\rho_t^1, G_t, \mu_t\right) - a^0\left(\rho_t^1, G_t, \mu_t\right)\right)\right)}{1 - \exp\left(-r\gamma\left(P\left(\rho_t^0, \rho_t^1, G_t, \mu_t\right) - \left(a^1\left(\rho_t^0, G_t, \mu_t\right) - a^0\left(\rho_t^0, G_t, \mu_t\right)\right)\right)\right)}.$$

Note that when an agent owns the indivisible asset, the volatility of the agent's wealth increases by $\sigma_D^2 + 2\rho_t\sigma_D\sigma_\eta$, which increases with ρ_t. Given that agents are risk-averse, I conjecture that the difference $a^1\left(\rho_t, G_t, \mu_t\right) - a^0\left(\rho_t, G_t, \mu_t\right)$ decreases with ρ_t. This suggests that the reservation value of asset owners (and the willingness to pay of non-owners) decreases with ρ_t, the correlation between the asset's payoff and their endowments. This conjecture will be validated through numerical exercises presented later. It implies that if an asset owner's type is ρ_t, they will prefer to trade with non-owners of type $\hat{\rho} < \rho_t$. Next, I will leverage this property to simplify the characterization of the law of motion for $G_t(\cdot, 0)$ and $G_t(\cdot, 1)$.

5.3 KFE

To characterize the law of motion for the cross-sectional distribution, I start with the KFE of density functions without matching, $g_t(\cdot, 0)$ and $g_t(\cdot, 1)$. They are represented as follows

$$\mathrm{d}g_t(\rho, j) = \frac{\partial}{\partial \rho}\left(\phi\left(\rho - \mu_t\right) g_t(\rho, j)\right) \mathrm{d}t + \frac{1}{2}\frac{\partial^2}{\partial \rho^2}\left(\sigma^2 g_t(\rho, j)\right) \mathrm{d}t. \tag{22}$$

As Hugonnier, Lester and Weill (2022) demonstrate, it is more convenient to work with the cumulative distribution function. For non-owners, denoted as $j = 0$, taking integration of both sides of equation (22) over the interval $[-1, \rho_n]$ yields:

$$\begin{aligned}\int_{-1}^{\rho_n} \mathrm{d}g_t(\rho, 0) &= \mathrm{d}\left(G_t(\rho_n, 0) - G_t(-1, 0)\right) \\ &= \phi\left(\rho_n - \mu_t\right) g_t(\rho_n, 0) \mathrm{d}t + \frac{\sigma^2}{2}\frac{\partial}{\partial \rho}\left(g_t(\rho_n, 0)\right) \mathrm{d}t.\end{aligned}$$

Let $G_t^{n,j}$ denote

$$\int_{-1}^{\rho_n} g_t(\rho, j)\,\mathrm{d}\rho, \quad j = 0, 1.$$

Taking into account the effect of matching, the law of motion for $G_t^{n,0}$ becomes:

$$\mathrm{d}G_t^{n,0} = \phi\left(\rho_n - \mu_t\right) g_t(\rho_n, 0) \mathrm{d}t + \frac{\sigma^2}{2}\frac{\partial}{\partial \rho}\left(g_t(\rho_n, 0)\right) \mathrm{d}t - 2\lambda G_t^{n,0}\left(\theta - G_t^{n,1}\right) \mathrm{d}t,$$

where $\theta - G_t^{n,1}$ is the measure of owners with type $\rho \geq \rho_n$ and $2\lambda G_t^{n,0}\left(\theta - G_t^{n,1}\right)$ represents the outflow of non-owners with types $\rho < \rho_n$, who become owners due to matching and trade.

For owners $j = 1$, taking the integration of both sides of equation (22) over the interval $[-1, \rho_n]$ yields:

$$\int_{-1}^{\rho_n} \mathrm{d}g_t(\rho, 1) = \mathrm{d}\left(G_t(1, 1) - G_t(\rho_n, 1)\right) = \phi\left(\rho_n - \mu_t\right) g_t(\rho_n, 1) \mathrm{d}t + \frac{\sigma^2}{2}\frac{\partial}{\partial \rho}\left(g_t(\rho_n, 1)\right) \mathrm{d}t.$$

Considering the effect of matching, it follows that

$$\mathrm{d}G_t^{n,1} = \phi\left(\rho_n - \mu_t\right) g_t(\rho_n, 1) \mathrm{d}t + \frac{\sigma^2}{2}\frac{\partial}{\partial \rho}\left(g_t(\rho_n, 1)\right) \mathrm{d}t + 2\lambda G_t^{n,0}(\theta - G_t^{n,1})\mathrm{d}t,$$

where $2\lambda G_t^{n,0}(\theta - G_t^{n,1})$ represents the inflow of non-owners of types $\rho < \rho_n$, who become owners due to matching and trade. The law of motion for G_t can thus be summarized by

$$\mathrm{d}G_t = \mu^g(\mu_t)\,\mathrm{d}t$$

$$\mathrm{d}\begin{bmatrix} G_t(\rho,0) \\ G_t(\rho,1) \end{bmatrix} = \begin{bmatrix} \phi(\rho-\mu_t)\,g_t(\rho,0) + \frac{\sigma^2}{2}\frac{\partial}{\partial\rho}(g_t(\rho,0)) - 2\lambda G_t(\rho,0)(\theta - G_t(\rho,1)) \\ \phi(\rho-\mu_t)\,g_t(\rho,1) + \frac{\sigma^2}{2}\frac{\partial}{\partial\rho}(g_t(\rho,1)) + 2\lambda G_t(\rho,0)(\theta - G_t(\rho,1)) \end{bmatrix}\mathrm{d}t.$$

5.4 Dynamic Optimization and BSDEs

This section will begin with HJB equations that illustrate the dynamic optimal choices of both owners and non-owners, while also highlighting the challenges associated with numerically solving HJB equations that contain the infinite-dimensional state variable G_t. Subsequently, I will introduce the probabilistic formulation of the value functions for owners and non-owners.

5.4.1 Analytic Formulation

The HJB equation for owners is given by

$$\begin{aligned}
&\beta J^1(w_t,\rho_t,G_t,\mu_k) \\
&= \max_c\left\{-e^{-\gamma c} + J^1_w(rw_t - c + \mu_D + \mu_\eta)\right\} + \frac{1}{2}J^1_{ww}\left(\sigma_D^2 + \sigma_\eta^2 + 2\rho_t\sigma_D\sigma_\eta\right) \\
&\quad + \frac{2\lambda}{1-\theta}\int_{-1}^{\rho_t}\left(J^0\left(w_t + P\left(\rho^0,\rho_t,G_t,\mu_k\right),\rho_t,G_t,\mu_k\right) - J^1(w_t,\rho_t,G_t,\mu_k)\right)\mathrm{d}G_t\left(\rho^0,0\right) \\
&\quad - \phi(\rho_t-\mu_k)\,J_\rho(\rho_t,G_t,\mu_k) + \frac{1}{2}\sigma^2 J_{\rho\rho}(\rho_t,G_t,\mu_k) + \left\langle \nabla_g J^1(\rho_t,G_t,\mu_k),\mu^g(\mu_k)\right\rangle \\
&\quad + \delta\left(J^1(w_t,\rho_t,G_t,\mu_{1-k}) - J^1(w_t,\rho_t,G_t,\mu_k)\right),\ \text{where } k = 0,1.
\end{aligned}$$

The second line of the HJB equation captures the expected change in an owner's continuation value due to transactions resulting from random matching with non-owners, whose willingness to pay is high enough. The term $\langle\nabla_g J^1, \mu^g(\mu_k)\rangle$ in the third line characterizes the instantaneous impact of changes in the cross-sectional distribution G_t, which is stochastic due to the randomness of the aggregate state μ_t. If no aggregate shock occurs, i.e., μ_t remains constant over time, the change in the cross-sectional distribution becomes deterministic, allowing the infinite-dimensional state variable G_t to be replaced by the unidimensional variable t.

The HJB equation for non-owners is given by

$$
\begin{aligned}
&\beta J^0\left(w_t,\rho_t,G_t,\mu_k\right)\\
&=\max_c\left\{-e^{-\gamma c}+J_w^0\left(rw_t-c+\mu_\eta\right)\right\}+\frac{1}{2}J_{ww}^0\sigma_\eta^2\\
&\quad+\frac{2\lambda}{\theta}\int_\rho^1\left(J^1\left(w_t-P\left(\rho_t,\rho^1,G_t,\mu_k\right),\rho_t,G_t,\mu_k\right)-J^0\left(w_t,\rho_t,G_t,\mu_k\right)\right)\mathrm{d}G_t\left(\rho^1,1\right)\\
&\quad-\phi\left(\rho_t-\mu_k\right)J_\rho^0\left(\rho_t,G_t,\mu_k\right)+\frac{1}{2}\sigma^2J_{\rho\rho}^0\left(\rho_t,G_t,\mu_k\right)+\left\langle\nabla_gJ^0\left(\rho_t,G_t,\mu_k\right),\mu^g\left(\mu_k\right)\right\rangle\\
&\quad+\delta\left(J^0\left(w_t,\rho_t,G_t,\mu_{1-k}\right)-J^0\left(w_t,\rho_t,G_t,\mu_k\right)\right),\ \text{ for } k=0,1.
\end{aligned}
$$

The second line of the HJB equation captures the expected impact on non-owners' welfare due to random matching with asset owners, provided their reservation values are sufficiently low.

Given equation (20) and the functional form conjecture of $J^j\left(w,\rho,G,\mu\right)$, the first-order condition with respect to consumption c is

$$
c=-\frac{\log\left(r\right)}{\gamma}+r\left(w+a^j\left(\rho,G,\mu\right)\right),
$$

which implies that the two HJB equations can be reduced to differential equations for the functions $a^j\left(\rho,G,\mu\right)$:

$$
\begin{aligned}
-\beta&=-r+r\gamma\left(\frac{\log\left(r\right)}{\gamma}-ra^1\left(\rho_t,G_t,\mu_k\right)+\mu_D+\mu_\eta\right)-\frac{1}{2}\left(r\gamma\right)^2\left(\sigma_D^2+\sigma_\eta^2+2\rho_t\sigma_D\sigma_\eta\right) \qquad (23)\\
&\quad+\frac{2\lambda}{1-\theta}\int_{-1}^{\rho_t}\left(1-e^{-r\gamma\left(P\left(\rho^0,\rho_t,G_t,\mu_k\right)+a^0\left(\rho_t,G_t,\mu_k\right)-a^1\left(\rho_t,G_t,\mu_k\right)\right)}\right)\mathrm{d}G_t\left(\rho^0,0\right)\\
&\quad-\phi\left(\rho_t-\mu_k\right)\left(r\gamma\frac{\partial a^1}{\partial\rho}\left(\rho_t,G_t,\mu_k\right)\right)+\frac{1}{2}\sigma^2\left(r\gamma\frac{\partial^2a^1}{\partial\rho^2}\left(\rho_t,G_t,\mu_k\right)-\left(r\gamma\right)^2\left(\frac{\partial a^1}{\partial\rho}\left(\rho_t,G_t,\mu_k\right)\right)^2\right)\\
&\quad+r\gamma\left\langle\nabla_ga^1\left(\rho_t,G_t,\mu_k\right),\mu_t^g\left(\mu_k\right)\right\rangle+\delta\left(1-e^{-r\gamma\left(a^1\left(\rho_t,G_t,\mu_{1-k}\right)-a^1\left(\rho_t,G_t,\mu_k\right)\right)}\right)\qquad k=0,1,
\end{aligned}
$$

and

$$-\beta = -r + r\gamma\left(\frac{\log(r)}{\gamma} - ra^0(\rho_t, G_t, \mu_k) + \mu_\eta\right) - \frac{1}{2}(r\gamma)^2\sigma_\eta^2 \tag{24}$$
$$+ \frac{2\lambda}{\theta}\int_{\rho_t}^1 \left(1 - e^{-r\gamma\left(-P\left(\rho_t,\rho^1,G_t,\mu_k\right)+a^1(\rho_t,G_t,\mu_k)-a^0(\rho_t,G_t,\mu_k)\right)}\right)\mathrm{d}G_t\left(\rho^1, 1\right)$$
$$-\phi(\rho_t - \mu_i)\left(r\gamma\frac{\partial a^0}{\partial\rho}(\rho_t, G_t, \mu_k)\right) + \frac{1}{2}\sigma^2\left(r\gamma\frac{\partial^2 a^0}{\partial\rho^2}(\rho_t, G_t, \mu_k) - (r\gamma)^2\left(\frac{\partial a^0}{\partial\rho}(\rho_t, G_t, \mu_k)\right)^2\right)$$
$$+ r\gamma\left\langle\nabla_g a^0(\rho_t, G_t, \mu_k), \mu_t^g(\mu_k)\right\rangle + \delta\left(1 - e^{-r\gamma\left(a^0(\rho_t,G_t,\mu_{1-k})-a^0(\rho_t,G_t,\mu_k)\right)}\right) \qquad k = 0, 1,$$

which verifies the conjecture regarding the functional form of $J^j(w, \rho, G, \mu)$. The analytic formulation's numerical challenge lies in evaluating the high-dimensional object $\nabla_g a^j(\rho, G, \mu)$.

5.4.2 Probabilistic Formulation

The functional form of $J^j(w, \rho, G, \mu)$ indicates that it is sufficient to solve for $a^j(\rho, G, \mu)$. Hence, I will focus on the BSDE of a_t^j

$$\mathrm{d}a_t^j = \mu_t^{a,j}\mathrm{d}t + \sigma_t^{a,j}\mathrm{d}B_t^1 + \left(a^j(\rho_t, G_t, \mu_{1-k}) - a_t^j(\rho_t, G_t, \mu_k)\right)\mathrm{d}M_t^k$$

where $\mathrm{d}M_t^k$ denotes the Markov switch from μ_k to $\mu_{1-k}, k = 0, 1$. Next, I will derive $\mu_t^{a,j}$, which identifies the BSDE. For the owners' continuation value function, applying Ito's formula to to

$$J_t^1 = -\exp\left(-r\gamma\left(w_t + a_t^1\right)\right),$$

$$\mathrm{d}J_t^1 = r\gamma e^{-r\gamma\left(w_t+a_t^1\right)}(rw - c + \mu_D + \mu_\eta)\,\mathrm{d}t - \frac{1}{2}(r\gamma)^2 e^{-r\gamma\left(w_t+a_t^1\right)}\left(\sigma_D^2 + \sigma_\eta^2 + 2\rho\sigma_D\sigma_\eta + \left(\sigma_t^{a,1}\right)^2\right)\mathrm{d}t$$
$$+ \frac{2\lambda}{1-\theta}\int_{-1}^{\rho}\left(-e^{-r\gamma\left(w_t+P\left(\rho^0,\rho_t,G_t,\mu_t\right)+a^0(\rho_t,G_t,\mu_t)\right)} + e^{-r\gamma\left(w_t+a^1(\rho_t,G_t,\mu_t)\right)}\right)\mathrm{d}G_t\left(\rho^0, 0\right)\mathrm{d}t$$
$$+ r\gamma e^{-r\gamma\left(w_t+a_t^1\right)}\mu_t^{a,1}\mathrm{d}t + \cdots,$$

and the BSDE that J_t^1 follows is

$$\mathrm{d}J_t^1 = -\left(-e^{-\gamma c} + \beta e^{-r\gamma\left(w_t+a_t^1\right)}\right)\mathrm{d}t + \cdots,$$

where I omit the stochastic terms for both $\mathrm{d}J_t^1$ expressions. Since the drift terms of $\mathrm{d}J_t^1$ under the two representations above are supposed to be identical,

$$\begin{aligned}\mu_t^{a,1} = & \frac{r-\beta}{r\gamma} - \left(\frac{\log(r)}{\gamma} - ra_t^1 + \mu_D + \mu_\eta\right) + \frac{1}{2} r\gamma \left(\sigma_D^2 + \sigma_\eta^2 + 2\rho_t\sigma_D\sigma_\eta + \left(\sigma_t^{a,1}\right)^2\right) \\ & - \frac{2\lambda}{r\gamma(1-\theta)} \int_{-1}^{\rho} \left(1 - e^{-r\gamma\left(P\left(\rho^0,\rho_t,G_t,\mu_t\right) + a_t^0 - a_t^1\right)}\right) \mathrm{d}G_t\left(\rho^0, 0\right),\end{aligned} \tag{25}$$

where the optimal consumption is plugged in. Similarly, for non-owners

$$\begin{aligned}\mu_t^{a,0} = & \frac{r-\beta}{r\gamma} - \left(\frac{\log(r)}{\gamma} - ra_t^0 + \mu_\eta\right) + \frac{1}{2} r\gamma \left(\sigma_\eta^2 + \left(\sigma_t^{a,0}\right)^2\right) \\ & - \frac{2\lambda}{r\gamma\theta} \int_{\rho}^{1} \left(1 - e^{-r\gamma\left(-P\left(\rho_t,\rho^1,G_t,\mu_t\right) + a_t^1 - a_t^0\right)}\right) \mathrm{d}G_t\left(\rho^1, 1\right)\end{aligned} \tag{26}$$

5.5 Numerical Implementation

The numerical procedure, which primarily involves simulating forward- and backward-looking stochastic processes, can be divided into two parts: 1) updating the cross-sectional distribution of agents' types, denoted as G_t; and 2) updating forward-looking variables, represented by a_t^j, where $j = 0, 1$.

To approximate the infinite-dimensional $G_t(\cdot, j)$, I divide the state space of ρ_t, denoted by $[-1, 1]$, into $N+1$ intervals: $-1 = \rho^0 < \rho^1 < \cdots < \rho^N < \rho^{N+1} = 1$. The cumulative probability functions over the N points, i.e., $G_t^{n,j} = G_t(\rho^n, j)$, serve to approximate the cross-sectional distribution. The law of motion for $G_t^{n,j}$ is given by:

$$\begin{bmatrix} G_{t+\Delta}^{n,0} \\ G_{t+\Delta}^{n,1} \end{bmatrix} = \begin{bmatrix} G_t^{n,0} \\ G_t^{n,1} \end{bmatrix} + \begin{bmatrix} \phi(\rho_n - \mu_t)\, g_t(\rho_n, 0) + \frac{\sigma^2}{2}\frac{\partial}{\partial\rho}\left(g_t(\rho_n, 0)\right) - 2\lambda G_t^{n,0}\left(\theta - G_t^{n,1}\right) \\ \phi(\rho_n - \mu_t)\, g_t(\rho_n, 1) + \frac{\sigma^2}{2}\frac{\partial}{\partial\rho}\left(g_t(\rho_n, 1)\right) + 2\lambda G_t^{n,0}\left(\theta - G_t^{n,1}\right) \end{bmatrix} \Delta,$$

given the asset owners and non-owners' policy functions and condition (21). To solve for $a^j(\rho, G, \mu)$, I simulate the forward-looking process a_t^j according to its BSDE:

$$\mathrm{d}a_t^j = \mu_t^{a,j}\mathrm{d}t + \sigma_t^{a,j}\mathrm{d}B_t^1 + \left(a^j(\rho_t, G_t, \mu_{1-k}) - a_t^j(\rho_t, G_t, \mu_k)\right)\mathrm{d}M_t^k.$$

The key to uncovering $a^j(\rho, G, \mu)$ lies in calculating the drift term $\mu_t^{a,j}$, $j = 0, 1$, which, in turn, hinges on the transaction prices from random matching, as indicated by equations (25) and (26). Since there are infinitely many possible matches, I consider the outcome of the matching with the midpoint of an interval as the average outcome across all types of agents within that interval. For instance, for the matching between an owner of type ρ_t and a non-owner from the interval $[\rho^{n-1}, \rho^n)$, the matching outcome is considered between the owner and the non-owner of type $\hat{\rho}^n = 0.5\left(\rho^{n-1} + \rho^n\right)$, representing the average outcome across all non-owners within the interval. Given the owner's a_t^1, the corresponding $\mu_t^{a,1}$ is calculated as

$$\frac{r-\beta}{r\gamma} - \left(\frac{\log(r)}{\gamma} - ra_t^1 + \mu_D + \mu_\eta\right) + \frac{1}{2}r\gamma\left(\sigma_D^2 + \sigma_\eta^2 + 2\rho_t\sigma_D\sigma_\eta + \left(\sigma_t^{a,1}\right)^2\right)$$
$$- \frac{2\lambda}{r\gamma(1-\theta)} \sum_{n=1}^{n(\rho_t)-1} \left(1 - e^{-r\gamma\left(P(\hat{\rho}^n, \rho_t, G_t, \mu_t) + a_t^0 - a_t^1\right)}\right)\left(G_t^{n,0} - G_t^{n-1,0}\right),$$

where $n(\rho_t)$ is the minimum n such that $\hat{\rho}^n > \rho_t$.

5.6 Numerical Solution

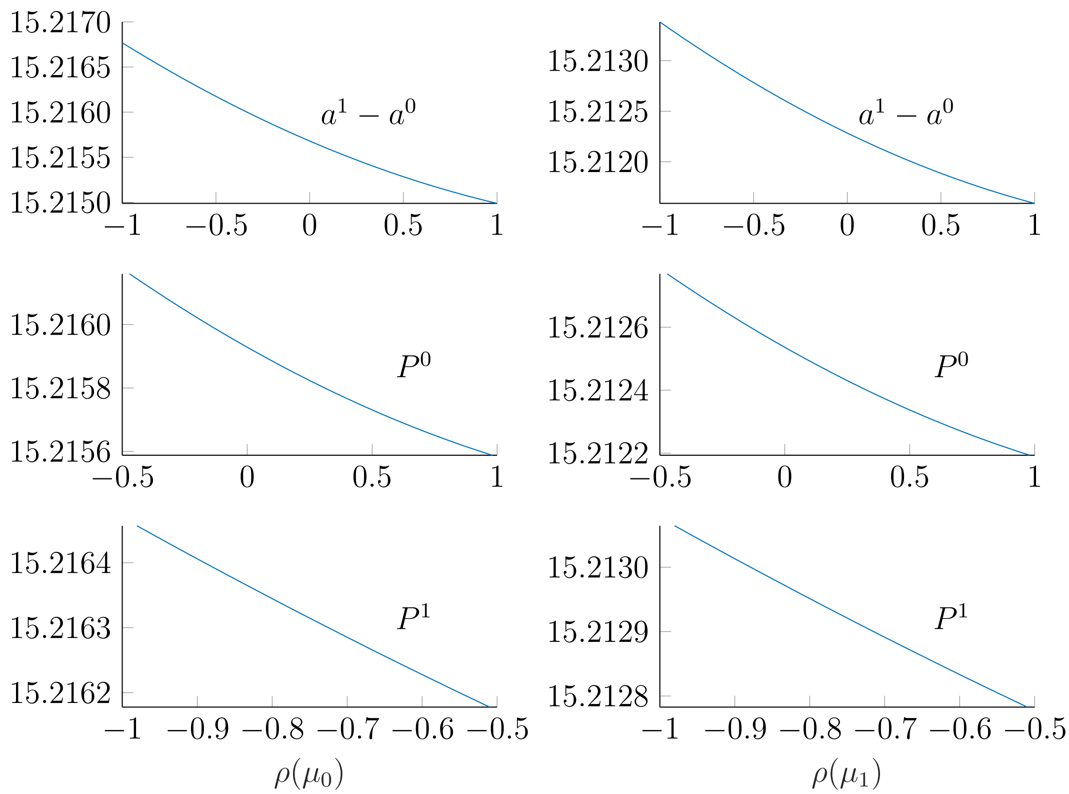

Figure 3: Bargaining Price

In this section, I present some basic properties of the model's numerical solution. The parameter values are chosen from Duffie et al. (2007), except for those I introduce: $\gamma = 200$, $\lambda = 625$, $\beta = 0.05$, $r = 0.05$, $q = 0.5$, $\mu_\eta = 0.5$, $\sigma_\eta = 0.5$, $\mu_D = 1$, $\sigma_D = 0.5$, and $\theta = 0.8$. Regarding idiosyncratic risk, $\phi = 0.25$ and $\sigma = 0.1$ are specified; for parameters related to aggregate risks, $\mu_0 = -0.5$, $\mu_1 = -0.25$, and $\delta = 10$ are set.

Figure 3 displays some features of the model, given the same cross-sectional distribution G_t under two possible aggregate states: μ_0 and μ_1. The x-axis of all plots represents an agent's type ρ. The two plots in the first row indicate that an owner's reservation value or a non-owner's willingness to pay decreases as the correlation between his or her endowment risk and the asset's risk increases. The comparison between the two plots reveals that asset owners' reservation values are higher when the asset payoff is less correlated with the average agents' endowment risks.

The plots in the second row of Figure 3 show that a non-owner of type $\rho_t = -0.5$ pays a lower price if he or she encounters an owner with a lower reservation value, that is, when the owner's type ρ is higher. Similarly, under the state $\mu_t = -0.25$, non-owners tend to pay lower prices to acquire the asset. The bottom two plots illustrate the relationship between the price at which an owner sells the asset and the buyers' types ρ. As expected, when buyers' willingness to pay increases — i.e., when a buyer's type ρ decreases — the seller can charge a higher price.

6 Final Remarks

The applications of the deep learning-based probabilistic approach go beyond dynamic heterogeneous-agent models. To the best of my knowledge, all dynamic economic models can be cast in the Forward-Backward SDE framework or its variants. Dynamic optimization with transaction costs and optimal stopping decisions can be characterized by reflected BSDEs. The set of equilibrium payoffs in dynamic games could be captured by set-valued BSDEs. Even the static combinatorial discrete choice can be approximated with dynamic optimization in a continuous-time setting.

When the number of state variables becomes a secondary concern for model construction, economists can treat parameters as time-invariant state variables, which paves the road for confronting models with data. Although the continuous-time setting currently is not the most favorable choice for quantitative models, it does have the advantage of accommodating data of various frequencies within a single model, which is still under-exploited in the literature.

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Appendix

A Algebra of the Finite Volume Method

In this section, I will first outline the law of motion for the boundary cases $G_t^{0,0}, G_t^{1,0}, G_t^{N,0}$ of type-0 households and $G_t^{1,1}, G_t^{N,0}$ of type-1 households. Second, I will illustrate the matrix operations of updating vectors

$$\mathbf{G_t^0} = \begin{bmatrix} G_t^{0,0} \\ G_t^{1,0} \\ G_t^{2,0} \\ \vdots \\ G_t^{N,0} \end{bmatrix}, \mathbf{G_t^1} = \begin{bmatrix} G_t^{1,1} \\ G_t^{2,1} \\ G_t^{3,1} \\ \vdots \\ G_t^{N,1} \end{bmatrix}.$$

The law of motion for $G_t^{0,0}$, i.e., the probability mass of type-0 at the boundary $x_0 = -\underline{b}$, is

$$\begin{aligned} G_{t+\Delta}^{0,0} - G_t^{0,0} = & -\mu_t\left(x_0, 0\right)\Delta\left(\frac{G_t^{1,0}}{x_1 - x_0}\mathbf{1}\left\{\mu_t\left(x_0, 0\right) < 0\right\} + G_t^{0,0}\mathbf{1}\left\{\mu_t\left(x_0, 0\right) \geq 0\right\}\right) \\ & - \sigma k_t\left(x_0, 0\right)\Delta W_t\left(\frac{G_t^{1,0}}{x_1 - x_0}\mathbf{1}\left\{k_t\left(x_0, 0\right)\Delta W_t < 0\right\} + G_t^{0,0}\mathbf{1}\left\{k_t\left(x_0, 0\right)\Delta W_t \geq 0\right\}\right) \\ & + \sigma^2\left[\frac{k_t^2\left(\hat{x}_1, 0\right)G_t^{1,0}}{\left(x_1 - x_0\right)^2} - \frac{k_t^2\left(x_0, 0\right)G_t^{0,0}}{x_1 - x_0}\right]\Delta - \lambda^0 G_t^{0,0}\Delta, \end{aligned}$$

where $\nabla_x\left\{k_t^2\left(x_0, 0\right)g_t\left(x_0, 0\right)\right\}$ is evaluated according to the following procedure.

$$\begin{aligned} \nabla_x\left\{k_t^2\left(x_0, 0\right)g_t\left(x_0, 0\right)\right\} = & \frac{k_t^2\left(0.5\left(x_1 + x_0\right), 0\right)g_t\left(0.5\left(x_1 + x_0\right), 0\right) - k_t^2\left(x_0, 0\right)G_t^{0,0}}{0.5\left(x_1 - x_0\right)} \\ = & \frac{2k_t^2\left(\hat{x}_1, 0\right)G_t^{1,0}}{\left(x_1 - x_0\right)^2} - \frac{2k_t^2\left(x_0, 0\right)G_t^{0,0}}{x_1 - x_0} \end{aligned}$$

The law of motion for $G_t^{1,0}$, the probability of type-0 households over $(x_0, x_1]$, is

$$
\begin{aligned}
& G_{t+\Delta}^{1,0} - G_t^{1,0} \\
& = -\mu_t(x_1,0)\,\Delta\left(\frac{G_t^{2,0}}{x_2-x_1}\mathbf{1}\{\mu_t(x_1,0)<0\} + \frac{G_t^{1,0}}{x_1-x_0}\mathbf{1}\{\mu_t(x_1,0)\geq 0\}\right) \\
& \quad + \mu_t(x_0,0)\,\Delta\left(\frac{G_t^{1,0}}{x_1-x_0}\mathbf{1}\{\mu_t(x_0,0)<0\} + G_t^{0,0}\mathbf{1}\{\mu_t(x_0,0)\geq 0\}\right) \\
& \quad - \sigma k_t(x_1,0)\,\Delta W_t\left(\frac{G_t^{2,0}}{x_2-x_1}\mathbf{1}\{k_t(x_1,0)\,\Delta W_t<0\} + \frac{G_t^{1,0}}{x_1-x_0}\mathbf{1}\{k_t(x_1,0)\,\Delta W_t\geq 0\}\right) \\
& \quad + \sigma k_t(x_0,0)\,\Delta W_t\left(\frac{G_t^{1,0}}{x_1-x_0}\mathbf{1}\{k_t(x_0,0)\,\Delta W_t<0\} + G_t^{0,0}\mathbf{1}\{k_t(x_0,0)\,\Delta W_t\geq 0\}\right) \\
& \quad + \sigma^2\left[\frac{k_t^2(\hat{x}_2,0)\,G_t^{2,0}}{(x_2-x_1)(x_2-x_0)} - \frac{k_t^2(\hat{x}_1,0)\,G_t^{1,0}}{(x_1-x_0)(x_2-x_0)}\right]\Delta \\
& \quad - \sigma^2\left[\frac{k_t^2(\hat{x}_1,0)\,G_t^{1,0}}{(x_1-x_0)^2} - \frac{k_t^2(x_0,0)\,G_t^{0,0}}{x_1-x_0}\right]\Delta - \lambda^0 G_t^{1,0}\Delta + \lambda^1 G_t^{1,1}\Delta.
\end{aligned}
$$

The law of motion for $G_t^{1,1}$, the probability of type-1 households over $(x_0, x_1]$, is

$$
\begin{aligned}
& G_{t+\Delta}^{1,1} - G_t^{1,1} \\
& = -\mu_t(x_1,1)\,\Delta\left(\frac{G_t^{2,1}}{x_2-x_1}\mathbf{1}\{\mu_t(x_1,1)<0\} + \frac{G_t^{1,1}}{x_1-x_0}\mathbf{1}\{\mu_t(x_1,1)\geq 0\}\right) \\
& \quad - \sigma k_t(x_1,1)\,\Delta W_t\left(\frac{G_t^{2,1}}{x_2-x_1}\mathbf{1}\{k_t(x_1,1)\,\Delta W_t<0\} + \frac{G_t^{1,1}}{x_1-x_0}\mathbf{1}\{k_t(x_1,1)\,\Delta W_t\geq 0\}\right) \\
& \quad + \sigma^2\left[\frac{k_t^2(\hat{x}_2,1)\,G_t^{2,1}}{(x_2-x_1)(x_2-x_0)} - \frac{k_t^2(\hat{x}_1,1)\,G_t^{1,1}}{(x_1-x_0)(x_2-x_0)}\right]\Delta - \lambda^1 G_t^{1,1}\Delta + \lambda^0\left(G_t^{0,0} + G_t^{1,0}\right)\Delta.
\end{aligned}
$$

The law of motion for $G_t^{N,j}, j = 0, 1$, the probability of both types of households over $(x_{N-1}, x_N]$, is

$$
\begin{aligned}
& G_{t+\Delta}^{N,j} - G_t^{N,j} \\
& = \mu_t(x_{N-1},j)\,\Delta\left(\frac{G_t^{N,j}}{x_N-x_{N-1}}\mathbf{1}\{\mu_t(x_{N-1},j)<0\} + \frac{G_t^{N-1,j}}{x_{N-1}-x_{N-2}}\mathbf{1}\{\mu_t(x_{N-1},j)\geq 0\}\right) \\
& \quad + \sigma k_t(x_{N-1},j)\,\Delta W_t\left(\frac{G_t^{N,j}}{x_N-x_{N-1}}\mathbf{1}\{k_t(x_{N-1},j)\,\Delta W_t<0\} + \frac{G_t^{N-1,j}}{x_{N-1}-x_{N-2}}\mathbf{1}\{k_t(x_{N-1},j)\,\Delta W_t\geq 0\}\right) \\
& \quad - \sigma^2\left[\frac{k_t^2(\hat{x}_N,j)\,G_t^{N,j}}{(x_N-x_{N-1})(x_N-x_{N-2})} - \frac{k_t^2(\hat{x}_{N-1},j)\,G_t^{N-1,j}}{(x_{N-1}-x_{N-2})(x_N-x_{N-2})}\right]\Delta - \lambda^j G_t^{N,j}\Delta + \lambda^{1-j} G_t^{N,1-j}\Delta.
\end{aligned}
$$

Next, I rewrite the law of motions for these probabilities in a compact vector-matrix format to fully take advantage of GPU's high performance on matrix operations. Let $\mu_{n,j}$ denote $\mu(x_n, j)$, $\mu^+_{n,j}$ denote $\max\{0, \mu_{n,i}\}$, $\mu^-_{n,i}$ denote $\min\{0, \mu_{n,i}\}$, $k_{n,j}$ denote $k_t(x_n, j)$, and $\hat{k}_{n,j}$ denote $k_t(\hat{x}_n, j)$.

$$G^{0,0}_{t+\Delta} = G^{0,0}_t + \left(-M^{0,0}_+\right) G^{0,0}_t + \left(-M^{0,0}_-\right) G^{1,0}_t - \lambda^0 \Delta G^{0,0}_t, \text{ where}$$

$$M^{0,0}_+ = \mu^+_{0,0}\Delta + \{k_{0,0}\Delta W_t\}^+ \sigma + \frac{k^2_{0,0}\sigma^2\Delta}{x_1 - x_0}$$

$$M^{0,0}_- = \frac{\mu^-_{0,0}\Delta}{x_1 - x_0} + \frac{\{k_{0,0}\Delta W_t\}^-}{x_1 - x_0}\sigma - \frac{\hat{k}^2_{1,0}\sigma^2\Delta}{(x_1 - x_0)^2}$$

$$G^{1,0}_{t+\Delta} = G^{1,0}_t + \left(M^{0,0}_+\right) G^{0,0}_t + \left(-M^{1,0}_+ + M^{0,0}_-\right) G^{1,0}_t + \left(-M^{1,0}_-\right) G^{2,0}_t - \lambda^0 \Delta G^{1,0}_t + \lambda^1 \Delta G^{1,1}_t, \text{ where}$$

$$M^{1,0}_+ = \frac{\mu^+_{1,0}}{x_1 - x_0}\Delta + \frac{\{k_{1,0}\Delta W_t\}^+}{x_1 - x_0}\sigma + \frac{\hat{k}^2_{1,0}\sigma^2\Delta}{(x_1 - x_0)(x_2 - x_0)}$$

$$M^{1,0}_- = \frac{\mu^-_{1,0}}{x_2 - x_1}\Delta + \frac{\{k_{1,0}\Delta W_t\}^-}{x_2 - x_1}\sigma - \frac{\hat{k}^2_{2,0}\sigma^2\Delta}{(x_2 - x_1)(x_2 - x_0)}$$

$$G^{1,1}_{t+\Delta} = G^{1,1}_t + \left(-M^{1,1}_+\right) G^{1,1}_t + \left(-M^{1,1}_-\right) G^{2,1}_t - \lambda^1 \Delta G^{1,1}_t + \lambda^0 \Delta \left(G^{0,0}_t + G^{1,0}_t\right), \text{ where}$$

$$M^{1,1}_+ = \frac{\mu^+_{1,1}}{x_1 - x_0}\Delta + \frac{\{k_{1,1}\Delta W_t\}^+}{x_1 - x_0}\sigma + \frac{\hat{k}^2_{1,1}\sigma^2\Delta}{(x_1 - x_0)(x_2 - x_0)}$$

$$M^{1,1}_- = \frac{\mu^-_{1,1}}{x_2 - x_1}\Delta + \frac{\{k_{1,1}\Delta W_t\}^-}{x_2 - x_1}\sigma - \frac{\hat{k}^2_{2,1}\sigma^2\Delta}{(x_2 - x_1)(x_2 - x_0)}$$

$$G^{n,j}_{t+\Delta} = G^{n,j}_t + \left(M^{n-1,j}_+\right) G^{n-1,j}_t + \left(-M^{n,j}_+ + M^{n-1,j}_-\right) G^{n,j}_t + \left(-M^{n,j}_-\right) G^{n+1,j}_t$$
$$- \lambda^j \Delta G^{n,j}_t + \lambda^{|j-1|} \Delta G^{n,|j-1|}_t, \text{ where}$$

$$M^{n,j}_+ = \frac{\mu^+_{n,j}}{x_n - x_{n-1}}\Delta + \frac{\{k_{n,j}\Delta W_t\}^+}{x_n - x_{n-1}}\sigma + \frac{\hat{k}^2_{n,j}\sigma^2\Delta}{(x_n - x_{n-1})(x_{n+1} - x_{n-1})}$$

$$M^{n,j}_- = \frac{\mu^-_{n,j}}{x_{n+1} - x_n}\Delta + \frac{\{k_{n,j}\Delta W_t\}^-}{x_{n+1} - x_n}\sigma - \frac{\hat{k}^2_{n+1,j}\sigma^2\Delta}{(x_{n+1} - x_n)(x_{n+1} - x_{n-1})}$$

$$G^{N,j}_{t+\Delta} = G^{N,j}_t + \left(M^{N-1,j}_+\right) G^{N-1,j}_t + \left(M^{N-1,j}_-\right) G^{N,j}_t - \lambda^j \Delta G^{N,j}_t + \lambda^{1-j} \Delta G^{N,1-j}_t$$

To group the coefficients of $G^{n,j}_t$,

$$\mathbf{M^0_+} \equiv \begin{bmatrix} M^{0,0}_+ \\ M^{1,0}_+ \\ \vdots \\ M^{n,0}_+ \\ \vdots \\ M^{N-1,0}_+ \end{bmatrix}, \mathbf{M^0_-} \equiv \begin{bmatrix} M^{0,0}_- \\ M^{1,0}_- \\ \vdots \\ M^{n,0}_- \\ \vdots \\ M^{N-1,0}_- \end{bmatrix}, \mathbf{M^1_+} \equiv \begin{bmatrix} M^{1,1}_+ \\ \vdots \\ M^{n,1}_+ \\ \vdots \\ M^{N-1,1}_+ \end{bmatrix}, \mathbf{M^1_-} \equiv \begin{bmatrix} M^{1,1}_- \\ \vdots \\ M^{n,1}_- \\ \vdots \\ M^{N-1,1}_- \end{bmatrix},$$

the update of $G_t^{n,j}$ has a compact form

$$\mathbf{G^0_{t+\Delta}} = \mathbf{G^0_t} + \begin{pmatrix} 0 \\ \mathbf{M^0_+} \end{pmatrix} \odot \mathbf{G^0_{t+}} + \left(- \begin{pmatrix} \mathbf{M^0_+} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \mathbf{M^0_-} \end{pmatrix} - \lambda^0 \Delta \right) \odot \mathbf{G^0_t}$$
$$- \begin{pmatrix} \mathbf{M^0_-} \\ 0 \end{pmatrix} \odot \mathbf{G^0_{t-}} + \lambda^1 \Delta \begin{bmatrix} 0 \\ \mathbf{G^1_t} \end{bmatrix},$$

$$\mathbf{G^1_{t+\Delta}} = G^1_t + \begin{pmatrix} 0 \\ \mathbf{M^1_+} \end{pmatrix} \odot \mathbf{G^1_{t+}} + \left(- \begin{pmatrix} \mathbf{M^1_+} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \mathbf{M^1_-} \end{pmatrix} - \lambda^1 \Delta \right) \odot \mathbf{G^1_t}$$
$$- \begin{pmatrix} \mathbf{M^1_-} \\ 0 \end{pmatrix} \odot \mathbf{G^1_{t-}} + \lambda^0 \Delta \begin{bmatrix} G_t^{0,1} + G_t^{1,0} \\ G_t^{2,0} \\ \vdots \\ G_t^{N,0} \end{bmatrix},$$

where $\odot$ denotes element-wise product, and

$$\mathbf{G^0_{t+}} \equiv \begin{bmatrix} 0 \\ G_t^{0,0} \\ G_t^{1,0} \\ G_t^{2,0} \\ \vdots \\ G_t^{N-2,0} \\ G_t^{N-1,0} \end{bmatrix}, \mathbf{G^0_{t-}} \equiv \begin{bmatrix} G_t^{1,0} \\ G_t^{2,0} \\ G_t^{3,0} \\ \vdots \\ G_t^{N-1,0} \\ G_t^{N,0} \\ 0 \end{bmatrix}, \mathbf{G^1_{t+}} \equiv \begin{bmatrix} 0 \\ G_t^{1,1} \\ G_t^{2,1} \\ G_t^{3,1} \\ \vdots \\ G_t^{N-2,1} \\ G_t^{N-1,1} \end{bmatrix}, \mathbf{G^1_{t-}} \equiv \begin{bmatrix} G_t^{2,1} \\ G_t^{3,1} \\ G_t^{4,1} \\ \vdots \\ G_t^{N-1,1} \\ G_t^{N,1} \\ 0 \end{bmatrix}.$$

B Krusell and Smith (1997) with Asset Pricing

In Section 4, the capital adjustment cost is zero, which implies that households can always convert one unit of final consumption goods into one unit of physical capital, and vice versa. With the consumption good as the numeraire, the price of physical capital is one, due to the no-arbitrage condition. In this section, I introduce technological illiquidity to physical capital, transforming it into durable goods. Consequently, its price, q_t, becomes a forward-looking stochastic process.

B.1 Model

Building on the model considered in Section 4, I introduce a capital adjustment cost. A household holding k_t efficiency units of physical capital can convert ιk_t units of consumption goods into $\Phi(\iota)k_t$ units of physical capital, and vice versa, where $\Phi(\iota) = \frac{1}{\psi}\ln(\psi\iota+1)$. Suppose the market price of physical capital is q_t, i.e., a household can exchange 1 unit of consumption goods for $\frac{1}{q_t}$ units of physical capital in the market. Since the opportunity cost of capital production is $\frac{1}{q_t}$ (in units of physical capital), the household's optimal investment rate, ι_t, satisfies

$$\begin{aligned}\iota_t &= \arg\max_{\iota} \Phi(\iota) - \frac{\iota}{q_t},\\ &= \frac{q_t - 1}{\psi}.\end{aligned}$$

Given the optimal investment rate ι_t, the capital stock held by a household follows

$$\frac{\mathrm{d}k_t}{k_t} = (\Phi(\iota_t) - \delta)\,\mathrm{d}t + \sigma \mathrm{d}W_t.$$

The price of physical capital, q_t, and its dynamics affect the demand and supply of both consumption goods and physical capital goods in the current and future markets. The current capital price determines the conversion of final goods to physical capital. The expected return on capital and its risks influence households' portfolio choices and their demands for the risky asset. Hence, the asset pricing of physical capital is an intricate subject. At this moment, I conjecture that q_t follows a BSDE

$$\frac{\mathrm{d}q_t}{q_t} = \mu_t^q \mathrm{d}t + \sigma_t^q \mathrm{d}W_t.$$

Given the law of motion for q_t, the return of holding physical capital is

$$\underbrace{\left(\frac{r_t^k - \iota_t}{q_t} + \Phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q\right)}_{\equiv R_t} \mathrm{d}t + (\sigma + \sigma_t^q)\mathrm{d}W_t.$$

Note that the financial market is incomplete; there is endogenous risk as households who take leverage to hold physical capital are disproportionately more exposed to aggregate risks, which is captured by σ_t^q.

Suppose the dollar amount invested in physical capital is κ_t, the consumption rate is c_t, and the risk-free rate is r_t. Then, the law of motion for a household's wealth, x_t, is given by

$$\mathrm{d}x_t = (r_t x_t + (R_t - r_t)\,\kappa_t + \epsilon_t w\,(K_t, L_t) + (1-\epsilon_t)\,g - c_t)\,\mathrm{d}t + (\sigma + \sigma_t^q)\,\kappa_t \mathrm{d}W_t.$$

Note that $\kappa_t = q_t k_t$ if k_t represents the efficiency units of physical capital held by the household. There are three differences from the physical capital holding described in Section 4: (1) the production of capital $-\frac{\iota_t}{q_t}+\Phi(\iota_t)$, (2) capital gain $\mu_t^q+\sigma\sigma_t^q$, and (3) the endogenous risk σ_t^q. As in Section 4, households are not allowed to short physical capital, i.e.,

$$\kappa_t \geq \underline{\kappa};$$

and there is an upper bound for borrowing:

$$\kappa_t - x_t \leq \underline{b}.$$

These two restrictions imply a constraint on the net worth of a household: $x_t \geq -\underline{b}$.

B.2 Markov Equilibrium

As in Section 4, I will solve for the Markov equilibrium of the model with the aggregate state variable $G_t(\cdot, j), j = 0, 1$, representing the distribution of households' wealth x_t. In this section, I will first outline a household's optimal dynamic decisions, then discuss the market clearing conditions, as well as the law of motion for $G_t(\cdot, j)$.

The Hamiltonian function for an individual household's dynamic optimization is

$$H(t, x, c, \kappa, y, z) = \ln(c) + (r_t x + (R_t - r_t)\kappa + \epsilon_t \underline{l} w(K_t, L_t) + (1-\epsilon_t)g - c)\, y + (\sigma + \sigma_t^q)\kappa z,$$

when the state constraint $x_t \geq -\underline{b}$ is not binding. The co-state variable of a type-j household is denoted by $Y_t^j, j = 0, 1$, which follows the same BSDE depicted by equation (14). The first-order conditions are

$$c_t^j = \frac{1}{Y_t^j}$$

$$\frac{r_t^k - \iota_t}{q_t} + \Phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q - r_t = -\frac{(\sigma + \sigma_t^q) Z_t^{Y,j}}{Y_t^j} \quad \text{if} \quad \underline{\kappa} < \kappa < x_t + \underline{b}$$

Note the Euler equation with respect to κ_t is more involved than the corresponding one in Section 4 because of capital production, capital gain, and the endogenous risk. Next, I characterize the portfolio choice κ_t. Similar to Section 4, I apply Ito's formula and decompose the volatility term $Z_t^{Y,j}$ into two components: $\frac{\partial Y^j}{\partial x}(x_t, G_t)(\sigma + \sigma_t^q)\kappa_t$, the contributions of x_t, and $\tilde{Z}_t^{Y,j}$, the contribution of G_t

$$Z_t^{Y,j} = \frac{\partial Y^j}{\partial x}(x_t, G_t)(\sigma + \sigma_t^q)\kappa_t + \tilde{Z}_t^{Y,j}.$$

Then, when the excess return $R_t - r_t$ is sufficiently large,

$$R_t - r_t > -\frac{(\sigma + \sigma_t^q)}{Y_t^j} \left(\frac{\partial Y^j}{\partial x}(x_t, G_t)(\sigma + \sigma^q)(x_t + \underline{b}) + \tilde{Z}_t^{Y,j} \right),$$

the optimal $\kappa_t = x_t + \underline{b}$. If the excess return is too low,

$$R_t - r_t < -\frac{(\sigma + \sigma_t^q)}{Y_t^j} \left(\frac{\partial Y^j}{\partial x}(n_t, G_t)(\sigma + \sigma^q)\underline{\kappa} + \tilde{Z}_t^{Y,j} \right),$$

the optimal $\kappa_t = \underline{\kappa}$. In between, the optimal κ_t satisfies

$$\kappa_t^j = -\frac{Y_t^j(R_t - r_t) + (\sigma + \sigma_t^q)\tilde{Z}_t^{Y,j}}{(\sigma + \sigma_t^q)^2 \frac{\partial Y^j}{\partial x}(x_t, G_t)}.$$

Overall, the policy function of a household's portfolio choice $\kappa(x_t, G_t)$ satisfies

$$\kappa^j(x_t, G_t) = \min\left\{\max\left\{\tilde{\kappa}^j(x_t, G_t), \underline{\kappa}\right\}, x_t + \underline{b}\right\}, \text{ where}$$
$$\tilde{\kappa}^j(x_t, G_t) \equiv -\frac{Y_t^j(R_t - r_t) + (\sigma + \sigma_t^q)\tilde{Z}_t^{Y,j}}{(\sigma + \sigma_t^q)^2 \frac{\partial Y^j}{\partial x}(x_t, G_t)}.$$

With the optimal consumption and portfolio choices, a household's value function $V^j(x_t, G_t)$, $j = 0, 1$, is characterized by the same BSDE as equation (16), and Y_t^j is still equal to $\nabla_x V^j(x_t, G_t)$.

Given a household's demand for risk-free bonds $\kappa^j(x_t, G_t) - x_t$, the condition of zero net bond supply is

$$\sum_{j=0,1} \int (\kappa^j(x, G_t) - x) \mathrm{d}G_t(x, j) = 0.$$

The aggregate stock of physical capital satisfies

$$q_t K_t = \sum_{j=0,1} \int \kappa^j(x, G_t) \mathrm{d}G_t(x, j) = \sum_{j=0,1} \int x \mathrm{d}G_t(x, j).$$

In this model, the bond market clearing condition depends not only on the risk-free rate r_t but also on the instant return and risk of physical capital, μ_t^q and σ_t^q. The consumption goods market clearing condition is

$$\sum_{j=0,1} \int c^j(x, G_t) \mathrm{d}G_t(x, j) + \frac{q_t - 1}{\psi} K_t = z K_t^\alpha L_t^{1-\alpha},$$

where $\frac{q_t - 1}{\psi} K_t$ represents the aggregate investment expenditure. In the model of Section 4, the consumption goods market clears automatically as households can convert consumption goods into physical capital on a one-for-one basis and vice versa. However, in the current model, the investment expenditure depends on the endogenous asset price q_t.

The stochastic Kolmogorov Forward Equation, which captures the dynamics of the aggregate state variable G_t or its density function $g_t(x, j), j = 0, 1$, is given by:

$$\begin{aligned} \mathrm{d}g_t(x, j) = & - \nabla_x(\mu_t(x, j)g_t(x, j))\mathrm{d}t - \frac{\partial}{\partial x}\left((\sigma + \sigma_t^q)\kappa_t(x, j)\mathrm{d}W_t\right) g_t(x, j) \\ & + \frac{1}{2}\nabla_{xx}\left\{(\sigma + \sigma_t^q)^2\kappa_t^2(x, j)g_t(x, j)\right\}\mathrm{d}t - \lambda^j g_t(x, j)\mathrm{d}t + \lambda^{1-j} g_t(x, 1 - j)\mathrm{d}t. \end{aligned}$$

The term $\mu_t(x, j)$ and $(\sigma + \sigma_t^q)\kappa_t(x, j)$ represent the drift and volatility terms of an individual household with wealth x_t, respectively.

B.3 Numerical Implementation

There are two major differences between the numerical procedure for solving the current model and the one for the model considered in Section 4, whose detailed numerical steps are displayed in Appendix C. First, since the price of physical capital is forward-looking, the algorithm ensures that its corresponding BSDE is upheld. As a by-product of solving the BSDE of q_t, I need to approximate $\sigma^q(\mathbf{G}_t; \Theta)$ with a neural network. Second, I directly capture the risk-free rate r_t as an approximated function of the aggregate state variable, i.e., $r(\mathbf{G}_t; \Theta)$, instead of solving for r_t via the consumption goods market clearing condition. Next, I outline the construction of the loss function concerning q_t's BSDE.

Given the aggregate state variable $G_t(\cdot, j)$ and the forward-looking variable $Y^j(x, G_t)$ approximated by a neural network, the market clearing condition for consumption goods yields the price of physical capital q_t. Let X_t denote:

$$\sum_{j=0,1} \int x \mathrm{d}G_t(x, j).$$

Then,

$$\begin{aligned} \sum_{j=0,1} \int \left(Y^j\left(x, G_t\right)\right)^{-1} \mathrm{d}G_t\left(x, j\right) + \frac{q_t - 1}{\psi}\frac{X_t}{q_t} &= z\left(\frac{X_t}{q_t}\right)^{\alpha} L_t^{1-\alpha}, \\ \sum_{j=0,1} \int \left(Y^j\left(x, G_t\right)\right)^{-1} \mathrm{d}G_t\left(x, j\right) + \frac{X_t}{\psi} &= \frac{1}{\psi}\frac{X_t}{q_t} + z\left(\frac{X_t}{q_t}\right)^{\alpha} L_t^{1-\alpha}. \end{aligned}$$

In particular, I use the Newton method to solve for the root of the above nonlinear equation with respect to q_t. Let $q(G_t;\Theta)$ denote the mapping from G_t to q_t.

Starting with the initial q_0 implied by $q(\mathbf{G}_0;\Theta)$, I calculate the dynamics of q_t with its BSDE:

$$q_{i+1} = q_i + q_i(\mu_i^q \Delta + \sigma_i^q w_i),$$

where σ_i^q is given by the neural network $\sigma^q(\mathbf{G}_i;\Theta)$, and μ_i^q is solved for by clearing the bond market. Given that the households' demand for physical capital increases with μ_t^q, the value of μ_i^q is found using the bisection method. The finite volume method updates $\mathbf{G}_i$ according to the KFE and the policy function. The BSDE loss with respect to q_t is

$$\sum_{i=1}^{\mathbf{I}} (q_i - q(\mathbf{G}_i;\Theta))^2.$$

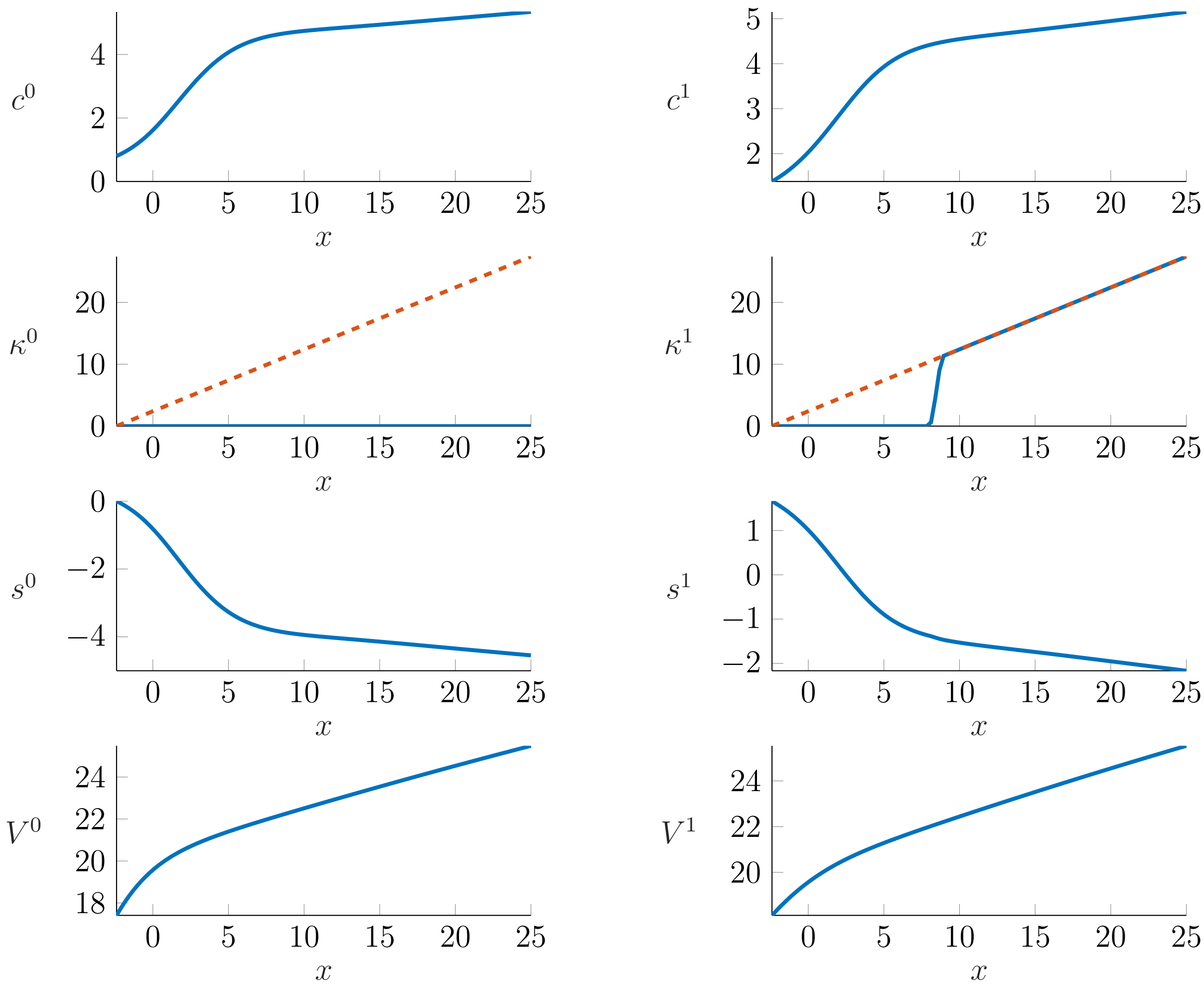

Figure 4: Policy and Value Functions

Figure 4 shows the properties of a few endogenous variables given an aggregate state. The average price of the durable asset is 1.249, the average volatility of the asset price is 0.5%, and the average risk-free rate is 3.3%. Due to idiosyncratic risk and market incompleteness, employed households choose to hold risky assets when their wealth level is sufficiently high, while unemployed households opt not to hold risky assets.

C Algorithm: The Construction of Loss Function

This algorithm section covers the details of how to construct the loss function for a deep learning package to minimize. Subsection C.1 considers the continuous-time version of Krusell and Smith (1997), and Subsection C.2 is dedicated to the example of the search and matching model.

C.1 Loss Function of Krusell and Smith (1997) in Section 4

C.1.1 Generating Initial Wealth Distribution

I generate the initial wealth distribution of households based on Beta distribution, Beta (α, β). Taking M as the number of economies, I select M random draws of α_0 and α_1 uniformly from $[1.5, 2.5]$, and M random draws of β_0 and β_1 uniformly from $[2, 4]$. Given the density functions of two Beta distributions $g_0 \sim \beta(\alpha_0, \beta_0)$ and $g_1 \sim \beta(\alpha_1, \beta_1)$, I construct

$$G^{n,0} = \int_{x_{n-1}}^{x_n} g_0(x)\,\mathrm{d}x, G^{n,1} = \int_{x_{n-1}}^{x_n} g_1(x)\,\mathrm{d}x, \quad n = 1, \cdots, N.$$

Since the distribution of type-0 households has a mass point at $x = -\underline{b}$, I use the maximum point of the series $G^{n,0}$, denoted by $(n_{\max}, G^{n_{\max},0})$ and construct the mass point $G^{0,0}$ at $x_0 = 0$ as $G^{0,0} = 0.8G^{n_{\max},0}$. Accordingly, the overall $G^{n,0}$'s are modified according to the following scheme: 1) for the n-th interval, $1 \leq n < n_{\max}$, increase the probability over it as $G^{n,0} = G^{n,0} + 0.3\left(G^{n_{\max},0} - G^{n,0}\right)$; 2) for the n-th interval, $n \geq n_{\max}$, do no adjustment. Given the adjustment, I rescale the series $G^{n,0}$ to have their sum equal one,

$$\frac{G^{n,0}}{\sum_{n=0}^{N} G^{n,0}} \Rightarrow G^{n,0}, n = 0, \cdots, N.$$

In the end, I adjust $G^{n,0}$ and $G^{n,1}$

$$\frac{\lambda_1}{\lambda_0 + \lambda_1} G^{n,0} \Rightarrow G^{n,0}, \quad \frac{\lambda_0}{\lambda_0 + \lambda_1} G^{n,1} \Rightarrow G^{n,1}$$

so that the proportion of type-0 households is at the stationary level ${}^{\lambda_1}/\left(\lambda_0+\lambda_1\right)$.

C.1.2 Initializing Exogenous Shocks

In a heterogeneous-agent model, I need to simulate both aggregate Brownian shocks and idiosyncratic employment shocks. Given a simulated path $\left\{W_{i,m}, i = 0, \cdots, \mathbf{I}-1\right\}$, let $w_{i,m} \equiv W_{i+1,m} - W_{i,m}$, where m is the economy index. Along the time path of an economy, I always focus on households of both types at certain fixed wealth levels, i.e., $x_h, h = 0, 1, \cdots, H$, in any period. In other words, within period i I consider the dynamics of household h from the beginning of the period to the end. And, in Period $h + 1$, I will go back to a household whose wealth level is x_h. Since there are two types of households, the sample size of overall households in an economy is $2H$. The idiosyncratic shock to household h of economy m in period i is denoted as $I^j_{i,h,m}$ if she is of type-j, $j = 0, 1$. $I^j_{i,h,m}$ is set to 1 with probability $\Delta\lambda^j$, where Δ is the length of a period; otherwise, $I^j_{i,h,m} = 0$. When $I^0_{i,h,m} = 1$, household h of type-0 switches to type-1 in period i.

C.1.3 Computation within Period i

1. Calculate aggregate capital supply K_i

$$K_i = \sum_{n=0}^{N} \hat{x}_n G_i^{n,0} + \sum_{n=1}^{N} \hat{x}_n G_i^{n,1}$$

   Recall that $x_n, n = 0, \cdots, N$ are points that discretizing the individual state space of the finite volume method, $\hat{x}_0 = x_0$, and $\hat{x}_n = 0.5\left(x_n + x_{n-1}\right)$ for $n \geq 1$. Given K_i, calculate the labor wage

$$w_i = (1-\alpha)\, z \left(\frac{K_i}{L}\right)^{\alpha}$$

   and the return to physical capital

$$r_i^K = \alpha z \left(\frac{K_i}{L}\right)^{\alpha-1}$$

2. Given inputs (x_n, G_i), $(\hat{x}_n, G_i)$, (x_h, G_i), the network generate outputs $Y^j_{n,i}$, $\tilde{Z}^{Y,i}_{n,t}$, $Y_{i,\hat{n}}$, $\tilde{Z}^{Y,j}_{i,\hat{n}}$, and $Y^j_{h,i}$, $V^j_{h,i}$, $\tilde{Z}^{Y,j}_{h,i}$, $U^{Y,j}_{h,i}$, $Z^{V,j}_{h,i}$, $U^{V,j}_{h,i}$.
3. Use the finite difference method to evaluate $\nabla_x V\left(x, G\right)$ along x_h

- for $h = 0$, use forward difference

$$\nabla_x V^j_{i,0} = \frac{V^j_{i,1} - V^j_{i,0}}{x_1 - x_0}$$

- for $1 \leq h \leq H - 1$, use 3 points difference:

$$\begin{aligned}\nabla_x V^j_{i,h} =& V^j_{i,h-1} \frac{x_h - x_{h+1}}{(x_{h-1} - x_h)(x_{h-1} - x_{h+1})} + V^j_{i,h} \frac{2x_h - x_{h-1} - x_{h+1}}{(x_h - x_{h-1})(x_h - x_{h+1})} \\ &+ V^i_{i,h+1} \frac{x_h - x_{h-1}}{(x_{h+1} - x_{h-1})(x_{h+1} - x_h)}\end{aligned}$$

- for $n = N$, use backward difference

$$\nabla_x V^j_{i,N} = \frac{V^j_{i,H} - V^j_{i,H-1}}{x_H - x_{H-1}}$$

Define the loss that corresponds to condition $Y^j_t = \nabla_x V^j(x_t, G_t)$

$$\text{Loss}_{\text{FD}} = \frac{1}{2H} \sum_{h=0}^{H} \sum_{j=0}^{1} \left(Y^j_{i,h} - \nabla_x V^j_{i,h}\right)^2.$$

4. Calculate

$$c^j_{i,h} = \left(Y^j_{i,h}\right)^{-1}$$

Fix $\tilde{h}$ that $x_{\tilde{h}}$ is large enough. Calculate

$$\text{Loss}_s = \frac{1}{2\left(H - \tilde{h}\right)} \sum_{h=\tilde{h}+1}^{H} \sum_{j=0}^{1} \left(\frac{c^j_{i,h} - c^j_{i,h-1}}{x_h - x_{h-1}} - \rho\right)^2$$

and

$$\text{Loss}_H = \sum_{j=0}^{1} \left(\max\left\{c^j_{i,H} - 1.2\rho\left(x_H + \frac{w_i \bar{l}}{\rho}\right), 0\right\}\right)^2$$

Loss_s captures the linearity of the policy function when a household's wealth level is so high that the borrowing constraint has negligible effects. And, Loss_H ensures that the policy functions are bounded from above.

5. Evaluate $\bigtriangledown_x Y(x, G)$ for both x_n, $\hat{x}_n$ and x_h,

- for $n = 0$ or $h = 0$, use forward difference

$$\bigtriangledown_x Y_{i,0}^j = \frac{Y_{1,t}^i - Y_{0,t}^i}{x_1 - x_0}$$

- for $1 \leq n \leq N-1$ or $1 \leq h \leq H-1$, use 3 points difference

$$\begin{aligned}\bigtriangledown_x Y_{i,n}^j =& Y_{i,n-1}^j \frac{x_n - x_{n+1}}{(x_{n-1} - x_n)(x_{n-1} - x_{n+1})} + Y_{i,n}^j \frac{2x_n - x_{n-1} - x_{n+1}}{(x_n - x_{n-1})(x_n - x_{n+1})} \\ &+ Y_{i,n+1}^j \frac{x_n - x_{n-1}}{(x_{n+1} - x_{n-1})(x_{n+1} - x_n)}\end{aligned}$$

- for $n = N$ or $h = H$, use backward difference

$$\bigtriangledown_x Y_{N,0}^j = \frac{Y_{N,t}^i - Y_{N-1,t}^i}{x_N - x_{N-1}}$$

For midpoints $\hat{x}_n$, the corresponding derivative terms are denoted as $\bigtriangledown_x Y_{i,\hat{n}}^j$.

6. Use the bisection method to find the risk-free rate r_i, whose upper bound is $\bar{r}$ and lower bound $\underline{r}$. Since $\bigtriangledown_x Y_{i,\hat{n}}^j$ is supposed to be negative and $Y_{i,\hat{n}}^j$ is supposed to be positive, each household's capital holding should be monotonically decreasing in the risk-free rate r_i. Iterate 25 rounds of the following bisection
   - set $r_i = \frac{\bar{r}+\underline{r}}{2}$ at the start of each round
   - calculate

$$\hat{k}_{i,n}^j = \min\left\{\max\left\{-\frac{1}{\sigma^2 \bigtriangledown_x \hat{Y}_{i,n}^j}\left(\hat{Y}_{n,t}^i\left(r_i^K - \delta - r_i\right) + \sigma \tilde{Z}_{i,\hat{n}}^{Y,j}\right), \underline{k}\right\}, \hat{x}_n - \underline{b}\right\}$$

   - calculate aggregate capital demand $K_{\text{demand}} = k_{i,0}^0 G_i^{0,0} + \sum_{n=1}^N \hat{k}_{i,n}^0 G_i^{n,0} + \sum_{n=1}^N \hat{k}_{i,n}^1 G_i^{n,1}$
   - if $K_{\text{demand}} \geq K_i$, the risk free rate is too low, set $\underline{r} = r_i$
   - if $K_{\text{demand}} < K_i$, the risk free rate is too high, set $\bar{r} = r_i$
7. Given r_i, calculate

$$k_{i,h}^j = \min\left\{\max\left\{-\frac{1}{\sigma^2 \bigtriangledown_x Y_{i,h}^j}\left(Y_{i,h}^j\left(r_i^K - \delta - r_i\right) + \sigma \tilde{Z}_{i,h}^{Y,j}\right), \underline{k}\right\}, x_h - \underline{b}\right\}$$

8. Calculate the drift of households' wealth $\mu_{i,h}^{w,j}$

$$\mu_{i,h}^{w,j} = r_i x_h + \left(r_i^K - \delta - r_i\right) k_{i,h}^j + j w_i + (1-j) g - c_{i,h}^j$$

9. Update $\mathbf{G}_{i+1}$ according to Section A, and update individual wealth level

$$x_h^j = x_h + \left(1 - I_{i,h}^j\right)\left(\mu_{i,h}^{w,j}\Delta + \sigma k_{i,h}^j w_i\right)$$

10. Calculate $Z_{i,h}^{Y,j}$

$$Z_{i,h}^{Y,j} = \triangledown_x Y_{i,h}^j \sigma k_{i,h}^j + \tilde{Z}_{h,i}^{Y,j}$$

Update the BSDEs at point x_h

$$\begin{aligned}
Y_{i+1,h}^j &= Y_{i,h}^j + \left(1 - I_{i,h}^j\right)\left(-\left(r_i - \rho\right) Y_{i,h}^j \Delta + Z_{i,h}^{Y,j} w_i\right) + I_{i,h}^j U_{i,h}^{Y,j} \\
V_{i+1,h}^j &= V_{i,h}^j + \left(1 - I_{i,h}^j\right)\left(-\left(\ln\left(c_{i,h}^j\right) - \rho V_{i,h}^j\right)\Delta + Z_{i,h}^{V,j} w_i\right) + I_{i,h}^j U_{i,h}^{Y,j}
\end{aligned}$$

Taking $\left(x_h^j, \mathbf{G}_{i+1}\right)$ as the inputs of network, generate $\tilde{Y}_{i+1,h}^j, \tilde{V}_{i+1,h}^i$ according to

$$\begin{aligned}
\tilde{Y}_{i+1,h}^0 &= \left(1 - I_{i,h}^0\right) Y^0\left(x_h^0, \mathbf{G}_{i+1}\right) + I_{i,h}^0 Y^1\left(x_h^0, \mathbf{G}_{i+1}\right) \\
\tilde{Y}_{i+1,h}^1 &= \left(1 - I_{i,h}^1\right) Y^1\left(x_h^1, \mathbf{G}_{i+1}\right) + I_{i,h}^1 Y^0\left(x_h^1, \mathbf{G}_{i+1}\right)
\end{aligned}$$

$$\begin{aligned}
\tilde{V}_{i+1,h}^0 &= \left(1 - I_{i,h}^0\right) V^0\left(x_h^0, \mathbf{G}_{i+1}\right) + I_{i,h}^0 V^1\left(x_h^0, \mathbf{G}_{i+1}\right) \\
\tilde{V}_{i+1,h}^1 &= \left(1 - I_{i,h}^1\right) V^1\left(x_h^1, \mathbf{G}_{i+1}\right) + I_{i,h}^1 V^0\left(x_h^1, \mathbf{G}_{i+1}\right)
\end{aligned}$$

where $Y^j(\cdot,\cdot)$ and $V^j(\cdot,\cdot)$ denote the function approximated by the network.

11. Calculate the loss of period i

$$\begin{aligned}
\text{Loss} =& \text{Loss} + \frac{1}{2(H+1)}\sum_{h=0}^{H}\sum_{j=0}^{1}\left[\left(V_{i+1,h}^j - \tilde{V}_{i+1,h}^j\right)^2 + \omega_Y\left(Y_{i+1,h}^j - \tilde{Y}_{i+1,h}^j\right)^2\right] \\
&+ \omega_0^0\left(g - c_{i,n=0}^0\right)^2 + \omega_0^1\left(\left\{w_t\bar{l} - c_{i,n=0}^1\right\}^-\right)^2 + \omega_{\text{FD}} \cdot \ \text{Loss}_{\text{FD}} \\
&+ \omega_s \cdot \text{Loss}_s + \omega_H \cdot \text{Loss}_H
\end{aligned}$$

Losses $\left(g - c_{i,n=0}^0\right)^2$ and $\left(\left\{w_t\bar{l} - c_{i,n=0}^1\right\}^-\right)^2$ ensures that boundary conditions for households with wealth level $x_t = -\underline{b}$.

C.2 Loss Function of Search-and-Matching Model in Section 5

In this section, I outline the construction of the loss function for a deep learning algorithm. This function is used to find the approximations of $a^j(\rho, G, \mu)$ and their volatility terms

$\sigma^{a,j}(\rho, G, \mu)$, where $j = 0, 1$. These are denoted as $a^j(\rho, G, \mu; \Theta)$ and $\sigma^{a,j}(\rho, G, \mu; \Theta)$, respectively, for $j = 0, 1$.

C.2.1 Initialization

I will simulate the cross-sectional distribution of types and ownerships for 1,000 years, based on the decision rule that an owner (or a non-owner) of type ρ_t would trade with counterparties of types $\rho < \rho_t$ ($\rho > \rho_t$), and save the distribution in the final period for the subsequent construction of the loss function. In particular, I randomly generate the initial cross-sectional distributions, $G^{n,j}_{t=0,m}$, $j = 0, 1$ for 10,000 economies, where each m refers to an economy, $m = 1, \ldots, 10,000$. For each economy, I simulate the Markov chain of $\mu_{t,m}$ over 1,000 years. Given the path of the aggregate shock, I simulate $G^{n,j}_{t,m}$ for 1,000 years and retain $G^{n,j}_{t=1000,m}$ for the computations that follow. Hereafter, I will suppress the subscript m since the construction of the loss function is parallel for each economy.

For a given economy, I consider H agents, who could be either owners or non-owners. Ownership does not affect the law of motion for an agent's type ρ_t. The initial $\rho_{0,h}$ values of the H agents are uniformly distributed between -1 and 1. For agent h, I simulate the path of idiosyncratic shocks to his or her type, denoted as $\left\{B^1_{i,h}, i = 0, \ldots, \mathbf{I} - 1\right\}$, and let $b^1_{i,h} = B^1_{i+1,h} - B^1_{i,h}$. The aggregate state of an economy is set to be constant over $\mathbf{I}$ periods. This setting does not eliminate aggregate risk because its impact on the continuation value is taken into account by the analytic formulation, the details of which are deferred to Subsection C.2.3.

C.2.2 Match and Bargaining

Given that an agent of type $\rho_{i,h}$ meets a counterparty in period i, the probability that the counterparty's type falls within the interval $[\rho^n, \rho^{n-1})$ and they own (do not own) an asset is $G^{n,1}_i - G^{n-1,1}_i$ ($G^{n,0}_i - G^{n-1,0}_i$). I will use the bargaining outcome with the midpoint agent $\hat{\rho}^n = 0.5\left(\rho^n + \rho^{n-1}\right)$ of the interval as the average outcome over all counterparties coming from the interval. Suppose the agent in question is a non-owner and $\rho_{i,h} < \hat{\rho}^n$; then, the transaction price, denoted as $p^{0,n}_{i,h}$, is solved by the first-order condition

$$0 = q\left(1 - e^{-r\gamma\left(p^{0,n}_{i,h} - \left(a^1_{i,h} - a^0_{i,h}\right)\right)}\right) - (1-q)\left(1 - e^{-r\gamma\left(-p^{0,n}_{i,h} + a^1(\hat{\rho}^n, G_i, \mu) - a^0(\hat{\rho}^n, G_i, \mu)\right)}\right),$$

where $a^1_{i,h}$ and $a^0_{i,h}$ are obtained through the simulation of corresponding BSDEs. I employ 15 rounds of Newton-Raphson iterations to obtain $p^{0,n}_{i,h}$, with the initial value given by $(1-q)\left(a^1_{i,h} - a^0_{i,h}\right) + q\left(a^1(\hat{\rho}^n, G_i, \mu) - a^0(\hat{\rho}^n, G_i, \mu)\right)$. Using a similar procedure, I obtain

$p_{i,h}^{1,n}$ when the agent is an owner and the counterparty's type $\hat{\rho}^n < \rho_{i,h}$.

C.2.3 The Loss of Analytic Formulation

It is more efficient to use the analytic formulation to account for the impact of jump risks on the conditional expectation. Taking the jump from μ_0 to μ_1 as an example, a large number of paths would need to be simulated to yield an accurate average effect of the jump following the probabilistic formulation. With the analytic formulation, a single evaluation at the new state μ_1 is sufficient. This evaluation is then weighted by the arrival rate δ to account for the average impact. Next, I will rewrite the PDEs for $a^j(\rho, G, \mu)$ as displayed by equations (23) and (24) for constructing the loss of the analytic formulation

$$\begin{aligned} 0 = &\mu_t^{a,j} - \frac{1}{2} r\gamma \left(\sigma_t^{a,j}\right)^2 - \frac{1}{r\gamma\Delta}\left(1 - E_t\left[e^{-r\gamma\left(a^j(\rho_{t+\Delta}, G_{t+\Delta}, \mu_k) - a^j(\rho_t, G_t, \mu_k)\right)}\right]\right) \\ &- \frac{\delta}{r\gamma}\left(1 - e^{-r\gamma\left(a^j(\rho_t, G_t, \mu_{1-k}) - a^j(\rho_t, G_t, \mu_k)\right)}\right), j = 0, 1; k = 0, 1, \end{aligned}$$

where

$$\begin{aligned} &\frac{1}{\Delta}\left(1 - E_t\left[e^{-r\gamma\left(a^j(\rho_{t+\Delta}, G_{t+\Delta}, \mu_k) - a^j(\rho_t, G_t, \mu_k)\right)}\right]\right) \\ = &-\phi(\rho_t - \mu_k)\left(r\gamma\frac{\partial a^0}{\partial \rho}(\rho_t, G_t, \mu_k)\right) + \frac{1}{2}\sigma^2\left(r\gamma\frac{\partial^2 a^0}{\partial \rho^2}(\rho_t, G_t, \mu_k) - (r\gamma)^2\left(\frac{\partial a^0}{\partial \rho}(\rho_t, G_t, \mu_k)\right)^2\right) \\ &+ r\gamma\left\langle \nabla_g a^0(\rho_t, G_t, \mu_k), \mu_t^g(\mu_k)\right\rangle + o(\Delta) \end{aligned}$$

and $\mu_t^{a,0}$ and $\mu_t^{a,1}$ are given by equations (26) and (25). To evaluate the conditional expectation, I employ Gauss-Hermite quadrature, i.e.,

$$E_t\left[e^{-r\gamma a^j(\rho_{t+\Delta}, G_{t+\Delta}, \mu_k)}\right] = \frac{1}{2}\left(e^{-r\gamma a^j\left(\rho_{t+\Delta}^u, G_{t+\Delta}, \mu_k\right)} + e^{-r\gamma a^j\left(\rho_{t+\Delta}^l, G_{t+\Delta}, \mu_k\right)}\right),$$

where

$$\begin{aligned} \rho_{t+\Delta}^u &= \rho_t - \phi(\rho_t - \mu_k)\Delta + \sigma\sqrt{\Delta}, \\ \rho_{t+\Delta}^l &= \rho_t - \phi(\rho_t - \mu_k)\Delta - \sigma\sqrt{\Delta}, \\ G_{t+\Delta} &= G_t + \mu_t^g(\mu_k)\Delta. \end{aligned}$$

C.2.4 Computation

For a given economy, the initial aggregate state is specified by $G_{i=0}^{n,j}, n = 1, \ldots, N, j = 0, 1$, and $\mu_k, k = 0, 1$, where μ_k does not change over the following $\mathbf{I}$ periods. Let $\mathbf{G}_i$ denote the vector $\left[G_i^{n,j}, n = 1, \ldots, N, j = 0, 1\right]$. The initial state of agent h is $\rho_{0,h}, h = 1, \ldots, H$, and the initial $a_{0,h}^j, j = 0, 1$ are given by the network $a^j\left(\rho_{0,h}, \mathbf{G}_0, \mu_k; \Theta\right)$.

Given the aggregate and individual initial states, I calculate the loss based on the analytic formulation:

$$\text{Loss} = \frac{1}{2H}\sum_{h=1}^{H}\sum_{j=0,1}\left(\mu_{0,h}^{a,j} - \frac{1}{2}r\gamma\left(\sigma_{0,h}^{a,j}\right)^2 - \frac{1}{r\gamma\Delta}\left(1 - E_0\left[e^{-r\gamma\left(a^j\left(\rho_{1,h},\mathbf{G}_1,\mu_k\right) - a^j\left(\rho_{0,h},\mathbf{G}_0,\mu_k\right)\right)}\right]\right)\right.$$
$$\left. - \frac{\delta}{r\gamma}\left(1 - e^{-r\gamma\left(a^j\left(\rho_{1,h},\mathbf{G}_1,\mu_{1-k}\right) - a^j\left(\rho_{0,h},\mathbf{G}_0,\mu_k\right)\right)}\right)\right)^2,$$

where $\sigma_{0,h}^{a,j}$ are generated by the network $\sigma^{a,j}\left(\rho_{0,h}, \mathbf{G}_0, \mu_k; \Theta\right)$ and $\mu_{0,h}^{a,j}, j = 0, 1$ are calculated based on equations (27) and (28) below. Next, I start the iteration over $\mathbf{I}$ periods. Entering period i, three steps of calculation follow.

1. Calculate $\mu_{i,h}^{a,j}, j = 0, 1$,

$$\mu_{i,h}^{a,0} = \frac{r-\beta}{r\gamma} - \left(\frac{\log(r)}{\gamma} - ra_{i,h}^0 + \mu_\eta\right) + \frac{1}{2}r\gamma\left(\sigma_\eta^2 + \left(\sigma_{i,h}^{a,0}\right)^2\right)$$
$$- \frac{2\lambda}{r\gamma\Theta}\sum_{n=n(h,i)}^{N}\left(1 - e^{-r\gamma\left(-p_{i,h}^{0,n} + a_{i,h}^1 - a_{i,h}^0\right)}\right)\Delta G_{i,n}^1 \tag{27}$$

$$\mu_{i,h}^{a,1} = \frac{r-\beta}{r\gamma} - \left(\frac{\log(r)}{\gamma} - ra_{i,h}^1 + \mu_D + \mu_\eta\right) + \frac{1}{2}r\gamma\left(\sigma_D^2 + \sigma_\eta^2 + 2\rho_{i,h}\sigma_D\sigma_\eta + \left(\sigma_{i,h}^{a,1}\right)^2\right)$$
$$- \frac{2\lambda}{r\gamma(1-\Theta)}\sum_{n=0}^{n(h,0)-1}\left(1 - e^{-r\gamma\left(p_{0,h}^{1,n} + a_{i,h}^0 - a_{i,h}^1\right)}\right)\Delta G_{i,n}^0 \tag{28}$$

$$n(h,i) \equiv \min\left\{n : \hat{\rho}^n > \rho_{i,h}\right\},$$

   where $\sigma_{i,h}^{a,j}, j = 0, 1$, are generated by the network $\sigma^{a,j}\left(\rho_{0,h}, \mathbf{G}_0, \mu_k; \Theta\right)$.

2. Calculate

$$\rho_{i+1,h} = \rho_{i,h} - \phi\left(\rho_{i,h} - \mu_k\right)\Delta + \sigma b_{i,h}^1$$
$$a_{i+1,h}^j = a_{i,h}^j + \mu_{i,h}^{a,j}\Delta + \sigma_{i,h}^{a,j}b_{i,h}^1, j = 0, 1$$

3. Calculate $\mathbf{G}_{i+1}$ according to its law of motion specified in Subsection 5.5, and calculate

the loss of the probabilistic formulation

$$\text{Loss} = \text{Loss} + \frac{1}{2H\mathbf{I}} \sum_{h=1}^{H} \sum_{j=0,1} \left(a_{i+1,h}^{j} - a^{j} \left(\rho_{i+1,h}, \mathbf{G}_{i+1}, \mu_k; \Theta \right) \right)^2 .$$

Then, move to the next period until reaching period **I**.

The calculations above are for a single economy. The ultimate loss function is the average loss across all $10,000$ economies.